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SOLUTIONS OF THE MULTITIME DIRAC EQUATION

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A simple method for solving the Dirac equation in space with a three-dimensional vector time and properties of wave functions for free particles moving along the arbitrary directed trajectories in six-dimensional space-time are considered. If the time multidimensionality exists indeed in Nature, each of currently known fermion families must contain several not yet discovered particles.

Рассматриваются простой метод решения уравнения Дирака в пространстве с трехмерным вектором времени и свойства волновых функций частиц, движущихся по произвольно направленным траекториям в шестимерном пространстве-времени. Если многомерность времени действительно существует в природе, то каждое из известных в настоящее время фермионных семейств должно содержать несколько еще не открытых частиц.

The six-dimensional theory with an equal number of space and time co-ordinates $\hat{\mathbf{x}} = (\mathbf{x}, \hat{\mathbf{t}})^{\mathrm{T}}$ [1] becomes apparent as a development of the theory of relativity taking into account a more complete symmetrization of space and time and as an attempt to overcome the difficulties of Lorentz transformations in the theories with faster-than-light velocities [2].

The analysis of a possible manifestation of the macroscopic multitime phenomena is presented in the review [7]. One can wait for becoming apparent of the hidden time dimensions only in some hardly accessible now regions, for example, in extremely strong gravitation fields [9]. Greater hopes to observe multitime effects are related to microscopic processes. Patty and Smalley [10] proposed a multitime generalization of the Dirac equation, Boyling and Cole [11] considered some interesting features of the new equation.

The goal of our paper is to present a simple method for solving the multitime spinor equation for a free particle with an arbitrary six-dimensional momentum-energy $\mathbf{p}, \hat{\mathbf{E}}$, moving along any space-time trajectory $\mathbf{x}(t), \hat{\mathbf{t}}(t)$ with a proper time t counted off along this trajectory, and to consider the properties of the obtained wave functions.

We proceed from Patty and Smalley's generalization

$$(i\hat{\gamma}\hat{\boldsymbol{\nabla}} - m)\Psi = 0, \tag{1}$$

[1] In what follows, the three-dimensional vectors in x- and t-subspaces will be denoted, respectively, by bold symbols and by a hat, six-dimensional vectors will be marked by bold symbols with a hat. In manuscripts it is convenient to use the notations $\bar{x}$, $\hat{x}$, and $\hat{\bar{x}}$. The «six-dimensional nabla» $\hat{\boldsymbol{\nabla}} = (\nabla, \hat{\nabla})$, where the time operator $\hat{\nabla} = (-\partial/\partial t_1, -\partial/\partial t_2, -\partial/\partial t_3)$. Scalar product is defined as $\hat{\mathbf{a}}\hat{\mathbf{b}} = \hat{a}\hat{b} - \mathbf{ab}$. As a rule, we shall suppose that the Latin and Greek indices take values $k = 1, \dots, 3$, $\mu = 1, \dots, 6$, and the constants $\hbar = c = 1$.

[2] One can find the detailed bibliography and a discussion of these problems, particularly, the six-dimensional Lorentz transformations and the physical meaning of two hidden time axes in papers [1–8].

where we present the $8 \times 8\,\gamma$ matrices in the form

$$\gamma_4 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \quad \text{and} \quad \gamma_\mu = \begin{pmatrix} O & \Sigma_\mu \\ -\Sigma_\mu & O \end{pmatrix} \quad \text{for} \quad \mu \neq 4$$

with

$$\Sigma_i = \begin{pmatrix} O & \sigma_i \\ \sigma_i & O \end{pmatrix}, \; i = 1,2,3, \quad \Sigma_5 = \begin{pmatrix} -iI_2 & O \\ O & iI_2 \end{pmatrix}, \quad \Sigma_6 = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix},$$

σ_i are the known Pauli matrices; I_n is an $n \times n$ unit matrix.

Let us search for solution of Eq. (1) in the form of a plane wave

$$\Psi(\hat{\mathbf{x}}) = \Phi\, e^{i\hat{\mathbf{x}}\hat{\mathbf{p}}},$$

with a space- and time-independent eight-component spinor $\Phi = (\phi_1, \dots, \phi_8)^T$; $\hat{\mathbf{p}} = (\mathbf{p}, \hat{E})$ is a six-dimensional particle momentum with vector energy $\hat{E} = E\hat{\tau} = (p^2 + m^2)^{1/2}\hat{\tau}$; and the constant unit vector $\hat{\tau}(\hat{\tau}^2 = 1)$ defines the direction of the particle time trajectory. Eq. (1) can be rewritten now as

$$(\gamma\mathbf{p} - \Theta E + m)\Phi = 0, \tag{2}$$

with matrices

$$\Theta = \hat{\gamma}\hat{\tau} = \begin{pmatrix} \tau_1 I_4 & \Theta_{23} \\ -\Theta_{23} & -\tau_1 I_4 \end{pmatrix}, \quad \Theta_{23} = \begin{pmatrix} -i\tau_2 I_2 & \tau_3 I_2 \\ -\tau_3 I_2 & i\tau_2 I_2 \end{pmatrix}, \quad \gamma\mathbf{p} = \begin{pmatrix} 0 & \sigma\mathbf{p} \\ -\sigma\mathbf{p} & 0 \end{pmatrix},$$

where $\tau_i = t_i/t$.

Splitting the wave function into two four-dimensional components: $\Phi = (\Phi_1, \Phi_2)^T$, one can present Eq. (2) as two matrix equations

$$\Phi_1 = (E\Theta_{23} - \Sigma\mathbf{p})\Phi_2/(m - E_1), \tag{3}$$

$$\Phi_2 = -(E\Theta_{23} - \Sigma\mathbf{p})\Phi_1/(m + E_1) \tag{4}$$

with a zero determinant

$$(E\Theta_{23} - \Sigma\mathbf{p})^2/(m^2 - E_1^2) + I_2 = 0$$

defining two-sign energy $\pm E$.

Rewriting matrix equations (3) as a system of eight algebraic equations, one can easily satisfy oneself that for every energy sign only four from them are linearly independent. So, each solution $(\Phi_1, \Phi_2)^T$ has four arbitrary components. Analogously to the usual one-time theory let us choose Φ_1 and Φ_2 in the right-hand sides of equations (3) and (4) as a unity matrix I_4. Then we get two independent solutions

$$\Phi_+ \equiv \Phi(E > 0) = (I_4, -g\Omega_-)^T, \tag{5}$$

$$\Phi_- \equiv \Phi(E < 0) = (-g\Omega_+, I_4)^T \tag{6}$$

with

$$\Omega_\pm = (E\Theta_{23} \pm \Sigma\mathbf{p}),$$

Table 1. **Four eight-component linearly independent solutions forming the matrices Φ_+ $(E>0)$ and Φ_- $(E<0)$**

Φ_+ $(E>0)$				
No.	I	II	III	IV
ϕ_1	1	0	0	0
ϕ_2	0	1	0	0
ϕ_3	0	0	1	0
ϕ_4	0	0	0	1
ϕ_5	igE_2	0	$g(p_3-E_3)$	gp_{12}^-
ϕ_6	0	igE_2	gp_{12}^+	$-g(p_3+E_3)$
ϕ_7	$g(p_3+E_3)$	gp_{12}^-	$-igE_2$	0
ϕ_8	gp_{12}^+	$-g(p_3-E_3)$	0	$-igE_2$
Φ_- $(E<0)$				
ϕ_1	igE_2	0	$-g(p_3+E_3)$	$-gp_{12}^-$
ϕ_2	0	igE_2	$-gp_{12}^+$	$-g(p_3-E_3)$
ϕ_3	$-g(p_3-E_3)$	$-gp_{12}^-$	$-igE_2$	0
ϕ_4	$-gp_{12}^+$	$g(p_3+E_3)$	0	$-igE_2$
ϕ_5	1	0	0	0
ϕ_6	0	1	1	0
ϕ_7	0	0	0	0
ϕ_8	0	0	0	1

$g=1/(m+E_1)$ and $E_i=|E_i|$. The components of these solutions are presented in Table 1. Here and in the following $p_{12}^{\pm}=p_1\pm ip_2$.

In contrast to the one-time theory, where the scalars

$$\bar{\Phi}_\pm\Phi_\pm=\Phi_\pm^+\gamma_4\Phi_\pm=(m/E)\Phi_\pm^+\Phi_\pm\neq 0,$$

the corresponding frame independent quantity in six-dimensional space-time $\bar{\Phi}_\pm\Phi_\pm=\Phi_\pm^+\Gamma\Phi_\pm$ with the matrix

$$\Gamma=i\gamma_4\gamma_5\gamma_6=\begin{pmatrix}-\Sigma_0 & 0\\ 0 & \Sigma_0\end{pmatrix},\quad \Sigma_0=\begin{pmatrix}O & I_2\\ I_2 & O\end{pmatrix}$$

is zero:

$$\bar{\Phi}_\pm\Phi_\pm=\mp 2mg\begin{pmatrix}O & I_2\\ I_2 & O\end{pmatrix},\quad \bar{\Phi}_+\Phi_-=0.$$

Therefore, it is more convenient to use the linear combinations Ψ_+ and Ψ_- with the components

$$\psi_{1,3}=N(\phi_1\pm\phi_3),\quad \psi_{2,4}=N(\phi_2\pm\phi_4),\quad N=(4mg)^{-1/2}$$

presented in Table 2. Taking into account the relations (5) and (6) one can prove that their relativistically invariant scalar products

$$\bar{\Psi}_{+i}\Psi_{+j}(E>0)=-\bar{\Psi}_{-i}\Psi_{-j}(E>0)=(-1)^i\delta_{ij}$$

and the total probabilities of physical states

$$\Psi_\pm^+ \Psi_\pm = 1 + gp^2/m$$

conserve always positive values.

Solutions with a fixed sign of energy in Table 2 differ by the helicities, i.e., by the eigenvalues λ_x and λ_τ for the projections of the spatial and temporal spin operators on the direction of vectors p and $\hat{p}$ considered as z axes and x axes in x and t subspaces:

$$S_3\Psi = \lambda_x \Psi, \quad S_4\Psi = \lambda_\tau \Psi,$$
$$S_3 = i\gamma_1\gamma_2 = \sigma_3 \times I_4, \quad S_4 = i\gamma_5\gamma_6 = \Gamma\gamma_4.$$

These expressions are the obvious generalization of the respective operators of the Dirac theory. The calculated values of $\lambda's$ are given in Table 2.

Table 2. **Matrices of solutions Ψ_+/N $(E>0)$ and Ψ_-/N $(E>0)$ for two-space and two-time spin states**

Ψ_+/N $(E>0)$				
No.	I	II	III	IV
ψ_1	1	0	1	0
ψ_2	0	1	0	1
ψ_3	1	0	–1	0
ψ_4	0	1	0	–1
ψ_5	$g(p_3+E_{23}^-)$	gp_{12}^-	$-g(p_3-E_{23}^+)$	$-gp_{12}^-$
ψ_6	gp_{12}^+	$-g(p_3-E_{23}^-)$	$-gp_{12}^+$	$g(p_3+E_{23}^+)$
ψ_7	$g(p_3-E_{23}^-)$	gp_{12}^-	$g(p_3+E_{23}^+)$	gp_{12}^-
ψ_8	gp_{12}^+	$-g(p_3+E_{23}^-)$	gp_{12}^+	$-g(p_3-E_{23}^+)$
λ_x	1	–1	1	–1
λ_τ	1	1	–1	–1
Ψ_-/N $(E>0)$				
ψ_1	$-g(p_3-E_{23}^-)$	$-gp_{12}^-$	$g(p_3+E_{23}^+)$	gp_{12}^-
ψ_2	$-gp_{12}^+$	$g(p_3+E_{23}^-)$	gp_{12}^+	$-g(p_3-E_{23}^+)$
ψ_3	$-g(p_3-E_{23}^-)$	$-gp_{12}^-$	$-g(p_3-E_{23}^+)$	$-gp_{12}^-)$
ψ_4	$-gp_{12}^+$	$g(p_3-E_{23}^-)$	$-gp_{12}^+$	$g(p_3+E_{23}^+)$
ψ_5	1	0	1	0
ψ_6	0	1	0	1
ψ_7	1	0	–1	0
ψ_8	0	1	0	–1
λ_x	1	–1	1	–1
λ_τ	1	1	–1	–1

Here $E_{23}^\pm = iE_2 \pm E_3$.

To get the customary one-time limit, one must suppose $\hat{\tau} = (1,0,0)$. In this case every eight-component spinor is decomposed into a pair of four-component ones: Ψ_1 and Ψ_2, and is a superposition of two customary Dirac spinors

$$\Psi = \left((\psi_1, \psi_2, \psi_5, \psi_6)^T, (\psi_3, \psi_4, \psi_7, \psi_8)^T\right),$$

Ψ_3 and Ψ_4 with the negative λ_τ get a superposition of spinors differing by signs of momentum and energy from thee preceding ones. One may think that these spinors describe new particles with yet unknown properties [1].

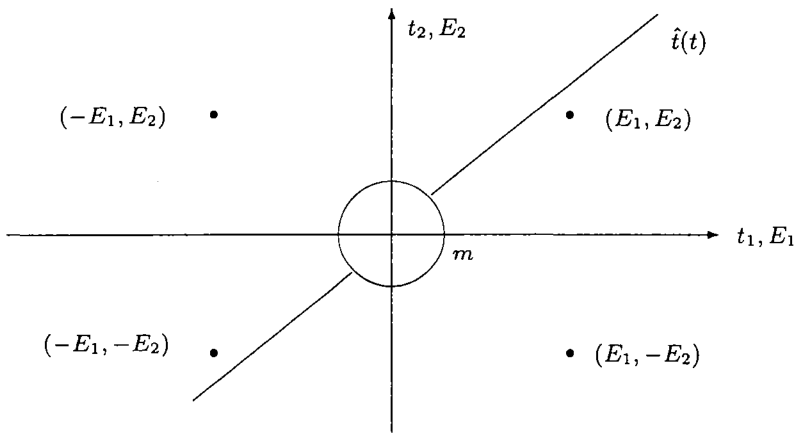

Fig. 1. Condition of time irreversibility $d\hat{t}/dt \geq 0$ permits only trajectories $\hat{t}(t)$ with all positive or all negative energies E_i. Trajectories in other regions correspond to processes going back in time and are excluded. The gap between particles ($E_i \geq m$) and antiparticles ($E_i \leq -m$) is conserved

Analogously to the usual theory, the multitime one has a forbidden energy region $E < m$, nevertheless at higher energies a possibility of continued transitions between positive and negative energy components, like the transitions between various momentum components p_i and $-p_i$, becomes apparent (Fig. 1), what seems to destroy the dichotomy between particles and antiparticles [11]. However, it is impossible for real physical processes, because due to time irreversibility $d\hat{t}/dt \geq 0$ all time trajectories are placed only in the regions with $E_i = E\tau_i \geq m$ (Fig. 1), and the distinction of matter and antimatter is always conserved. One must also stress that the time irreversibility prevents the creation of paricles with time vectors different from ours (Fig. 2), though the energy of our accelerators is enough for that. It may be that the particles with various time vectors take part in virtual quantum processes, however, the multitime quantum theory has not been developed yet. The considered solving of the multitime Dirac equation (1) is a step in this direction.

The discovery of additional members of the Dirac spinor families (for example, in the reactions of particle–antiparticle pairs production by γ quanta, by means of the investigation of splitting of hydrogen energy levels and so on) would be an indication on the multitime

[1] The particular solutions found by Boyling and Cole [11] for another one-time case $\hat{\tau} = (0, 0, 1)$ can be obtained as a linear superposition of the functions from the upper part of Table 1:

$$\Phi_{1,3}^{\text{Cole}} = \lambda^{-1}(\Phi_1 \mp \lambda\Phi_3), \quad \Phi_{2,4}^{\text{Cole}} = \lambda(\Phi_2 \mp \lambda^{-1}\Phi_4).$$

To go into our one-time world, these particular solutions must be turned in the t subspace by 90°.

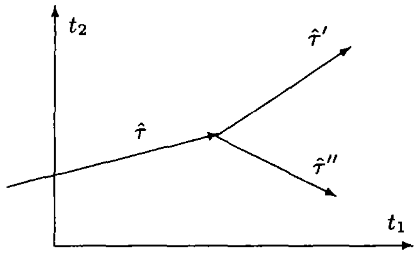

Fig. 2. The creation of a component with the energy $\hat{E}' = \hat{\tau}'E \geq 0$ is accompanied, without fail, by the creation of a compensating, moving back in time component with the energy $\hat{E}'' = \hat{\tau}''E \leq 0$. (The energy vector is parallel to the time vector: $\hat{E} = E\hat{\tau}$.)

structure of our world and existence of the hidden time dimensions. And oppositely — absence of such particles will testify for the time one-dimensionality.

REFERENCES

1. *Cole E. A. B.* // Nuovo Cim. A. 1977. V. 40. P. 171.
2. *Pavsič M. J.* // Phys. A. 1981. V. 14. P. 3217.
3. *Cole E. A. B., Buchanan S. A.* // J. Phys. A. 1982. V. 15. P. L255.
4. *Barashenkov V. S.* Six-Dimensional Space-Time Transformations. JINR Commun. E2-97-93. Dubna, 1997.
5. *Barashenkov V. S., Yuriev M. Z.* Sub- and Superlumina; Velocities in Space with Vector Time. JINR Preprint E2-2000-1. Dubna, 2000.
6. *Barashenkov V. S., Yuriev M. Z.* // Nuovo Cim. B. 1997. V. 112. P. 117.
7. *Barashenkov V. S.* // Found. Rhys. 1998. V. 29. P. 485.
8. *Barashenkov V. S., Yuriev M. Z.* Extensions of Quantum Physics. Montreal, 2002. P. 53–61.
9. *Barashenkov V. S., Pestov A. B., Yuriev M. Z.* // Gen. Relat.&Grav. 1997. V. 29. P. 1345.
10. *Patty C. E., Smalley L. I.* // Phys. Rev. D. 1985. V. 32. P. 891.
11. *Boyling J. B., Cole E. A. B.* // Intern. J. Theor. Phys. 1993. V. 3. P. 801.

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