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LOW ENERGY PION-NUCLEUS OPTICAL POTENTIALS

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A B S T R A C T

We discuss the local and non-local pion-nucleus optical potentials. We find that the local potential becomes non-local when two nucleon correlations are included. The two potentials (including correlations) can be made local through a transformation on the wave function. The new local potentials agree up to quadratic terms when expanded in powers of the density. The influence of finite range correlations and off-shell pion-nucleon form factors are also investigated.

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## 1. INTRODUCTION

The low energy pion-nucleon interaction is dominated by a conspicuous P wave resonance, the (3,3) resonance. Any theory attempting to describe the low energy pion-nucleus interaction must necessarily take this fact into account. The recipe that has been used in the past is that of Kisslinger <sup>1)</sup> where the P wave gives rise to a non-local term in the optical potential. This potential is inserted into the Klein-Gordon equation for the pion which is then solved numerically. This approach has been used by Sternheim <sup>2)</sup> to analyze low energy pion-nucleus scattering experiments. The most successful application, however, has been in the field of  $\pi$  mesic atoms where it forms the basis for the multiple scattering theory of Ericson and Ericson <sup>3)</sup>. For a presentation of the impressive results obtained in this area we refer to the paper by Krell and Ericson <sup>4)</sup>.

A recent precision experiment in the (3,3) resonance region by Binon et al. <sup>5)</sup> has caused a considerable theoretical activity and several models have been used for the analysis of their data. Two important novel observations were made. It was noted by Dedonder <sup>6)</sup> and Fäldt <sup>7)</sup> that a correct treatment of the kinematics leads to a considerable enhancement factor for the non-local part of the optical potential which is indeed supported by the data. In the present paper the corresponding kinematic problem is investigated for the case of mesic atoms. The second observation was independently made by Fäldt <sup>7)</sup>, Lee and McManus <sup>8)</sup> and Wilkin <sup>9)</sup>. They noted that the Glauber multiple scattering theory naturally suggested a local potential which was found to agree quite well with the data. It was also noted that the original derivation of Kisslinger could as well be used to derive the local potential, the two potentials being different for off-shell pion-nucleon scattering only. In view of this remark it was natural to assume that the local potential should be applicable also to mesic atoms. Preliminary calculations confirmed this hypothesis <sup>10)</sup>.

The present paper is a study of the pion-nucleus optical potential. Our basic theoretical tool will be the multiple scattering theory employed by Ericson and Ericson <sup>3)</sup>. We remind you that the success of the microscopic description depends crucially on the smallness of the pion-nucleon scattering lengths in comparison with inter-nucleon distances. We give a derivation of the multiple scattering equations, a system of A coupled integral equations, starting from the Schrödinger equation and discuss the necessary approximations involved. The equations are used in much the same way as the Ericsons. Furthermore, the general multiple scattering

formalism is valid at all energies. An application to pion-nucleus scattering at high energies has already been described in a previous paper <sup>11)</sup>.

The main part of the paper concerns the application to low energy scattering, in particular pionic atoms. We discuss the presently fashionable local and non-local potentials. In the non-local case we recover the results of the Ericsons, in particular the correlation induced Lorentz-Lorenz term. In the local case we find that the potential becomes non-local when correlation effects are taken into account. This result differs from that of Scheck and Wilkin <sup>12)</sup> whose potential remains local. The Schrödinger equations for the local and non-local potentials (including the correlation induced Lorentz-Lorenz terms) look very different. However, both potentials can be made local through a transformation of the wave function (different for the two potentials). When the new local potentials are expanded in powers of the density they agree in linear and quadratic terms. This property was first noticed by Scheck and Wilkin <sup>12)</sup> and assures similar numerical results for pionic atoms. We also discuss finite range corrections and the effect of off-shell pion-nucleon form factors. Here we find that the strength of the Lorentz-Lorenz term is a measure of the ratio (spatial extension of elementary pion-nucleon interaction)/(nucleon-nucleon correlation length). In this more general case the two potentials do not agree to second order in the density.

## 2. KINEMATICS

In the theoretical analysis of pion-nucleus scattering the strong interaction is introduced through a phenomenological optical potential which is derived from the elementary pion-nucleon interaction. It is important to realize that the optical potential describes the interaction in the pion-nucleus c.m. system. Due to the large nuclear mass we must therefore evaluate the elementary pion-nucleon interaction in a pion-nucleon lab. system.

For our considerations it is sufficient to retain the isoscalar part of the pion-nucleon amplitude. At low energies in the pion nucleon c.m. system

$$f_c(\bar{k}'_c, \bar{k}_c) = b_0 + c_0 \bar{k}'_c \cdot \bar{k}_c \quad ) \quad (2.1)$$

where  $b_0$  is the S wave scattering length and  $c_0$  the P wave scattering volume. When transforming to the lab. system

$$f_{\ell}(\bar{k}'_{\ell}, \bar{k}_{\ell}) = J f_c(\bar{k}'_c, \bar{k}_c) \quad , \quad (2.2)$$

an additional angular dependence is introduced through the Jacobian  $J$ . For non-relativistic pions  $J$  has its largest value in the forward direction  $J(\theta_c = 0) = 1 + \mu/m$  and its smallest value in the backward direction  $J(\theta_c = 180^\circ) = 1 - \mu/m$ . Since this variation cannot be taken into account in a conventional Klein-Gordon equation it is customary to use the forward value. At high energies this choice is easily understood since the nuclear form factor strongly favours forward scattering. At threshold no such argument can be applied. This is unfortunate since the variation of  $J$  introduces a 30% uncertainty.

The transformation from c.m. to lab. momenta is most easily achieved by considering the relativistically invariant momentum transfer,

$$-t = (\bar{k} - \bar{k}')^2 = (\bar{k}_{\ell} - \bar{k}'_{\ell})^2 - (\omega_{\ell} - \omega'_{\ell})^2 \simeq (\bar{k}_{\ell} - \bar{k}'_{\ell})^2 \quad . \quad (2.3)$$

The energy loss is completely negligible for low energy pions. As a result we obtain a lab. amplitude

$$f_{\ell}(\bar{k}'_{\ell}, \bar{k}_{\ell}) = J \left\{ b_0 + c_0 (\bar{k}'_c{}^2 - \bar{k}_c{}^2) + c_0 \bar{k}'_c \cdot \bar{k}_c \right\} \quad . \quad (2.4)$$

The middle term  $c_0(k_c^2 - k_{\ell}^2)$  can be retained as an energy dependent term in the optical potential but its importance for pionic atoms is really small. In the (3,3) resonance region, though, it must necessarily be retained. We shall therefore consider the following identical on-shell amplitudes

$$\begin{aligned} f_{\ell}(\bar{k}'_{\ell}, \bar{k}_{\ell}) &= b'_0 + c'_0 \bar{k}'_{\ell} \cdot \bar{k}_{\ell} \\ &= b'_0 + c'_0 k_{\ell}^2 - \frac{1}{2} c'_0 \bar{q}^2 \quad , \end{aligned} \quad (2.5)$$

where  $\bar{q} = \bar{k}'_{\ell} - \bar{k}_{\ell}$  and

$$b'_0 = \left(1 + \frac{\mu}{m}\right) b_0 \quad , \quad (2.6a)$$

$$c'_0 = \left(1 - \frac{\mu}{m}\right) c_0 \quad . \quad (2.6b)$$

This particular choice of the parameters  $b'_0$  and  $c'_0$  is the one favoured by experiment and corresponds to choosing the forward value of  $J$  for  $b'_0$  and the backward value of  $J$  for  $c'_0$ .

The multiple scattering theory that we shall develop is formulated in co-ordinate space. Therefore, we introduce the co-ordinate space scattering amplitude defined as

$$f(\vec{r}', \vec{r}) = \frac{1}{(2\pi)^6} \int d^3 k' d^3 k e^{-i\vec{k}' \cdot \vec{r}'} f_L(\vec{k}', \vec{k}) e^{i\vec{k} \cdot \vec{r}} . \quad (2.7)$$

Clearly, this definition assumes  $f_L(\vec{k}', \vec{k})$  to be known also for off energy shell scattering.

When the off-shell form of the pion-nucleon amplitude is assumed to be given by the  $\vec{k}' \cdot \vec{k}$  forms of the amplitude (2.5) we obtain the standard non-local Kisslinger amplitude

$$f_{NL}(\vec{r}', \vec{r}) = \{ b'_0 + c'_0 \vec{\nabla}_{r'} \cdot \vec{\nabla}_r \} \delta(\vec{r}') \delta(\vec{r}) . \quad (2.8)$$

On the other hand, if we assume the off-shell behaviour to be given by the  $\vec{q}^2$  form of (2.5) we obtain the local interaction which can be written in many different ways

$$\begin{aligned} f_L(\vec{r}' - \vec{r}_i, \vec{r} - \vec{r}_i) &= \{ b'_0 + c'_0 k^2 + \frac{1}{2} c'_0 (\vec{\nabla}_{r'} + \vec{\nabla}_r)^2 \} \delta(\vec{r}' - \vec{r}_i) \delta(\vec{r} - \vec{r}_i) \\ &= \{ b'_0 + c'_0 k^2 + \frac{1}{2} c'_0 \Delta_{r_i} \} \delta(\vec{r}' - \vec{r}_i) \delta(\vec{r} - \vec{r}_i) . \end{aligned} \quad (2.9)$$

The second form turns out to be the most convenient one. In the present paper we shall only consider interactions of the form  $f_{NL}$  and  $f_L$ .

### 3. THE OPTICAL POTENTIAL

We first observe that the pion-nucleon scattering lengths are a good deal smaller than the distance between nucleons in the nucleus. Therefore, it is natural to attempt a microscopic description of the pion-nucleus interaction based on the elementary pion-nucleon interaction. Our aim is to write down a phenomenological equation for the pion wave function  $\Psi(\vec{r})$ . For this purpose we shall use the approach of the Ericsons<sup>3)</sup> and refer to the Appendix for a derivation starting from the Schrödinger equation. The nucleus will be considered as a collection of nucleons with a prescribed density distribution and transitions between different nuclear states will be neglected.

The Schrödinger equation for scattering by a single non-local potential  $V(\bar{r}, \bar{r}')$  is given by

$$(\Delta + k^2) \Psi(\bar{r}) = -2\mu \int d^3 r'' V(\bar{r}, \bar{r}'') \Psi(\bar{r}'') \quad (3.1)$$

It can be rewritten as an integral equation

$$\Psi(\bar{r}) = \varphi_k(\bar{r}) + \frac{2\mu}{4\pi} \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r'' V(\bar{r}', \bar{r}'') \Psi(\bar{r}'') \quad (3.2)$$

where  $\varphi_k$  is the incoming plane wave and where the Green's function  $g(\bar{r}, \bar{r}')$  is given by

$$g(\bar{r}, \bar{r}') = \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \quad (3.3)$$

Introducing the co-ordinate space scattering amplitude  $f(\bar{r}, \bar{r}')$  instead of the potential  $V(\bar{r}, \bar{r}')$  we write the solution as

$$\Psi(\bar{r}) = \varphi_k(\bar{r}) + \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r'' f(\bar{r}', \bar{r}'') \varphi_k(\bar{r}'') \quad (3.4)$$

Considering the limit of large  $\bar{r}$  for the scattered wave we find that  $f(\bar{r}, \bar{r}')$  is related to the momentum space scattering amplitude  $f(\bar{k}, \bar{k}')$  through Eq. (2.7).

In the case of pion-nucleus interaction a hierarchy of equations to determine  $\Psi(\bar{r})$  can be constructed in the following way. The wave function  $\Psi(\bar{r})$  at any point  $\bar{r}$  is the sum of the incoming plane wave  $\varphi_k(\bar{r})$  and the scattering waves from all the nucleons. In analogy with Eq. (3.4)

$$\Psi(\bar{r}) = \varphi_k(\bar{r}) + \int d^3 r_j \rho(\bar{r}_j) \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r'' f(\bar{r}', \bar{r}'') \varphi(\bar{r}; \bar{r}_j) \quad (3.5)$$

where  $\rho(\bar{r}_j)$  is the nuclear density distribution and  $\varphi(\bar{r}; \bar{r}_j)$  the effective exciting wave for nucleon no.  $j$  which is being located at  $\bar{r}_j$ . We remark that  $\varphi(\bar{r}; \bar{r}_j)$  will in general differ from the incoming plane wave  $\varphi_k(\bar{r})$  since the pion could have scattered an arbitrary number of times before it finally hits nucleon no.  $j$  and leaves the nucleus without further scatterings. When the range of the pion-nucleon interaction is shorter than the nucleon-nucleon correlation length then the potentials will not overlap and

$$(\Delta_r + k^2) \varphi(\bar{r}; \bar{r}_j) \Big|_{\bar{r}=\bar{r}_j} = 0, \quad (3.6)$$

i.e., the effective exciting wave is a free wave. Applying the operator  $(\Delta + k^2)$  we obtain the Schrödinger equation for the pion

$$(\Delta + k^2) \Psi(\bar{r}) = -4\pi \int d^3r_j g(\bar{r}_j) \int d^3r'' f(\bar{r}-\bar{r}_j, \bar{r}''-\bar{r}_j) \varphi(\bar{r}; \bar{r}_j). \quad (3.7)$$

This is one of our basic equations.

The full wave function  $\Psi(\bar{r})$  differs from  $\varphi(\bar{r}; \bar{r}_j)$  only through the scattered wavelet from the nucleon no.  $j$ . Assuming that the addition of one scatterer will not appreciably change the interaction we may as a first approximation put  $\varphi(\bar{r}; \bar{r}_j) \approx \Psi(\bar{r})$ . Then for pointlike interactions the Schrödinger equation (3.7) reduces to

$$(\Delta + k^2) \Psi(\bar{r}) = -2\mu V(\bar{r}) \Psi(\bar{r}), \quad (3.8)$$

where the potential  $V(\bar{r})$  depends on the specific form of the pion-nucleon interaction  $f(\bar{r}, \bar{r}')$ . For the non-local interaction (2.8) we obtain the standard Kisslinger potential

$$V_{NL}(\bar{r}) = -\frac{4\pi}{2\mu} \{ b'_0 g(\bar{r}) - c'_0 \bar{\nabla} g(\bar{r}) \bar{\nabla} \} \quad (3.9)$$

and for the local interaction (2.9) a local potential

$$V_L(\bar{r}) = -\frac{4\pi}{2\mu} \{ (b'_0 + c'_0 k^2) g(\bar{r}) + \frac{1}{2} c'_0 \Delta g(\bar{r}) \} \quad (3.10)$$

These equations are often used in a phenomenological way<sup>2)</sup>, i.e., the parameters  $b'_0$  and  $c'_0$  are fitted to the experimental data.

In order to account for correlations we must proceed one step further in the iteration and construct an integral equation for  $\varphi(r; r_j)$ . But  $\varphi(\bar{r}; \bar{r}_j)$  is the sum of the incoming wave and the scattered waves from all the nucleons except  $j$  itself. Thus

$$\varphi(\bar{r}; \bar{r}_j) = \varphi_k(\bar{r}) +$$

$$\int d^3 r_i g(\bar{r}_i) [1 + G(\bar{r}_j, \bar{r}_i)] \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r'' f(\bar{r}' - \bar{r}_i, \bar{r}'' - \bar{r}_i) \varphi(\bar{r}; \bar{r}_j, \bar{r}_i) \quad (3.11)$$

where  $\varphi(\bar{r}; \bar{r}_j, \bar{r}_i)$  is now the effective exciting wave for nucleon no. i knowing that nucleon no. j is being located at  $\bar{r}_j$  and nucleon no. i itself at  $\bar{r}_i$ . Furthermore, since we know that there is a nucleon at  $r_j$  we have restricted the locations of  $\bar{r}_i$  by introducing the correlation function  $G(\bar{r}_j, \bar{r}_i)$  such that

$$g(\bar{r}_j, \bar{r}_i) = g(\bar{r}_j) g(\bar{r}_i) [1 + G(\bar{r}_j, \bar{r}_i)]$$

and

$$G(\bar{r}_j, \bar{r}_i) = \begin{cases} -1 & , |\bar{r}_j - \bar{r}_i| = 0 \\ \approx 0 & , |\bar{r}_j - \bar{r}_i| > 0 \end{cases} \quad (3.12)$$

This general procedure can be continued by constructing an integral equation for  $\varphi(\bar{r}; \bar{r}_i, \bar{r}_j)$  etc., giving rise to a set of A coupled integral equations.

An improved set of approximate equations is now obtained by breaking the iteration at the second step, putting  $\varphi(\bar{r}; \bar{r}_j, \bar{r}_i) \approx \varphi(\bar{r}; \bar{r}_i)$ . This means in particular that we are neglecting higher order correlations in the nuclear density distribution. In this approximation the solution for the pion wave function is obtained by solving Eq. (3.11) for  $\varphi(\bar{r}; \bar{r}_j)$  and substituting the result into Eq. (3.9). A better method is to subtract Eq. (3.5) from Eq. (3.11) thus yielding a direct connection between  $\varphi(\bar{r}; \bar{r}_j)$  and  $\Psi(\bar{r})$ ,

$$\varphi(\bar{r}; \bar{r}_j) = \Psi(\bar{r}) + \int d^3 r_i g(\bar{r}_i) G(\bar{r}_j, \bar{r}_i) \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r'' f(\bar{r}' - \bar{r}_i, \bar{r}'' - \bar{r}_i) \varphi(\bar{r}; \bar{r}_i). \quad (3.13)$$

We conclude that  $\varphi(\bar{r}; \bar{r}_j)$  and  $\Psi(\bar{r})$  only differ through terms linear in the correlation function G.

The basic theory is in our approximation given by Eqs. (3.9) and (3.13). In order to proceed further the interaction  $f(\bar{r}, \bar{r}')$  has to be specified. Furthermore, the actual form of the correlation correction depends crucially on the value of  $k\lambda$ ,  $\lambda$  being the correlation length. In a previous paper <sup>11)</sup> we demonstrated that the above approach very nicely



reproduced known results both at high and low energies. In the present paper we concentrate on pion-nucleus scattering near threshold. The presently fashionable non-local and local potentials will be discussed separately.

#### 4. CORRELATION FUNCTION

In the following we shall calculate corrections to the optical potential arising from nucleon-nucleon correlations. For this purpose it will be convenient to have a simple model for the correlation function. In general it has to satisfy the normalization condition

$$\int d^3 r_2 \rho(\bar{r}_2) G(\bar{r}_1, \bar{r}_2) = -1. \quad (4.1)$$

However, for ease of calculation we shall assume that the correlation function  $G(\bar{r}_1, \bar{r}_2)$  depends only on the relative distance, although it is not necessary to restrict ourselves to this special class, and shall put

$$G(\bar{r}_1, \bar{r}_2) = G(\bar{r}_1 - \bar{r}_2) = -e^{-|\bar{r}_1 - \bar{r}_2|^2 / 2\xi^2}. \quad (4.2)$$

For a general  $G(\bar{r})$  we define the correlation length  $\xi$  through

$$\int d^3 r \frac{1}{r} G(\bar{r}) = -4\pi \xi^2. \quad (4.3)$$

When the correlation length can be considered much smaller than all other lengths in the problem it is sufficient to calculate the correlation effects in the short range limit,  $\xi \rightarrow 0$ , where

$$\bar{\nabla} G(\bar{r}) = \hat{r} \left\{ \frac{r}{\xi^2} e^{-r^2/2\xi^2} \right\} = \hat{r} \delta(r) \quad (4.4)$$

$$\Delta G(\bar{r}) = \left\{ \frac{3}{\xi^2} - \frac{r^2}{\xi^4} \right\} e^{-r^2/2\xi^2} = \frac{2}{r} \delta(r) + \delta'(r) \quad (4.5)$$

These two relations will be of great help in the actual calculations.

The numerical value for the correlation length  $\xi$  will be taken from a comparison with the Fermi gas model where  $G(\bar{r})$  has a simple analytic form. From our definition (4.3), it follows that  $\xi k_F = 3/4$ , where  $k_F$  is the Fermi momentum. With a radius parameter  $r_0 = 1.1$  fm for the nucleus we find a correlation length  $\xi = 0.55$  fm. This value does not account for dynamical correlations but is quite sufficient for our purposes.

5. NON-LOCAL INTERACTION

The non-local interaction has been studied in great detail by the Ericsons<sup>3)</sup>. Our approach is almost identical to theirs and for the non-local interaction we recover their results. The Schrödinger equation in the absence of nucleon-nucleon correlations is given by Eq. (3.9).

In this Section we shall use the pointlike form of the non-local interaction

$$f_{NL}(\vec{r}', \vec{r}) = \{ b_0' + c_0' \bar{\nabla}_{r'} \cdot \bar{\nabla}_r \} \delta(\vec{r}') \delta(\vec{r}) \quad , \quad (5.1)$$

and shall only retain those terms which survive for short range correlations, i.e., in the limit  $\lambda \rightarrow 0$ . In a later section we shall discuss in more detail corrections due to the finite range of both correlation function and interaction. Substituting the interaction  $f_{NL}$  of Eq. (5.1) into the Schrödinger equation (3.7) gives

$$(\Delta + k^2) \Psi(\vec{r}) = -4\pi \{ b_0' g(\vec{r}) \Psi^{(1)}(\vec{r}) - c_0' \bar{\nabla} (g(\vec{r}) \bar{E}^{(1)}(\vec{r})) \} \quad , \quad (5.2)$$

with

$$\Psi^{(1)}(\vec{r}) \equiv \varphi(\vec{r}; \vec{r}) \quad , \quad (5.3)$$

$$\bar{E}^{(1)}(\vec{r}) \equiv \left[ \bar{\nabla}_r \varphi(\vec{r}; \vec{r}_j) \right]_{\vec{r}_j = \vec{r}} \quad . \quad (5.4)$$

Here,  $\Psi^{(1)}$  and  $\bar{E}^{(1)}$  differ from  $\Psi$  and  $\bar{\nabla} \Psi$  only through terms linear in the correlation function. The precise connection, in our approximation, is given by Eq. (3.13) and in the short range correlation limit we obtain

$$\begin{aligned} \Psi^{(1)}(\vec{r}) - \Psi(\vec{r}) &= \int d^3 r_j g(\vec{r}_j) G(\vec{r}, \vec{r}_j) \{ b_0' \Psi^{(1)}(\vec{r}_j) - c_0' \bar{E}^{(1)}(\vec{r}_j) \cdot \bar{\nabla}_r \} g(\vec{r}, \vec{r}_j) \\ &= 0 \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \bar{E}^{(1)}(\vec{r}) - \bar{\nabla} \Psi(\vec{r}) &= \int d^3 r_j g(\vec{r}_j) G(\vec{r}, \vec{r}_j) \{ b_0' \Psi^{(1)}(\vec{r}_j) - c_0' \bar{E}^{(1)}(\vec{r}_j) \cdot \bar{\nabla}_r \} \bar{\nabla}_r g(\vec{r}, \vec{r}_j) \\ &= -\frac{4\pi}{3} c_0' g(\vec{r}) \bar{E}^{(1)}(\vec{r}) \quad . \end{aligned} \quad (5.6)$$

These two relations complete the solution since we can now express the Schrödinger equation in the pion wave function  $\Psi(\bar{r})$  alone.

In the original paper by the Ericsons<sup>3)</sup> the multiple scattering equations are used in a different way. In Eq. (5.2) they replace  $\Psi(\bar{r})$  by  $\Psi^{(1)}(\bar{r})$  and get an equation for  $\Psi^{(1)}(\bar{r})$ . The connection between  $\Psi^{(1)}(\bar{r})$  and  $\bar{E}^{(1)}(\bar{r})$  is obtained directly from Eq. (3.1). For pionic atoms they argue that this equation should reproduce the eigenvalues correctly. The method that is used in the present paper shows that no additional assumption is necessary (cf., the Varenna Lectures of Ericson<sup>3)</sup>). This fact is of great importance in connection with the local potential.

Finally we express the result in the more conventional variables

$$g(\bar{r}) = -4\pi b_0' g(\bar{r}) \quad , \quad (5.7a)$$

$$\alpha(\bar{r}) = -4\pi c_0' g(\bar{r}) \quad , \quad (5.7b)$$

which give

$$(\Delta + k^2)\Psi(\bar{r}) = g(\bar{r})\Psi(\bar{r}) - \bar{\nabla} \left( \frac{\alpha(\bar{r})}{1 - \frac{1}{3}\alpha(\bar{r})} \bar{\nabla} \Psi(\bar{r}) \right) \quad (5.8)$$

Comparing with Eq. (3.9) we conclude that the introduction of short range correlations, renormalizes the P wave part of the interaction but leaves the S wave part unchanged. This phenomenon is identical to the Lorentz-Lorenz effect for light scattering in dense media.

The numerical solution of equations of the type (5.8) is achieved by a transformation that makes the potential local<sup>4)</sup>. With the transformation

$$\Psi(\bar{r}) = (1 + \alpha_1(\bar{r}))^{-1/2} \varphi(\bar{r}) \quad , \quad (5.9)$$

$$\alpha_1(\bar{r}) = \alpha(\bar{r}) / (1 - \frac{1}{3}\alpha(\bar{r})) \quad (5.10)$$

we get

$$(\Delta + k^2) \varphi(\bar{r}) = \tilde{V}_{NL}(\bar{r}) \varphi(\bar{r}) \quad (5.11)$$

and a local potential

$$\tilde{V}_{NL}(\bar{r}) = \frac{1}{1 + \alpha_1(\bar{r})} \left\{ q(\bar{r}) + k^2 \alpha_1(\bar{r}) + \frac{1}{2} \Delta \alpha_1(\bar{r}) - \frac{1}{4} \cdot \frac{1}{1 + \alpha_1(\bar{r})} (\bar{\nabla} \alpha_1(\bar{r}))^2 \right\}. \quad (5.12)$$

We repeat that this result neglects effects due to the finite range of correlation function and interaction. For finite  $\xi$  it is, therefore, only meaningful when  $k \xi \ll 1$ , a condition which is well satisfied for mesic atoms.

## 6. LOCAL INTERACTION

In the absence of correlations the local interaction gives a Schrödinger equation (3.10) with a local potential, hence the name. When correlations are taken into account the potential becomes more complicated. Moreover, non-local terms are generated.

As in the previous case we consider the pointlike form of the interaction. Of the many possible representations, the most convenient one is:

$$f_L(\bar{r}' - \bar{r}_j, \bar{r} - \bar{r}_j) = \left\{ b_0' + c_0' k^2 + \frac{1}{2} c_0' \Delta_{r_j} \right\} \delta(\bar{r}' - \bar{r}_j) \delta(\bar{r} - \bar{r}_j). \quad (6.1)$$

In analogy with the previous case we retain only those terms which survive in the short range correlation limit. The limitations mentioned in connection with the non-local interaction will thus apply also here. Substituting the interaction  $f_L$  of Eq. (6.1) into the Schrödinger equation (3.7) gives

$$\begin{aligned} (\Delta + k^2) \Psi(\bar{r}) &= -4\pi \left\{ (b_0' + c_0' k^2) \varphi(\bar{r}; \bar{r}_j) + \frac{1}{2} c_0' \Delta_{r_j} (\varphi(\bar{r}; \bar{r}_j)) \right\} \Big|_{\bar{r}_j = \bar{r}} \\ &= \left\{ q(\bar{r}) + k^2 \alpha_1(\bar{r}) + \frac{1}{2} \Delta \alpha_1(\bar{r}) \right\} \varphi(\bar{r}; \bar{r}) \\ &\quad + \frac{1}{2} \alpha_1(\bar{r}) (\Delta_{r_j} \varphi(\bar{r}; \bar{r}_j)) \Big|_{\bar{r}_j = \bar{r}} + (\bar{\nabla} \alpha_1(\bar{r})) \cdot (\bar{\nabla}_{r_j} \varphi(\bar{r}; \bar{r}_j)) \Big|_{\bar{r}_j = \bar{r}}. \end{aligned} \quad (6.2)$$

When correlations are neglected  $\varphi(\bar{r}; \bar{r}_j) \simeq \Psi(\bar{r})$  and the last two terms on the right-hand side vanish. In the presence of correlations they give rise to gradient dependent terms. For the further calculation it is again convenient to introduce

$$\Psi^{(0)}(\bar{r}) \equiv \varphi(\bar{r}; \bar{r}) \quad , \quad (6.3)$$

$$\bar{E}^{(0)}(\bar{r}) \equiv [\bar{\nabla}_r \varphi(\bar{r}; \bar{r}_j)]_{\bar{r}_j = \bar{r}} \quad , \quad (6.4)$$

$$D^{(0)}(\bar{r}) \equiv [\Delta_r \varphi(\bar{r}; \bar{r}_j)]_{\bar{r}_j = \bar{r}} = -k^2 \Psi^{(0)}(\bar{r}) \quad , \quad (6.5)$$

where the last equality is valid for non-overlapping potentials. The relations between these quantities and  $\Psi(\bar{r})$  are different from those in the non-local case. By a straightforward application of Eq. (3.13) we obtain

$$\begin{aligned} \Psi^{(0)}(\bar{r}) - \Psi(\bar{r}) &= \int d^3 r' g(\bar{r}, \bar{r}') \{ (b_0' + c_0' k^2) \rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) G(\bar{r}, \bar{r}_j) \\ &\quad + \frac{1}{2} c_0' \Delta_{r_j} (\rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) G(\bar{r}, \bar{r}_j)) \} \Big|_{\bar{r}_j = \bar{r}} \\ &= -\frac{1}{2} \alpha(\bar{r}) \Psi^{(0)}(\bar{r}) \quad , \end{aligned} \quad (6.6)$$

$$\begin{aligned} \bar{\nabla}_{r_j} \varphi(\bar{r}; \bar{r}_j) \Big|_{\bar{r}_j = \bar{r}} &\equiv \bar{\nabla} \Psi^{(0)}(\bar{r}) - \bar{E}^{(0)}(\bar{r}) \\ &= \int d^3 r' g(\bar{r}, \bar{r}') \{ (b_0' + c_0' k^2) \rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) \bar{\nabla}_r G(\bar{r}, \bar{r}_j) \\ &\quad + \frac{1}{2} c_0' \Delta_{r_j} (\rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) \bar{\nabla}_r G(\bar{r}, \bar{r}_j)) \} \Big|_{\bar{r}_j = \bar{r}} \\ &= -\frac{1}{3} \alpha(\bar{r}) \bar{E}^{(0)}(\bar{r}) \quad , \end{aligned} \quad (6.7)$$

$$\begin{aligned} \Delta_{r_j} \varphi(\bar{r}; \bar{r}_j) \Big|_{\bar{r}_j = \bar{r}} &\equiv \Delta \Psi^{(0)}(\bar{r}) + D^{(0)}(\bar{r}) - 2 \bar{\nabla} \cdot \bar{E}^{(0)}(\bar{r}) \\ &= \int d^3 r' g(\bar{r}, \bar{r}') \{ (b_0' + c_0' k^2) \rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) \Delta_r G(\bar{r}, \bar{r}_j) \\ &\quad + \frac{1}{2} c_0' \Delta_{r_j} (\rho(\bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) \Delta_r G(\bar{r}, \bar{r}_j)) \} \Big|_{\bar{r}_j = \bar{r}} \\ &= -\eta(\bar{r}) \Psi^{(0)}(\bar{r}) + \frac{1}{3} \bar{\nabla} (\alpha(\bar{r}) \bar{E}^{(0)}(\bar{r})) \quad . \end{aligned} \quad (6.8)$$

Here we have kept all terms which survive in the short range correlation limit, i.e., also those proportional to  $k^2$ .

The only real difficulty is encountered in Eq. (6.2) where the most singular term has the form

$$\int d^3 r' g(\bar{r}, \bar{r}') \Delta_{r'} (\Delta_r G(\bar{r}, \bar{r}')) = \int d^3 r' [\Delta_{r'} g(\bar{r}')] \cdot [\Delta_{r'} G(\bar{r}')] \quad . \quad (6.9)$$

In view of Eq. (4.5) it would give a delta function contribution. However, it must be interpreted as being zero. This is most easily seen by calculating  $(\Delta_r + k^2)\varphi(\bar{r}; \bar{r}_j)|_{\bar{r}_j = \bar{r}}$  which has to be zero for non-overlapping potentials. Here, exactly the same term is encountered and the limit  $\xi \rightarrow 0$  must be taken in such a way that it vanishes. This can, e.g., be done by introducing a finite hard core in the correlation function  $G(\bar{r})$ , which is then taken to zero at the end of the calculation.

We can now write down a Schrödinger equation in  $\Psi(\bar{r})$  alone. After some re-arrangements we find

$$(\Delta + k^2)\Psi(\bar{r}) = \left\{ g(\bar{r}) \left(1 - \frac{1}{2}\alpha(\bar{r})\right) + k^2\alpha(\bar{r}) + \frac{1}{2}\Delta\alpha(\bar{r}) \right\} \Psi^{(1)}(\bar{r}) - \frac{1}{12} \bar{\nabla}(\alpha^2(\bar{r})) \bar{E}^{(1)}(\bar{r}) + \frac{1}{4} \alpha^2(\bar{r}) \bar{\nabla} \cdot \bar{E}^{(1)}(\bar{r}) \quad (6.10)$$

and

$$\Psi^{(1)}(\bar{r}) = \frac{1}{1 + \frac{1}{2}\alpha(\bar{r})} \Psi(\bar{r}) \quad , \quad (6.11)$$

$$\bar{E}^{(1)}(\bar{r}) = \frac{1}{1 - \frac{1}{3}\alpha(\bar{r})} \bar{\nabla} \Psi^{(1)}(\bar{r}) \quad . \quad (6.12)$$

The presence of non-local terms is rather obvious but it must be admitted that the result looks less appealing than in the non-local case. We also conclude that in contrast to the non-local interaction also the local part of the potential is renormalized since  $\Psi^{(1)}(\bar{r}) \neq \Psi(\bar{r})$ . In this sense the local interaction is more singular than the non-local one. The non-local potential of Eq. (6.10) can also be transformed into an equivalent local potential but the transformation is rather complicated.

## 7. LOCAL OR NON-LOCAL POTENTIAL?

In the paper by Scheck and Wilkin<sup>12)</sup> the short range correlation corrections to the optical potential are obtained by a completely different method. They use a theorem by Bèg and others to deduce contributions quadratic in the density from the linear ones. This theorem concerns the scattering by two non-overlapping separable potentials and asserts that the double scattering term depends in a very particular way on the on-shell values of the elementary scattering amplitudes. This result is then applied to the second order Born approximation of the potentials (3.9) and (3.10),

to deduce the presence of certain counterterms quadratic in the density. Since the two potentials were derived from the same on-shell amplitude they should be identical when quadratic terms are included. Since we are using a completely different approach it is of some interest to check whether their theorem is satisfied by our potentials.

The localized form of the non-local interaction is given by Eq. (5.12) and a trivial expansion yields the second order terms. The non-local potential of Eq. (6.10) has to be localized before a comparison can be done. Keeping only quadratic terms we have

$$(\Delta + k^2) \Psi(\bar{r}) = \left\{ g(\bar{r})(1 - \alpha(\bar{r})) + k^2 \alpha(\bar{r})(1 - \frac{1}{2}\alpha(\bar{r})) + \frac{1}{2}(1 - \alpha(\bar{r})) \Delta \alpha(\bar{r}) \right\} \Psi(\bar{r}) - \frac{1}{12} \bar{\nabla} (\alpha^2(\bar{r}) \bar{\nabla} \Psi(\bar{r})) + \frac{1}{4} \alpha^2(\bar{r}) \Delta \Psi(\bar{r}), \quad (7.1)$$

which, in the quadratic approximation can be written as

$$(\Delta + k^2) \Psi(\bar{r}) = \left\{ g(\bar{r})(1 - \alpha(\bar{r})) + k^2 \alpha(\bar{r})(1 - \frac{3}{4}\alpha(\bar{r})) + \frac{1}{2}(1 - \alpha(\bar{r})) \Delta \alpha(\bar{r}) \right\} \Psi(\bar{r}) - \frac{1}{12} \bar{\nabla} (\alpha^2(\bar{r}) \bar{\nabla} \Psi(\bar{r})) \quad (7.2)$$

This non-local potential is transformed into an equivalent local potential by a transformation similar to that in Eq. (5.8) and still keeping only linear and quadratic terms we have

$$\tilde{V}_L(\bar{r}) = g(\bar{r})(1 - \alpha(\bar{r})) + k^2 \alpha(\bar{r})(1 - \frac{2}{3}\alpha(\bar{r})) + \frac{1}{2}(1 - \alpha(\bar{r})) \Delta \alpha(\bar{r}) + \frac{1}{24} \Delta \alpha^2(\bar{r}). \quad (7.3)$$

The expansion of  $\tilde{V}_{NL}(\bar{r})$  to second order gives exactly the same result. Thus we have shown that our localized potentials satisfy the theorem of Scheck and Wilkin. This is somewhat astonishing since our local potential becomes non-local when correlations are included in contrast to their local potential which remains local. In practical applications cubic terms in the density are not very important and the two potentials therefore give very similar numerical results.

In fact we can prove a more general result. From the theorem of Bèg one expects any two pion-nucleon amplitudes that agree on the mass shell to give the same optical potential to second order in the density. We consider the interaction

$$f_\lambda(\bar{r}', \bar{r}) = \lambda f_L(\bar{r}', \bar{r}) + (1 - \lambda) f_{NL}(\bar{r}', \bar{r}), \quad (7.4)$$

which is still derived from the same on-shell amplitude. In this case

$$\Psi^{(1)}(\vec{r}) = \frac{1}{1 + \lambda \frac{1}{2} \alpha(\vec{r})} \Psi(\vec{r}) \quad , \quad (7.5)$$

$$\bar{E}^{(1)}(\vec{r}) = \frac{1}{1 - \frac{1}{3} \alpha(\vec{r})} \bar{\nabla} \Psi^{(1)}(\vec{r}) \quad , \quad (7.6)$$

and the Schrödinger equation becomes

$$\begin{aligned} (\Delta + k^2) \Psi(\vec{r}) = & \left\{ q(\vec{r}) \left( 1 - \lambda \frac{1}{2} \alpha(\vec{r}) \right) + \lambda k^2 \alpha(\vec{r}) + \frac{1}{2} \lambda \Delta \alpha(\vec{r}) \right\} \Psi^{(1)}(\vec{r}) \\ & - \left\{ 1 - \lambda \left( 1 - \frac{1}{6} \alpha(\vec{r}) \right) \right\} \bar{\nabla} \left( \alpha(\vec{r}) \bar{E}^{(1)}(\vec{r}) \right) + \frac{1}{3} \lambda \alpha^2(\vec{r}) \bar{\nabla} \cdot \bar{E}^{(1)}(\vec{r}) . \end{aligned} \quad (7.7)$$

We observe that due to Eqs. (7.5) and (7.6) this equation has terms which are quadratic in both  $\lambda$  and the density. An independent check is therefore non-trivial. Nevertheless, after making the necessary transformations on the wave function we obtain a localized potential which is identical to that in Eq. (7.3). The theorem is thus satisfied.

We have thus proved that for short range correlations both the local and the non-local potentials, or any linear combination thereof, give the same localized optical potential to second order in the density. The true wave functions, however, may be vastly different for different off-shell behaviours.

## 8. OFF-SHELL FORM FACTORS AND FINITE RANGE CORRECTIONS

The scattering amplitudes  $f_L(\vec{r}', \vec{r})$  and  $f_{NL}(\vec{r}', \vec{r})$  that have been used up to now were obtained under the assumption that the on-shell scattering amplitude could be used also for off-shell scattering. In general additional form factors will be present. As an illustration consider the generalized non-local scattering amplitude

$$f_2(\vec{k}_2', \vec{k}_2) = \left\{ b_0' + c_0' \vec{k}_2' \cdot \vec{k}_2 \right\} \exp \left[ - \frac{\sigma^2}{4} (\vec{k}_2'^2 + \vec{k}_2^2) \right] \quad . \quad (8.1)$$

At threshold there is no need to renormalize the on-shell scattering lengths. In co-ordinate space the scattering amplitude has a finite spatial extension

$$f_{NL}(\vec{r}', \vec{r}) = \left\{ b_0' + c_0' \bar{\nabla}_{r'} \cdot \bar{\nabla}_r \right\} h_\sigma(\vec{r}') h_\sigma(\vec{r}) \quad , \quad (8.2)$$



with

$$h_{\sigma}(\vec{r}) = \frac{1}{(\sigma\sqrt{\pi})^3} e^{-r^2/\sigma^2} \quad (8.3)$$

Unfortunately, we do not know what the realistic value of  $\sigma$  might be. A comparison with the static model suggests  $\sigma \simeq 0.8$  fm in the (3,3) resonance region.

Our multiple scattering theory is valid for these more general interactions as well. When inserting  $f_{NL}(\vec{r}', \vec{r})$  of Eq. (8.2) into the Schrödinger equation (3.7) we obtain

$$(\Delta + k^2)\Psi(\vec{r}) = -4\pi \left\{ b_0' \int d^3r_i g(\vec{r}_i) h_{\sigma}(\vec{r}-\vec{r}_i) \Psi''(\vec{r}_i) - c_0' \bar{\nabla}_r \int d^3r_i g(\vec{r}_i) h_{\sigma}(\vec{r}-\vec{r}_i) \bar{E}''(\vec{r}_i) \right\} \quad (8.4)$$

with

$$\Psi''(\vec{r}) = \int d^3r' h_{\sigma}(\vec{r}-\vec{r}') \varphi(\vec{r}'; \vec{r}) \quad (8.5)$$

$$\bar{E}''(\vec{r}) = \int d^3r' h_{\sigma}(\vec{r}-\vec{r}') \bar{\nabla}_{r'} \varphi(\vec{r}'; \vec{r}) \quad (8.6)$$

For a general  $h_{\sigma}$  one thus encounters an integro-differential equation. Only for  $\sigma \rightarrow 0$ , i.e., for pointlike interactions does it reduce to a differential equation. Since  $\sigma$  might be comparable to the correlation length or the nuclear surface thickness it is important to account for the smearing. In view of the smooth behaviour of the wave function we propose

$$(\Delta + k^2)\Psi(\vec{r}) = \tilde{g}(\vec{r}) \Psi''(\vec{r}) - \bar{\nabla}(\tilde{\alpha}(\vec{r}) \bar{E}''(\vec{r})) \quad (8.7)$$

where smeared quantities are defined as

$$\tilde{\alpha}(\vec{r}) = \int d^3r' h_{\sigma}(\vec{r}-\vec{r}') \alpha(\vec{r}') \quad (8.8)$$

Apart from the smearing there is no change in our new Schrödinger equation.

Next we express  $\Psi^{(1)}(\vec{r})$  and  $\bar{E}^{(1)}(\vec{r})$  in terms of  $\Psi(\vec{r})$ . This is done through Eq. (2.16) which, however, first, must be smeared in accordance with Eqs. (8.5) and (8.6). We obtain

$$\Psi^{(1)}(\bar{r}) - \Psi(\bar{r}) = \int d^3r_i g(\bar{r}, \bar{r}_i) G(\bar{r}, \bar{r}_i) \{ b_0' \Psi^{(1)}(\bar{r}_i) - c_0' \bar{E}^{(1)}(\bar{r}_i) \cdot \bar{\nabla}_r \} g_{rr}(\bar{r}, \bar{r}_i) \quad (8.9)$$

where the twice smeared Green's function is defined as

$$g_{rr}(\bar{r}, \bar{r}') = \int d^3r_1 d^3r_2 h_r(\bar{r}' - \bar{r}_1) h_r(\bar{r} - \bar{r}_2) g(\bar{r}_1, \bar{r}_2) \quad (8.10)$$

Following our previous approximation scheme the smearing in  $\Psi(\bar{r})$  has been neglected. We only consider the long wave length limit and expand in powers of the correlation length. In leading order we obtain with our Gaussian correlation function (4.2)

$$\Psi^{(1)}(\bar{r}) - \Psi(\bar{r}) = \eta \xi^2 \left\{ \bar{q}(\bar{r}) \Psi^{(1)}(\bar{r}) - \frac{1}{3} \eta^2 \bar{\nabla} \bar{\alpha}(\bar{r}) \cdot \bar{E}^{(1)}(\bar{r}) \right\} \quad (8.11)$$

where

$$\eta = \left( 1 + \frac{\sigma^2}{\xi^2} \right)^{-1/2} \quad (8.12)$$

A similar calculation for  $\bar{E}^{(1)}(\bar{r})$  gives

$$\begin{aligned} \bar{E}^{(1)}(\bar{r}) - \bar{\nabla} \Psi(\bar{r}) &= \int d^3r_i g(\bar{r}, \bar{r}_i) G(\bar{r}, \bar{r}_i) \{ b_0' \Psi^{(1)}(\bar{r}_i) - c_0' \bar{E}^{(1)}(\bar{r}_i) \cdot \bar{\nabla}_r \} g_{rr}(\bar{r}, \bar{r}_i) \\ &= \frac{1}{3} \eta^3 \bar{\alpha}(\bar{r}) \bar{E}^{(1)}(\bar{r}) + \frac{1}{3} \xi^2 \eta^3 \bar{\nabla} (\bar{q}(\bar{r}) \Psi^{(1)}(\bar{r})) \quad (8.13) \end{aligned}$$

The solutions for  $\Psi^{(1)}$  and  $\bar{E}^{(1)}$  are obtained by putting  $\Psi^{(1)} \simeq \Psi$  and  $\bar{E}^{(1)} \simeq \bar{\nabla} \Psi$  in the finite range corrections. Numerically, they are anyway too small to be confirmed experimentally. We conclude that the parameter  $\eta$  plays an essential rôle. In particular

$$\begin{array}{ll} \sigma \gg \xi & , \quad \text{no Lorentz-Lorenz effect} \\ \sigma \simeq \xi & , \quad \text{weak Lorentz-Lorenz effect} \\ \sigma \ll \xi & , \quad \text{full Lorentz-Lorenz effect} \end{array}$$

i.e., the strength of the Lorentz-Lorenz effect is a measure of the ratio (spatial extension of elementary interaction)/(nucleon-nucleon correlation length) rather than a measure of the importance of short range correlations alone.

Similar calculations can be done for the local interaction but we no longer expect the two localized potentials to agree to second order in the density. This is most easily seen in the limit  $\eta \ll \sigma$  where  $\eta \approx 0$ . In this case, the two Schrödinger equations reduce to those of Eqs (3.9) and (3.10) which differ in the second order terms.

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A P P E N D I X

Our basic equations can also be derived from the non-relativistic Schrödinger equation. Such a derivation is more instructive than the intuitive set of equations used in the main text since all approximations can be clearly stated.

We write the Hamiltonian as

$$H = H_0 + V, \quad H_0 = K + H_T, \quad V = \sum_{i=1}^A v_i, \quad (A.1)$$

where  $K$  is the kinetic energy of the pion,  $H_T$  the target Hamiltonian and  $v_i$  the interaction between the pion and nucleon no.  $i$ . The Schrödinger equation is then equivalent to the following set of equations (see any textbook on scattering theory)

$$\Psi_i = \varphi_i + \frac{1}{E - H_0 + i\epsilon} \sum_{n=1}^A t_n \Psi_n, \quad (A.2)$$

$$\Psi_n = \varphi_i + \frac{1}{E - H_0 + i\epsilon} \sum_{m=1}^A t_m \Psi_m, \quad (A.3)$$

$$t_n = v_n + v_n \frac{1}{E - H_0 + i\epsilon} t_n, \quad (A.4)$$

where  $\Psi_i$  is the wave function for the complete pion-nucleus system and  $\varphi_i$  the initial state with energy  $E = E_k + \epsilon_0$ . The relation to our approach in the main text is obvious.  $\Psi_n$  corresponds to the effective exciting wave for nucleon no.  $n$  and differs from the complete wave  $\varphi_i$  only through the outgoing wavelet from nucleon no.  $n$  itself. The main difference is that Eqs. (A.2) and (A.3) together give the complete solution.

We limit ourselves to coherent interactions where the nucleus remains in its initial state  $|\eta_0\rangle$  and our aim is to derive a Schrödinger equation in the pion variables above. The pion wave function is defined as

$$\Psi(\vec{r}) = \langle \vec{r} \eta_0 | \Psi_i \rangle \quad (A.5)$$

The approximation which makes our exact equations tractable is the "closure approximation" whereby the many-particle Green's function is replaced by the free particle Green's function for the pion

$$\frac{1}{E - H_0 + i\epsilon} \approx \frac{1}{E_k - K + i\epsilon} \quad (A.6)$$

In the case of Eq. (A.4) this approximation is known as the impulse approximation and assures that free and bound nucleons scatter in the same way. For pions the closure approximation is expected to be valid at all energies <sup>3)</sup>.

In the impulse approximation the amplitude  $t_n$  depends only on the co-ordinates of nucleon no. n. Denoting the nucleon co-ordinates collectively by  $\vec{\xi}$  we have in the no-recoil limit,  $M \gg A$ ,

$$\langle \vec{r}' | t_n | \vec{r} \rangle = \prod_{k=1}^A \delta(\vec{\xi}'_k - \vec{\xi}_k) \langle \vec{r}' - \vec{\xi}_n | t | \vec{r} - \vec{\xi}_n \rangle \quad (A.7)$$

and, furthermore

$$\langle \vec{k}' | t | \vec{k} \rangle = - \frac{4\pi}{2\mu} f(\vec{k}', \vec{k}) \quad (A.8)$$

We are now ready to write down our equations. From Eq. (A.2) we project out the pion wave function  $\Psi(\vec{r})$  and obtain in the closure and no-recoil approximations

$$\Psi(\vec{r}) = \langle \vec{r} | \eta_0 \rangle + \int d^3 r' g(\vec{r}, \vec{r}') \int d^3 r_j g(\vec{r}_j) \int d^3 r'' f(\vec{r}' - \vec{r}_j, \vec{r}'' - \vec{r}_j) \varphi(\vec{r}''; \vec{r}_j) \quad (A.9)$$

and

$$g(\vec{r}_j) \varphi(\vec{r}; \vec{r}_j) \equiv \langle \vec{r} | \eta_0 | \sum_{k=1}^A \delta(\vec{\xi}_k - \vec{r}_j) | \psi_k \rangle \quad (A.10)$$

The Green's function  $g(\vec{r}, \vec{r}')$  and the co-ordinate space scattering amplitude  $f(\vec{r}', \vec{r})$  are defined as before

$$g(\vec{r}, \vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \quad (A.11)$$

$$f(\vec{r}', \vec{r}) = \frac{1}{(2\pi)^6} \int d^3 k' \int d^3 k e^{-i\vec{k}' \cdot \vec{r}'} f(\vec{k}', \vec{k}) e^{i\vec{k} \cdot \vec{r}} \quad (A.12)$$

The second of our equations is obtained from Eq. (A.3) in the closure and no-recoil approximation by multiplying with  $\sum_{k=1}^A \delta(\bar{\xi}_k - \bar{r}_i)$ . Projecting out the coherent part we obtain after division with  $\rho(\bar{r}_i)$

$$\varphi(\bar{r}; \bar{r}_j) = \langle \bar{r} \eta_0 | \varphi_i \rangle + \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r_j \rho(\bar{r}_j) [1 + G(\bar{r}_i, \bar{r}_j)] \int d^3 r'' f(\bar{r}' - \bar{r}_j, \bar{r}'' - \bar{r}_j) \varphi(\bar{r}''; \bar{r}_i, \bar{r}_j) \quad (\text{A.13})$$

and

$$\rho(\bar{r}_i) \rho(\bar{r}_j) [1 + G(\bar{r}_i, \bar{r}_j)] \varphi(\bar{r}; \bar{r}_i, \bar{r}_j) \equiv \langle \bar{r} \eta_0 | \sum_{m=1}^A \sum_{\substack{n=1 \\ n \neq m}}^A \delta(\bar{\xi}_m - \bar{r}_i) \delta(\bar{\xi}_n - \bar{r}_j) | \Psi_n \rangle \quad (\text{A.14})$$

with  $G(\bar{r}_i, \bar{r}_j)$  chosen to be the nuclear two-particle correlation function.

Equations (A.9) and (A.13) do not exhaust the information available from the wave functions  $\Psi_n$ . Multiplying Eq. (A.3) by

$$\sum_{m=1}^A \sum_{\substack{n=1 \\ n \neq m}}^A \delta(\bar{\xi}_m - \bar{r}_i) \delta(\bar{\xi}_n - \bar{r}_j)$$

we obtain an equation similar to Eq. (A.14) but which expresses  $\varphi(\bar{r}; \bar{r}_i, \bar{r}_j)$  in terms of  $\varphi(\bar{r}; \bar{r}_i, \bar{r}_j, \bar{r}_k)$ . This procedure can be continued until we get  $\varphi(\bar{r}; \bar{r}_{j_1}, \dots, \bar{r}_{j_{A-1}})$  expressed in terms of  $\varphi(\bar{r}; \bar{r}_{j_1}, \dots, \bar{r}_{j_A})$ . A knowledge of  $\varphi(\bar{r}; \bar{r}_{j_1}, \dots, \bar{r}_{j_A})$  is equivalent to knowing the complete solution to the scattering problem. Thus we have shown how the multiple scattering equations written down in the main text can be derived from the Schrödinger equation. The only necessary approximation was that of closure.

The assumption which is now being made is that  $\varphi(\bar{r}; \bar{r}_i, \bar{r}_j)$  has a weaker dynamical dependence on  $\bar{r}_i$  than on  $\bar{r}_j$  so that we can replace it by its average value

$$\varphi(\bar{r}; \bar{r}_i, \bar{r}_j) \approx \varphi(\bar{r}; \bar{r}_j) \quad (\text{A.15})$$

Our system of equations is now closed, at least for point interactions, and we write the result as

$$(\Delta + k^2) \Psi(\bar{r}) = -4\pi \int d^3 r_j \rho(\bar{r}_j) \int d^3 r' f(\bar{r} - \bar{r}_j, \bar{r}' - \bar{r}_j) \varphi(\bar{r}'; \bar{r}_j) \quad (\text{A.16})$$

$$\varphi(\bar{r}; \bar{r}_j) = \Psi(r) + \int d^3 r' g(\bar{r}, \bar{r}') \int d^3 r_j g(\bar{r}_j) G(\bar{r}_j, \bar{r}_j) \int d^3 r'' f(\bar{r}' - \bar{r}_j, \bar{r}'' - \bar{r}_j) \varphi(\bar{r}''; \bar{r}_j). \quad (\text{A.17})$$

Finally, we remark that Eq. (A.16) can be obtained from Eq. (A.2) without the closure approximation.

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