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BOOTSTRAP BASED ON THE VENEZIANO MODEL WITH UNITARITY

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ABSTRACT

Using a sum rule derived by Arbab and Slansky, the Veneziano model is combined with unitarity and the over-all scale of the $\pi\pi$ interaction determined in reasonable agreement with experiment.

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Recent studies of finite energy sum rule (FESR) bootstraps together with the assumption of straight line trajectories are leading to new ways of analysing the S matrix theory of strong interactions [1]. Of particular interest has been a proposal by Veneziano [2] for the construction of a simple Regge behaved, crossing symmetric amplitude, which is a solution of the FESR bootstrap. Recent evidence seems to show that this model provides a reasonable parametrization for many strong interaction processes [3], despite having some unsatisfactory features, especially in connection with unitarity. However, the model can only give ratios of coupling constants and hence widths (not over-all magnitudes), due to its homogeneous nature.

An additional constraint is therefore necessary to fix the scale of the interactions. We shall propose here a simple scheme in which unitarity is used to supply such a constraint [4]. As the basic illustration, we consider $\pi\pi$ scattering, but the essential aspects may be generalized to other processes. We start with the Veneziano model for the $\pi\pi$ system [5], which is a solution of the FESR bootstrap, linear in the Veneziano constant λ. We impose unitarity through a sum rule derived by Arbab and Slansky [6]. This gives us a non-linear equation for λ which enables us to fix the scale of the interaction.

Thus this dynamical scheme predicts not only ratios but also the magnitude of resonance widths and scattering lengths.

Let $A^I(\ell,t)$ be the signatured partial wave amplitude with isospin I in the t channel defined for all ℓ by the Froissart-Gribov continuation

$$A^I(\ell,t) = \frac{1}{\pi\nu_t}\int_{4\mu^2}^{\infty} ds\, \tilde{A}_s^I(s,t)\, Q_\ell\left(1+\frac{s}{2\nu_t}\right) \tag{1}$$

Here $\tilde{A}_s^I(s,t)$ is the s channel absorptive part with isospin I in the t channel and $\nu_t = \frac{t}{4} - \mu^2$, and μ is the pion mass. The generalized unitarity relation in the complex ℓ plane is then

$$A^I(\ell,t) - [A^I(\ell^*,t)]^* = 2i\rho(t) A^I(\ell,t)[A^I(\ell^*,t)]^* \tag{2}$$

with the normalization $\rho(t) = (\nu_t/\nu_t+\mu^2)^{\frac{1}{2}}$ in the elastic region. Arbab and Slansky [6] noticed that Eq. (2) implies

$$-\frac{1}{2i\rho(t)} = \lim_{\ell \to \alpha^*} A^I(\ell,t)$$

where $\alpha(t)$ is the Regge trajectory function, and proposed that the real part of this relation would be a useful tool, independent of FESR for calculating high energy parameters.

They thus obtained the sum rule *)

$$\mathrm{Re} \lim_{\ell \to \alpha^*} \int_{4\mu^2}^{\infty} ds\, \tilde{A}_s^I(s,t) Q_\ell\left(1+\frac{s}{2\nu_t}\right) = 0 \tag{3}$$

We shall use this sum rule to fix the value of λ. In the evaluation of the integral in Eq. (3), we divide the integration region into two parts at $s = N$, above which we assume the Regge asymptotic behaviour implied by the $\pi\pi$ Veneziano amplitude, and below which we saturate with resonances whose widths can be calculated from the same Veneziano amplitude, which is :

$$\left.\begin{aligned} A^{I=0} &= \tfrac{3}{2}[V(t,s)+V(t,u)] - \tfrac{1}{2}V(s,u) \\ A^{I=1} &= V(t,s) - V(t,u) \\ A^{I=2} &= V(s,u) \end{aligned}\right\} \tag{4}$$

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*) Since the integral in Eq. (3) is divergent for Re $\alpha \geq \ell$, it should be evaluated for Re $\alpha < \ell$ and continued to $\ell = \alpha^*$.

where

$$V(t,s) = -\lambda \frac{\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(s))}{\Gamma(1-\alpha(t)-\alpha(s))} \tag{5}$$

Here we take

$$\mathrm{Re}\,\alpha(t) = \frac{1}{2} + a(t-\mu^2) \quad \text{with} \quad a = \frac{1}{2(m_\rho^2-\mu^2)} \tag{6}$$

in order to satisfy PCAC [5] and $\mathrm{Re}\,\alpha(m_\rho^2) = 1$. By the Veneziano amplitude given by Eq. (5), we mean the following : when unitarity is imposed, the poles corresponding to resonances will be pushed onto the second Riemann sheet, i.e., $\alpha(t)$ will be modified by acquiring an imaginary part above the elastic threshold. Unfortunately one must solve at least part of the unitarity problem before we can guess how it will be modified [7]. Nevertheless it will be interesting to try to keep Eq. (5) valid for $\mathrm{Im}\,\alpha \neq 0$ as the first approximation.

One immediately notices that Eq. (5) for $\mathrm{Im}\,\alpha \neq 0$ has unwanted ancestors and implies that degenerate resonances have degenerate total widths.

However, in our approach we use Eq. (3) at the ρ meson pole $(I=1,\ t = m_\rho^2)$ which we assume to be elastic [8]. This enables us to express $\mathrm{Im}\,\alpha(m_\rho^2)$ in terms of λ, since

$$\Gamma_\rho^{\mathrm{elastic}} = \frac{\lambda(m_\rho^2-4\mu^2)^{3/2}}{3m_\rho^2} \quad \text{and} \quad \Gamma_\rho^{\mathrm{total}} = \frac{\mathrm{Im}\,\alpha(m_\rho^2)}{a\,m_\rho}$$

and our assumption of elasticity for the ρ implies $\Gamma_\rho^{\mathrm{total}} = \Gamma_\rho^{\mathrm{elastic}}$.

Thus the high energy part of the integral depends only on λ. Also the integration from $4\mu^2$ to N does not depend on the detailed behaviour of $\mathrm{Im}\,\alpha$ but only on the elastic widths of the resonances, which can be expressed in terms of λ. Thus it should be possible to extract information about λ from the sum rule independent of the detailed behaviour of $\mathrm{Im}\,\alpha$ *).

EVALUATION OF HIGH ENERGY PART ($s \geq N$)

For the interval $s \geq N$ we substitute the Regge asymptotic form obtained from Eqs. (4) and (5) and we can then do the integral. This gives

$$\mathrm{Re} \lim_{\ell \to \alpha^*(m_\rho^2)} \int_N^\infty ds\, \tilde{A}_s^1(s,t)\, Q_\ell(1 + s/2\nu_\rho)$$

$$= -\frac{\mathrm{Re}\,\beta(m_\rho^2)}{2\,\mathrm{Im}\,\alpha(m_\rho^2)} \sin[y(m_\rho^2)] \Big[1 + \cot[y(m_\rho^2)] \Big\{ \frac{\mathrm{Im}\,\beta(m_\rho^2)}{\mathrm{Re}\,\beta(m_\rho^2)} + \tag{7}$$

$$+ 2c_2\,\mathrm{Im}\,\alpha \Big\} + O(\mathrm{Im}\,\alpha)^2 + O\Big(\frac{\mathrm{Im}\,\alpha(m_\rho^2)}{z_N}\Big)\Big]$$

Here

$$\mathrm{Re}\,\beta(m_\rho^2) = \frac{4}{3}\nu_\rho a \lambda \tag{8}$$

and
$$y(m_\rho^2) = 2\,\mathrm{Im}\,\alpha(m_\rho^2)\,\ln(2z_N)$$

$$= \frac{16 a\lambda}{3 m_\rho}\nu_\rho^{3/2} \ln(2z_N) \tag{9}$$

with
$$z_N = 1 + N/2\nu_\rho$$
$$c_2 = \frac{2}{2\mathrm{Re}\,\alpha + 1} + \psi(\mathrm{Re}\,\alpha + 1/2) - \psi(\mathrm{Re}\,\alpha + 1)$$

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*) We assume that $\mathrm{Im}\,\alpha(s) \to \infty$ as $s \to \infty$ in order to have Regge asymptotic behaviour for our amplitudes.

It can be shown that we must drop the term $\mathrm{Im}\,\beta(m_\rho^2)/\mathrm{Re}\,\beta(m_\rho^2)$ as we have dropped the non-resonating background[9]. We may also drop the term $2c_2 \mathrm{Im}\,\alpha(m_\rho^2)\cot y(m_\rho^2)$ for realistic $\mathrm{Im}\,\alpha(m_\rho^2)$ [10] provided $\cot y(m_\rho^2)$ is not much larger than one, since $c_2 = 0.28$ for $\mathrm{Re}\,\alpha = 1$. The term $\sin(y(m_\rho^2))$ is non-linear in $\mathrm{Im}\,\alpha(m_\rho^2)$ (and hence λ) because Z_N is large enough for N in the intermediate energy region.

EVALUATION OF LOW ENERGY PART $(s < N)$

One can explicitly calculate elastic widths Γ_R linear in λ for all resonances with $S_R < N$ from the Veneziano formula[11], e.g.,

$$\Gamma_\rho = \frac{\lambda}{3}\frac{(m_\rho^2-4\mu^2)^{3/2}}{m_\rho^2}\,, \quad \Gamma_\varepsilon = \frac{3\lambda}{2}(m_\rho^2-4\mu^2)^{1/2}\,,$$
$$\Gamma_{f_0} = \frac{\lambda(m_f^2-4\mu^2)^{5/2}}{20\,m_f^2(m_\rho^2-\mu^2)}\,, \quad \cdots \tag{10}$$

and so on. [Equation (10) defines our normalization.]

The function $\tilde{A}_s^I(s,t)$ is then

$$\tilde{A}_s^I(s,t) = \sum_{I'} X_{II'} \sum_{\ell'=0} \pi(2\ell'+1)\Gamma_R m_R\,\delta(s-s_R) \times \left(\frac{\nu_R+\mu^2}{\nu_R}\right)^{1/2} P_{\ell'}(1+t/2\nu_R) \tag{11}$$

where

$$X_{II'} = \begin{pmatrix} 1/3 & 1 \\ 1/3 & 1/2 \end{pmatrix}$$

is a 2x2 submatrix of the isospin crossing matrix (there are no isospin two resonances in the model), and the elastic widths Γ_R are given by Eq. (10)

One can safely substitute the zero width approximation for $\tilde{A}_s^I(s,t)$, i.e., Eq. (11) into Eq. (3), and retain sufficient accuracy because $\mathrm{Im}\,\alpha$ is small.

Substituting Eqs. (7)-(11) into Eq. (3), we obtain the sum rule in the form

$$\sin(c\lambda) = \lambda \sum_i \gamma_i \cos(\lambda \delta_i) \tag{12}$$

where $c\lambda = y$ and γ_i and δ_i are calculable. The $\cos(\delta_i\lambda)$ factor comes from the relation

$$\mathrm{Re}\, Q_{\alpha^*}(z) \approx Q_{\mathrm{Re}\,\alpha}(z) \cos[\mathrm{Im}\,\alpha \ln(2z)] \tag{13}$$

for values of z corresponding to the resonances.

Equation (12) is thus a non-linear equation for λ which we solved for various values of the separation point $s = N$ and the results are shown in the Table. The values $N = N_1, N_2, \ldots, N_8$ are defined by the Figure.

We give the values of $\sum_i \gamma_i$ for each value of N. Since for our solutions $0.80 \leq \cos(\delta_i\lambda) \leq 1$. The values of $\sum \gamma_i$ give the reader an idea of how much each set of resonances contributes to the sum rule.

From the Table, it appears that λ is approaching some limit as N increases. We note here that $\lambda \approx 0.7$ corresponds to $\Gamma_\rho \approx 145$ MeV, $\Gamma_{f_0} \approx 125$ MeV for widths and for scattering lengths $a_0 \approx 0.28\,\mu^{-1}$, $a_1 \approx 0.06\,\mu^{-1}$. We chose each N_i at the point half-way between two sets of resonances but, of course there is some freedom to vary it. However, $\Delta N/N$ becomes smaller as N increases. Therefore it is better to take N large. This also improves our approximations to take N large as we assume Regge behaviour for $s > N$.

A more accurate calculation including the term $2c_2 \mathrm{Im}\alpha \cot y$ in Eq. (7) and improving our evaluation of the low energy part of the integral by including $\mathrm{Im}\beta(m_R^2)$ $(R \neq \rho)$, changes our results by less than 10%.

In conclusion, we would like to point out the following :

(i) In our approximate dynamical scheme, the smallness of the widths (which other bootstrap schemes have not reproduced), comes out automatically from a bootstrap condition based on unitarity.

(ii) In the case of linearly rising trajectories, one has to give up the usual N/D method of unitarizing the scattering amplitude, at least in its familiar form. The scheme outlined here is very simple and free from ambiguities like CDD poles. Also the generalization to more complicated processes would be straightforward.

(iii) In this note we have attempted to fix the scale of interactions given the trajectory function $\alpha(t)$. It would be possible to do a nearly complete bootstrap [i.e., to fix both λ and $\alpha(t)$] by imposing sum rules for both $I = 0$ and 1. This more ambitious case will be discussed in a later paper.

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N	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
c	0.67	0.83	0.94	1.03	1.07	1.12	1.16	1.20
$\sum_{i=1}^{n} \gamma_i$	0.54	0.72	0.84	0.92	0.98	1.03	1.07	1.10
λ	2.23	1.38	1.06	0.90	0.81	0.74	0.71	0.69
$y(m_\rho^2)$	1.49	1.15	0.99	0.91	0.87	0.83	0.82	0.82
Γ_ρ (in MeV)	459	283	218	185	167	152	147	142
Z_N	5.5	10	14.5	19	23.4	27.8	32.3	36.7

Solutions of Eq. (12) for various values of N

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8) If we apply the sum rule at another resonance, we must know the inelasticity and the relation between $\mathrm{Im}\,\alpha(m_R^2)$ and λ is changed.

9) Near $\ell = \alpha(t)$, we can write $A^I(\ell,t) = (\beta(t)/\ell - \alpha(t)) + c(t)$. Equation (2) applied at $\ell = \alpha(t)$ gives us the relation $\mathrm{Im}\,\beta(t) = 2\rho(t)\,\mathrm{Re}\,c(t)\,\mathrm{Re}\,\beta(t)$ and we now find that the $\mathrm{Im}\,\beta$ and $\mathrm{Re}\,c$ terms cancel in the relation

$$\lim_{\ell \to \alpha^*} \mathrm{Re}\,A^I(\ell,t) = 0$$

[Eq. (3)]. Therefore to be consistent we must drop $\mathrm{Im}\,\beta(m_\rho^2)$ as well as $\mathrm{Re}\,c$.

10) Note that inserting experimental values into $\mathrm{Im}\,\alpha(m_\rho^2) = a\Gamma_\rho m_\rho$ gives $\mathrm{Im}\,\alpha(m_\rho^2) \approx 0.09$.

11) There is a small problem here since the width of the ε' resonance (see the Figure) turns out to be small and negative. Therefore we ignore its contribution.

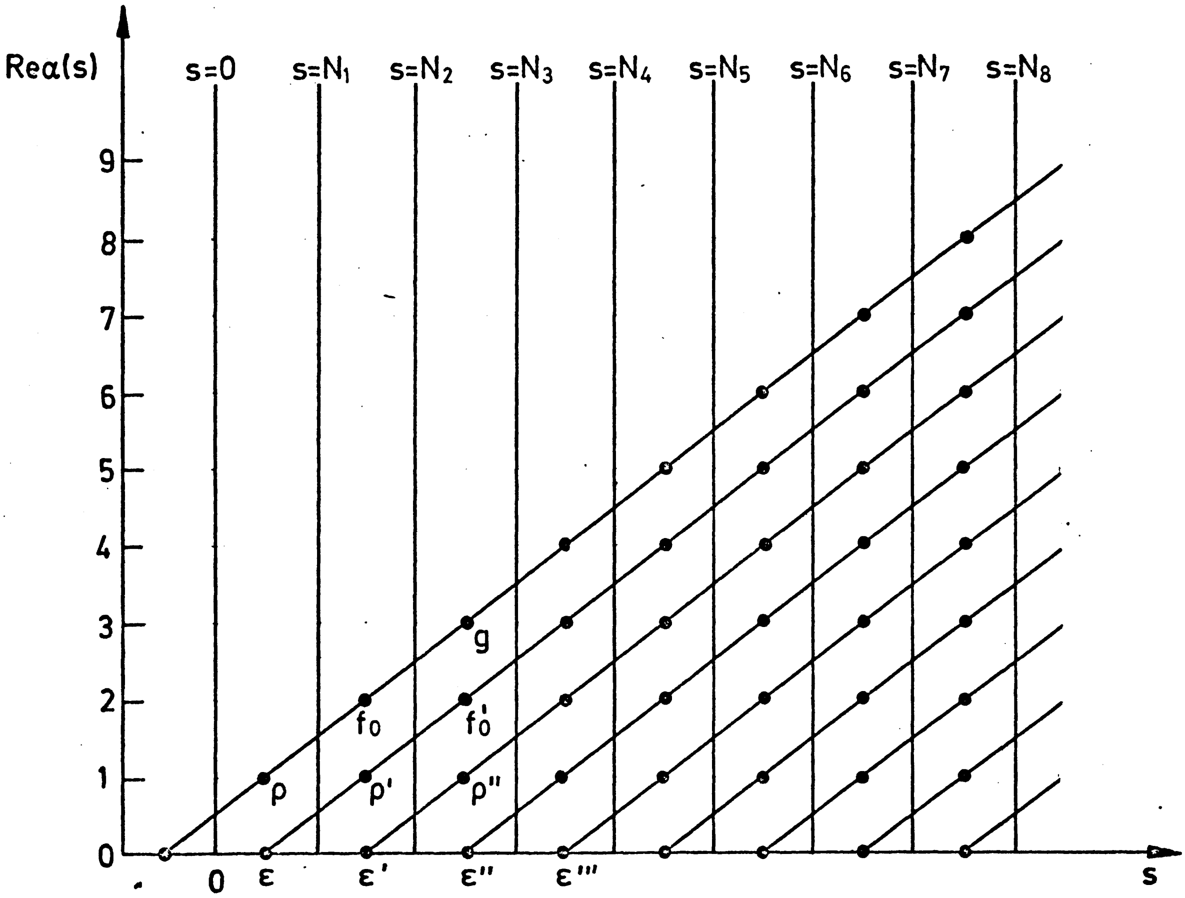

FIG. 1 Resonances used to saturate the sum rule (marked •) and definition of N_1, N_2 .....N_8