GALOIS SYMMETRIES OF FUNDAMENTAL GROUPOIDS AND NONCOMMUTATIVE GEOMETRY

A.B. GONCHAROV

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Août 2002

IHES/M/02/56

Galois symmetries of fundamental groupoids and noncommutative geometry

A. B. Goncharov

Contents

1 Introduction

Abstract. We give a simple proof of the formula for coproduct Δ in the Hopf algebra of motivic iterated integrals on the line. We show that it encodes the group law of the automorphism group of certain noncommutative variety. We relate the coproduct Δ with the coproduct in the Hopf algebra of decorated rooted planar trivalent trees, which is a planar decorated version of the one defined by Connes and Kreimer [CK]. As an application we derive explicit formulas for the coproduct in the motivic multiple polylogarithm Hopf algebra.

In chapter 7 we discuss some general principles relating Feynman integrals and mixed motives. They are suggested by chapter 4 and the Feynman integral approach for multiple polylogarithms on curves given in [G2].

1. The Hopf algebra of motivic iterated integrals. Consider the iterated integral

$$\mathrm{I}_\gamma(a; z_1, ..., z_m; b) := \int_{a \leq t_1 \leq ... \leq t_m \leq b} \frac{dt_1}{t_1 - z_1} \wedge \frac{dt_2}{t_2 - z_2} \wedge ... \wedge \frac{dt_m}{t_m - z_m} \qquad (1)$$

Here γ is a path between the points a and b in $\mathbb{C}-\{z_1\cup...\cup z_m\}$, and integration is over simplex consisting of all ordered m–tuples of points $(t_1,...,t_m)$ on γ.

The iterated integral (1) is a period of a mixed Hodge structure or, better, mixed Tate motive. Therefore we can upgrade (1) to a more sophisticated object, the framed mixed Tate motive (see chapter 3 of [G3] for the background, and section 3.2 for the precise description of our set-up).

$$\mathrm{I}^{\mathcal{M}}(a;z_1,...,z_m;b);\qquad a,b,z_i\in S \tag{2}$$

By its very definition it lies in a certain Hopf algebra with coproduct Δ, the Hopf algebra of regular functions on the motivic Galois group. One of our results is a simple derivation of the following explicit formula for the coproduct:

Theorem 1.1 *The coproduct is computed by the formula*

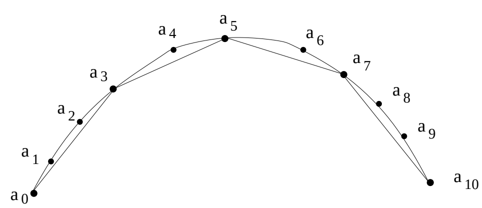
$$\Delta\mathrm{I}^{\mathcal{M}}(a_0;a_1,a_2,...,a_m;a_{m+1})= \tag{3}$$

$$\sum_{0=i_0<i_1<...<i_k<i_{k+1}=m+1}\mathrm{I}^{\mathcal{M}}(a_0;a_{i_1},...,a_{i_k};a_{m+1})\otimes\prod_{p=0}^{k}\mathrm{I}^{\mathcal{M}}(a_{i_p};a_{i_p+1},...,a_{i_{p+1}-1};a_{i_{p+1}})$$

Here $0\leq k\leq m$. For example the term

$$\mathrm{I}^{\mathcal{M}}(a_0;a_3,a_5;a_7)\otimes\mathrm{I}^{\mathcal{M}}(a_0;a_1,a_2;a_3)\mathrm{I}^{\mathcal{M}}(a_3;a_4;a_5)\mathrm{I}^{\mathcal{M}}(a_5;a_6;a_7)$$

is illustrated by the following picture:



The terms in formula (3) are in one–to–one correspondence with the subsets

$$\{a_{i_1},...,a_{i_k}\}\subset\{a_1,...,a_m\} \tag{4}$$

If we locate the ordered set of elements $\{a_0,...,a_{m+1}\}$ on a hemicircle, as on the picture, then the terms in (3) correspond to the polygons with vertices at the points a_i, containing a_0 and a_{m+1}, inscribed into the hemicircle.

Formula (3) in the Hodge realization was proved in chapter 5 of [G3]. Using the injectivity of the regulators it can be transformed to the motivic setting when the points a_i are defined over a number field. The proof given below is more direct and penetrates the algebraic structures staying behind this formula.

These structures include automorphism groups of certain noncommutative varieties discussed in chapters 2 and 3, and a Hopf algebra of decorated rooted planar trees in chapter 4.

In fact we prove a stronger version of theorem 1.1, where no restrictions on the points $\{a_0, ..., a_{m+1}\}$ is assumed. (From the analytic point of view this means that the iterated integral (1) can be divergent, so it has to be regularized.) The precise formulation of our result is given in theorem 3.3. It allows us to calculate explicitly the coproduct of the motivic multiple polylogarithms, see theorem 5.4. The latter result plays the central role in the mysterious correspondence between the structure of the Lie coalgebra of multiple polylogarithms at N-th roots of unity and modular varieties, see [G2-4].

2. The algebraic structures underlying formula (3). It is well known that integrals (1) satisfy the following basic properties:

The shuffle product formula. Let $\Sigma_{m,n}$ be the set of all shuffles of the ordered sets $\{1, ..., m\}$ and $\{m+1, ..., m+n\}$. Then

$$\mathrm{I}_\gamma(a; z_1, ..., z_m; b) \cdot \mathrm{I}_\gamma(a; z_{m+1}, ..., z_{m+n}; b) = \sum_{\sigma \in \Sigma_{m,n}} \mathrm{I}_\gamma(a; z_{\sigma(1)}, ..., z_{\sigma(m+n)}; b)$$

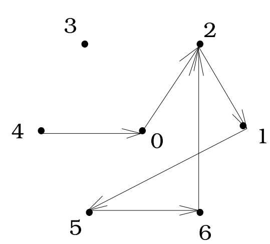
The path composition formula. If $\gamma = \gamma_1 \gamma_2$, where γ_1 is a path from a to x, and γ_2 is a path from x to b in $\mathbb{C} - \{z_1 \cup ... \cup z_n \cup a \cup b\}$, then

$$\mathrm{I}_\gamma(a; z_1, ..., z_m; b) = \sum_{k=0}^{m} \mathrm{I}_{\gamma_1}(a; z_1, ..., z_k; x) \cdot \mathrm{I}_{\gamma_2}(x; z_{k+1}, ..., z_m; b)$$

Let S be an arbitrary set. In chapter 2 we define a commutative, graded Hopf algebra $\mathcal{I}_\bullet(S)$. As a commutative algebra it is generated by symbols $\mathbb{I}(s_0; s_1, ..., s_n; s_{n+1})$ with $s_i \in S$ obeying the relations as above. We use formula (3) to *define* the coproduct in $\mathcal{I}_\bullet(S)$.

If S is a subset of points of the affine line $\mathbb{A}^1$ the Hopf algebra $\mathcal{I}_\bullet(S)$ reflects the basic properties of the iterated integrals on $\mathbb{A}^1_S := \mathbb{A}^1 - S$ between the tangential base points at S.

To show that $\mathcal{I}_\bullet(S)$ is indeed a Hopf algebra we interpret it as the algebra of regular functions on a certain pro–unipotent group scheme. Namely, let Γ_S be the graph whose vertices are elements of S, and every two vertices connected by unique edge. Let $\mathrm{P}(S)$ be the completed path algebra of this graph.



A path 4–0–2–1–5–6–2
is an element of the path algebra.

It is a composition of the
path 4–0–2 with the path
2–1–5–6–2

It has some additional structures: two algebra structures $\circ$ and $*$, plus a coproduct Δ. Let $G(S)$ be the group of automorphisms of $\mathrm{P}(S)$ preserving these structures, and acting as the identity on $\mathrm{P}_+(S)/\mathrm{P}_+^2(S)$. Here $\mathrm{P}_+(S)$ is given by paths of positive length. In other words it is the group of automorphism of the corresponding noncommutative variety preserving some natural structures on it. We prove in theorem 2.5 that

$$G(S) = \mathrm{Spec}(\mathcal{I}_\bullet(S))$$

In chapter 3 we show that the algebra $\mathrm{P}(S)$ is provided by the motivic fundamental groupoid of path

$$\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1_S, S) \tag{5}$$

on the affine line punctured at S, between the tangential base points at S.

We consider (5) as a pro–object in the abelian category of mixed Tate motives or one of its realizations. Such a category is equipped with a canonical fiber functor ω. The motivic Galois group acts on $\omega(\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1_S, S))$. Let $G_{\mathcal{M}}(S)$ be its image.

The motivic fundamental groupoid carries some additional structures provided by the composition of paths and "canonical loops" near the punctures. Using these structures we identify $\omega(\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1_S, S))$ with the path algebra $\mathrm{P}(S)$. Therefore $G_{\mathcal{M}}(S)$ is realized as a subgroup of $G(S)$:

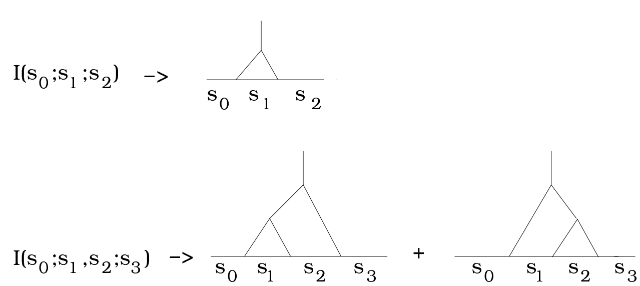
$$G_{\mathcal{M}}(S) \hookrightarrow G(S) \tag{6}$$

Theorem 1.1 follows immediately from this.

In chapter 4 we show how the collection of all planar trivalent rooted trees decorated by the ordered set $\{a_0, a_1, ..., a_m, a_{m+1}\}$ governs the fine structure of the motivic object (2). Let $\mathcal{T}_\bullet(S)$ be the commutative Hopf algebra of S–decorated planar rooted trivalent trees. We relate the groups $G(S)$ and $\mathrm{Spec}(\mathcal{T}_\bullet(S))$. Precisely, consider the map

$$t : \mathbb{I}(s_0; s_1, ..., s_m; s_{m+1}) \longmapsto \text{sum of all planar rooted trivalent trees} \tag{7}$$
$$\text{decorated by the ordered set } \{s_0, s_1, ..., s_m, s_{m+1}\}$$

Example. Here is how the map t looks in the two simplest cases



We prove in chapter 4 that the map t transforms the coproduct in $\mathcal{I}(S)$ to the one in $\mathcal{T}_\bullet(S)$

3. An application: the motivic multiple polylogarithm Hopf algebras. Recall the multiple polylogarithms ([G0], [G6]) where $n_i \in \mathbb{N}_+$, $|x_i| < 1$:

$$\mathrm{Li}_{n_1,\ldots,n_m}(x_1,...,x_m) \quad = \sum_{0<k_1<k_2<\ldots<k_m} \frac{x_1^{k_1}x_2^{k_2}...x_m^{k_m}}{k_1^{n_1}k_2^{n_2}...k_m^{n_m}} \tag{8}$$

According to theorem 2.2 in [G3] one can write them as iterated integrals. Namely, if $|x_i| < 1$, then setting

$$a_1 := (x_1...x_m)^{-1}, a_2 := (x_2...x_m)^{-1}, ..., a_m := x_m^{-1}$$

we get for a path γ inside of the unit disc

$$\mathrm{Li}_{n_1,\ldots,n_m}(x_1,...,x_m) = (-1)^m \mathrm{I}_{n_1,\ldots,n_m}(a_1,...,a_m) :=$$

$$(-1)^m \mathrm{I}_\gamma(0; a_1, \underbrace{0,...,0}_{n_1}, a_2, \underbrace{0,...,0}_{n_2}, a_3, ..., a_m, \underbrace{0,...,0}_{n_m}; 1) \tag{9}$$

Let $\mathcal{M}$ be the category of mixed Tate motives over a number field F or one of the mixed Tate categories described in section 3.1. Let G be a subgroup of the multiplicative group F^* of the field F. A particularly interesting example is when $G = \mu_N$ is the group of all N-th roots of unity. Upgrading iterated integrals (9) to their motivic counterparts (2), and using identity (9) as a definition, we arrive to motivic multiple polylogarithms

$$\mathrm{Li}^{\mathcal{M}}_{n_1,\ldots,n_m}(x_1,...,x_m)$$

We show in chapter 5 that, adding the motivic logarithms $\log^{\mathcal{M}}(x)$ to them, we get a graded, commutative Hopf algebra $\mathcal{Z}^{\mathcal{M}}_\bullet(G)$. In the Hodge setting the corresponding result was proved in [G3]. Chapters 5 and 6 contain a detailed discussion of explicit formulae for the coproduct in these Hopf algebras. It replaces chapter 6 in [G3], where the same results were presented in the Hodge setting.

In particular we get the cyclotomic Hopf algebra $\mathcal{Z}^{\mathcal{M}}_\bullet(\mu_N)$. It is the algebra of regular functions on the group scheme $G_{\mathcal{M}}(0 \cup \mu_N)$. We show that the semidirect product of $\mathbb{G}_m$ and $G_{\mathcal{M}}(0 \cup \mu_N)$ is isomorphic to the image of the motivic Galois group acting on the motivic fundamental group $\pi_1^{\mathcal{M}}(\mathbb{G}_m - \mu_N, v_0)$.

The following conjecture is a reformulation of conjecture 7.4 in [G3].

Conjecture 1.2 *Let F be an arbitrary field. Then the semidirect product of $\mathbb{G}_m$ and $G_{\mathcal{M}}(F)$ is isomorphic to the motivic Galois group of the category of all mixed Tate motives over F.*

Acknowledgment. I am grateful to Maxim Kontsevich for several stimulating discussions. This paper was written during my stay at IHES (Bures sur Yvette) in July of 2002 and at MPI(Bonn) in August of 2002. I am very grateful to IHES and MPI for hospitality and support. I was supported by the NSF grant DMS-0099390.

2 Iterated integrals Hopf algebras

In this chapter we introduce the Hopf algebras $\mathcal{I}(S)$ and $\widetilde{\mathcal{I}}(S)$ and show that they appear naturally as the algebras of functions on the automorphism groups of certain non commutative varieties.

1. The Hopf algebra $\widetilde{\mathcal{I}}_\bullet(S)$ and $\mathcal{I}_\bullet(S)$. Let S be a set. We will define a commutative Hopf algebra $\mathcal{I}_\bullet(S)$ over $\mathbb{Q}$, graded by the integers $n \geq 0$.

As a *commutative* $\mathbb{Q}$*-algebra*, $\mathcal{I}_\bullet(S)$ is generated by the elements

$$\mathbb{I}(s_0; s_1, ..., s_m; s_{m+1}), \quad s_i \in S \tag{10}$$

The generator (10) is homogeneous of degree m. The relations are the following. For any $s_i, a, b, x \in S$ one has:

i) *The unit*: for any $a, b \in S$ one has $\mathbb{I}(a; b) := \mathbb{I}(a; \emptyset; b) = 1$.

ii) *The shuffle product formula*

$$\mathbb{I}(a; s_1, ..., s_m; b) \cdot \mathbb{I}(a; s_{m+1}, ..., s_{m+n}; b) = \sum_{\sigma \in \Sigma_{m,n}} \mathbb{I}(a; s_{\sigma(1)}, ..., s_{\sigma(m+n)}; b)$$

iii) *The path composition formula*

$$\mathbb{I}(s_0; s_1, ..., s_m; s_{m+1}) = \sum_{k=0}^{m} \mathbb{I}(s_0; s_1, ..., s_k; x) \cdot \mathbb{I}(x; s_{k+1}, ..., s_m; s_{m+1})$$

iv) $\mathbb{I}(a; s_1, ..., s_m; a) = 0$ for $m > 0$.

To define a Hopf algebra structure on $\mathcal{I}_\bullet(S)$ we use the following formula for the coproduct Δ on the generators:

$$\Delta \mathbb{I}(a_0; a_1, a_2, ..., a_m; a_{m+1}) = \tag{11}$$

$$\sum_{0=i_0<i_1<...<i_k<i_{k+1}=m+1} \mathbb{I}(a_0; a_{i_1}, ..., a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} \mathbb{I}(a_{i_p}; a_{i_p+1}, ..., a_{i_{p+1}-1}; a_{i_{p+1}})$$

Thanks to i) the empty subset and the subset $\{a_1, ..., a_m\}$ contribute the terms

$$1 \otimes \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1}) \quad \text{and} \quad \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1}) \otimes 1$$

The counit is determined by the condition that it kills $\mathcal{I}_{>0}(S)$.

Proposition 2.1 *One has*

$$\mathbb{I}(a_0; a_1, ..., a_m; a_{m+1}) = (-1)^m \mathbb{I}(a_{m+1}; a_m, ..., a_1; a_0)$$

Proof. We use the induction on m. When $m = 1$ the path composition formula iii) plus i) gives

$$0 = \mathbb{I}(a_0; a_1; a_0) = \mathbb{I}(a_0; a_1; a_2) + \mathbb{I}(a_2; a_1; a_0)$$

Let us assume we proved the claim for all $k < m$. The path composition formula with $x := a_{m+1}$ gives

$$0 = \mathbb{I}(a_0; a_1, ..., a_m; a_0) = \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1})+$$

$$\sum_{1 \leq k \leq m-1} \mathbb{I}(a_0; a_1, ..., a_k; a_{m+1}) \cdot \mathbb{I}(a_{m+1}; a_{k+1}, ..., a_m; a_0) + \mathbb{I}(a_{m+1}; a_1, ..., a_m; a_0)$$

Applying the induction assumption to the second factors in the sum we get

$$0 = \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1})+$$

$$\sum_{1 \leq k \leq m-1} (-1)^{m-k} \mathbb{I}(a_0; a_1, ..., a_k; a_{m+1}) \cdot \mathbb{I}(a_0; a_m, ..., a_{k+1}; a_{m+1})+$$

$$\mathbb{I}(a_{m+1}; a_1, ..., a_m; a_0)$$

We claim that

$$\mathbb{I}(a_0; a_1, ..., a_m; a_{m+1})+ \tag{12}$$

$$\sum_{1 \leq k \leq m-1} (-1)^{m-k} \mathbb{I}(a_0; a_1, ..., a_k; a_{m+1}) \cdot \mathbb{I}(a_0; a_m, ..., a_{k+1}; a_{m+1})$$

$$= (-1)^{m-1} \mathbb{I}(a_0; a_m, ..., a_1; a_{m+1})$$

Indeed, this equality is equivalent to

$$\mathbb{I}(a_0; a_1, ..., a_m; a_{m+1})+ \tag{13}$$

$$\sum_{0 \leq k \leq m-1} (-1)^{m-k} \mathbb{I}(a_0; a_1, ..., a_k; a_{m+1}) \cdot \mathbb{I}(a_0; a_m, ..., a_{k+1}; a_{m+1}) = 0$$

To prove (13) we use the shuffle product formula to rewrite the sum. Then the claim is rather obvious (see the beginning of the proof of theorem 4.1 in [G1] for details). It remains to notice that proposition follows immediately from (12). The proposition is proved.

Let $\widetilde{\mathcal{I}}_\bullet(S)$ be the similar object where the relations ii)– iv) are omitted.

Proposition 2.2 *The coproduct* Δ *provides* $\mathcal{I}_\bullet(S)$*, as well as* $\widetilde{\mathcal{I}}_\bullet(S)$*, with a structure of a commutative, graded Hopf algebra.*

The algebra of regular functions $\mathcal{O}(G)$ on a group scheme G is a commutative Hopf algebra with the coproduct $\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ induced by the group multiplication map $G \times G \longrightarrow G$.

To prove proposition 2.2 we interpret $\mathcal{I}_\bullet(S)$ and $\widetilde{\mathcal{I}}_\bullet(S)$ as the algebras of regular functions on a certain pro–unipotent group schemes, defined as automorphisms groups of certain non commutative objects.

2. The path algebra $P(S)$. Let S be a set. Let K be a field. Let $P(S)$ be the K-vector space with basis

$$p_{s_0,\ldots,s_n}, \quad n \geq 1, s_k \in S \tag{14}$$

It has a grading such that the degree of (14) is $-2(n-1)$. Let us equip it with the following structures.

i) *The* $\circ$*-product.* There is an associative product

$$\circ : P(S) \otimes_K P(S) \longrightarrow P(S)$$

$$p_{a,X,b} \circ p_{c,Y,d} = \begin{cases} p_{a,X,Y,d} & : \quad b = c; \\ 0 & : \quad b \neq c \end{cases}$$

In particular $p_{a,b} = p_{a,x} \circ p_{x,b}$. The element $e_0 := \sum_{i \in S} p_{i,i}$ is the unit for this product. The algebra $P(S)$ is decomposed into a sum

$$P(S) = \oplus_{i,j \in S} P(S)_{i,j} \tag{15}$$

where $P(S)_{i,j}$ is spanned by the elements (14) with $s_0 = i, s_n = j$. Below we consider only the automorphisms F of $P(S)$ preserving this decomposition:

$$F(P(S)_{i,j}) \subset P(S)_{i,j} \tag{16}$$

Algebra $P(S)$ has an interpretation as a tensor algebra in a certain monoidal category, see section 2.4, which makes this restriction on F natural.

i1) *The* $*$*-product.* We define *another* associative product

$$* : P(S)(1) \otimes P(S)(1) \longrightarrow P(S)(1)$$

by the formula

$$p_{X,b} * p_{c,Y} = \begin{cases} p_{X,b,Y} & : \quad b = c; \\ 0 & : \quad b \neq c \end{cases} \tag{17}$$

where X and Y are ordered collections of elements of S, possibly empty. Here $M(1)$ means M with the grading shifted down by 2.

Then $P(S)(1)$ is an associative algebra generated by the elements $p_{i,j}$.

One can add to this algebra the elements p_i, $i \in S$, composed with the other elements just as in (17). Then p_i are orthogonal projectors: $p_i^2 = p_i$, $p_i * p_j = 0$ if $i \neq j$, and $e := \sum_{i \in S} p_i$ is a unit. Using these projectors we can describe decomposition (15) as as

$$P(S) = \oplus_{i,j \in S} P(S)_{i,j}; \quad P(S)_{i,j} := p_i * P(S) * p_j$$

The automorphisms of the algebra $P(S)$ preserving the projectors p_i respect this decomposition.

Recall the graph Γ_S from s. 1.2. The $*$–algebra $P(S)$ is the path algebra of this graph. Namely, $p_{i_0,\ldots,i_n}$ corresponds to the path $i_0 \to i_1 \to \ldots \to i_n$ in Γ_S. It turns out to be isomorphic to the free product of the algebra $\mathbb{Q}[S]$ of functions on S with finite support with the polynomial algebra $\mathbb{Q}[x]$:

$$P(S) \xrightarrow{\sim} \mathbb{Q}[S] *_{\mathbb{Q}} \mathbb{Q}[x]$$

iii) *The coproduct* δ. Let us define a coproduct

$$\delta : P(S) \longrightarrow P(S) \otimes P(S)$$

such that

$$\delta : P(S)_{a,b} \longmapsto P(S)_{a,b} \otimes P(S)_{a,b} \qquad (18)$$

by the formula

$$\delta(p_{a,x_1,\ldots,x_n,b}) := \sum p_{a,x_{i_1},\ldots,x_{i_k},b} \otimes p_{a,x_{j_1},\ldots,x_{j_{n-k}},b}$$

where the sum is over all decompositions

$$\{1, \ldots, n\} = \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\}, \quad i_1 < \ldots < i_k; \quad j_1 < \ldots < j_{n-k}$$

It is easy to see that it is cocommutative and coassociative.

The compatibilities. The expression $A * B \circ C * D \circ E$ does not depend on the bracketing.

δ is an algebra morphism for the $\circ$–algebra structure:

$$\delta(X \circ Y) = \delta(X) \circ \delta(Y) \qquad (19)$$

The compatibility with the $*$–structure is this:

$$\delta(X * Y) = \delta(X) *_2 \delta(Y); \qquad (20)$$

where

$$(X_1 \otimes Y_1) *_2 (X_2 \otimes Y_2) := (X_1 * Y_1) \otimes (X_2 \circ Y_2) + (X_1 \circ Y_1) \otimes (X_2 * Y_2)$$

It follows that δ is uniquely determined by (19), (20) and

$$\delta(p_{a,b}) = p_{a,b} \otimes p_{a,b} \qquad (21)$$

Observe that

$$\delta(P(S)_{a,b}) = \delta(p_{a,a} \circ P(S) \circ p_{b,b}) \subset$$
$$(p_{a,a} \otimes p_{a,a}) \circ (P(S) \otimes P(S)) \circ (p_{b,b} \otimes p_{b,b}) = \mathrm{P}(S)_{a,b} \otimes \mathrm{P}(S)_{a,b}$$

So (19) + (21) => (18).

Remark. $\delta(e_0) \neq e_0 \otimes e_0$. Otherwise it has all the properties of the Hopf $\circ$–algebra.

For every $a \in S$ the $\circ$–algebra $P(S)_{a,a}$ is a cocommutative Hopf algebra with the unit $p_{a,a}$. It is a universal enveloping algebra of a Lie algebra $L(S)_a$, reconstructed as the subspace of primitives in $P(S)_{a,a}$. The Lie algebra $L(S)_a$ is free with generators labeled by the set S. The canonical Lie algebra isomorphism $i_{a,b} : L(S)_a \longrightarrow L(S)_b$ is given by $l \longmapsto p_{b,a} \circ l \circ p_{a,b}$.

3. The path algebra as a tensor algebra in a monoidal category. Let S be a finite set. Let $\mathcal{C}$ be a tensor category. Consider the category $Q_{\mathcal{C}}(S)$ whose objects

$$V = \{V_{i,j}\}, \quad (i,j) \in S \times S$$

are objects $V_{i,j}$ of $\mathcal{C}$ labeled by elements of $S \times S$. Let $V = \{V_{i,j}\}$ and $W = \{W_{i,j}\}$. Then

$$\mathrm{Hom}_{Q_{\mathcal{C}}(S)}(V, W) := \oplus_{(i,j)\in S\times S}\mathrm{Hom}_{\mathcal{C}}(V_{i,j}, W_{i,j})$$

We define $V \otimes W$ by setting

$$(V \otimes W)_{i,j} = \oplus_{k\in S} V_{i,k} \otimes_{\mathcal{C}} W_{k,j}$$

From now on we assume that $\mathcal{C}$ is the category of K–vector spaces, and denote by $Q(S)$ the corresponding monoidal category. There is a unit object $\mathbb{I}$ defined as

$$\mathbb{I}_{i,j} := \begin{cases} 0 & : \quad i \neq j; \\ K & : \quad i = j \end{cases}$$

Consider an object $\mathbb{E}_S$ such that

$$\dim(\mathbb{E}_S)_{i,j} = 1 \quad \text{for any } (i,j) \in S \times S$$

Let $T(\mathbb{E}_S)$ be the free associative algebra with unit in the category $Q(S)$ generated by $\mathbb{E}_S$.

Lemma 2.3 *$T(\mathbb{E}_S)$ is isomorphic to the path $*$–algebra $P(S)(1)$.*

Proof. Choose a nonzero vector $p_{i,j}$ in $(\mathbb{E}_S)_{i,j}$. Set

$$p_{i_0,\dots,i_n} := p_{i_0,i_1} \otimes p_{i_1,i_2} \otimes \dots \otimes p_{i_{n-1},i_n} \tag{22}$$

Then

$$T(\mathbb{E}_S) = \oplus_{n\geq 0}\mathbb{E}_S^{\otimes n} = \mathbb{I} \oplus \oplus_{n\geq 1}\Big(\oplus_{i_0,\dots,i_n\in S} K \cdot p_{i_0,\dots,i_n}\Big)$$

Writing $\mathbb{I}_{i,i} = K \cdot p_i$ we get the needed isomorphism. The lemma is proved.

Below we identify $p_{i,j}$ with the object of $Q(S)$ whose (i', j')–component is zero unless $i = i', j = j'$, when it is K. Then we have the objects (22).

There is the second product

$$\circ : T(\mathbb{E}_S) \otimes T(\mathbb{E}_S) \longrightarrow T(\mathbb{E}_S); \quad \deg(\circ) = -2$$

4. The pro–unipotent algebraic groups scheme $G(S)$. Let $\mathrm{P}(S)$ be the completion of the path algebra $P(S)$ with respect to the powers of the augmentation ideal $P_+(S)$.

Definition 2.4 *a)* $\mathrm{Aut}_0(\mathrm{P}(S))$ *is the pro–unipotent group scheme of automorphisms of the $*$–algebra* $\mathrm{P}(S)$ *which act as the identity on the quotient*

$$\mathrm{P}_+(S)/(\mathrm{P}_+(S))^2 \tag{23}$$

b) $G(S) \subset \mathrm{Aut}_0(\mathrm{P}(S))$ *is the subgroup of all automorphisms F such that:*
i) F commutes with δ.
ii) F is an automorphism of the $\circ$–algebra structure.

The grading of $\mathrm{P}(S)$ provides natural gradings of the algebras of regular functions on $G(S)$ and $\mathrm{Aut}_0(\mathrm{P}(S))$.

Theorem 2.5 *a) The commutative, graded algebra* $\mathcal{O}\Big(\mathrm{Aut}_0(\mathrm{P}(S))\Big)$ *is identified with the polynomial algebra in an infinite number of variables* $I_{s_0,\ldots,s_{n+1}}$ *with $n \geq 0$ and $s_i \in S$:*

$$\mathcal{O}\Big(\mathrm{Aut}_0(\mathrm{P}(S))\Big) = K[I_{s_0,\ldots,s_{n+1}}]; \quad \deg(I_{s_0,\ldots,s_{n+1}}) = -2n$$

b) The identification in a) provides an isomorphism of commutative, graded Hopf algebras

$$\mathcal{O}\Big(\mathrm{Aut}_0(\mathrm{P}(S))\Big) \stackrel{\sim}{\longrightarrow} \widetilde{\mathcal{I}}_\bullet(S) \tag{24}$$

c) The isomorphism (24) induces an isomorphism of commutative, graded Hopf algebras

$$\mathcal{O}(G(S)) \stackrel{\sim}{\longrightarrow} \mathcal{I}_\bullet(S) \tag{25}$$

Proof. We consider first the case when S is finite, and then take the inductive limit over finite subsets of S.

An automorphism F satisfying (16), being an $*$–algebra automorphism, is uniquely determined by its values on the generators $p_{a,b}$. Indeed, condition (16) plus lemma 2.3 imply that it is an automorphism of the free $*$–algebra in the monoidal category $Q(S)$ generated by the elements $p_{a,b}$.

Let us write

$$F(p_{a,b}) = p_{a,b} + \sum I_{a,s_1,\ldots,s_m,b}(F) \cdot p_{a,s_1,\ldots,s_m,b}$$

where the summation is over all nonempty collections of elements $s_1, ..., s_m$ of S. Here $I_{a,s_1,\ldots,s_m,b}(F) \in K$ are the coefficients, providing a regular function

$$I_{a,s_1,\ldots,s_m,b} \in \mathcal{O}(\mathrm{Aut}_0(\mathrm{P}(S))$$

Observe that $I_{a,b}(F) = 1$ just means that F acts as the identity on (23).

We claim that the map

$$I_{a,s_1,\dots,s_m,b} \longrightarrow \mathbb{I}(a; s_1, ..., s_m; b)$$

provides the isomorphism (24). Indeed, it is obviously an isomorphism of commutative, graded F–algebras. So we get a).

The crucial fact that the coproduct in $\mathcal{O}\Big(\mathrm{Aut}_0(\mathrm{P}(S))\Big)$ is identified with Δ follows easily from the very definitions. So we have b).

c) The condition that F commutes with the $\circ$–product is equivalent to iii) plus iv) in the definition of the shuffle algebra. Indeed, the path composition formula iii) is equivalent to $F(p_{a,b}) = F(p_{a,x}) \circ F(p_{x,b})$. The $\circ$–unit is given by $\sum_{i \in S} p_{i,i}$. The condition iv) just means that F preserves the $\circ$–unit: $F(\sum p_{i,i}) = \sum p_{i,i}$.

Given that F preserves the both products, the fact that F commutes with δ is equivalent to the shuffle product formula i). Indeed, the condition

$$\delta F(p_{a,b}) = F\delta(p_{a,b}) \stackrel{(21)}{=} F(p_{a,b} \otimes p_{a,b}) = F(p_{a,b}) \otimes F(p_{a,b})$$

is just equivalent to the shuffle product formula. Since δ is completely determined by (21) and compatibilities with $*$ and $\circ$, the statement follows.

The part c) and hence the theorem are proved.

Proposition 2.2 follows from theorem 2.5.

3 The motivic fundamental groupoid and its Galois group

In this chapter we show that the basic properties of the motivic fundamental groupoid $\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; S)$ in one of the mixed Tate categories $\mathcal{C}$ described in s. 3.1 imply that the canonical fiber functor on $\mathcal{C}$ sends it to the path algebra $\mathrm{P}(S)$. This immediately implies theorem 1.1.

1. The Betti realization of the fundamental groupoid. Let S be a subset of $\mathbb{C}$. Choose a tangent vector v_s at every $s \in S$ such that $< dt, v_s >= 1$.

Let $\mathbb{C}_S := \mathbb{C} - S$. Let us recall the pronilpotent completion of the topological torsors of path. Let I_a be the augmentation ideal of the group ring $\mathbb{Z}[\pi_1(\mathbb{C}_S; a)]$ of the topological fundamental group $\pi_1(\mathbb{C}_S; a)$. Denote by $\mathcal{P}(\mathbb{C}_S; a, b)$ the set of homotopy classes of path between the tangential base points v_a, v_b. Then

$$\pi_1^{\mathrm{nil}}(\mathbb{C}_S; a) := \lim_{\longleftarrow} \mathbb{Z}[\pi_1(\mathbb{C}_S; a)]/I_a^n; \quad \mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; a, b) := \lim_{\longleftarrow} I_a^n \backslash \mathcal{P}(\mathbb{C}_S; a, b)$$

There are the path composition morphisms

$$\circ : \mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; a, b) \otimes \mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; b, c) \longrightarrow \mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; a, c)$$

They provide $\mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$ with a structure of a principal homogeneous space over $\pi_1^{\rm nil}(\mathbb{C}_S; a)$. There is a coproduct map

$$\delta : \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b) \longrightarrow \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b) \otimes \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$$

provided by the map $\gamma \longmapsto \gamma \otimes \gamma$. It is obviously compatible with the composition of path. It makes $\pi_1^{\rm nil}(\mathbb{C}_S; a)$ into a cocommutative Hopf algebra.

We have an increasing filtration $W_\bullet$ of $\mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$ indexed by $0, -2, -4, \ldots$:

$$W_{-2n}\mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b) := I_a^n \circ \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$$

There are canonical isomorphism

$$\mathrm{Gr}_0^W \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b) = \mathbb{Z}(0); \quad \mathrm{Gr}_{-2}^W \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b) = H_1(\mathbb{C}_S, \mathbb{Z}) = \mathbb{Z}[S]$$

Denote by $p_{a,b}$ the canonical generator of $\mathrm{Gr}_0^W \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$. The second isomorphism is provided by the map

$$s \in S \longmapsto p_{a,s,b} := p_{a,s} \circ ([\gamma_s] - [1]) \circ p_{s,b} \in \mathrm{Gr}_{-1}^W \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$$

where γ_s is a simple loop around s based at v_s, and 1 is the identity loop at v_s

Set

$$\mathcal{P}^{\rm nil}(\mathbb{C}_S; S) := \oplus_{a,b \in S} \mathcal{P}^{\rm nil}(\mathbb{C}_S; a, b)$$

The $*$–algebra structure on $\mathcal{P}^{\rm nil}(\mathbb{C}_S; S)(1)$ is given by

$$\alpha_{a,b} * \alpha_{b,c} := \alpha_{a,b} \circ ([\gamma_b] - [1]) \circ \alpha_{b,c}$$

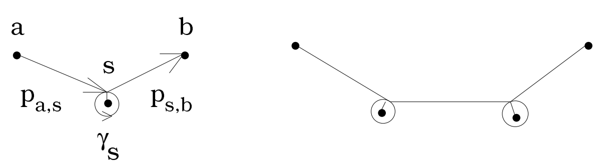
Lemma 3.1 *There is canonical isomorphism*

$$\mathrm{Gr}_\bullet^W \mathcal{P}^{\rm nil}(\mathbb{C}_S; S) = P(S)$$

It respects the grading, $*$–and $\circ$–algebra structures and the coproduct δ.

Proof. The isomorphism is given by

$$p_{a;s_1,\ldots,s_n;b} \longmapsto p_{a,s_1} \circ ([\gamma_{s_1}] - [1]) \circ p_{s_1,s_2} \circ ([\gamma_{s_2}] - [1]) \circ \ldots \circ p_{s_n,b}$$



The proof follows easily from the fact that, if S is finite, $\pi_1(\mathbb{C}_S; a)$ is a free group with $|S|$ generators, and hence $\pi_1^{\rm nil}(\mathbb{C}_S; a)]$ is the completion of the tensor algebra generated by $p_{a,s,a}$. The lemma is proved.

Now let us discuss the motivic fundamental groupoids.

2. The set–up. Let F be a field. Below we work in one of the following categories $\mathcal{C}$:

i) the abelian category of mixed Tate motives over a number field F ([G5], [L1]).

ii) $F = \mathbb{C}$, and $\mathcal{C}$ is the category of Hodge-Tate structures.

iii) F is an arbitrary field such that $\mu_{l^\infty} \notin F$, and $\mathcal{C}$ is the mixed Tate category of l-adic Tate $\mathrm{Gal}(\overline{F}/F)$–modules.

There is also one hypothetical set–up:

iv) F is an arbitrary field, $\mathcal{C}$ is the hypothetical abelian category of mixed Tate motives over F.

Any category $\mathcal{C}$ from the list above is a mixed Tate K–category, where $K = \mathbb{Q}$ in i), ii) and $K = \mathbb{Q}_l$ in iii) (see [BD] or chapter 3 of [G3] for the background). In particular it is generated as a tensor category by a simple object $K(1)$. Each object carries canonical weight filtration $W_\bullet$, and morphisms in $\mathcal{C}$ are strictly compatible with this filtration. There is canonical fiber functor to the category of finite dimensional graded vector spaces:

$$\omega : \mathcal{C} \longrightarrow \mathrm{Vect}_\bullet, \quad X \longmapsto \oplus_n Hom_{\mathcal{C}}(K(-n), \mathrm{gr}^W_{2n} X)$$

The space $\mathrm{End}(\omega)$ of the endomorphisms of the fiber functor is a graded Hopf algebra. We set

$$\mathcal{A}_\bullet(\mathcal{C}) := \quad \text{graded dual } \mathrm{End}(\omega)^\vee$$

The functor ω provides canonical equivalence between the category $\mathcal{C}$ and the category of finite dimensional graded $\mathcal{A}_\bullet(\mathcal{C})$–comodules.

$\mathrm{Spec}(\mathcal{A}_\bullet(\mathcal{C}))$ is a pro–unipotent group scheme. The grading on $\mathcal{A}_\bullet(\mathcal{C})$ encodes a natural semidirect product of $\mathbb{G}_m$ and $\mathrm{Spec}(\mathcal{A}_\bullet(\mathcal{C}))$.

3. The fundamental groupoid. Let S be any subset of $F = \mathbb{A}^1(F)$. A particular interesting case is $S = F$. Choose a tangent vector v_s at every point $s \in S$. We assume v_s is defined over F. The differential dt provides a canonical choice which is used below.

Let $\mathbb{A}^1_S := \mathbb{A}^1 - S$ and $\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; S)$ be the fundamental groupoid of paths on $\mathbb{A}^1_S$ between the tangential base points v_s. It is a pro–object in $\mathcal{C}$.

The fundamental groupoid in the situations ii) and iii) were defined by Deligne [D]. In the situation i) it is defined in [DG], (for another construction see chapter 6 of [G4]). In particular there are path composition morphisms in $\mathcal{C}$

$$\circ : \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b) \otimes \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; b, c) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, c) \tag{26}$$

They provide $\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b)$ with a structure of principal homogeneous space over the fundamental group $\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, a)$, understood as a Hopf algebra in $\mathcal{C}$.

4. The structures on the fundamental groupoid. Let us assume for a while that S is finite. Recall the symmetric monoidal category $Q_{\mathcal{C}}(S)$, see s. 2.3. We define an object $\mathcal{P}^{\mathcal{C}}(S)$ in $Q_{\mathcal{C}}(S)$ by

$$\mathcal{P}^{\mathcal{C}}(S)_{a,b} := \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b)$$

There are the following structures on this object.

i) The path composition morphisms (26) provide $\mathcal{P}^{\mathcal{C}}(S)$ with a structure of an algebra in $Q_{\mathcal{C}}(S)$, called the $\circ$–algebra structure.

ii) There are canonical "loop around s" morphisms

$$\gamma_s : \mathbb{Q}(1) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; s, s)$$

They provide morphisms

$$* : \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b)(1) \otimes \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; b, c)(1) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, c)(1)$$

$$\alpha_{a,b} * \alpha_{b,c} := \alpha_{a,b} \otimes \gamma_b \otimes \alpha_{b,c}$$

These morphisms make $\mathcal{P}^{\mathcal{C}}(S)(1)$ into an algebra in the category $Q_{\mathcal{C}}(S)$. We call it the $*$–algebra structure.

iii) For any $a, b \in S$ there is a coproduct given by a morphism in $\mathcal{C}$

$$\delta : \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b) \longrightarrow \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b) \otimes \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b)$$

It is an $\circ$–algebra morphism. One has

$$\delta(\gamma_b) = \gamma_b \otimes 1 + 1 \otimes \gamma_b; \tag{27}$$

It follows from this that the compatibility of δ with the $*$–algebra product is given by (20).

5. The path algebra provided by the fundamental groupoid and $G_{\mathcal{C}}(S)$. Let us apply the fiber functor ω to $\mathcal{P}^{\mathcal{C}}(S)$, getting

$$\mathcal{P}_\omega(S) := \omega(\mathcal{P}^{\mathcal{C}}(S))$$

Remark. Since $\mathcal{P}^{\mathcal{C}}(S)$ is a pro–object in $\mathcal{C}$, $\mathcal{P}_\omega(S)$ is a projective limit of finite dimensional K–vector spaces. However its graded components are finite dimensional.

Then $\mathcal{P}_\omega(S)$ is an algebra in the monoidal category $Q_{\mathcal{A}_\bullet(\mathcal{C})-mod}(S)$, and $\mathcal{P}_\omega(S)(1)$ is an $*$–algebra in the same category.

The space

$$\mathrm{Hom}(K(0), \mathrm{gr}^W_0 \mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b))$$

is one dimensional. Choose a non zero element $p_{a,b}$ of this space. One has

$$\delta(p_{a,b}) = p_{a,b} \otimes p_{a,b} \tag{28}$$

Set

$$p_{a;s_1,...,s_m;b} := p_{a,s_1} * p_{s_1,s_2} * ... * p_{s_{m-1},s_m} * p_{s_m,b} \tag{29}$$

It follows from (27) and (28) that δ is given by formula (18).

Proposition 3.2 *There is a natural isomorphism*

$$\mathcal{P}_\omega(S) \longrightarrow \mathrm{P}(S)$$

respecting the grading, $*-$ and $\circ-$algebra structures, and the coproduct on both objects.

Proof. The statement boils down to the fact that elements (29), when $\{s_1, ..., s_m\}$ run through all elements of S^m, form a basis in $\mathrm{gr}^W_{2m}\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b)$. Since $\mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; S)$ is the Betti realization of the Hodge version $\mathcal{P}^{\mathcal{H}}(\mathbb{A}^1_S; S)$, there is an isomorphism

$$\mathcal{P}_\omega(S)_{(n)} \cong \mathrm{Gr}^W_{-2n}\mathcal{P}^{\mathrm{nil}}(\mathbb{C}_S; S)$$

This plus lemma 3.1 implies the statement in the Hodge setting. Hence (by the injectivity of the regulators) it is true in the motivic situation i). The l-adic case is handled similarly using the comparison theorem with the Betti realization. The proposition is proved.

We define an element

$$\mathrm{I}^{\mathcal{C}}(a_0; a_1, a_2, ..., a_m; a_{m+1}) \in \mathcal{A}_m(\mathcal{C}) \tag{30}$$

as the linear functional on $\mathrm{End}(\omega)$ given by the matrix element

$$F \in \mathrm{End}(\omega) \longmapsto < F(p_{a,b}), p_{a;s_1,\ldots,s_m;b} >$$

One can describe the elements of $\mathcal{A}_\bullet(\mathcal{C})$ by framed objects in $\mathcal{C}$, see chapter 3 of [G3]. Then element (30) is represented by the following framed object in $\mathcal{C}$:

$$\Big(\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; a, b); p_{a,b}, p^*_{a;s_1,\ldots,s_m;b}\Big)$$

Here $p^*_{\ldots}$ are the elements of the basis dual to $p_{\ldots}$.

Recall the group $G(S)$ defined in chapter 2.

The group scheme $\mathrm{Spec}(\mathcal{A}_\bullet(\mathcal{C}))$ acts on $\mathcal{P}_\omega(S)$ through its quotient, denoted $G_{\mathcal{C}}(S)$. The semidirect product of $\mathbb{G}_m$ and $G_{\mathcal{C}}(S)$ is called the Galois group of the fundamental groupoid $\mathcal{P}^{\mathcal{C}}(\mathbb{A}^1_S; S)$.

Theorem 3.3 *a) $G(S)$ is the group of all automorphisms of the $*-$algebra $\mathcal{P}_\omega(S)(1)$ in the monoidal category $Q_{\mathcal{A}_\bullet(\mathcal{C})-mod}(S)$ preserving the $\circ-$algebra structure on $\mathcal{P}_\omega(S)$, commuting with the coproduct δ and acting as the identity on*

$$\mathcal{P}_\omega(S)(1)/(\mathcal{P}_\omega(S)(1))^2 \tag{31}$$

b) The map

$$\mathrm{I}(a_0; a_1, a_2, ..., a_m; a_{m+1}) \longrightarrow \mathrm{I}^{\mathcal{C}}(a_0; a_1, a_2, ..., a_m; a_{m+1}) \tag{32}$$

provides a surjective morphism of the Hopf algebras $\mathcal{O}(G(S)) \twoheadrightarrow \mathcal{O}(G_{\mathcal{C}}(S))$, and hence an inclusion of pro–unipotent group schemes

$$G_{\mathcal{C}}(S) \hookrightarrow G(S)$$

c) The coproduct is computed by the formula

$$\Delta \mathrm{I}^{\mathcal{C}}(a_0; a_1, a_2, ..., a_m; a_{m+1}) = \tag{33}$$

$$\sum_{0=i_0<i_1<...<i_k<i_{k+1}=m+1} \mathrm{I}^{\mathcal{C}}(a_0; a_{i_1}, ..., a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} \mathrm{I}^{\mathcal{C}}(a_{i_p}; a_{i_p+1}, ..., a_{i_{p+1}-1}; a_{i_{p+1}})$$

Proof. Part c) means simply that (32) commutes with the coproduct. Part b) follows immediately from a). And a) is a direct consequence of theorem 2.5 and proposition 3.2. The theorem is proved.

There is an equivalent version of theorem 3.3a):

$G(S)$ is the group of all automorphisms of the $\circ$–algebra $\mathcal{P}_\omega(S)$ in the monoidal category $Q_{\mathcal{A}_\bullet(\mathcal{C})-mod}(S)$ preserving the elements γ_s, the coproduct δ, and acting as identity on (31).

Indeed, the $*$–product is determined by the $\circ$–product and the elements γ_s.

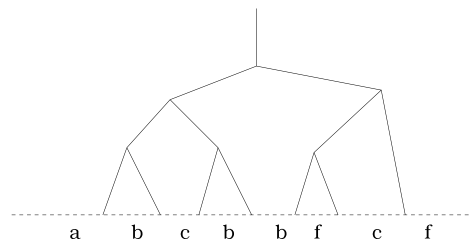
4 Iterated integrals and planar trivalent rooted trees

In this chapter we show how the Hopf algebra of decorated rooted planar trivalent trees encodes the properties of the motivic iterated integrals.

1. Terminology. A tree can have two types of edges: internal edges and legs. Both vertices of an internal edge are of valence ≥ 3. A valence 1 vertex of a leg is called an external vertex.

A planar tree is a tree located on the plane. We may picture planar trees inscribed into a circle. The external vertices divide this circle into a union of arcs. Let S be a set. An S–decoration of a planar tree is an S–valued function on the set of arcs.

A rooted tree is a tree with one distinguished leg called the root. We picture planar rooted trees growing down from the root, and put the external vertices of all the legs but the root on a line. These vertices divide this line into a union of arcs. They correspond to the arcs defined above, so we can talk about S–decorated planar rooted trees.



2. The Hopf algebra $\mathcal{T}_\bullet(S)$ of S–decorated planar rooted trivalent trees. It is a commutative, graded, Hopf algebra. As a vector space it is generated by disjoint unions of S–decorated planar rooted trivalent trees.

Precisely, consider a graded vector space with a basis given by S–decorated planar rooted trivalent trees with $n+2$ arcs, where $n \geq 1$ provides the grading. The vector space $\mathcal{T}_\bullet(S)$ is its symmetric algebra.

In other words $\mathcal{T}_\bullet(S)$ is the commutative algebra generated by S–decorated planar rooted trivalent trees with the only relation that a decorated planar tree with just one edge equals to 1 (regardless of the decoration).

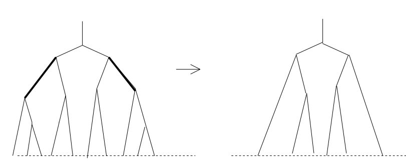
Let us define the coproduct $\Delta_{\mathcal{T}}$. Since $\Delta_{\mathcal{T}}$ has to be an algebra morphism, we have to define it only on the generators.

Let T be an S–decorated planar rooted trivalent tree. Consider the set

$$\mathcal{E}_T := \quad \{ \text{ all internal edges of } T \ \} \cup \{ \text{ the root of } T \ \}$$

An element $E \in \mathcal{E}_T$ determines a planar trivalent rooted tree T_E growing down from E: the edge E serves as the root of this tree. The tree T_E inherits a natural S–decoration: take the arcs containing the endpoints of the tree T_E and keep their decorations.

A subset $\{E_1, ..., E_k\}$ of $\mathcal{E}_T$ is called *admissible* if for any $i \neq j$ the edges E_i and E_j do not lie on the same path going down from the root. In other words E_i is not contained in the tree T_{E_j} if $i \neq j$. Such an admissible subset determines a connected S–decorated planar trivalent tree $T/(T_{E_1} \cup ... \cup T_{E_k})$. Namely, this tree is obtained by shrinking each of the rooted trees $T_{E_1}, ..., T_{E_k}$ into legs of a new tree: we shrink the domain encompassed by each of these trees and the bottom line. In particular we shrank to points all the arcs on the bottom line located under these trees. The remaining arcs with the inherited decorations are the arcs of the new tree. See the picture where we shrank the two thick edges:



Now we set

$$\Delta_{\mathcal{T}}(T) := \sum \frac{T}{T_{E_1} \cup ... \cup T_{E_k}} \otimes \prod_{i=1}^{k} T_{E_i}$$

where the sum is over all admissible subsets of $\mathcal{E}(T)$, including the empty subset and the subset formed by the root.

Lemma 4.1 *$\Delta_{\mathcal{T}}$ provides $\mathcal{T}_\bullet(S)$ with a structure of a commutative, graded Hopf algebra.*

Proof. Essentially the same as in [CK].

Remark. The Hopf algebra $\mathcal{T}_\bullet(S)$ is a Hopf subalgebra of the similar Hopf algebra of all decorated rooted planar trees (not necessarily trivalent). It is the planar decorated version of the one defined by Connes and Kreimer [CK].

We say that a planar rooted tree is *decorated by an ordered set* $\{s_0, s_1, ..., s_m, s_{m+1}\}$ if the tree has $m+1$ bottom legs, and the corresponding $m+2$ arcs are decorated, in their natural order from left to right, by the set above.

We decorate the arcs, not legs – see however the remark below.

Theorem 4.2 *The map*

$$t : \mathbb{I}(s_0; s_1, ..., s_m; s_{m+1}) \longmapsto \text{sum of all planar rooted trivalent trees}$$

$$\text{decorated by the ordered set } \{s_0, s_1, ..., s_m, s_{m+1}\}$$

provides an injective homomorphism of commutative, graded Hopf algebras

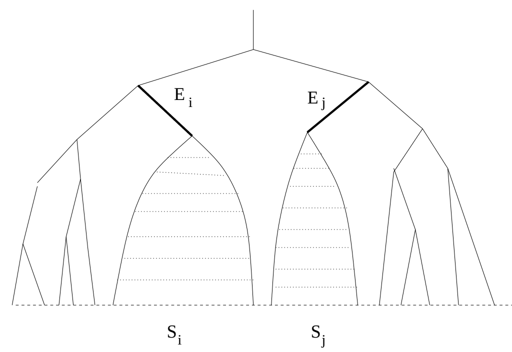
$$t : \widetilde{\mathcal{I}}_\bullet(S) \hookrightarrow \mathcal{T}_\bullet(S) \tag{34}$$

Proof. We need to check only that our map commutes with the coproducts. Let T be a generator of $\mathcal{T}_\bullet(S)$, decorated by an ordered set $\{s_0, ..., s_{m+1}\}$.

Observe that an edge E determines a subset $S_E \subset \{s_0, ..., s_{m+1}\}$ consisting of the labels of all arcs located between the very left and right bottom legs of the tree T_E, i.e. just under this tree. Different edges produce different subsets.

Let $\{E_1, ..., E_k\}$ be an admissible subset of $\mathcal{E}_T$. The subsets S_i corresponding to the edges E_i are disjoint – this is a reformulation of the definition of an admissible subset. In fact they are even separated: there is at least one arc between the arcs labeled by S_i and S_{i+1}.

We claim that the sum over all trees decorated by $\{s_0, ..., s_{m+1}\}$, with such subsets $S_1, .., S_k$ given to us, corresponds to a single term of the formula $\Delta\mathbb{I}(s_0; s_1, ..., s_m; s_{m+1})$. Namely, if we enumerate the terms in the formula using (4), this term corresponds to the subset $\{s_0, ..., s_{m+1}\} - (S_1 \cup ... \cup S_k)$.



To verify this claim observe two things:

i) given such a tree T we can alter the subtree T_{E_i} to any other subtree with the same root E_i and the same set of arcs under this tree, labeled by S_i.

ii) if we shrink the trees $T_{E_1}, ..., T_{E_k}$ the result can be any tree labeled by the ordered set $\{s_0, ..., s_{m+1}\} - (S_1 \cup ... \cup S_k)$.

The theorem is proved.

Remark. The element $\mathrm{I}^{\mathcal{C}}(s_0; s_1, ..., s_m; s_{m+1})$ is invariant under the action of the translation group of the affine line. Therefore it is natural to encode it by orbits of the action

$$\{s_0, s_1, ..., s_m, s_{m+1}\} \longrightarrow \{s_0 + a, s_1 + a, ..., s_m + a, s_{m+1} + a\}$$

of the additive group $\mathbb{A}^1$ on the set of $(m+2)$–tuples $\{s_0, s_1, ..., s_m, s_{m+1}\}$. These orbits are described by the $(m+2)$–tuples

$$\{s_0 - s_{m+1}, s_1 - s_0, s_2 - s_1, ..., s_{m+1} - s_m\}$$

which naturally sit not at the arcs, but rather at the legs of the corresponding trees.

3. The $\otimes^m$–invariant of variations of mixed Tate structures ([G5], s. 5.1). Let $\mathcal{V}$ be a variation of mixed Tate objects over a smooth base B in one of our set–ups ii) – iv). So $\mathcal{V}$ is a unipotent variation of Hodge–Tate structures in ii), a lisse l-adic mixed Tate sheaf in iii), and a (yet hypothetical) variation of mixed Tate motives in iv). We assume that $\mu_{l^\infty} \not\subset \mathcal{O}^*_M$ in iii).

Then $\mathcal{V}$ is an object of an appropriate mixed Tate category $\mathcal{C}_B$ of variations of mixed Tate objects over B. The Tate object $K(m)_M$ is given by $K(m)_B := p^*K(m)$ where $p : B \longrightarrow \mathrm{Spec}(F)$ is the structure morphism. We can use the standard formalism of mixed Tate categories (see [BD] or [G3], chapter 3). In particular there is a commutative, graded Hopf K–algebra $\mathcal{A}^{\mathcal{C}}_{\bullet}(B)$ with the coproduct Δ, the fundamental Hopf algebra of this category. One has

$$\mathcal{A}^{\mathcal{C}}_1(B) = \mathcal{O}^*(B)_K := \mathcal{O}^*(B) \otimes_{\mathbb{Q}} K$$

For any positively graded (coassociative) Hopf algebra $\mathcal{A}_\bullet$ there is canonical map

$$\Delta^{[m]} : \mathcal{A}_m \longrightarrow \otimes^m \mathcal{A}_1$$

Namely, it is dual to the multiplication map $\otimes^m \mathcal{A}_1^\vee \longrightarrow \mathcal{A}_m^\vee$, and can be defined as the composition

$$\mathcal{A}_m \stackrel{\Delta}{\longrightarrow} \mathcal{A}_{m-1} \otimes \mathcal{A}_1 \stackrel{\Delta\otimes\mathrm{Id}}{\longrightarrow} \mathcal{A}_{m-2} \otimes \mathcal{A}_1 \otimes \mathcal{A}_1 \stackrel{\Delta\otimes\mathrm{Id}\otimes\mathrm{Id}}{\longrightarrow} \ldots \stackrel{\Delta\otimes\mathrm{Id}\otimes\ldots\otimes\mathrm{Id}}{\longrightarrow} \otimes^m \mathcal{A}_1$$

Let us choose a $K(m)_B$–framing of the variation $\mathcal{V}$. Then, by the very definition, we get an element

$$[\mathcal{V}] \in \mathcal{A}_m^{\mathcal{C}}(B)$$

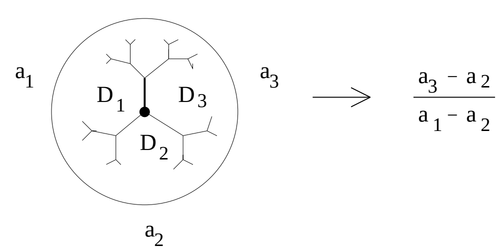
Definition 4.3 *The element*

$$\Delta^{[m]}\Big([\mathcal{V}]\Big) \in \otimes^m \mathcal{O}^*(B)_K$$

is called the $\otimes^m$–invariant of a framed variation of mixed Tate objects over B.

4. The $\otimes^m$–invariant of the multiple logarithm variation and planar rooted trivalent trees. Let $\mathcal{M}_{m+2}(\mathbb{A}^1)$ be the space of $m+2$ ordered distinct points $(a_0, ..., a_{m+1})$ on the affine line. Every internal vertex v of a planar trivalent rooted tree T provides an invertible function f_v^T on $\mathcal{M}_{m+2}(\mathbb{A}^1)$. Indeed, if we picture a planar trivalent tree inscribed into a circle, the complement to the tree in the disc bounded by this circle is a union of several connected domains. These domains are in bijective correspondence with the arcs on the circle defined by the tree.

An internal vertex v determines three such domains D_1^v, D_2^v, D_3^v, so that v shares the boundaries of these domains. The plane structure of the tree plus the orientation of the plane provides a cyclic order of these domains, and the root of the tree provides a natural order of these domains.



The function corresponding to an internal vertex of a decorated rooted planar trivalent tree

Precisely, among the edges sharing the vertex v one edge is closer to the root than the other. We count the domains going clockwise from this edge. We will assume that the numeration D_1^v, D_2^v, D_3^v reflects this order. Let a_1^v, a_2^v, a_3^v be the labels of the arcs assigned to the domains D_1^v, D_2^v, D_3^v. Set

$$f_v^T := \frac{a_3^v - a_2^v}{a_1^v - a_2^v} \in \mathcal{O}^*(\mathcal{M}_{m+2}(\mathbb{A}^1))$$

The internal vertices of a planar trivalent rooted tree have *canonical partial order* $<$: we have $v_1 < v_2$ if and only if there is a path going down from the root such that both vertices are located on this path, and v_1 is closer to the root than v_2. We say that an ordering $(v_1, ..., v_m)$ of internal vertices of a planar rooted tree T is *compatible* with canonical order if $v_i < v_j$ implies $i < j$.

Definition 4.4

$$\Omega_m := \sum_T \sum_{\{v_1,...,v_m\}} f_{v_1}^T \otimes ... \otimes f_{v_m}^T \in \otimes^m \mathcal{O}^*(\mathcal{M}_{m+2}(\mathbb{A}^1))$$

Here the first sum is over all different planar trivalent rooted trees T with $m+1$ leaves, and the second sum is over all orderings $\{v_1, ..., v_m\}$ of the set of internal vertices of the tree T compatible with the canonical partial order on this set.

Proposition 4.5

$$\Omega_m = \Delta^{[m]} \mathrm{I}(a_0; a_1, ..., a_m; a_{m+1})$$

Proof. Follows easily from formula (33) by induction on m.

5 The motivic multiple polylogarithms Hopf algebras

In this section we use formula (33) to generalize some of the results from chapter 6 in [G3] from the Hodge setting to the more general settings described in s. 3.1. Then we define the cyclotomic Hopf and Lie algebras. This and the next chapters replace sections 6.3 - 6.7 in [G3].

1. The basic formula for the coproduct. In this section we work with the Hopf algebra $\mathcal{I}_\bullet(S)$ where S is any set containing a distinguished element 0. Set

$$\mathbb{I}_{n_0,n_1+1,...,n_m+1}(a_0; a_1, .., a_m; a_{m+1}) := \tag{35}$$

$$\mathbb{I}(a_0; \underbrace{0, ..., 0,}_{n_0 \text{ times}}\ a_1, \underbrace{0, ..., 0,}_{n_1 \text{ times}}\ a_2, \quad ..., \quad \underbrace{0, ..., 0}_{n_m \text{ times}}\ ; a_{m+1})$$

Let V be a vector space. Denote by $V[[t_1, ..., t_m]]$ the vector space of the formal power series in t_i whose coefficients are vectors of V. Let us make the generating series

$$\mathbb{I}(a_0; a_1, ..., a_m; a_{m+1}|t_0; t_1; ...; t_m) := \tag{36}$$

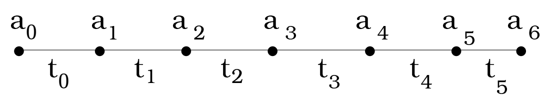
$$\sum_{n_i \geq 0} \mathbb{I}_{n_0, n_1+1, \ldots, n_m+1}(a_0; a_1, ..., a_m; a_{m+1}) t_0^{n_0} ... t_m^{n_m} \in \quad \mathcal{I}_\bullet(S)[[t_0, ..., t_m]]$$

To visualize them consider a line segment with the following additional data, called *decoration*:

i) The beginning of the segment is labeled by a_0, the end by a_{m+1}.

ii) There are m points *inside* of the segment labeled by $a_1, ..., a_m$ from the left to the right.

iii) These points cut the segment on $m+1$ arcs labeled by $t_0, t_1, ..., t_m$.



A decorated segment

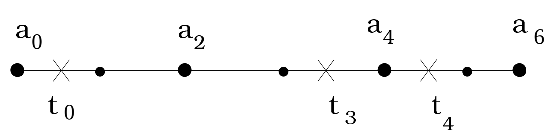
Remark. The way t's sit between the a's reflects the shape of the iterated integral $\mathrm{I}_{n_0,...,n_m}(a_0; a_1, ..., a_m; a_{m+1})$.

As we show in theorem 5.1 below, the terms of the coproduct of the element (36) correspond to the decorated segments equipped with the following additional data, called marking:

a) Mark (by making them boldface on the picture) points $a_0; a_{i_1}, ..., a_{i_k}; a_{m+1}$ so that

$$0 = i_0 < i_1 < ... < i_k < i_{k+1} = m+1 \tag{37}$$

b) Mark (by cross) segments $t_{j_0}, ..., t_{j_k}$ such that there is just one marked segment between any two neighboring marked points.



A marked decorated segment

The conditions on the crosses for a marked decorated segment just mean that

$$i_\alpha \leq j_\alpha < i_{\alpha+1} \quad \text{for any } 0 \leq \alpha \leq k \tag{38}$$

The marks provide a new decorated segment:

$$(a_0 | t_{j_0} | a_{i_0} | t_{j_1} | a_{i_1} | \quad ... \quad | a_{i_k} | t_{j_k} | a_{m+1}) \tag{39}$$

Theorem 5.1

$$\Delta \mathbb{I}(a_0; a_1, ..., a_m; a_{m+1} | t_0; ...; t_m) = \tag{40}$$

$$\sum \mathbb{I}(a_0; a_{i_1}, ..., a_{i_k}; a_{m+1} | t_{j_0}; t_{j_1}; ...; t_{j_k}) \otimes$$

$$\prod_{\alpha=0}^{k}\Big(\mathbb{I}(a_{i_\alpha}; a_{i_\alpha+1}, ..., a_{j_\alpha}; 0|t_{i_\alpha}; ...; t_{j_\alpha})\cdot$$

$$\mathbb{I}(0; a_{j_\alpha+1}, ..., a_{i_{\alpha+1}-1}; a_{i_{\alpha+1}}|t_{j_\alpha}; t_{j_\alpha+1}; ...; t_{i_{\alpha+1}-1})\Big)$$

where the sum is over all marked decorated segments, i.e. over all sequences $\{i_\alpha\}$ *and* $\{j_\alpha\}$ *satisfying inequality (38).*

Proof. Recall that by iv) in section 2.1 and theorem 3.3b) one has

$$\mathbb{I}(0; a_1, ..., a_m; 0) = 0 \tag{41}$$

To calculate (40) we apply the coproduct formula (33) to the element (35) and then keep track of the nonzero terms using (41).

The left hand side factors of the nonzero terms in the formula for the coproduct correspond to certain subsets

$$A \subset \{a_0;\ \underbrace{0, ..., 0,}_{n_0 \text{ times}}\ a_1,\ \underbrace{0, ..., 0,}_{n_1 \text{ times}}\ \ \ ...\ \ \ , a_m,\ \underbrace{0, ..., 0}_{n_m \text{ times}}\ ; a_{m+1}\}$$

containing a_0 and a_{m+1}, and called the admissible subsets. Such a subset A determines the subset $I = \{i_1 < ... < i_k\}$ where $a_0, a_{i_1}, ..., a_{i_k}, a_{m+1}$ are precisely the set of all a_i's containing in A. A subset A is called *admissible* if it satisfies the following properties:

i) A contains a_0 and a_{m+1}.

ii) The set of 0's in A located between a_{i_α} and $a_{i_{\alpha+1}}$ must be a string of *consecutive* 0's located between a_{j_α} and $a_{j_\alpha+1}$ for some $i_\alpha \leq j_\alpha < i_{\alpha+1}$.

In other words, the factors in the coproduct are parametrized by

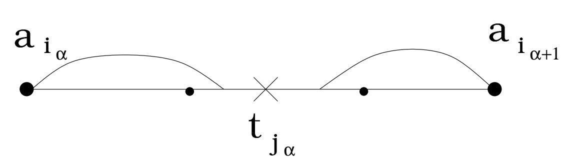
a marked decorated segment and *a connected string of* 0*'s in each of the crossed arcs.*

The connected string of 0's in some of the crossed arcs might be empty.

The string of zero's between a_{i_α} and $a_{i_\alpha+1}$ satisfying ii) looks as follows:

$$\{a_{i_\alpha}, \widehat{a}_{j_\alpha}, \underbrace{\widehat{0}, ..., \widehat{0},}_{p_{j_\alpha}} \underbrace{0, ..., 0,}_{s_{j_\alpha}} \underbrace{\widehat{0}, ..., \widehat{0},}_{q_{j_\alpha}} \widehat{a}_{j_\alpha+1}, a_{i_\alpha}\} \tag{42}$$

where $p_{j_\alpha} + q_{j_\alpha} + s_{j_\alpha} = n_{j_\alpha}$. This notation emphasizes that all 0's located between a_{i_α} and $a_{i_{\alpha+1}}$ are in fact located between a_{j_α} and $a_{j_\alpha+1}$, and form a connected segment of length s_{j_α}.



An admissible subset A provides the following element, which is the left hand side of the corresponding term in the coproduct:

$$\mathbb{I}(a_0; \widehat{a}_{j_0}, \underbrace{0, ..., 0,}_{s_{j_0} \text{ times}} \widehat{a}_{j_0+1}, a_{i_1}, \widehat{a}_{j_1}, \underbrace{0, ..., 0,}_{s_{j_1} \text{ times}} \widehat{a}_{j_1+1}, \quad ... \quad ; a_{m+1})$$

The right hand side of the term in the coproduct corresponding to the subset A is a product over $0 \leq \alpha \leq k$ of elements of the following shape:

$$\mathbb{I}(a_{i_\alpha}; \underbrace{0, ..., 0,}_{n_{i_\alpha}} a_{i_\alpha+1}, \underbrace{0, ..., 0,}_{n_{i_\alpha+1}} \quad ... \quad a_{j_\alpha}, \underbrace{0, ..., 0}_{p_{j_\alpha}}; 0) \cdot \underbrace{\mathbb{I}(0;0) \cdot ... \cdot \mathrm{I}(0;0)}_{s_{j_\alpha}}$$

$$\mathbb{I}(0; \underbrace{0, ..., 0,}_{q_{j_\alpha}} a_{j_\alpha+1}, \underbrace{0, ..., 0,}_{n_{j_\alpha+1}} \quad ... \quad a_{i_{\alpha+1}-1}, \underbrace{0, ..., 0}_{n_{i_{\alpha+1}-1}}; a_{i_{\alpha+1}})$$

where the middle factor $\mathbb{I}(0;0) \cdot ... \cdot \mathbb{I}(0;0)$ is equal to 1 since $\mathbb{I}(0;0) = 1$ according to (41). Translating this into the language of the generating series we get the promised formula for the coproduct. The theorem is proved.

A geometric interpretation of formula (40) . It is surprisingly similar to the one for the multiple logarithm element. Recall that the expression (36) is encoded by a decorated segment

$$(a_0|t_0|a_1|t_1| \quad ... \quad |a_m|t_m|a_{m+1}) \tag{43}$$

The terms of the coproduct are in the bijection with the marked decorated segments S obtained from a given one (43). Denote by $L_S \otimes R_S$ the term in the coproduct corresponding to S. The left factor L_S is encoded by the decorated segment (39) obtained from the marked points and arcs. For example for the marked decorated segment on the picture we get $L_S = \mathbb{I}(a_0|t_0|a_2|t_3|a_4|t_4|a_6)$.

The marks (which consist of $k+1$ crosses and $k+2$ boldface points) determine a decomposition of the segment (43) into $2(k+1)$ little decorated segments in the following way. Cutting the initial segment in all the marked points and crosses we get $2(k+1)$ little segments. For instance, the very right one is the segment between the last cross and point a_m, and so on. Each of these segments either starts from a marked point and ends by a cross, or starts from a cross and ends by a marked point.

There is a natural way to make a *decorated* segment out of each of these little segments: mark the "cross endpoint" of the little segment by the point 0, and for the arc which is just near to this marked point use the letter originally attached to the arc containing it. For example the marked decorated segment on the picture above produces the following sequence of little decorated segments:

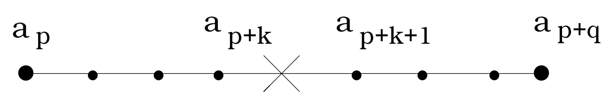
$$(a_0|t_0|0), \quad (0|t_0|a_1|t_1|a_2) \quad (a_2|t_2|a_3|t_3|0) \quad (0|t_3|a_4) \quad (a_4|t_4|0) \quad (0|t_4|a_5|t_5|a_6)$$

Then the factor R_S is the product of the generating series for the elements corresponding to these little decorated segments. For example for the marked decorated segment on the picture we get

$$R_S = \mathbb{I}(a_0; 0|t_0) \cdot \mathbb{I}(0; a_1; a_2|t_0; t_1) \cdot$$

$$\mathbb{I}(a_2; a_3; 0|t_2; t_3) \cdot \mathbb{I}(0; a_4|t_3) \cdot \mathbb{I}(a_4; 0|t_4) \cdot \mathbb{I}(0; a_5; a_6|t_4; t_5)$$

To check that formula for the coproduct of the multiple logarithms fits into this description we use the path composition formula iii) provided by theorem 3.3b), together with the fact that each term on the right hand side of this formula correspond to a marked colored segment shown on the picture:



2. The multiple polylogarithms Hopf algebra of a subgroup $G \subset F^*$. Now let us suppose that we work in one of the set ups i)–iv), and F is the corresponding field. Let $S := F$. We define the $\mathrm{I}^{\mathcal{C}}$–elements as the images of the corresponding $\mathbb{I}$–elements by the homomorphism established in theorem 32. Below we choose a coordinate t on the affine line, and use the standard tangent vectors v_s dual to dt. By theorem 32. the results of the previous section are valid for the corresponding $\mathrm{I}^{\mathcal{C}}$–elements.

The $\mathrm{I}^{\mathcal{C}}$–analogs of elements (35) are clearly invariant under the translations. In general they are not invariant under the action of $\mathbb{G}_m$ given by $a_i \longmapsto \lambda a_i$. However, if the iterated integral $\mathrm{I}_{n_0,\ldots,n_m}(a_0; a_1, ..., a_m; a_{m+1})$ is convergent, then the corresponding element is invariant under the action of the affine group, see lemma 5.3 below.

Let $a \in F^*$. We set

$$a := \log^{\mathcal{C}}(a) := \mathrm{I}^{\mathcal{C}}(0; 0; a) \in \mathcal{A}_1^{\mathcal{C}}$$

It is easy to check that in any of the set ups i) -iii), and hypothetically in iv), it provides an extension class in

$$\mathrm{Ext}^1_{\mathcal{C}}(K(0), K(1)) \stackrel{\sim}{=} F^* \otimes_{\mathbb{Q}} K$$

which corresponds to the element $a \in F^*$. Set

$$a^t := \exp(\log^{\mathcal{C}}(a) \cdot t) \in \mathcal{A}^{\mathcal{C}}_{\bullet} \widehat{\otimes} k[[t]]$$

The shuffle product formula implies

$$\mathrm{I}^{\mathcal{C}}(0; a|t) = a^t$$

One has

$$\Delta(1) = 1 \otimes 1, \quad \Delta(a) = a \otimes 1 + 1 \otimes a$$

It follows that

$$\Delta(a^t) = a^t \otimes a^t \tag{44}$$

Suppose that $a_i \neq 0$. Set

$$\mathrm{I}^{\mathcal{C}}_{n_1,\ldots,n_m}(a_1, ..., a_m) :=$$
$$\mathrm{I}^{\mathcal{C}}(0; a_1, \underbrace{0, ..., 0}_{n_1 \text{ zeros}}, a_2, \underbrace{0, ..., 0}_{n_2 \text{ zeros}}, ..., a_m, \underbrace{0, ..., 0}_{n_m \text{ zeros}}; 1)$$

Let us package them into the generating series

$$\mathrm{I}^{\mathcal{C}}(a_1, ..., a_m | t_1, ..., t_m) := \tag{45}$$
$$\sum_{n_i \geq 1} \mathrm{I}^{\mathcal{C}}_{n_1,\ldots,n_m}(a_1, ..., a_m) t_1^{n_1 - 1} ... t_m^{n_m - 1} \in \quad \mathcal{A}_{\bullet}(\mathcal{C})[[t_1, ..., t_m]]$$

Lemma 5.2 *Suppose that $a_i \neq 0$. Then*

$$\mathrm{I}^{\mathcal{C}}(0; a_1, ..., a_m; a_{m+1} | t_0; ...; t_m) = \tag{46}$$
$$a_{m+1}^{t_0} \mathrm{I}^{\mathcal{C}}(\frac{a_1}{a_{m+1}}, ..., \frac{a_m}{a_{m+1}} | t_1 - t_0, ..., t_m - t_0)$$

Proof. This is a motivic version of proposition 2.15 in [G3]. It follows by induction using the shuffle product formula for

$$\sum_{n \geq 0} \mathrm{I}^{\mathcal{C}}(0; \underbrace{0, ..., 0}_{n \text{ times}}; a_{m+1}) t_0^n * \sum_{n_i > 0} \mathrm{I}^{\mathcal{C}}_{n_1,\ldots,n_m}(0; a_1, ..., a_m; a_{m+1}) t_1^{n_1 - 1} ... t_m^{n_m - 1}$$

just as in the second proof of proposition 2.15 in loc. cit.. The lemma is proved.

Lemma 5.3 *One has*

$$\mathrm{I}^{\mathcal{C}}(a_1; a_2, ..., a_m; 0 | t_1; ...; t_m) =$$
$$(-1)^{m-1} \mathrm{I}(0; a_m, ..., a_2; a_1 | -t_m; -t_{m-1}; ...; -t_1) \tag{47}$$

In particular $\mathrm{I}^{\mathcal{C}}(a; 0 | t) = a^{-t}$.

Proof. This is a special case of the equality

$$\mathrm{I}^{\mathcal{C}}(a_1; a_2, ..., a_m; 0) = (-1)^{m-1} \mathrm{I}^{\mathcal{C}}(0; a_m, ..., a_2; a_1)$$

provided by proposition 2.1. The lemma is proved.

A marked decorated segment is *special* if the first cross is marking the segment t_0. A decorated segment with such a data is called a *special marked decorated segment*. See an example on the picture above. The conditions on the crosses for a marked decorated segment just mean that

$$i_\alpha \leq j_\alpha < i_{\alpha+1} \quad \text{for any } 0 \leq \alpha \leq k, \qquad j_0 = i_0 = 0 \tag{48}$$

We will employ the notation

$$\mathrm{I}^{\mathcal{C}}(a_1 : ... : a_{m+1} | t_0 : ... : t_m) := \quad \mathrm{I}^{\mathcal{C}}(0; a_1, ..., a_m; a_{m+1} | t_0; ...; t_m)$$

Theorem 5.4 *One has*

$$\Delta \mathrm{I}^{\mathcal{C}}(a_1 : ... : a_{m+1}|t_0 : ... : t_m)$$

$$\sum \mathrm{I}^{\mathcal{C}}(a_{i_1} : ... : a_{i_k} : a_{m+1}|t_{j_0} : t_{j_1} : ... : t_{j_k})\otimes$$

$$\prod_{\alpha=0}^{k}\Big((-1)^{j_\alpha - i_\alpha}\mathrm{I}^{\mathcal{C}}(a_{j_\alpha} : a_{j_\alpha-1} : ... : a_{i_\alpha}| - t_{j_\alpha} : -t_{j_\alpha-1} : ... : -t_{i_\alpha})\cdot \qquad (49)$$

$$\mathrm{I}^{\mathcal{C}}(a_{j_\alpha+1} : ... : a_{i_{\alpha+1}-1} : a_{i_{\alpha+1}}|t_{j_\alpha} : t_{j_\alpha+1} : ... : t_{i_{\alpha+1}-1})\Big)$$

where the sum is over all special marked decorated segments, i.e. over all sequences $\{i_\alpha\}$ and $\{j_\alpha\}$ such that

$$i_\alpha \leq j_\alpha < i_{\alpha+1} \quad \textit{for any } 0 \leq \alpha \leq k, \qquad j_0 = i_0 = 0, i_{k+1} = i_{m+1} \qquad (50)$$

Proof. It follows immediately from theorem 5.1 using lemmas 5.2 and 5.3.

Now let G be a subgroup of F^*. Denote by $\mathcal{Z}^{\mathcal{C}}_w(G) \subset \mathcal{A}^{\mathcal{C}}_w(F)$ the $\mathbb{Q}$-vector subspace generated by the elements

$$\mathrm{I}^{\mathcal{C}}_{n_1,...,n_m}(a_1, ..., a_m), \quad a_i \in G \quad w = n_1 + ... + n_m \qquad (51)$$

Set

$$\mathcal{Z}^{\mathcal{C}}_\bullet(G) := \oplus_{w\geq 1}\mathcal{Z}^{\mathcal{C}}_w(G)$$

It is equipped with a depth filtration $\mathcal{F}^{\mathcal{D}}_\bullet$ defined as follows:

$\mathcal{F}^{\mathcal{D}}_0\mathcal{Z}^{\mathcal{C}}_\bullet(G)$ is spanned by products of $\log^{\mathcal{C}}(a)$, $a \in G$, and

$\mathcal{F}^{\mathcal{D}}_k\mathcal{Z}^{\mathcal{C}}_\bullet(G)$ for $k \geq 1$ is spanned by the elements (51) with $m \leq k$.

Theorem 5.5 *Let G be any subgroup of F^*. Then $\mathcal{Z}^{\mathcal{C}}_\bullet(G)$ is a graded Hopf subalgebra of $\mathcal{A}^{\mathcal{C}}_\bullet$. The depth provides a filtration on this Hopf algebra.*

Proof. The graded vector space $\mathcal{Z}^{\mathcal{C}}_\bullet(G)$ is closed under the coproduct by theorem 5.4. The statement about the depth filtration is evident from the formula for coproduct given in proposition 6.1. It is a graded algebra by the shuffle product formula. The theorem is proved.

Remark. The depth filtration is not defined by a grading of the algebra $\mathcal{Z}^{\mathcal{C}}_\bullet(G)$ because of the relations like

$$\mathrm{Li}^{\mathcal{C}}_n(x)\cdot \mathrm{Li}^{\mathcal{C}}_m(y) \quad = \quad \mathrm{Li}^{\mathcal{C}}_{n,m}(x,y) + \mathrm{Li}^{\mathcal{C}}_{m,n}(y,x) + \mathrm{Li}^{\mathcal{C}}_{n+m}(xy)$$

Conjecture 5.6 *The Hopf algebra $\mathcal{Z}^{\mathcal{M}}_\bullet(F^*)$ is isomorphic to the motivic Hopf algebra $\mathcal{A}^{\mathcal{M}}_\bullet(F)$ of F.*

3. The cyclotomic Lie algebras. Let μ_N be the group of N-th roots of unity. Set

$$\mathcal{C}^{\mathcal{M}}_\bullet(\mu_N) := \frac{\mathcal{Z}^{\mathcal{M}}_{>0}(\mu_N)}{\mathcal{Z}^{\mathcal{M}}_{>0}(\mu_N)\cdot \mathcal{Z}^{\mathcal{M}}_{>0}(\mu_N)}$$

Corollary 5.7 *a)* $\mathcal{C}_\bullet^{\mathcal{M}}(\mu_N)$ *is a graded Lie coalgebra.*

b). $\mathcal{C}_1^{\mathcal{M}}(\mu_N) = \mathcal{Z}_1^{\mathcal{M}}(\mu_N) \stackrel{\sim}{=} \Big($*the group of cyclotomic units in* $\mathbb{Z}[\zeta_N][1/N]\Big) \otimes \mathbb{Q}$.

Proof. a). Clear.

b) Since

$$\mathrm{I}_1^{\mathcal{M}}(a) = \log^{\mathcal{M}}(1-a) \quad \text{and} \quad \log^{\mathcal{M}}(1-a) - \log^{\mathcal{M}}(1-a^{-1}) = \log^{\mathcal{M}}(a)$$

the weight 1 component $\mathcal{Z}_1^{\mathcal{M}}(G)$ is generated by $\log^{\mathcal{M}}(1-a)$ and $\log^{\mathcal{M}}(a)$. Notice that if $a^N = 1$ then $N \cdot \log^{\mathcal{M}}(a) = 0$. This proves b). The corollary is proved.

We call $\mathcal{C}_\bullet^{\mathcal{M}}(\mu_N)$ the *cyclotomic Lie coalgebra*. Its dual $C_\bullet^{\mathcal{M}}(\mu_N)$ is the *cyclotomic Lie algebra*. The dual to the universal enveloping algebra of the cyclotomic Lie algebra is isomorphic to $\mathcal{Z}_\bullet^{\mathcal{M}}(\mu_N)$.

Let $\mathcal{C}_m^{\Delta}(\mu_N)$ be the $\mathbb{Q}$–subspace of $\mathcal{C}_m^{\mathcal{M}}(\mu_N)$ generated by $\mathrm{I}^{\mathcal{M}}(0; a_1, ..., a_m; 1)$ where $a_i^N = 1$.

Corollary 5.8 $\mathcal{C}_\bullet^{\Delta}(\mu_N) := \oplus_{m\geq 1}\mathcal{C}_m^{\Delta}(\mu_N)$ *is a graded Lie coalgebra.*

We call it the *diagonal cyclotomic Lie coalgebra.*

Proof. Clear from the theorem 5.4.

6 Examples

In this chapter we calculate the coproduct on the generators corresponding to multiple polylogarithms, and give several examples.

1. The $\mathrm{Li}^{\mathcal{C}}$**–generators of the multiple polylogarithm Hopf algebra**. Let us define several other generating series for the multiple polylogarithms elements. Consider the following two pairs of sets of variables:

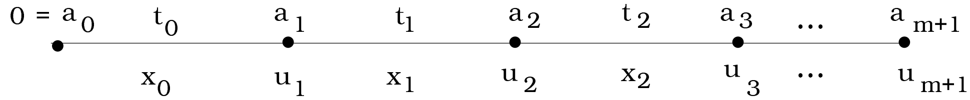
$$i) \quad (x_0, ..., x_m) \quad \text{such that } x_0 ... x_m = 1; \qquad ii) \quad (a_1 : ... : a_{m+1})$$

$$iii) \quad (t_0 : ... : t_m); \qquad iv) \quad (u_1, ..., u_{m+1}) \quad \text{such that } u_1 + ... + u_{m+1} = 0$$

The relationship between them is given by

$$x_i := \frac{a_{i+1}}{a_i}, \quad i = 1, ..., m+1; \quad x_0 = \frac{a_1}{a_{m+1}}$$
$$u_i := t_i - t_{i-1}, \quad u_{m+1} := t_0 - t_m$$

the indices are modulo $m+1$, and is illustrated on the picture below



Observe that x_i, a_i are the multiplicative variables, and t_i, u_i are additive variables.

We introduce the $\mathrm{Li}^{\mathcal{C}}$-generating series

$$\mathrm{Li}^{\mathcal{C}}(*, x_1, ..., x_m | t_0 : ... : t_m) := \quad \mathrm{Li}^{\mathcal{C}}(x_0, ..., x_m | t_0 : ... : t_m) := \tag{52}$$

$$(-1)^m \mathrm{I}^{\mathcal{C}}((x_1 ... x_m)^{-1} : (x_2 ... x_m)^{-1} : ... : x_m^{-1} : 1 | t_0 : ... : t_m)$$

$$=: (-1)^m \mathrm{I}^{\mathcal{C}}(a_1 : a_2 : ... : a_{m+1} | t_0 : ... : t_m) \quad =: \mathrm{I}^{\mathcal{C}}(a_1 : a_2 : ... : a_{m+1} | u_1, ... u_{m+1})$$

Remark. The $(,)$-notation is used for the variables which sum to zero (under the appropriate group structure), and the $(:)$-notation is used those sets of variables which are essentially homogeneous with respect to the multiplication by a common factor, see lemma 5.3.

2. The coproduct in terms of the $\mathrm{Li}^{\mathcal{C}}$–generating series. In this section we rewrite the formula for the coproduct using the $\mathrm{Li}^{\mathcal{C}}$–generators instead of the $\mathrm{I}^{\mathcal{C}}$–generators. To state the formula set

$$X_{a \to b} := \quad \prod_{s=a}^{b-1} x_s$$

Proposition 6.1 *Let us suppose that $x_i \neq 0$. Then*

$$\Delta \mathrm{Li}^{\mathcal{C}}(x_0, x_1, ..., x_m | t_0 : t_1 : ... : t_m) \quad =$$

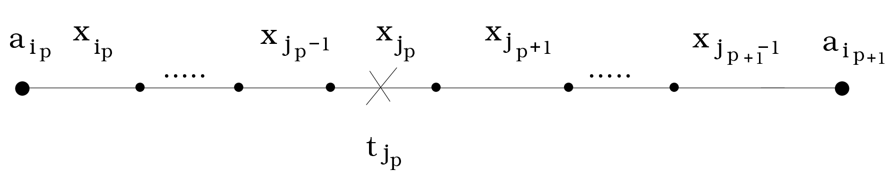
$$\sum \mathrm{Li}^{\mathcal{C}}(X_{i_0 \to i_1}, X_{i_1 \to i_2}, ..., X_{i_k \to m} | t_{j_0} : t_{j_1} : ... : t_{j_k}) \otimes \tag{53}$$

$$\prod_{p=0}^{k} \Big((-1)^{j_p - i_p} X_{i_p \to i_{p+1}}^{t_{j_p}} \mathrm{Li}^{\mathcal{C}}(*, x_{j_p - 1}^{-1}, x_{j_p - 2}^{-1}, ..., x_{i_p}^{-1} | -t_{j_p} : -t_{j_p - 1} : ... : -t_{i_p}) \cdot$$

$$\mathrm{Li}^{\mathcal{C}}(*, x_{j_p + 1}, x_{j_p + 2}, ..., x_{i_{p+1} - 1} | t_{j_p} : t_{j_p + 1} : ... : t_{i_{p+1} - 1}) \Big) \tag{54}$$

Here the sum is over special marked decorated segments, i.e. sequences $\{i_p\}$, $\{j_p\}$ satisfying (50).

The p-th factor in the product on the right is encoded by the data on the p-th segment:



Namely, $X_{i_p \to i_{p+1}}$ is the product of all x_i on this segment. The first term in (54) is encoded by the segment between t_{j_p} and t_{i_p}, which we read from the right to the left. The factor (54) is encoded by the segment between t_{j_p} and $t_{i_{p+1}}$ which we read from left to right.

Proof. Follows from lemma 5.2 and theorem 5.4.

3. The coproduct in the classical polylogarithm case. Recall that $\mathcal{A}_\bullet^{\mathcal{C}}$ is the commutative Hopf algebra of the framed mixed Tate objects with the coproduct Δ and the product $*$. Recall the restricted coproduct: $\Delta'(X) := \Delta(X) - (X \otimes 1 + 1 \otimes X)$. Notice that Δ is a homomorphism of algebras and Δ' is not.

Corollary 6.2

$$\Delta : \mathrm{Li}^{\mathcal{C}}(x|t) \longmapsto \mathrm{Li}^{\mathcal{C}}(x|t) \otimes x^t + 1 \otimes \mathrm{Li}^{\mathcal{C}}(x|t) \tag{55}$$

Proof. This is a special case of proposition 6.1.

This formula just means that

$$\Delta' \mathrm{Li}_n^{\mathcal{C}}(x) = \mathrm{Li}_{n-1}^{\mathcal{C}}(x) \otimes \log^{\mathcal{C}} x + \mathrm{Li}_{n-2}^{\mathcal{C}}(x) \otimes \frac{(\log^{\mathcal{C}} x)^2}{2} + ... + \mathrm{Li}_1^{\mathcal{C}}(x) \otimes \frac{(\log^{\mathcal{C}} x)^{n-1}}{(n-1)!}$$

4. The coproduct for the depth two multiple polylogarithms. We will use both types of the I–notations for multiple polylogarithm elements, so for instance

$$\mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \quad = \quad \mathrm{I}^{\mathcal{C}}(0; a_1, a_2; 1|t_1; t_2)$$

Set $\zeta^{\mathcal{C}}(t_1, ..., t_m) := \mathrm{I}^{\mathcal{C}}(1 : ... : 1|t_1, ..., t_m)$.

Proposition 6.3 *a) One has*

$$\Delta \mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \quad = 1 \otimes \mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2)$$

$$\mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \otimes a_1^{-t_1} * a_2^{t_1 - t_2} \quad + \quad \mathrm{I}^{\mathcal{C}}(a_1 : 1|t_1) \otimes a_1^{-t_1} * \mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2 - t_1)$$

$$-\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_2) \otimes a_1^{-t_2} * \mathrm{I}^{\mathcal{C}}(a_2 : a_1|t_2 - t_1) \quad + \quad \mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_1 : a_2|t_1) * a_2^{-t_2}$$

b) Let us suppose that $a_1^N = a_2^N = 1$. Then modulo the N-torsion one has

$$\Delta' \mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \quad = \quad \mathrm{I}^{\mathcal{C}}(a_1 : 1|t_1) \otimes \mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2 - t_1)$$

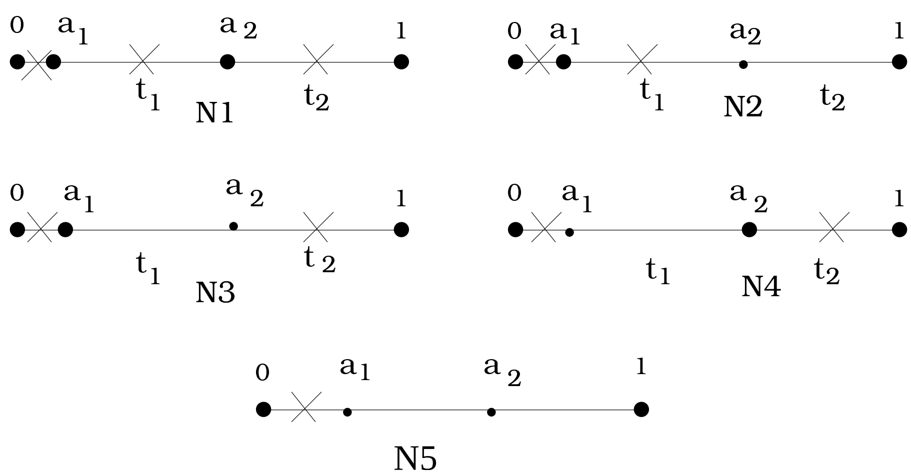
$$-\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_2 : a_1|t_2 - t_1) \quad + \quad \mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_1 : a_2|t_1)$$

In particular

$$\Delta' \zeta^{\mathcal{C}}(t_1, t_2) \quad = \quad \zeta^{\mathcal{C}}(t_1) \otimes \zeta^{\mathcal{C}}(t_2 - t_1) - \zeta^{\mathcal{C}}(t_2) \otimes \zeta^{\mathcal{C}}(t_2 - t_1) + \zeta^{\mathcal{C}}(t_2) \otimes \zeta^{\mathcal{C}}(t_1) \tag{56}$$

Proof. a) Since in our case $a_0 = 0$ and $t_0 = 0$, the nonzero contribution can be obtained only from those marked decorated segments where t_0-arc is not marked. Let us call the pictures where $a_0 = 0$, $a_{m+1} = 1$, $t_0 = 0$ and the t_0-arc is not marked by *special* marked decorated segments.

The five terms in the formula above correspond to the five special marked decorated segments presented in the picture.



Using the formulas from lemma 5.3 we get the following four terms corresponding to the terms N1-N4 on the picture:

$$\mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_1; 0|t_1) \cdot \mathrm{I}^{\mathcal{C}}(0; a_2|t_1) \cdot \mathrm{I}^{\mathcal{C}}(a_2; 0|t_2) \cdot \mathrm{I}^{\mathcal{C}}(0; 1|t_2) =$$
$$\mathrm{I}^{\mathcal{C}}(a_1 : a_2 : 1|t_1, t_2) \otimes a_1^{-t_1} \cdot a_2^{t_1-t_2}$$

$$\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_1) \otimes \mathrm{I}(a_1; 0|t_1) \cdot \mathrm{I}^{\mathcal{C}}(0; a_2; 1|t_1; t_2) \quad = \quad \mathrm{I}^{\mathcal{C}}(a_1 : 1|t_1) \otimes a_1^{-t_1} \cdot \mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2 - t_1)$$

$$\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_1; a_2; 0|t_1; t_2) \cdot \mathrm{I}(0; 1|t_2) =$$
$$-\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_2) \otimes \mathrm{I}(0; a_2; a_1|-t_2; -t_1) \quad = \quad -\mathrm{I}^{\mathcal{C}}(a_1 : 1|t_2) \otimes a_1^{-t_2} \cdot \mathrm{I}(a_2 : a_1|t_2 - t_1)$$

$$\mathrm{I}^{\mathcal{C}}(0; a_2; 1|0; t_2) \otimes \mathrm{I}^{\mathcal{C}}(0; a_1; a_2|0; t_1) \cdot \mathrm{I}^{\mathcal{C}}(a_2; 0|t_2) \cdot \mathrm{I}^{\mathcal{C}}(0; 1|t_2) =$$
$$\mathrm{I}^{\mathcal{C}}(a_2 : 1|t_2) \otimes \mathrm{I}^{\mathcal{C}}(a_1 : a_2|t_1) \cdot a_2^{-t_2}$$

Part b) follows from a) if we notice that $a^N = 1$ provides $a^t = 1$ modulo N-torsion. The proposition is proved.

Remark. In theorem 4.5 of [G2] the reader can find a different way to write the formulas for the coproduct in the depth 2 case. It is easy to see that the formulas given there are equivalent to the formulas above.

5. Explicit formulas for the coproduct of the weight three, depth two multiple polylogarithm elements. Applying proposition 6.3 or proposition 6.1 we get

$$\Delta' : \mathrm{Li}^{\mathcal{C}}_{2,1}(x,y) \quad \longmapsto \quad \mathrm{Li}^{\mathcal{C}}_{1,1}(x,y)\otimes x + \mathrm{Li}^{\mathcal{C}}_{1}(y)\otimes \mathrm{Li}^{\mathcal{C}}_{2}(x) + \mathrm{Li}^{\mathcal{C}}_{2}(xy)\otimes \mathrm{Li}^{\mathcal{C}}_{1}(y)$$

$$-\mathrm{Li}^{\mathcal{C}}_{1}(xy)\otimes \Big(\mathrm{Li}^{\mathcal{C}}_{2}(x) + \mathrm{Li}^{\mathcal{C}}_{2}(y) - \mathrm{Li}^{\mathcal{C}}_{1}(y)\cdot (xy + \frac{x^2}{2})\Big)$$

$$\Delta' : \mathrm{Li}^{\mathcal{C}}_{1,2}(x,y) \quad \longmapsto \quad \mathrm{Li}^{\mathcal{C}}_{1,1}(x,y)\otimes y - \mathrm{Li}^{\mathcal{C}}_{2}(xy)\otimes x - \mathrm{Li}^{\mathcal{C}}_{1}(xy)\otimes x\cdot y$$

$$+\mathrm{Li}^{\mathcal{C}}_{2}(y)\otimes \mathrm{Li}^{\mathcal{C}}_{1}(x) + \mathrm{Li}^{\mathcal{C}}_{1}(y)\otimes \mathrm{Li}^{\mathcal{C}}_{1}(x)\cdot y + \mathrm{Li}^{\mathcal{C}}_{1}(xy)\otimes \mathrm{Li}^{\mathcal{C}}_{2}(y)$$

$$-\mathrm{Li}^{\mathcal{C}}_{2}(xy)\otimes \mathrm{Li}^{\mathcal{C}}_{1}(x) - \mathrm{Li}^{\mathcal{C}}_{1}(xy)\otimes \mathrm{Li}^{\mathcal{C}}_{1}(x)\cdot xy - \mathrm{Li}^{\mathcal{C}}_{1}(xy)\otimes \mathrm{Li}^{\mathcal{C}}_{2}(x)$$

6. The Lie coalgebra structure in the depth two (see [G0], p. 13). Recall that the space of the indecomposables

$$\mathcal{L}^{\mathcal{C}}_{\bullet} := \quad \frac{\mathcal{A}^{\mathcal{C}}_{>0}}{\mathcal{A}^{\mathcal{C}}_{>0}\cdot \mathcal{A}^{\mathcal{C}}_{>0}}$$

inherits a structure of graded Lie coalgebra with the cobracket δ. I will describe

$$\sum_{n,m>0} \delta \mathrm{I}^{\mathcal{C}}_{n,m}(a_1,a_2)\cdot t_1^{n-1}t_2^{m-1} \quad \in \quad \wedge^2 \mathcal{A}^{\mathcal{C}}_{\bullet}\widehat{\otimes}\mathbb{Z}[[t_1,t_2]]$$

In the formulas below δ acts on the first factor in $\mathcal{A}^{\mathcal{C}}_{\bullet}\widehat{\otimes}\mathbb{Z}[[t_1,t_2]]$. Using proposition 6.3 we have

$$\delta\Big(\sum_{m>0,n>0} \mathrm{I}_{m,n}(a:b:c)\cdot t_1^{m-1}t_2^{n-1}\Big) \quad =$$

$$\sum_{m>0,n>0}\Big(\mathrm{I}_{m,n}(a:b:c)\cdot t_1^{m-1}t_2^{n-1})\wedge(\frac{b}{a}\cdot t_1+\frac{c}{b}\cdot t_2) \quad - \quad \mathrm{I}_m(a:b)\cdot t_1^{m-1}\wedge\widetilde{\mathrm{I}}_n(b:c)\cdot t_2^{n-1}$$

$$+ \quad \mathrm{I}_m(a:c)\cdot t_1^{m-1}\wedge\mathrm{I}_n(b:c)\cdot(t_2-t_1)^{n-1} \quad - \quad \mathrm{I}_n(a:c)\cdot t_2^{n-1}\wedge\mathrm{I}_m(b:a)\cdot(t_2-t_1)^{m-1}\Big)$$

Set $\mathrm{I}_{0,n} = \mathrm{I}_{n,0} = 0$. Here is a more concrete formula for δ:

$$\delta \mathrm{I}_{m,n}(a,b) \quad =$$

$$\mathrm{I}_{m-1,n}(a,b)\wedge\frac{b}{a} \quad + \quad \mathrm{I}_{m,n-1}(a,b)\wedge\frac{1}{b} \quad - \quad \mathrm{I}_m(\frac{a}{b})\wedge\mathrm{I}_n(b) \quad +$$

$$\sum_{i=0}^{m-1}(-1)^i\binom{n+i-1}{i}\mathrm{I}_{m-i}(a)\wedge\mathrm{I}_{n+i}(b)$$

$$-(-1)^{m+j-1}\sum_{j=0}^{n-1}(-1)^j\binom{m+j-1}{j}\mathrm{I}_{n-j}(a)\wedge\mathrm{I}_{m+j}(\frac{b}{a})$$

7 Feynman integrals, Feynman diagrams and mixed motives

1. Motivic correlators. Let us return to chapter 4. Why plane trivalent trees appear in the description of motivic iterated integrals, and what is the general framework for this relationship?

Further, we defined in chapters 8 and 9 of [G2] a real valued version of multiple polylogarithms as correlators of a Feynman integral. In fact we gave there a more general construction which provides a *definition* of a real valued version of multiple polylogarithms on arbitrary smooth curve. Moreover the Feynman diagram construction given in [G2] provides much more then just functions – it can be lifted to a construction of framed mixed motives whose periods are the correlators of the corresponding Feynman integral. A version of conjecture 1.2 claims that all framed mixed Tate motives appear this way.

So why Feynman integrals and Feynman diagrams appear in theory of mixed motives, and what role do they play there? I think we see in the two examples above a manifestation of the following general principal.

Feynman integrals are described by their correlators. Correlators of Feynman integrals are often periods of framed mixed motives. If so, the correlators must be upgraded to more sophisticated objects: the equivalence classes of the corresponding framed mixed motives (see chapter 3 of [G8] for the background). These objects lie in a certain commutative Hopf algebra $\mathcal{H}_{\mathrm{Mot}}$ with the coproduct Δ_{Mot}. One can (loosely) think about $\mathcal{H}_{\mathrm{Mot}}$ as of the algebra of regular functions on the motivic Galois group. Precisely, $\mathcal{H}_{\mathrm{Mot}}$ is the Hopf algebra in the (hypothetical) abelian tensor category $\mathcal{P}_{\mathcal{M}}$ of all pure motives, see [G8]. The motivic Galois group is a pro–affine group scheme in this category, see [D2]. Although $\mathcal{H}_{\mathrm{Mot}}$ is still a hypothetical mathematical object, we can see it in different existing realizations, e.g. in the Hodge realization.

Climbing up the road

correlators of Feynman integrals (numbers) $\longrightarrow$

motivic correlators (elements of the Hopf algebra $\mathcal{H}_{\mathrm{Mot}}$)

we gain a new perspective: one can now raise the question

$$\text{what is the coproduct of motivic correlators in } \mathcal{H}_{\mathrm{Mot}}? \qquad (57)$$

Reflections on this theme occupy the rest of the chapter.

2. The correspondence principle. Let us formulate precisely question (57). Let

$$\mathrm{Cor} = \langle \varphi(s_1), ..., \varphi(s_{m+1}) \rangle \tag{58}$$

be a correlator of a certain Feynman integral $\mathcal{F}$. If we understand our Feynman integral by its perturbation series expansion then according to the Feynman rules correlator (58) is defined as a sum of finite dimensional integrals:

$$\mathrm{Cor} := \sum_{\Gamma \in \mathcal{S}_{\mathcal{F}}(\mathrm{Cor})} \int_{X_\Gamma} \omega_\Gamma \tag{59}$$

The sum is over a (finite) set $\mathcal{S}_{\mathcal{F}}(\mathrm{Cor})$ of certain combinatorial objects Γ, given by decorated graphs. It is determined by the Feynman integral $\mathcal{F}$ and type of the correlator we consider. Such a Γ provides us a real algebraic variety X_Γ and a differential form of top degree ω_Γ on $X_\Gamma(\mathbb{R})$.

Let us assume that the integrals in (59) are convergent. This is often not the case in physically interesting examples. However it is the case for the Feynman integral considered in [G2]. Then correlator (58) is a well defined number.

Let us assume further that this number is a period of a mixed motive. This means that it is given by a sum of integrals of rational differential forms on certain varieties (which may differ from $X_\Gamma \otimes \mathbb{C}$) over certain chains whose boundary lie in a union of divisors. Below we consider only Feynman integrals whose correlators have this property, called Feynman integrals of algebraic geometric type. A nontrivial example of such a Feynman integral is given in [G2].

Then we conjecture that one can uniquely upgrade this number to the corresponding motivic correlator

$$\mathrm{Cor}_{\mathcal{M}} \in \mathcal{H}_{\mathrm{Mot}} \tag{60}$$

It is a framed mixed motive. The period of its Hodge realization is given by (58). We ask in (57) about the coproduct

$$\Delta_{\mathcal{M}}\Big(\mathrm{Cor}_{\mathcal{M}}\Big) \in \mathcal{H}_{\mathrm{Mot}} \otimes \mathcal{H}_{\mathrm{Mot}} \tag{61}$$

We suggest that the answer should be given combinatorially in terms of the decorated graphs Γ used in the definition (59). Here is a more precise version of this guess.

For a given Feynman integral $\mathcal{F}$ there exists a combinatorially defined commutative Hopf algebra $\mathcal{H}_{\mathcal{F}}$ in a tensor category $T_{\mathcal{F}}$ such that the right hand side of (59) provides an element

$$\gamma := \sum_{\Gamma \in \mathcal{S}_{\mathcal{F}}(\mathrm{Cor})} [\Gamma] \in \mathcal{H}_{\mathcal{F}} \tag{62}$$

Denote by $\Delta_{\mathcal{F}}$ the coproduct in $\mathcal{H}_{\mathcal{F}}$.

Let H_1 and H_2 be Hopf algebras in tensor categories T_1 and T_2. A Hopf algebra homomorphism $H_1 \longrightarrow H_2$ is given by a tensor functor $F : T_1 \longrightarrow T_2$ and a Hopf algebra homomorphism $F(H_1) \longrightarrow H_2$.

The correspondence principle. *The map*

$$c_{\mathcal{M}} : \gamma \longmapsto \mathrm{Cor}_{\mathcal{M}} \tag{63}$$

gives rise to a Hopf algebra homomorphism:

$$c_{\mathcal{M}} : \mathcal{H}_{\mathcal{F}} \longrightarrow \mathcal{H}_{\mathrm{Mot}}$$

provided by a tensor functor $F : T_{\mathcal{F}} \longrightarrow \mathcal{P}_{\mathrm{Mot}}$

In particular it is compatible with the coproducts:

$$(c_{\mathcal{M}} \otimes c_{\mathcal{M}})\Big(\Delta_{\mathcal{F}}(\gamma)\Big) = \Delta_{\mathcal{M}}(\mathrm{Cor}_{\mathcal{M}}) \tag{64}$$

This allows us to calculate the coproduct (61) combinatorially as the left hand side in (64), providing an answer to the question (57).

3. An example. The story described in chapter 4 should serve as an example of such a situation. In this case the motivic iterated integrals

$$\mathrm{I}^{\mathcal{M}}(s_0; s_1, ..., s_m; s_{m+1}) \tag{65}$$

should be seen as the motivic correlators. So far there is no Feynman integral providing these correlators, so we work directly with Feynman diagrams. Thanks to theorem 4.2 there is a Hopf algebra map

$$t : \widetilde{\mathcal{I}}_\bullet(S) \hookrightarrow \mathcal{T}_\bullet(S)$$

where t is as in (7). The Hopf subalgebra

$$t(\widetilde{\mathcal{I}}_\bullet(S)) \subset \mathcal{T}_\bullet(S)$$

should be considered as the combinatorially defined Hopf algebra $\mathcal{H}_{\mathcal{F}}$ responsible for the motivic correlators (65). Then the map $c_{\mathcal{M}}$ is given by

$$\sum \text{planar rooted trivalent trees decorated by } \{s_0, s_1, ..., s_m, s_{m+1}\} \longmapsto$$

$$\mathrm{I}^{\mathcal{M}}(s_0; s_1, ..., s_m; s_{m+1})$$

Since $\mathcal{I}_\bullet(S)$ is a quotient of $\widetilde{\mathcal{I}}_\bullet(S)$, we arrive at the diagram

$$\mathcal{O}_{G_{\mathcal{M}}(S)} \twoheadleftarrow \mathcal{I}_\bullet(S) \twoheadleftarrow \widetilde{\mathcal{I}}_\bullet(S) =: \mathcal{H}_{\mathcal{F}}$$

Going to the corresponding group schemes we get

$$G_{\mathcal{M}}(S) \stackrel{(6)}{\hookrightarrow} G(S) \stackrel{\mathrm{def}}{\hookrightarrow} \mathrm{Spec}(\widetilde{\mathcal{I}}_\bullet(S)) =: G_{\mathcal{F}}$$

Observe that the coproduct in $\mathcal{H}_{\mathcal{F}} := t(\widetilde{\mathcal{I}}_{\bullet}(S))$ is inherited from a bigger Hopf algebra $\mathcal{T}_{\bullet}(S)$.

4. The motivic Galois group of a Feynman integral and the correspondence principle. We define the affine group scheme $G_{\mathcal{F}}$ as the spectrum of $\mathcal{H}_{\mathcal{F}}$. Let us formulate a relationship between the $G_{\mathcal{F}}$–groups and motivic Galois groups more precisely.

i) A Feynman integral $\mathcal{F}$ can be understood as a collection of correlators. Correlators of Feynman integrals of algebraic geometric type can be lifted to their motivic avatars. The commutative algebra they generate *should* form a Hopf subalgebra $\mathcal{H}_{\mathrm{Mot}}(\mathcal{F})$ of the motivic Hopf algebra:

$$\mathcal{H}_{\mathrm{Mot}}(\mathcal{F}) \hookrightarrow \mathcal{H}_{\mathrm{Mot}} \tag{66}$$

We call it the *motivic Hopf algebra of a Feynman integral.*

ii) A Feynman integral $\mathcal{F}$ should determine its combinatorially defined Hopf algebra $\mathcal{H}_{\mathcal{F}}$.

iii) These two Hopf algebras are related by a canonical homomorphism of Hopf algebras, the *motivic correlator homomorphism*

$$c_{\mathcal{M}} : \mathcal{H}_{\mathcal{F}} \twoheadrightarrow \mathcal{H}_{\mathrm{Mot}}(\mathcal{F})$$

provided by a tensor functor $F : T_{\mathcal{F}} \longrightarrow \mathcal{P}_{\mathcal{F}}$. The map $c_{\mathcal{M}}$ is surjective by the very definition of $\mathcal{H}_{\mathrm{Mot}}(\mathcal{F})$.

Just like in s. 7.3, the coproduct in $\mathcal{H}_{\mathcal{F}}$ is probably inherited from a bigger object: although only the sum $\sum[\Gamma]$ belongs to $\mathcal{H}_{\mathcal{F}}$, we should be able to make sence of $\Delta_{\mathcal{F}}[\Gamma]$ for every summand $[\Gamma]$.

We define the *motivic Galois group* $G_{\mathrm{Mot}}(\mathcal{F})$ *of a Feynman integral* $\mathcal{F}$ as the spectrum of $\mathcal{H}_{\mathrm{Mot}}(\mathcal{F})$. Then $c_{\mathcal{M}}$ provides an injective homomorphism

$$c^*_{\mathcal{M}} : G_{\mathrm{Mot}}(\mathcal{F}) \hookrightarrow G_{\mathcal{F}}$$

It seems that the Feynman integral itself should be upgraded to an infinite dimensional mixed motive. Its matrix coefficients corresponding to different framings (see ch. 3 of [G3]) provide us all its correlators.

5. An example: the mixed Tate case. In the particularly interesting mixed Tate case, when all motivic correlators of a Feynman integral $\mathcal{F}$ are mixed Tate motives, the canonical fiber functor allows to think about $\mathcal{H}_{\mathrm{Mot}}(\mathcal{F})$ as of a commutative, graded Hopf algebra. Then $G_{\mathrm{Mot}}(\mathcal{F})$ is a semidirect product of $\mathbb{G}_m$ and $\mathrm{Spec}(\mathcal{H}_{\mathrm{Mot}}(\mathcal{F}))$, with the action of $\mathbb{G}_m$ provided by the grading.

In the mixed Tate case $\mathcal{H}_{\mathcal{F}}$ should also be a graded Hopf algebra, and the semidirect product of $\mathbb{G}_m$ and its spectrum provides us the $G_{\mathcal{F}}$–group of a Feynman integral.

Let us show that the correspondence principle almost determines the motivic correlator. Let $\mathrm{Cor}_{\mathcal{M}} = c_{\mathcal{M}}(\gamma)$. Calculation of $\Delta_{\mathcal{H}}(\gamma)$ is a combinatorial problem. So we may assume $\Delta_{\mathcal{H}}(\gamma)$ is given to us. Denote by $\mathcal{H}^T_{\rm Mot}$ the motivic Hopf algebra of the category of mixed Tate motives over $\mathbb{Q}$. Let us assume that

$$(c_{\mathcal{M}} \otimes c_{\mathcal{M}})(\Delta'_{\mathcal{H}}(\gamma)) \in \mathcal{H}^T_{\rm Mot} \otimes \mathcal{H}^T_{\rm Mot}$$

Then thanks to (64)

$$\Delta'_{\mathcal{M}}({\rm Cor}_{\mathcal{M}}) \in \mathcal{H}^T_{\rm Mot} \otimes \mathcal{H}^T_{\rm Mot} \qquad (67)$$

Suppose ${\rm Cor}_{\mathcal{M}}$ is a framed mixed motive of weight $2n$ defined over $\mathbb{Q}$. Since we used the restricted coproduct in (67), all factors there are of smaller weight, and are mixed Tate. Therefore, as follows from the following easy lemma, if the framing on ${\rm Cor}_{\mathcal{M}}$ is Tate, we conclude that ${\rm Cor}_{\mathcal{M}}$ is itself a framed mixed Tate motive over $\mathbb{Q}$!

Lemma 7.1 *Let X be a mixed motive over F framed by $\mathbb{Q}(0)$ and $\mathbb{Q}(n)$. Denote by $\overline{X}$ the equivalence class of this framed mixed motive, so $\overline{X} \in \mathcal{H}_{\rm Mot}$. Then if and $\Delta'\overline{X} \in \mathcal{H}^T_{\rm Mot} \otimes \mathcal{H}^T_{\rm Mot}$ then $\overline{X} \in \mathcal{H}^T_{\rm Mot}$. If $\overline{Y} \in \mathcal{H}_{\rm Mot}$ and $\Delta'\overline{X} = \Delta'\overline{Y}$ then $\overline{X} - \overline{Y} \in {\rm Ext}^1_{\rm Mot/F}(\mathbb{Q}(0), \mathbb{Q}(n))$*

Therefore ${\rm Cor}_{\mathcal{M}}$ is determined by its restricted coproduct $\Delta'({\rm Cor}_{\mathcal{M}})$ up to an element of

$${\rm Ext}^1_{{\rm Mot}/\mathbb{Q}}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$$

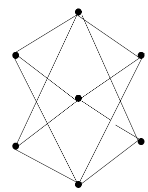
This vector space is zero for even n. For odd n it is one dimensional and spanned by $\zeta^{\mathcal{M}}(n)$. Therefore we can determine the motivic correlator up to a rational multiple of $\zeta^{\mathcal{M}}(n)$, which is zero for even n. Finally, if the element (67) is defined over $\mathbb{Z}$ this plus the conjecture ([G6]) that all framed mixed Tate motives over $\mathbb{Z}$ are the multiple ζ–motives would imply that ${\rm Cor}_{\mathcal{M}}$ is a multiple ζ–motive.

An example: $\zeta^{\mathcal{M}}(3,5)$. Physicists computed many correlators. D. Kreimer and D. Broadhurst discovered that, in small weights, these correlators are often expressed by the multiple ζ–values. The simplest multiple ζ–value which should not be expressed by the classical ζ–values appears in the weight 16 (the weights in the Tate case are even numbers, so it is customary to divide them by two; then the weight is 8). For example one can take $\zeta(3,5)$. Formula (56) tells us that

$$\Delta'\zeta^{\mathcal{M}}(3,5) = -5 \cdot \zeta^{\mathcal{M}}(3) \otimes \zeta^{\mathcal{M}}(5) \qquad (68)$$

Since $\zeta^{\mathcal{M}}(2n+1) \neq 0$ this implies that $\zeta^{\mathcal{M}}(3,5) \neq 0$.

According to [BGK] $\zeta(3,5)$ appears as the correlator corresponding to the following remarkable Feynman diagram:



A Feynman diagram corresponding, according to [BGK], to ζ(3,5).

This graph has several combinatorial properties which can not be found in any simpler graph. For instance this is the simplest graph which does not have Hamiltonian circles, i.e. cycles without selfintersections going through all vertices. So in this respect it is like $\zeta(3,5)$, which is the simplest non classical ζ–value.

It would be very interesting to recover formula (68) for the coproduct from the combinatorics of this graph.

Summarizing, the correspondence principle explains why the multiple ζ–values appear in calculations of correlators, and predicts the values of such correlators modulo the ideal generated by π^2 in the multiple ζ–algebra up to a rational multiple of $\zeta(2n+1)$.

On the other hand results of the paper [BB] suggest (although do not prove) that correlators of physically interesting Feynman integrals can be any periods, and not only very specific multiple ζ–values. It is not clear whether this immediately contradicts to the correspondence principle: the non mixed Tate periods of smallest weight can appear because of non Tate framing corresponding to the correlator, and then spread out into higher weights. However it probably suggests that we can not expect that the correspondence principle works for an arbitrary Feynman integral.

We assumed before the correlators are given by convergent integrals, and thus are well defined numbers/motives. If these integrals are divergent the renormalization group enters into the game, acting on possible regularizations. Therefore the Hopf algebra $\mathcal{H}_{\mathcal{F}}$ should be related to, although bigger then, the renormalization Hopf algebra.

REFERENCES

[BD] Beilinson A.A., Deligne P.: *Motivic polylogarithms and Zagier's conjecture* Manuscript, version of 1992.

[BGK] Broadhurst D., Gracey J., Kreimer D.: *Beyond the triangle and uniqueness relations: non zeta counterterms of large N at positive knots.* Z. Phys. C75(1997) 559–574. hep-th/9607174.

[BB] Belkale P, Brosnan P.: *Matroids, motives and conjecture of Kontsevich,* math.AG/0012198.

[CK] Connes A., Kreimer D.: *Hopf Algebras, Renormalization and Noncommutative Geometry* hep-th/9808042.

[D] Deligne P.: *Le group fondamental de la droite projective moine trois points*, In: Galois groups over $\mathbb{Q}$. Publ. MSRI, no. 16 (1989) 79-298.

[D2] Deligne P.: *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, Birkhauser Boston, Boston, MA, 1990.

[DG] Deligne P., Goncharov A.B.: *Groupes fondamentaux motiviques de Tate mixte* To appear.

[Dr] Drinfeld V.G.: *On quasi-triangular quasi-Hopf algebras and some group related to* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Leningrad Math. Journal, 1991. (In Russian).

[G0] Goncharov A.B.: *Multiple ζ-numbers, hyperlogarithms and mixed Tate motives*, Preprint MSRI 058-93, June 1993.

[G1] Goncharov A.B.: *The dihedral Lie algebras and Galois symmetries of* $\pi_1^{(l)}(\mathbb{P}^1 - 0, \mu_N, \infty)$. Duke Math. J. vol 100, N3, (2001), pp. 397-487. math.AG/0009121.

[G2] Goncharov A.B.: *Multiple ζ-values, Galois groups and geometry of modular varieties* Proc. of the Third European Congress of Mathematicians. Progress in Mathematics, vol. 201, p. 361-392. Birkhauser Verlag. (2001) math.AG/0005069.

[G3] Goncharov A.B.: *Multiple polylogarithms and mixed Tate motives.* math.AG/0103059.

[G4] Goncharov A.B.: *Periods and mixed Tate motives.* math.AG/0202154.

[G5] Goncharov A.B.: *Volumes of hyperbolic manifolds and mixed Tate motives* JAMS 12 (1999), n 2, 569-618. math.AG/9601021.

[G6] Goncharov A.B.: *Polylogarithms in arithmetic and geometry* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 374–387, Birkhuser, Basel, 1995.

[G7] Goncharov A.B.: *Multiple L-values, geometry of symmetric spaces and Feynman diagrams.* To appear.

[K] Kreimer D.: *On the Hopf algebra structure of perturbative quantum field theories.* q-alg/9707029.

[L1] Levine, M: *Tate motives and the vanishing conjectures for algebraic K-theory.* Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), 167–188, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer Acad. Publ., Dordrecht, 1993.

Adress: Dept. of Mathematics, Brown University, Providence, RI 02912, USA. e-mail sasha@math.brown.edu