Group Representations with an Arbitrary Invariant Metric

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ABSTRACT

Group representations are discussed from the point of view of an arbitrary invariant form $(\psi, \Gamma\psi)$ in the representation space. The relations between the metric tensor Γ and the corresponding reducible or irreducible representations D and the criteria for their existence is investigated. Finally the invariant metric used in quantumelectrodynamics and in certain wave equations is critically examined in terms of group representations.

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I. Introduction

There are many examples in physics where a general and in particular an indefinite form (rather than the usual positive definite form) is invariant under a group of transformations. The purpose of this paper is to treat these problems in a unified manner from the point of view of group representations. First we illustrate the problem in terms of simple examples:

(1) Consider the group of the canonical transformations of the classical particle mechanics. Let $q_k = x_{2k}$, $p_k = x_{2k-1}$, $k = 1, 2, \ldots n$, then the canonical equations can be written in the form

$$\dot{x}_k = \Gamma_{kj} \frac{\partial H}{\partial x_j} , \tag{1}$$

where

$$\Gamma = \begin{pmatrix} \omega & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Since the equations (1) and (2) are valid in any canonical coordinates, the matrix Γ is invariant under canonical transformations. If x represents a point of the 2n-dimensional phase space, the "norm" $(x, \Gamma x)$ is invariant and always equal to zero. Furthermore, for the subgroup of linear (hence unimodular) canonical transformations C the bilinear form $(x, \Gamma y)$, which always transforms into $(x', \Gamma y')$, is itself invariant:

$$(x', \Gamma y') = (x, C^+ \Gamma C y) = (x, \Gamma y),$$

where C^+ denotes the usual hermitian conjugate of C. The last step follows from equations (1) and (2), i.e. $C^+ \Gamma C = \Gamma$. The metric Γ

is unitary but not hermitian. The phase space is the representation space of the group of canonical transformations. They contain as a subgroup the 3-dimensional rotation group. Thus, we have here, for example, representations of the rotation group which leave the metric Γ invariant (reducible representations, as it will turn out).

(2) As a second example consider the Dirac equation. The bilinear scalar form $\bar{\psi}\psi = (\psi, \gamma^0 \psi)$ is invariant under homogeneous Lorentz transformations. ψ transforms according to $\psi'(Lx) = S(L)\psi(x)$, where S(L) are nonunitary 4x4 representations of the Lorentz group corresponding to the metric γ^0. Hence $S^+ \gamma^0 S = \gamma^0$.

(3) There are other examples, where an indefinite metric plays a more fundamental role in that the invariant bilinear form has a direct physical meaning. We mention first certain theories of spin 0 [1] and spin one-half [2] particles where the indefinite invariant norm in Hilbert space has the meaning of charge density. This example will be treated more fully at the end.

(4) An indefinite metric has also been introduced in Hilbert space of the usual quantum theory despite the fact that the norm here has the physical meaning of probability and hence must be positive definite. With this interpretation of the norm the theory must be such that no negative probabilities occur for the final observable quantities. To this group belong notably the use of the indefinite metric in connection with the elimination of some of the divergence difficulties in field theories [3], with the quantization of the electro-

1) S. Sakata, M. Taketani, Proc. Phys. Math. Soc. Japan 22, 757 (1940)
H. Feshbach, F. Villars, Rev. Mod. Phys. 30, 24 (1958)

2) A.O. Barut, Ann. of Physics, 5, 95 (1958)

3) P.A.M. Dirac, Proc. Roy. Soc. London A 180, 1 (1942); W. Pauli, Rev. Mod. Phys. 15, 175 (1943)

magnetic field [4], with some certain field theoretical models [5], with the non-linear spinor equations [6] and others [7]. We shall come back to some of these problems at the end.

The question, therefore, arises under what conditions one can use representations of the symmetry groups which leave an arbitrary metric in the representation space invariant [8]. Recently this problem has also been investigated by Shirkov [9] and in particular by Schlieder [10] for the important case of the Lorentz group. In the following we offer a different and general point of view valid for any symmetry group. In Section II the general theory is discussed and in Section III the results are applied to several examples.

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4) S.N. Gupta, Proc. Roy. Soc. London 63, 681 (1950); K. Bleuler, Helv. Phys. Acta, 23, 567 (1950); K. Bleuler, W. Heitler, Prog. Theor. Phys. 5, 600 (1950)

5) G. Källen, W. Pauli, Mat.Fy.Medd.Dan.Vid.Selsk. 30, Nr. 7 (1955) M. Markov, Nuclear Physics, 10, 140 (1959); 12, 190 (1959)

6) W. Heisenberg, Nucl. Phys. 4, 532 (1957); H.P. Dürr, W. Heisenberg H. Mitter, S. Schlieder, K. Yamazaki, Z. Naturf. 14a, 441 (1959)

7) There are many other papers on the indefinite metric. Not attempt will be made to give an exhaustive list of these papers.

8) A preliminary report on this problem was presented earlier: A.O. Barut, Bull. Amer. Phys. Soc. 4, 30 (1959)

9) In. M. Shirkov, Soviet Physics JETP, 6, 684 (1958)

10) S. Schlieder, Z. Naturf. 15a, 448 (1960)

II. General Theory

1. Formulation of the problem

The problem to be investigated is the following: Given a group $\mathcal{G}$ with elements a, b, ..., in particular, a group of transformations leaving a quadratic form $(x, G x)$ invariant, we are looking for representations of $\mathcal{G}$ such that in the representation space a quadratic form (norm) $(\psi, \Gamma \psi)$ is invariant. Γ is arbitrary and in particular not positive definite. The problem has a priori two aspects: (1) given an arbitrary representation D what is the invariant metric Γ, if it exists ?, (2) given a metric Γ, are there representations D_Γ which leave Γ invariant ?

We start from a linear space with the usual positive definite metric (ψ, ψ) and then introduce the arbitrary quadratic form $(\psi, \Gamma \psi)$. Hence all the usual rules of the linear algebra are valid. A^*, A^+ and A^T will denote the conjugate complex, the hermitian conjugate and the transpose of the linear operator A, respectively. We shall refer to Γ simply as metric (or fundamental tensor [10]). The representation space of the vectors ψ is a finite or infinite dimensional Hilbert space. The norm $(\psi, \Gamma \psi)$ does not necessarily have the meaning of probability as the examples mentioned in the Introduction show. Clearly, the use of arbitrary metric in quantum theory depends on the existence and properties of such representations of the underlying symmetry groups. The knowledge of Γ is equivalent to constructing the invariants of the theory.

2. General Properties of Representations

We assume first the metric Γ to be non-singular. An operation of conjugation (adjointness), denoted by $\hat{}$, can be defined in two ways: Either by $(A \psi, \Gamma \psi) = (\psi, \Gamma \hat{A} \psi)$, or by

$(\hat{A}\psi, \Gamma\psi) = (\psi, \Gamma A\psi)$ which we may call right and left conjugation, respectively. It is easy to see that

(a) $\hat{A}_{right} = \hat{A}_{left}$ if and only if $\Gamma = \Gamma^{+}$,

(b) for both right and left conjugation, $\hat{\hat{A}} = A$ if and only if $\Gamma = \Gamma^{+}$.

In general, however, Γ is not hermitian. In this case we shall choose one of the conjugations and remember (b). We take

$$\hat{A} \equiv \Gamma^{-1} A^{+} \Gamma \quad . \tag{3}$$

The self-adjoint operator satisfies $\hat{A} = A$ and has real expectation values. Further algebraic properties (eigenvalues, eigenvectors, orthogonality, etc.) are well-known and will not be discussed[11].

A representation D which leaves $(\psi, \Gamma\psi)$ invariant satisfies the relation

$$D^{+} \Gamma D = \Gamma \; , \text{ or } \hat{D} = D^{-1}, \tag{4}$$

and will be called a norm-preserving (or pseudounitary) operator.

The concept of equivalence of two representations remains unchanged. However, instead of unitary equivalence we have to define in the sense of preserving the norm $(\psi, \Gamma\psi)$: Two representations D_1, D_2 are equivalent in this physical sense if they are connected by a norm preserving operator V:

$$D_1 = V^{-1} D_2 V, \text{ where } V^{-1} = \hat{V} = \Gamma^{-1} V^{+} \Gamma \tag{5}$$

Then we see that (a) if D_1 satisfies Eq. (4) , so does D_2, i.e. pseudo-unitarily equivalent representations have the same metric Γ associated with them, (b) conversely, if two representations leave the same metric Γ invariant, then they are connected by a norm preserving operator V, (c) D_1 and D_2 have the same expectation values.

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11) See, for example, Appendix of reference 2, or L.K. Pandit, Nuov. Cim. Suppl. 11, 157 (1959).

Given a representation D, then D^*, $D^{+^{-1}}$ and D^{-1^T} are also representations as can be seen by taking the complex conjugate or hermitian conjugate, etc. of the equation $D(a)D(b) = D(ab)$. It follows from Eq. (4) that for non-singular metrics which we are considering, D and $D^{+^{-1}}$ are equivalent:

$$D = \Gamma^{-1} D^{+^{-1}} \Gamma \qquad (6)$$

Furthermore, D^* and $D^{T^{-1}}$ are equivalent: $D^* = \Gamma^{T^{-1}} D^{T^{-1}} \Gamma^T$. This follows from the transpose of Eq. (4). Or,

$$D = (\Gamma^{T^{-1}} D^{T^{-1}} \Gamma^T)^* = \Gamma^{-1+} D^{+^{-1}} \Gamma^+$$

Comparing this with Eq. (6) we find $\Gamma \Gamma^{+^{-1}} D^{+^{-1}} = D^{+^{-1}} \Gamma \Gamma^{+^{-1}}$. Then by Schur's lemma we arrive at the following result: If $D^{+^{-1}}$ or D is an irreducible representation, then Γ must be (up to factor) hermitian. The converse of this theorem, i.e. if D and $D^{+^{-1}}$ are equivalent and D is irreducible then there exists a hermitian metric has been proved by Schlieder [10]. Thus, for irreducible representation there is a unique operation of conjugation and $\hat{D} = D$.

Actually, D and $D^{+^{-1}}$ are not pseudounitarily equivalent, unless $\Gamma = \Gamma^{-1}$ which in the case of irreducible representations, i.e. $\Gamma = \Gamma^+$, implies $\Gamma^2 = I$ and hence, in diagonal form, Γ contains only ± 1. In the examples (2) and (3) of the Introduction Γ has this property, but the linear canonical transformations preserve a non-hermitian metric hence they are reducible representations.

3. Infinitesimal Operators

Let $\mathcal{G}$ be a Lie group. If we write the representation in the form

$$D(a) = e^{i a_i K_i}$$

where a_i are the real parameters of the group element a, then Eq. (4) gives, by comparing the coefficients of the parameters,

$$K_i = \Gamma^{-1} K_i^{+} \Gamma = \hat{K}_i \tag{7}$$

i.e. the infinitesimal operators are self-adjoint. They will have real eigenvalues, if Γ is hermitian (irreducible representations) and if the corresponding eigenvectors have non-zero norms [11].

Consider a coordinate system in which a commuting subset, $\{K_i\}_s$, of the infinitesimal operators is in diagonal form. For irreducible representations their diagonal elements will therefore be real. In such a basis $\{K_i\}_s^+ = \{K_i\}_s$ and hence from Eq. (7) we see that Γ commutes with all the commuting diagonal operators $\{K\}_s$, and can be made itself diagonal up to a factor. If this factor is so chosen that Γ has diagonal elements ± 1, then D and D^{+-1} are pseudounitarily equivalent. These general statements has been discussed explicitely for the inhomogeneous Lorentz group in reference 10.

4. Existence of Nonunitary Representations

It is well-known that if a group is finite, or more generally compact (i.e. if the finite number of parameters which describe the group elements run over a closed set in the parameter space), then any representation (reducible or irreducible) by matrices with non-vanishing determinants can be transformed into a unitary representation by a similarity transformation [12]. (Schur-Auerbach Theorem). The unitary representation corresponding to a given representation D is explicitly given by,

$$D_I = d^{-\frac{1}{2}} U^{-1} D_{\Gamma} U d^{\frac{1}{2}} \tag{8}$$

12) See, for example, E.P. Wigner, Group Theory, Academic Press, New York 1959, p. 74 and p. 101

where U is the unitary matrix which diagonalizes the hermitian expression $Z = \sum_a D_\Gamma(a)\ D^+_\Gamma(a)$, where the summation (or the Hurwitz integral in the case of continuous compact groups) is over all group elements; d is the positive definite diagonal form of Z and $d^{\frac{1}{2}}$ the positive square root of d.

We must first see what types of metrics Γ can be transformed into $\Gamma = I$. In the previous section we have shown that pseudounitarily equivalent representations have the metric Γ. Consider now two arbitrarily equivalent representations $D_1 = S^{-1}D_2S$ (the infinitesimal operators being also related by $K_1 = S^{-1}K_2S$), where S is arbitrary. Then we can prove that

(a) If D_2 has the metric Γ_2 in the sense of equation (4), then D_1 has the metric $\Gamma_1 = S^+\Gamma_2S$. Indeed $(\emptyset, \Gamma_1\emptyset)$ is invariant under D_1 if $(\psi, \Gamma_2\psi)$ is invariant under D_2, where $\emptyset = S^{-1}\psi$;

(b) the positive definite character and hermiticity of the metric is preserved under the equivalence i.e. if Γ_2 is of the form $\Gamma_2 = M^+M$, so is $\Gamma_1 = (MS)^+(MS)$. Note that although D_1 and D_2 have the same eigenvalues Γ_1, Γ_2 have not. Nevertheless, we can classify and compare all normal Γ's by bringing them into diagonal form.

(c) all metrics which can be put into the form $\Gamma_1 = I$ by a similarity transformation are of the form $\Gamma_2 = M^+M$, i.e. positive definite. Hence an indefinite metric cannot occur among the irreducible representations of the compact groups. We shall show that it can exist for reducible representations. $\Gamma = I$ implies by Eq. (6) that $D = D^{+^{-1}}$.

(d) the above Schur-Auerbach theorem can be generalized to any positive definite metric: Given any representation D_{Γ_2} we can obtain a representation D_{Γ_1} by

$$D_{\Gamma_1} = d^{-\frac{1}{2}}\ V^{-1}\ D_{\Gamma_2}\ V\ d^{\frac{1}{2}} \qquad (9)$$

where V is the pseudounitary matrix, $\hat{V} = V^{-1}$, which diagonalizes the self adjoint expression $Y = \sum_a D_{\Gamma_2}(a)\, \hat{D}_{\Gamma_2}(a)$ and d is again the diagonalized form of Y. Of course, if Γ_2 is known to start with it may be easier to determine S from $\Gamma_1 = S^+ \Gamma_2 S$, then $D_{\Gamma_1} = S^{-1} D_{\Gamma_2} S$.

The situation is different for non-compact (such as Lorentz) groups. Here the Hurwitz integral in Z or Y does not, in general, converge. Therefore, the metric is not restricted to be positive definite. Let us first show that indefinite metrics must occur. Consider the group of transformations, $\mathcal{G}$, leaving (x, Gx) invariant; G non-singular, hermitian and $G^2 = I$. Any group element $a \in \mathcal{G}$ satisfies $a^+ G a = G$. G itself belongs to the group, hence $D(a^+)D(G)D(a) = D(G)$. If now a metric Γ exists, then $D^+(a)\, \Gamma\, D(a) = \Gamma$ and $D^+(G)\, \Gamma\, D(G) = \Gamma$. For irreducible representations satisfying $D(a^+) = D^+(a)$ we find then that $\Gamma = k\, D(G)$ (since the only matrix commuting with all D(a) is a multiple of identity), $\Gamma^+ = \Gamma = \Gamma^{-1}$. Therefore if $G \neq I$ i.e. indefinite, so is Γ, since for faithful representations $D(G) \neq I$. The transformation group $\mathcal{G}$ with the above properties of G is essentially the n-dimensional Lorentz group which has thus no unitary ($\Gamma = I$) finite-dimensional irreducible representations, a result well-known for n = 4.

It follows from an application of Schur's lemma that an irreducible representation cannot admit two different metrics (up to a factor). On the other hand a non-unitary representation may not admit any metric at all. This is the case when D and $D^{+^{-1}}$ are not equivalent. The criterion for this to happen is the following: If the representation $\{ D \}$ contains an irreducible subset on matrices $\{ d \}$ which allows a metric Γ, then D cannot allow any metric, not even Γ itself. For example, the proper homogeneous Lorentzgroup is homomorphic to 2x2 unimodular matrices ($D^{\frac{1}{2}0}$-representation); the latter contains an irreducible subset, namely the unitary unimodular matrices. Hence the non-unitary representations $D^{jj'}$ do not have a definite metric.

5. Reducible Representations

In contrast to irreducible representations the reducible ones may admit, in general, infinitely many metrics or invariant forms. For example, let D, D_1, D_2 and D_3 be unitary irreducible representations, then one can verify from Eq. (4) that

$D = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$ admits arbitrary metrics of the form $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$ admits arbitrary metrics of the form $\Gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$D = \begin{pmatrix} D_1 & & 0 \\ & D_2 & \\ 0 & & D_3 \end{pmatrix}$ admits arbitrary metrics of the form $\Gamma = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

etc. Therefore, the Kroenecker product of representations, $D_1 X D_2$, reduced out to its irreducible parts, will also admit various arbitrary metrics.

If D is non-unitary then the only possible way of introducing a metric is by forming [9,10]

$$D = \begin{pmatrix} D & 0 \\ 0 & D^{+^{-1}} \end{pmatrix} \tag{10}$$

Whether or not D and $D^{+^{-1}}$ are equivalent. The representation (10) admits metrics of the form $\Gamma = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, which if it is hermitian is always of the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ (up to factor). The example (2) of the Introduction belongs precisely to this class: D is the $D^{\frac{1}{2}0}$ representation of the proper homogeneous Lorentz group. Hence D is reducible for this group, but irreducible for the larger improper homogeneous Lorentz group. Dirac wave function transforms according to $S^{-1}DS$ where S is such that $S^{+} \begin{pmatrix} o & b \\ c & o \end{pmatrix} S$ is diagonal.

III. Applications

a) Wave Equations for Charged Particles with an indefinite Metric[1),2)]

In these theories the norm of the Hilbert space state vector is identified with the charge of the single particle state: $(\psi, \Gamma \psi) = \pm 1$. Probabilities and expectation values are calculated in the usual way by $(\psi, \Gamma A \psi)$, irrespective of the norm of the state: The Hamiltonian, for example, has positive and negative eigenvalues, but the indefinite metric gives quite naturally always a positive value for the expectation value of the energy, i.e. the almost fully occupied negative energy states need not be introduced. For the case of charged spin $\frac{1}{2}$ particles the Hamiltonian obtained is

$$H = \frac{p^2}{2m} \gamma^o (1 + \gamma^5) - m \gamma^5, \quad H = \hat{H} = \gamma^o H^+ \gamma^o \tag{11}$$

with the normalization

$$(\psi, \Gamma \psi) = \int \psi^* \gamma^o \psi \, d^3x = \pm 1 \tag{12}$$

Indeed the ∞-dimensional representation in question of the Lorentz group is not unitary but satisfies

$$D^+(K_i) \gamma^o D(K_i) = \gamma^o$$

and is explicitely given in reference 2. Here D is a function of the infinitesimal operators K_i of the inhomogeneous Lorentz group and is also a 4x4-matrix. This representation separates spin and angular momentum so that they are separately constants of the motion.

This example shows that it may be possible to introduce an indefinite metric in a second quantized theory such that due to the existence of superselection rules, the norm of the state vector will be identified with a physical quantity like charge, baryon and lepton number, etc.

Not much is known about the non-unitary infinite dimensional representations of the Lorentz group. The above is an example of such a representation. Another simple example is given by the quantization of scalar fields with anti-commutators and spinor fields by commutators using an indefinite metric.

b) Quantumelectrodynamics

In this section we discuss the indefinite metric in the quantization of the electromagnetic field as there has been some questions about the Lorentz-covariance of the procedure [13]. From the point of view of the present formulation there can be no question about the relativistic invariance since this has been built in from the beginning. All we have to do is to identify the invariant metric and the representation satisfying Eq. (4). However, the metric Γ in quantum electrodynamics is usually introduced by the equations [14]

$$A_\mu \Gamma = \Gamma A^\mu \quad , \text{ or } \quad a_\mu \Gamma = \Gamma a^\mu \qquad (13)$$

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13) S. Sunakawa, Prog. Theor. Phys. Japan, 19, 221 (1958), in his investigation on the covariance of indefinite metric transforms also the metric tensor Γ which of course destroys the norm.
S.N. Gupta, Prog. Theor. Phys. Japan 21, 581 (1959) shows the invariance of the norm, but not of the commutation relations, Eq. (13) .

where $A_\mu(x)$ are the field operators and $a_\mu^{(k)}$ the corresponding annihilation and creation operators. The relations (13) are not covariant. Furthermore, since Γ is fixed, under a Lorentz transformation (13) transforms into $A^1 \Gamma' = \Gamma' A^1$, $A^1 = DAD^{-1}$ hence.

$$DAD^{-1} \Gamma = \Gamma \, DAD^{-1}.$$

This equation is not compatible with (13) . In fact, any relation between operators A, B, which transform under D, and the invariant metric Γ of the form $A \Gamma = \lambda \Gamma B$ does not exist unless trivially $[D, \Gamma] = 0$ which means that D is unitary (eq. 4). Thus, one must give up Eq. (13). Actually, it is only necessary in the quantization to require that the number operators have real eigenvalues and this is accomplished with arbitrary (non-self adjoint) operator a, since $N = E \hat{a} a$ is self-adjoint. Observable quantities or real quantities in the classical limit depend on $\hat{a} a$, or $a + \hat{a}$, and hence are self-adjoint. For example, the so-called "anomalous oscillator" (Pauli [3]) is defined by the self-adjoint Hamiltonian $H = - E a \hat{a}$ and $[a, \hat{a}] = -1$.
The indefinite metric

$$\Gamma = \begin{pmatrix} +1 & & & & & \\ & -1 & & & 0 & \\ & & +1 & & & \\ & & & -1 & & \\ & 0 & & & +1 & \\ & & & & & \ddots \end{pmatrix}$$

gives positive energy values 0, E, 2E, ... with the eigenvectors $|0\rangle$, $\hat{a}|0\rangle$, $\hat{a}^2|0\rangle$,..... . Furthermore, a hermitian field can be expanded in the form

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2k^0} \left(e^{ikx} a(k) + e^{-ikx} \hat{a}(k) \right)$$

where a need not be hermitian, or antihermitian.

The relativistic invariance being clarified the elimination of the longitudinal photons with the subsidiary condition can be carried out in the usual fashion [14].

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10) G. Källen, Encyclopedia of Physics, Vol V, part 1, p. 200. Springer Verlag 1958, N.N. Bogoliubov, D.V. Shirkov, Introduction to the Theory of quantized Fields, p. 132, Interscience publishers, New York 1959.