A CATEGORIAL CHARACTERIZATION OF VARIETIES

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ABSTRACT. A simple, direct proof of the following characterization of varieties of (finitary) algebras is presented: a cocomplete category is equivalent to a variety iff it has an algebraic generator, i.e., a regular generator which is exactly projective and finitely generated. This improves somewhat a recent restatement, due to Pedicchio and Wood, of the classical characterization theorem of Lawvere. A bijective correspondence between algebraic theories and algebraic generators is established.

I. INTRODUCTION

Varieties of (finitary, one-sorted) algebras have been characterized in Lawvere's dissertation [L] as precisely the categories which have

(i) finite limits

(ii) coequalizers of equivalence relations

(iii) effective equivalence relations

and

(iv) a regular generator with copowers which is

a. regulary projective

and

b. abstractly finite.

The last condition, abstract finiteness, is proved below to be equivalent to being finitely generated in the sense of Gabriel and Ulmer. (The terminology here is unfortunate, but quite standard: "generator" and "finitely generated" have no connection.) Pedicchio and Wood have recently observed that if the generator G, instead of being a regular projective (i.e., such that its hom-functor preserves regular epimorphisms) is requested to

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be an *effective projective*, i.e., the hom-functor of G preserve reflexive coequalizers, than one can drop the condition that equivalence relations be effective. In the present note we introduce an *exact projective* as an object such that the hom-functor preserves coequalizers of equivalence relations. And we call a regular generator which is a finitely generated, exact projective an *algebraic generator*. The main purpose of our paper is to present a simple proof of the following

Characterization Theorem. *A cocomplete category is equivalent to a variety iff it has an algebraic generator.*

This improves somewhat the formulation of Pedicchio and Wood: we do not request the existence of finite limits, and exact projectives seem more natural in this context than effective projectives. For example, in any variety, we have

$$\text{regular projective} = \text{exact projective}$$

(and both are precisely the retracts of free algebras). Whereas effective projectives are, typically, just retracts of finitely generated free algebras, thus, in the characterization theorem of Pedicchio and Wood finite generation seems to be requested twice. The cocompleteness in the Characterization Theorem can be weakened to the existence of (a) copowers of the algebraic generator and (b) coequalizers of *reflexive pairs*, i.e., pairs of parallel morphisms $f_1, f_2 : A \longrightarrow B$ for which there exists $d : B \longrightarrow A$ with $f_1 d = id = f_2 d$. We are going to show that by applying Birkhoff Variety Theorem one obtains a simple proof of the Characterization Theorem. To achieve this, we introduce all the necessary categorical concepts in the next section, and prove some preliminary results. After proving the Characterization Theorem in Section III, we attend, in Section IV, to the relationship between algebraic theories and algebraic generators. Throughout the paper categories are supposed to be *locally small* (i.e., with small hom-sets).

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II. CATEGORICAL PRELIMINARIES

II.1 Varieties. Throughout the paper, a variety is an equationally defined full subcategory of the category

$$Alg\ \Sigma$$

of Σ-algebras and homomorphisms, where

$$\Sigma = (\Sigma_n)_{n \in \omega}$$

is a given (one-sorted) finitary signature. Our aim is to characterize such categories abstractly.

II.2 Forgetful Functor of $Alg\ \Sigma$ is denoted by

$$U : Alg\ \Sigma \longrightarrow Set.$$

II.3 Limits and Colimits. It is well-known that U creates limits, i.e., for every diagram $D : \mathcal{D} \longrightarrow Alg\ \Sigma$ of Σ-algebras, if we form a limit $(L \xrightarrow{\pi_d} UDd)_{d \in \mathcal{D}}$ of UD in Set, then L carries a unique structure of a Σ-algebra for which all π_d are homomorphisms. Morever, that Σ-algebra together with the cone $(\pi_d)_{d \in \mathcal{D}}$ is a limit of D in $Alg\Sigma$.

Analogously, U creates directed colimits (the reason is that directed colimits commute with finite products in Set). And U also creates coequalizers of equivalence relations - more generally, coequalizers of reflexive pairs. (Analogous reason: reflexive coequalizers commute with finite products in Set.)

The category $Alg\Sigma$ is in fact cocomplete, but U does not create (nor preserve) colimits in general. Let us also recall that for every variety $\mathcal{V} \subseteq Alg\Sigma$ the forgetful functor $U_{\mathcal{V}} : \mathcal{V} \longrightarrow Set$ preserves

(a) limits

(b) filtered colimits

and

(c) regular epimorphisms (i.e. morphisms which are coequalizers of parallel pairs)

In fact, (a) and (b) follow from the fact that $\mathcal{V}$ is closed in $Alg\Sigma$ under limits and filtered colimits. And for (c) use the fact that $\mathcal{V}$ is closed under regular quotients (= homomorphic images) and U preserves reflexive coequalizers.

II.4 Relations. One usually defines (binary) relations on an object K of a category as subobjects of the product $K \times K$; each relation is, thus, represented by a monomorphism $w : W \longrightarrow K \times K$. Or equivalently, by a parallel pair

$$w_1,\ w_2 : W \longrightarrow K$$

which is *monic* (i.e., given distinct morphisms $f, g : V \longrightarrow W$ then $w_1 f \neq w_1 g$ or $w_2 f \neq w_2 g$). Here we want to work with relations not assuming that finite products exist:

A *relation* on an object K is represented by a monic pair $w_1, w_2 : W \longrightarrow K$. Given another monic pair $v_1, v_2 : V \longrightarrow K$, it represents the same relation iff there is a *factorization* of it through (w_1, w_2), i.e., a morphism

$$(\star) \qquad i : V \longrightarrow W \text{ with } v_1 = w_1 i \text{ and } v_2 = w_2 i$$

such that i is an isomorphism. Using the concept of factorization $(\star)$ in general (not necessarily isomorphic) we obtain the following:

II.5 Definition. By an *equivalence relation* on an object K is meant a relation $w_1, w_2 : W \longrightarrow K$ which is

(i) *reflexive*, i.e., the pair $id, id : K \longrightarrow K$ factors through (w_1, w_2)

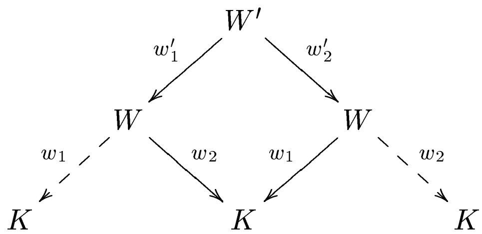
(ii) *symmetric*, i.e., the pair (w_2, w_1) factors through (w_1, w_2)

and

(iii) *transitive*, i.e., given morphisms $v_1, v_2, v_3 : V \longrightarrow K$ such that both (v_1, v_2) and (v_2, v_3) factor through (w_1, w_2), then also (v_1, v_3) factors through (w_1, w_2).

Remark.(1) A pair of morphisms $w_1, w_2 : W \longrightarrow K$ is *reflexive* (i.e., a morphism $d : K \longrightarrow U$ with $u_1 d = id = u_2 d$ exists) iff (i) above holds.

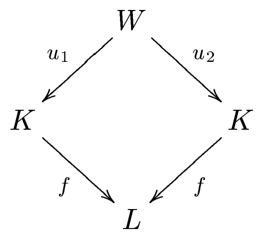
(2) In categories with pullbacks, transitivity is equivalent to the following condition on the pullback



of w_2 and w_1: the pair $(w_1 w_1', w_2 w_2')$ factors through (w_1, w_2).

Examples. (1) In a variety $\mathcal{K}$, equivalence relations on an algebra K (in the categorical sense above) are precisely the congruences, i.e., the equivalence relations (in the set-theoretical sense) that are subalgebras of $K \times K$.

(2) Let $\mathcal{K}$ be a category with pullbacks. For every morphism $f : K \longrightarrow L$ a *kernel pair* $u_1, u_2 : U \longrightarrow K$ i.e., a pullback



of f with itself is an equivalence relation.

II.6 Definition. A category is said to have *effective equivalence relations* if every equivalence relation is a kernel pair of some morphism.

Example. (1) Every variety $\mathcal{V}$ of Σ-algebras has effective equivalence relations: given an equivalence relation W on an algebra K of $\mathcal{V}$, form the quotient algebra $f : K \longrightarrow K/W$ modulo the congruence W. Since $\mathcal{V}$ is closed under quotients in Alg Σ, it follows that W is a kernel equivalence of f in $\mathcal{V}$.

(2) No quasivariety $\mathcal{V}$ which is not a variety has effective equivalence relations: if W is a congruence on $K \in \mathcal{V}$ such that K/W does not lie in $\mathcal{V}$, then W is a non-effective equivalence relation on K.

II.7 Definition. An object G of a category $\mathcal{K}$ is called

(i) a *regular projective* provided that its hom-functor

$$hom(G, -) : \mathcal{K} \longrightarrow Set$$

preserves regular epimorphisms,

(ii) an *exact projective* provided that its hom-functor

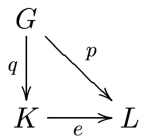
preserves coequalizers of equivalence relations

and

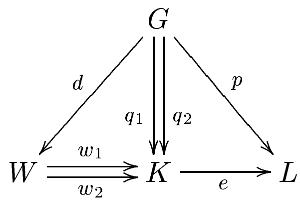
(iii) an *effective projective* provided that its hom-functor preserves

coequalizers of reflexive pairs.

Remark. Explicitly, G is a regular projective iff for every regular quotient $e : K \longrightarrow L$ and every morphism $p : G \longrightarrow L$ there exists a factorization q through e:



And if kernel pairs exist, then G is an exact projective iff it is a regular projective and for every coequalizer $e : K \longrightarrow L$ of an equivalence relation $w_1, w_2 : W \longrightarrow K$ the above factorizations q are "essentially unique".That is, given $q_1, q_2 : G \longrightarrow K$ with $p = eq_1 = eq_2$ there exists a unique $d : G \longrightarrow W$ with $q_1 = u_1 d$ and $q_2 = u_2 d$.



Finally, G is an effective projective iff it is a regular projective and for every coequalizer $e : K \longrightarrow L$ of a reflexive pair $w_1, w_2 : U \longrightarrow K$, given $q_1, q_2 : G \longrightarrow K$ with $p = eq_1 = eq_2$ there exists a zig-zag connecting q_1 with q_2 via the functions

$$w_1(-), w_2(-) : hom(G, W) \longrightarrow hom(G, K).$$

II.8 Observation. In categories with kernel pairs and effective equivalence relations we have

$$\text{regular projective} \Leftrightarrow \text{exact projective}.$$

In fact, let $w_1, w_2 : W \longrightarrow K$ be an equivalence relation with a coequalizer $c : K \longrightarrow L$. Then every $hom(G, -)$, where G is a regular projective, preserves this coequalizer. Indeed, w_1, w_2 is a kernel pair (of some morphism, thus) of c. Since $hom(G, -)$ preserves kernel pairs and regular epimorphisms, it maps c to a regular epimorphism $\bar{c}$ which has kernel pair $\bar{w}_i = hom(G, w_i)$; it follows that $\bar{c}$ is a coequalizer of $\bar{w}_1, \bar{w}_2$.

Corollary. *In every variety $\mathcal{K}$, regular (= exact) projectives are precisely the retracts of $\mathcal{K}$-free algebras.*

In fact, necessity follows from the fact that every algebra of $\mathcal{K}$ is a regular quotient of a $\mathcal{K}$-free algebra. For the sufficiency, observe that if A is a $\mathcal{K}$-free algebra on α generators, then $hom(A,-)$ is equivalent to the α power of the forgetful functor $U_{\mathcal{K}} : \mathcal{K} \longrightarrow Set$. Now $U_{\mathcal{K}}$ preserves coequalizers of equivalence relations, and in Set such coequalizers commute with products. Thus, $hom(A,-) \cong U_{\mathcal{K}}^{\alpha}$ preserves coequalizers of equivalence relations.

II.9 Open problem. Characterize effective projectives in all varieties.

Remark. (1) A free algebra on finitely many generators is always an effective projective, and so are all retracts. Maybe that these are all the effective projectives.

(2) In every quasivariety $\mathcal{K}$ which is not a variety there exist regular projectives which are not exact. Namely, $U_{\mathcal{K}} : \mathcal{K} \longrightarrow Set$ does not preserve coequalizers of equivalence relations (see Example II.6(2)) thus, a $\mathcal{K}$-free algebra on one generator fails to be an effective projective.

II.10 Finitely Generated Object is an object G whose hom-functor

$$hom(G,-)$$

preserves directed unions. Explicitly, G is finitely generated iff given a directed diagram $(B_i)_{i\in I}$ of subobjects $b_i : B_i \longrightarrow B$ $(i \in I)$ whose union is B (i.e., no proper subobject contains all b_i's), then every morphism from G to B factorizes through b_i for some $i \in I$. (This definition slightly differs from that of Gabriel and Ulmer [GU] who call G finitely generated iff $hom(G,-)$ preserves colimits of directed diagrams of monomorphisms. The two concepts coincide in all the usual categories — namely, whenever colimits of directed diagrams of monomorphisms are characterized as those cocones of monomorphisms which are collectively extremally epimorphic.)

Examples. (1) In a variety or a quasivariety, "finitely generated" has the usual algebraic meaning of being generated by a finite subset.

(2) In the category Pos of posets, "finitely generated" means "finite".

II.11 Lemma. *Let $\mathcal{K}$ be a finitely complete, cocomplete and well-powered category with regular factorization of morphisms. Then for all finitely generated objects we have*

$$\textit{effective projective} \Leftrightarrow \textit{exact projective.}$$

Proof. In II.5 of [AK] for every reflexive relation $w_1, w_2 : W \longrightarrow K$ an equivalence relation ${w_1}^*, {w_2}^* : W^* \longrightarrow K$ is constructed which is coequalized by the same morphisms with domain K. The construction uses only

(a) composition of relations

and

(b) directed colimits of monomorphisms.

Let G be a finitely generated exact projective. The hom-functor $hom(G,-)$ preserves composition of relations because it preserves finite limits (obviously) and regular factorizations (since G is a regular projective). And $hom(G,-)$ preserves directed colimits of monomorphisms (since G is finitely generated and in Set directed unions are directed colimits). Therefore, $hom(G,-)$ preserves the construction $W \mapsto W^*$ of "equivalence hulls". Thus, preservation of coequalizers of equivalence relations implies preservations of reflexive coequalizers.

II.12 Regular Generators. For every object G we denote by

$$I \bullet G \qquad \text{(the } I\text{-copower of } G)$$

a coproduct of copies of G indexed by I, whenever it exists. We call G a *regular generator* if every object A is a regular quotient of the copower $I \bullet G$ where $I = hom(A,G)$. More precisely: that copower exists and the canonical morphism

$$e : hom(G,A) \bullet G \longrightarrow A$$

whose f-component is f (for all $f : G \longrightarrow A$) is a regular epimophism.
More generally, a small set $\mathbb{G}$ of objects is called *regularly generating* provided that every object A is a canonical regular quotient of the coproduct

$$\coprod_{G \in \mathbb{G}} hom(G,A) \bullet G.$$

This is, in particular, true whenever $\mathbb{G}$ is a *dense* set (or dense full subcategory of $\mathcal{K}$), which means that every object A of $\mathcal{K}$ is a canonical colimit of the diagram $\mathcal{D}_A$ of all arrows from objects of $\mathbb{G}$ into A:

$$\mathcal{D}_A : \mathbb{G} \downarrow A \longrightarrow \mathcal{K} \text{ with } \mathcal{D}_A(G \xrightarrow{g} A) = G.$$

Here $\mathbb{G} \downarrow A$ denotes the comma-category of all morphisms $g : G \longrightarrow A$ with $G \in \mathbb{G}$, and to say that A is a canonical colimit of $\mathcal{D}_A$ means that the morphisms $g : G \longrightarrow A$ themselves form a colimit cocone of $\mathcal{D}_A$.

II.13 Examples. (i) Let $\mathcal{K}$ be a variety or quasivariety of algebras. Then a $\mathcal{K}$-free algebra G on one generator is a regular generator of $\mathcal{K}$: every algebra in $\mathcal{K}$ is a regular quotient of a $\mathcal{K}$-free algebra on α generators (for some cardinal α) which is $\alpha \bullet G$. But unless $\mathcal{K}$ is equivalent to a quasivariety of unary algebras, G is not dense. A $\mathcal{K}$-free algebra on infinitely many generators is dense.

(ii) In the category Pos of posets, the 2-chain is dense.

(iii) In the category Gra of graphs (i.e., sets with a binary relation) no single object is a regular generator. The set consisting of (1) a single arrow and (2) a single node is dense.

II.14 Abstract Finiteness. P. Freyd has introduced the concept of an abstractly finite object which we generalize to that of an abstractly finite set of objects. We then prove that for regularly projective regular generators we have

$$\text{abstractly finite} \Leftrightarrow \text{finitely generated.}$$

See [H] for a similar result.

Definition. A set $\mathbb{G}$ of objects in a category is called *abstractly finite* if coproducts of objects of $\mathbb{G}$ exist, and given any morphism

$$f : G \longrightarrow \coprod_{i \in I} G_i \qquad G \in \mathbb{G} \text{ and } \mathrm{G_i} \in \mathbb{G}\ (\mathrm{i} \in \mathrm{I})$$

there exists a finite set $J \subseteq I$ such that f factorizes through the canonical morphism

$$\coprod_{j \in J} G_j \longrightarrow \coprod_{i \in I} G_i.$$

Example. Every finitely generated object G is abstractly finite. In fact, for $\emptyset \neq J \subseteq I$ the canonical morphism $J \bullet G \longrightarrow I \bullet G$ is a (split) monomorphism. And if I is infinite then $I \bullet G$ is a directed union of all $J \bullet G \longrightarrow I \bullet G$ where J ranges through nonempty finite subsets of I.

II.15 Proposition. *Let $\mathcal{K}$ be a category with regular factorizations. If $\mathbb{G}$ is an abstractly finite, regularly generating set of regular projectives, then all objects of $\mathbb{G}$ are finitely generated.*

Proof. Let $G_0 \in \mathbb{G}$ and a directed union of monomorphisms $(B_i \xrightarrow{b_i} B)_{i \in I}$ be given. We are to show that every morphism $f : G_0 \longrightarrow B$ factorizes through some b_i.

Since $\mathbb{G}$ is a regular generator, for the object

$$C_i = \coprod_{G \in \mathbb{G}} hom(G, B_i) \bullet G \qquad (i \in I)$$

we have a canonical regular ephimorphism

$$e_i : C_i \longrightarrow B_i \qquad (i \in I).$$

Moreover, the objects C_i form a directed diagram: given $i \leq j$ the connecting morphism $B_i \longrightarrow B_j$ induces a function $hom(G, B_i) \longrightarrow hom(G, B_j)$ yielding a connecting morphism $C_i \longrightarrow C_j$. We can describe a colimit of the diagram $(C_i)_{i \in I}$ as follows: for each $G \in \mathbb{G}$ let T_G be the set of all morphisms $t : G \longrightarrow B$ which factor through some b_i, $i \in I$, and let

$$r_{G,i} : hom(G, B_i) \longrightarrow T_G \qquad (i \in I)$$

be the function of composition with $b_i : B_i \longrightarrow B$. Then a colimit of $(C_i)_{i \in I}$ is the object

$$C = \coprod_{G \in \mathbb{G}} T_G \bullet G$$

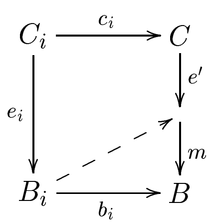
with the colimit morphisms

$$c_i : \coprod_{G \in \mathbb{G}} hom(G, B_i) \bullet G \longrightarrow \coprod_{G \in \mathbb{G}} T_G \bullet G$$

given by $r_{G,i}$. Moreover, the canonical regular epimorhpisms e_i above form a natural transformation, thus, they yield a unique

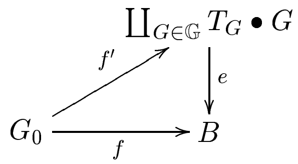
$$e : \coprod_{G \in \mathbb{G}} T_B \bullet G \longrightarrow B, \quad e = \mathrm{colim}_{i \in I}\, e_i,$$

which is a regular epimorphism. In fact, let $e = me'$ be a regular factorization of e and use the diagonal fill-in to show that the (colimit) cocone $b_i : B_i \longrightarrow B$ factors through m:



This proves that m is an isomorphism, thus, e is a regular epimorphism.

We are ready to prove that every morphism $f : G_0 \longrightarrow B$ factorizes through some b_i. Since G_0 is a regular projective, f factorizes through e:



Abstract finiteness implies that f' factorizes through a finite sub-coproduct of C. Now every summand of C corresponds to some morphism $t \in T_G$, and since the poset I is directed, for every finite set of summands of C there exists $i \in I$ such that all these summands factor through b_i. It follows that f' (and thus f) factors through b_i, as requested.

II.16 Corollary. *Let G be a regularly projective regular generator in a category with regular factorizations. Then*

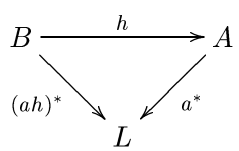
$$G \text{ finitely generated} \Leftrightarrow G \text{ abstractly finite.}$$

Necessity is II.15 applied to $\{G\}$. Sufficiency follows from II.14 (Example).

II.17 Proposition. *Let $\mathcal{A}$ be an abstractly finite, regularly generating set for which coproducts of collections of objects from $\mathcal{A}$ exist. Then the closure of $\mathcal{A}$ under finite coproducts is dense.*

Remark. A similar (but actually weaker) statement is Satz 7.4 in [GU].

Proof. It is sufficient to prove that if $\mathcal{A}$ is an abstractly finite, regularly generating set closed under finite coproducts, then $\mathcal{A}$ is dense. Consider $\mathcal{A}$ as a full subcategory of $\mathcal{K}$. Density means that given objects K and L of $\mathcal{K}$ and given a function mapping any $A \xrightarrow{a} K$ in $\mathcal{A} \downarrow K$ to some $A \xrightarrow{a^*} L$ in $\mathcal{A} \downarrow K$ such that the following *naturality condition* is fulfiled (for all morphisms $h : B \longrightarrow A$ of $\mathcal{A}$): $(ah)^* = a^*h$,



then there exists a unique morphism $f : K \longrightarrow L$ with $a^* = fa$ for all $A \xrightarrow{a} K$ in $\mathcal{A} \downarrow K$. Uniqueness is clear since $\mathcal{A}$ is generating. To prove the existence, put

$$B = \coprod_{A \in \mathcal{A}} \hom(A, K) \bullet A$$

and denote by

$$e : B \longrightarrow K$$

the canonical regular epimorphism. If $\hat{a} : A \longrightarrow B$ denotes the coproduct injection corresponding to $a \in hom(A, K)$, we have

$$e\hat{a} = a.$$

Let $u, v : W \longrightarrow B$ be a pair of morphisms whose coequalizer is e. Since for any quotient $d : W' \longrightarrow W$ the pair ud, vd has the same coequalizer, we can assume that W is a coproduct of objects C_i ($i \in I$) of $\mathcal{A}$, i.e., we have $u, v : \coprod_{i \in I} C_i \longrightarrow B$. Define

$$g : B \longrightarrow L \qquad \text{by} \qquad g\hat{a} = a^* : B \longrightarrow L$$

(for all $A \xrightarrow{a} K$ in $\mathcal{A} \downarrow K$). It is sufficient to prove $gu = gv$: then g factors as $g = fe$, and we conclude

$$a^* = g\hat{a} = fe\hat{a} = fa \qquad (\textit{for all } A \xrightarrow{a} K)$$

as requested. Thus, if $u_i, v_i : C_i \longrightarrow B$ are the components of u and v, respectively, it is our task to prove

$$gu_i = gv_i : C_i \longrightarrow L \qquad (i \in I).$$

Since $C_i \in \mathcal{A}$ and $\mathcal{A}$ is abstractly finite, there exists a finite subcoproduct of B through which both u_i and v_i factor. However, since $\mathcal{A}$ is closed under finite coproducts, the above canonical coproduct B has all finite subcoproducts equal to singleton subcoproducts. In other words, u_i and v_i factor through $\hat{a}_0$ for some $a_0 : A_0 \longrightarrow K$ in $\mathcal{A} \downarrow \mathcal{K}$:

$$u_i = \hat{a}_0 u'_i \quad \text{and} \quad v_i = \hat{a}_0 v'_i \quad \text{for} \quad u'_i, v'_i : C_i \longrightarrow A_0$$

The naturality condition yields

$$\begin{aligned} gu_i &= g\hat{a}_0 u'_i \\ &= a_0^* u'_i \\ &= (a_0 u'_i)^* \end{aligned}$$

$$= (e\hat{a}_0 u'_i)^*$$
$$= (eu_i)^*$$
analogously $gv_i = (ev_i)^*$. Thus, $eu_i = ev_i$ implies $gu_i = gv_i$, as requested.

II.18 Corollary. *Let $\mathcal{K}$ be a category with reflexive coequalizers and an abstractly finite regularly generating set $\mathcal{A}$ having coproducts of collections of objects from $\mathcal{A}$. Then $\mathcal{K}$ has limits and colimits.*

In fact, by II.17 we can assume that $\mathcal{K}$ has an abstractly finite dense set $\mathcal{A}$. Density implies that the restricted Yoneda embedding

$$E : \mathcal{K} \longrightarrow Set^{\mathcal{A}^{op}}$$

given by

$$K \longmapsto hom(-, K)/\mathcal{A}^{op}$$

is full and faithful. And E has a left adjoint L: to a functor $H : \mathcal{A}^{op} \longrightarrow Set$ it assigns a colimit

$$L(H) = colim(El\ H)$$

of the diagram of elements of H. (That is, we form the category $\mathcal{E}_H$ whose objects are all pairs (X, x) with $X \in \mathcal{A}$ and $x \in HX$ and whose morphisms $f : (X, x) \longrightarrow (Y, y)$ are those morphisms $f : Y \longrightarrow X$ of $\mathcal{K}$ for which $Hf(y) = x$. Then $El\ H : \mathcal{E}_H \longrightarrow \mathcal{K}$ is defined by $(X, x) \longmapsto X$.) These colimits exist: recall that $\mathcal{K}$ has (a) reflexive coequalizers and (b) coproducts of objects in $\mathcal{A}$. Now our diagram $El\ H$ uses only objects of $\mathcal{A}$, and when we apply the standard procedure of computing a colimit as a coequalizer of a parallel pair between two coproducts (see [M], V.2, Theorem 1), it is easy to see that the parallel pair is actually reflexive.

Consequently, $\mathcal{K}$ is equivalent to the full subcategory $E[\mathcal{K}]$ of $Set^{\mathcal{A}^{op}}$ which is reflective. Since $Set^{\mathcal{A}^{op}}$ has limits and colimits, so do all reflective subcategories.

II.19 Remark. A category with kernel pairs, their coequalizers and a generating set formed by regular projectives has regular factorizations. In fact, let $f : A \longrightarrow B$ have kernel pair $e_1, e_2 : E \longrightarrow A$ with a coequalizer $c : A \longrightarrow A'$, and let us prove that the unique $m : A' \longrightarrow B$ with $f = mc$ is a monomorphism. Given $w_1, w_2 : W \longrightarrow A'$ with $mw_1 = mw_2$, we can suppose that W is a regular projective. Consequently, there are $w'_1, w'_2 : W \longrightarrow A$ with $w_i = w'_i c$ $(i = 1, 2)$. Since $fw'_1 = fw'_2$, we have $r : W \longrightarrow E$ with $w'_i = e_i r$. This proves

$$w_1 = cw'_1 = ce_1 r = ce_2 r = cw'_2 = w_2.$$

II.20 Corollary. *Let $\mathcal{K}$ be a category with reflexive coequalizers which has a finitely generated regular generator with copowers. Then $\mathcal{K}$ has limits and colimits.*

In fact, the generator is abstractly finite, see II.16. Thus, II.18 applies.[2ex]

III. CHARACTERIZATION THEOREM

III.1 Definition. By an *algebraic generator* in a category is meant a regular generator which is finitely generated and exactly projective (i.e. whose hom-functor preserves directed unions and coequalizers of equivalences).

III.2 Theorem. *A category is equivalent to a variety iff it is cocomplete and has an algebraic generator.*

Remark. (1) Instead of cocompleteness, it is sufficient to assume the existence of

(a) reflexive coequalizers

and

(b) copowers of the algebraic generator.

This follows from Corollary II.20.

(2) The algebraic generator G will also be proved to be *finitely presentable* in $\mathcal{K}$, i.e., the hom-functor of G preserves filtered colimits.

(3) Quasivarieties, i.e., implicational classes of Σ-algebras, have an analogous characterization: they are precisely the categories which are cocomplete and have a finitely generated, regularly projective regular generator. This, also follows from the proof below.

Proof. Necessity is clear: every variety $\mathcal{V}$ is cocomplete, and a $\mathcal{V}$-free algebra on one generator, V, is a regular generator since every algebra in $\mathcal{V}$ is a regular quotient of a free $\mathcal{V}$-algebra, i.e., of a copower of V. And V is an exact projective, see II.8.

To prove the sufficiency, let $\mathcal{K}$ be a cocomplete category with an algebraic generator G. Denote by Σ the signature whose n-ary operation symbols are the morphisms from G to $n \bullet G$ for all $n \in \omega$:

$$\Sigma_n = hom(G, n \bullet G).$$

We define a functor $E : \mathcal{K} \longrightarrow Alg\Sigma$ which is full and faithful and whose image is an HSP-class thus, a variety (by Birkhoff Variety Theorem).

(i) Functor E. We define it so that the following triangle

$$(1) \qquad \begin{array}{ccc} \mathcal{K} & \xrightarrow{\;E\;} & Alg\,\Sigma \\ & {\scriptstyle \hom(G,-)} \searrow \quad \swarrow {\scriptstyle U} & \\ & Set & \end{array}$$

commutes. For $K \in \mathcal{K}$ the algebra EK has the underlying set $hom(G, K)$, and the n-ary operation σ_{EK} corresponding to $\sigma : G \longrightarrow n \bullet G$ in Σ_n is defined as follows:

$$\sigma_{EK}(f_1, ..., f_n) \equiv G \xrightarrow{\sigma} n \bullet G \xrightarrow{[f_1,...,f_n]} K.$$

It is easy to see that E is well-defined, i.e., that for every morphism $h : K \longrightarrow K'$ the function $hom(G, h)$ is a Σ-homomorphism from EK to EK'. E is faithful because $hom(G, -)$ is (since G is a generator).

E is full because $\{n \bullet G; n \in \omega\}$ is dense by II.17 and for every homomorphism

$$h : EK \longrightarrow EK' \quad (K, K' \in \mathcal{K})$$

we obtain a natural function (in the sense of the proof of II.17) mapping $f = [f_1, \ldots, f_n] : n \bullet G \longrightarrow K$ to $[h(f_1), \ldots, h(f_n)] : n \bullet G \longrightarrow K'$. Consequently, there exists $g : K \to K'$ with $g \cdot f = [h(f_1), \ldots, h(f_n)]$ for all f. The case $n = 1$ yields $h = Eg$.

(ii) An HSP-class $\bar{\mathcal{K}}$. The image of $E : \mathcal{K} \longrightarrow Alg\,\Sigma$ is a full subcategory of Alg Σ, and we denote by $\bar{\mathcal{K}}$ its closure under isomorphism. Clearly, $\mathcal{K}$ is equivalent to $\bar{\mathcal{K}}$; it is sufficient to prove that $\bar{\mathcal{K}}$ is an HSP class of Σ-algebras.

The functor E preserves limits (which exist: see II.20) and directed unions. This follows from the diagram (1) above and the fact that $hom(G, -)$ preserves both and U creates both. Consequently, $\bar{\mathcal{K}}$ is closed under limits and directed unions in Alg Σ. In particular:

(ii a) $\bar{\mathcal{K}}$ is closed under products.

(ii b) $\bar{\mathcal{K}}$ is closed under subalgebras. In fact, since $\bar{\mathcal{K}}$ is closed under directed unions, it is sufficient to prove that $\bar{\mathcal{K}}$ is closed under finitely generated subalgebras. Thus, let K be an object of $\mathcal{K}$ and

$$B_0 \subseteq hom(G, K)$$

be a finite set. We will prove that the subalgebra B of EK generated by B_0 lies in $\bar{\mathcal{K}}$. For $B_0 = \{b_1, ..., b_n\}$ consider $[b_1..., b_n] : n \bullet G \longrightarrow K$, then the (underlying set of the) subalgebra B is

$$B = \{[b_1, ..., b_n] \cdot \sigma;\ \sigma \in \Sigma_n\}$$

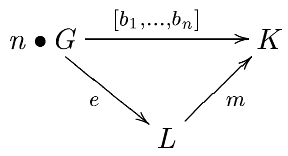
This follows easily from $\sigma_{EK}(b_1,...,b_n) = [b_1,...,b_n] \cdot \sigma$. (Thus the right-hand side is indeed a subalgebra of EK: for any $\rho \in \Sigma_k$, given k elements $[b_i]\sigma_1,...,[b_i]\sigma_k$ we have

$$\rho_{EK}([b_i]\sigma_1,...,[b_i]\sigma_k) = [b_i]\sigma$$

where

$$\sigma \equiv G \xrightarrow{\rho} k \bullet G \xrightarrow{[\sigma_1,...,\sigma_k]} n \bullet G.)$$

By II.19 we can form a regular factorization



and we prove that the subobject Em of EK represents the subalgebra B, i.e., that for all $f \in hom(G,K)$ we have

$$f \in B \qquad \text{iff f factorizes through m.}$$

(This proves $B \cong EL \in \bar{\mathcal{K}}$.) In fact, every $f \in B$ has the form

$$f = [b_1,...,b_n]\sigma = m(e\sigma).$$

Conversely, let $f = mg$, then $g : G \longrightarrow L$ factorizes (since G is a regular projective) through $e : n \bullet G \longrightarrow L$, i.e., there exists $\sigma \in \Sigma_n$ with $g = e\sigma$. This shows that

$$f = me\sigma = [b_1,...,b_n]\sigma \in B.$$

(ii c) $\bar{\mathcal{K}}$ is closed under regular quotients (or "homomorphic images"). Let $e : K \longrightarrow L$ be a regular epimorphism in $Alg\Sigma$ with $K \in \bar{\mathcal{K}}$. Then e is a coequalizer of its kernel pair $u_1, u_2 : U \longrightarrow K$. Since U is a subalgebra of $K \times K$, it follows from (ii a) and (ii b) that $U \in \bar{\mathcal{K}}$. Since u_1, u_2 is an equivalence relation in $Alg\Sigma$, it is one in $\bar{\mathcal{K}}$ too, and from the fact that E preserves coequalizers of equivalence relations (since G is an exact projective), we conclude that $L \in \bar{\mathcal{K}}$.

Finally, G is finitely presentable because EG is finitely presentable in $\bar{\mathcal{K}}$–in fact, EG is easily seen to be a free algebra on one generator, viz, $id_G \in UEG$, of $\bar{\mathcal{K}}$.

III.3 Corollary *(Lawvere Theorem). A category is equivalent to a variety iff it has* *(i) finite limits*

(ii) effective equivalence relations

(iii) coequalizers of reflexive pairs,

and

(iv) an abstractly finite, regularly projective regular generator.

This follows from III.2 due to II.8 and II.14.

Remark. In the formulation in Lawvere's dissertation [L] the assumption (iii) is missing. However, coequalizers are used in the proof. The truth is that in Lawvere's (more technical) proof only coequalizers of equivalence relations are needed. In that sense his result is stronger than the formulation above.

III.4 Remark. As mentioned in the introduction, this paper has been initiated by a reformulation of III.3 in [PW], where effectivity of equivalence relations is left out, and the regular generator is supposed to be a finitely presentable, effective projective. Instead of "finitely presentable" the argument of Pedicchio and Wood only requests "abstractly finite". But does it? Maybe this can be left out completely:

Open problem. Is every effective projective in a cocomplete category abstractly finite?

The answer would, of course, be affirmative if the following implication would be valid for all endofunctors H of Set:

$$H \text{ preserves reflexive coequalizers} \Longrightarrow H \text{ finitary.}$$

(Then we denote by H the composite of the functor assigning to every set I the copower $I \bullet G$ and $hom(G, -)$, for the given effective projective G. We conclude that H is finitary, from which it immediately follows that G is abstractly finite.) Now the above implication does not hold, unfortunately, if measurable cardinals exist. But we suspect that if holds under the assumption of nonexistence of measurable cardinals.

III.5 Many-sorted varieties. So far we have only worked with one-sorted signatures. However, the situation with many-sorted signatures is completely analogous – we just need abstract finiteness rather than finite generation here (because we have not found a proof of a generalization of Example II.14).

Recall that a many-sorted signature is a pair (S, Σ) where S is a set of sorts and Σ is a set of operation symbols σ with prescribed arities $ar(\sigma)$ which are nonempty words over Σ. An operation σ of arity $s_0 s_1 ... s_n$ is

n-ary, with the i-th variable of sort s_i ($i = 0, ..., n-1$) and with result of sort s_n. We form again the category $Alg\ \Sigma$ of Σ-algebras and homomorphisms, and we have a forgetful functor

$$U : Alg\ \Sigma \longrightarrow Set^S$$

assigning to every Σ-algebra the collection of its underlying sets (indexed by S).

A *many-sorted variety* is an equationally defined full subcategory of $Alg\ \Sigma$ – or, equivalently, an HSP class of Σ-algebras.

III.6 Characterization of Many-Sorted Varieties. *A cocomplete category is equivalent to a many-sorted variety iff it has an abstractly finite, regularly generating set of exact projectives.*

Remark. Cocompleteness can be weakened to the existence of reflexive coequalizers and coproducts of members of the generating set, see II.18.

Proof. Necessity is clear: for every variety $\mathcal{V}$ the collection of all $\mathcal{V}$-free algebras on one generator of sort s ($s \in S$) has all the required properties.

The proof of sufficiency is analogous to that in III.2. Let $\mathbb{G} = \{G_s; s \in S\}$ be the generating set above, and assume (without loss of generality) that it is closed under finite coproducts. Define an S-sorted signature Σ to have operation symbols of arity $s_0 ... s_n$ given by

$$\Sigma_{s_0 ... s_n} = hom(G_{s_n}, G_{s_0} + G_{s_1} + ... + G_{s_{n-1}}).$$

Then we obtain a full and faithful functor $E : K \longrightarrow Alg\ \Sigma$ whose composite with the forgetful functor $U : Alg\ \Sigma \longrightarrow Set^S$ is $(hom(G_p, -))_{s \in S} : \mathcal{K} \longrightarrow Set^S$. (The functor E is full because $\{G_s\}$ is dense). By Proposition II.15 each G_s is finitely generated, thus, E preserves directed colimits of monomorphisms. The rest is completely analogous to III.2.

IV. ALGEBRAIC THEORIES

IV.1 One of the main results of Lawvere's thesis [L] is that varieties can be presented as categories of models of algebraic theories. We will now derive this from the Characterization Theorem and show that algebraic theories precisely correspond to algebraic generators.

Recall that a (one-sorted) *algebraic theory* is a category $\mathcal{T}$ whose objects are natural numbers, and such that $n = 1 \times 1 \times ... \times 1$ (n factors) holds for all $n \in \omega$. (In particular, $\mathcal{T}$ has finite products, corresponding to the usual sums in ω.) A *model* of $\mathcal{T}$ is a functor $M : \mathcal{T} \longrightarrow Set$ preserving finite products. We denote by

$$Mod\,\mathcal{T}$$

the category of all models, a full subcategory of $Set^{\mathcal{T}}$.

IV.2 Remark (a) The category $Mod\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under

(i) limits,

(ii) filtered colimits

and

(iii) coequalizers of equivalence relations.

This follows easily from the fact that there three types of construction commute in Set (hence, in $Set^{\mathcal{T}}$) with finite products.

(b) A coproduct of hom-functors $\mathcal{T}(n,-)+\mathcal{T}(m,-)$ as objects of $Mod\,\mathcal{T}$ is the hom-functor $\mathcal{T}(n+m,-)$. This follows via Yoneda Lemma from the fact that models preserve finite products.

(c) The hom-functor $\mathcal{T}(1,-)$ is an algebraic generator of $Mod\,\mathcal{T}$ having copowers. In fact, finite copowers exist by (b), and the infinite ones are filtered colimits (see (a)) of finite ones. Moreover, the collection of all finite copowers $\mathcal{T}(n,-) = n \bullet \mathcal{T}(1,-)$ is dense, therefore, $\mathcal{T}(1,-)$ is a regular generator. Next, since $\mathcal{T}(1,-)$ is naturally isomorphic to the functor $ev_1 : Mod\,\mathcal{T} \longrightarrow Set$ of evaluation at 1, and since $Mod\,\mathcal{T}$ is closed under filtered colimits and coequalizers of equivalence relations in $Set^{\mathcal{T}}$, it follows that ev_1 preserves these colimits. Consequently, $\mathcal{T}(1,-)$ is a finetely generated exact projective.

IV.3 Corollary *$Mod\,\mathcal{T}$ is equivalent to a variety for every algebraic theory $\mathcal{T}$.*

IV.4 Example of an algebraic theory: let G be an object of a category with finite copowers. Denote by

$$\mathcal{T}_G$$

the algebraic theory of all finite copowers of G as a full subcategory of $\mathcal{K}^{op}$. More precisely, the objects of $\mathcal{T}_G$ are natural numbers, and the morphisms from n to k are the morphisms from $k \bullet G$ to $n \bullet G$ in $\mathcal{K}$.

We denote by

$$E_G : \mathcal{K} \longrightarrow Mod\,\mathcal{T}_G$$

the functor assigning to every object K the domain restriction of $\mathcal{K}(-,K) : \mathcal{K}^{op} \longrightarrow Set$ to the theory $\mathcal{T}_G$. Observe that

$$E_G(G) \cong \mathcal{T}_G(1,-) \qquad \text{in } Mod\,\mathcal{T}_G.$$

IV.5 Remark Conversely to IV.3 we now prove that every variety is equivalent to $Mod\,\mathcal{T}$ for some algebraic theory:

IV.6 Proposition *An object G of a cocomplete category $\mathcal{K}$ is an algebraic generator iff the fuctor $E_G : \mathcal{K} \longrightarrow Mod\,\mathcal{T}_G$ is an equivalence of categories.*

Proof. If E_G is an equivalense of categories, then the fact that $\mathcal{T}_G(1,-)$ is an algebraic generator of $Mod\,\mathcal{T}_G$ implies that G is an algebraic generator of $\mathcal{K}$ because

$$E_G(G) \cong \mathcal{T}_G(1,-).$$

Conversely, let G be an algebraic generator. Due to II.16 and II.17 the subcategory of finite copowers of G is dense, thus, E_G is full and faithfull. It remains to prove that every model $M : \mathcal{T}_G \longrightarrow Set$ is isomorphic to $E_G K$ for some K in $\mathcal{K}$. Since M preserves finite products, it is a filtered colimit of hom-functors

$$M = \operatorname*{colim}_{i\in I} \mathcal{T}_G(n_i,-).$$

Let K be the corresponding filtered colimit of $n_i \bullet G$ in $\mathcal{K}$. Since G is finitely presentable by Remark (i) in III.2 , the functor E_G preserves filtered colimits, thus, from

$$K = \operatorname*{colim}_{i\in I}(n_i \bullet G)$$

it follows that

$$E_G K \cong \operatorname*{colim}_{i\in I} \mathcal{T}_G(n_i,-) = M.$$

IV.7 Recall the concept of an *algebraic functor* from $Mod\,\mathcal{T}$ to $Mod\,\mathcal{T}'$, where $\mathcal{T}$ and $\mathcal{T}'$ are algebraic theories: Given a functor $J : \mathcal{T} \longrightarrow \mathcal{T}'$ preserving finite products and being the identily function on objects, the compose-with-J functor

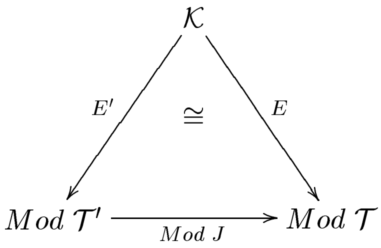
$$Mod\,J : Mod\mathcal{T}' \longrightarrow Mod\,\mathcal{T}$$

is called algebraic. And algebraic functors are precisely those of the form $Mod\,J$, see e.g. [B].

Definition. Let $\mathcal{K}$ be a category. By an *algebraic theory of* $\mathcal{K}$ is meant an algebraic theory $\mathcal{T}$ whose model category is equivalent to $\mathcal{K}$, together with an equivalence

$$E : \mathcal{K} \longrightarrow Mod\,\mathcal{T}.$$

Another algebraic theory for $\mathcal{K}$, $E' : \mathcal{K} \longrightarrow Mod\ \mathcal{T}'$, is said to be *isomorphic to* $(E, \mathcal{T})$ *over* $\mathcal{K}$ iff there exists an algebraic functor $Mod\ J : Mod\ \mathcal{T}' \longrightarrow Mod\ \mathcal{T}$, where $J : \mathcal{T} \longrightarrow \mathcal{T}'$ is an isomorphism, such that the following triangle



commutes up-to natural isomorphism.

IV.8 Theorem *Let $\mathcal{K}$ be a cocomplete category. Then algebraic theories of $\mathcal{K}$ correspond, up to isomorphism bijectively, to algebraic generators of $\mathcal{K}$. That is*

(1) for every algebraic generator G the pair $(\mathcal{T}_G, E_G)$ is an algebraic theory of $\mathcal{K}$,

(2) every algebraic theory of $\mathcal{K}$ is isomorphic over $\mathcal{K}$ to one of the form $(\mathcal{T}_G, E_G)$ where G is an algebraic generator

and

(3) two algebraic generators are isomorphic in $\mathcal{K}$ iff their algebraic theories are isomorphic over $\mathcal{K}$.

Proof. For (1) see IV.6.
To prove (2), let $(\mathcal{T}, E)$ be an algebraic theory of $\mathcal{K}$. Since $\mathcal{T}(1, -)$ is an algebraic generator of $Mod\ \mathcal{T}$, see IV.2, any object G with

$$E\ G \cong \mathcal{T}(1, -) \qquad \text{in } Mod\ \mathcal{T}$$

is an algebraic generator of $\mathcal{K}$. Choose a natural isomorphism

$$\delta : \mathcal{T}(1, -) \longrightarrow EG$$

and extend this to natural isomorphisms δ_n:

$$\mathcal{T}(n, -) \cong n \bullet \mathcal{T}(1, -) \xrightarrow{n \bullet \delta} n \bullet (EG) \cong E(n \bullet G)$$

for all $n \in \omega$. This yields the following natural bijections

$$\mathcal{T}_G(n, k) = \mathcal{K}(k \bullet G, n \bullet G)$$

$$Mod\,\mathcal{T}(E(k\bullet G),E(n\bullet G))\cong Mod\,\mathcal{T}(\mathcal{T}(k,-),\mathcal{T}(n,-))$$

$$\mathcal{T}(k,n)$$

which define an isomorphism functor

$$J:\mathcal{T}_G\longrightarrow\mathcal{T}.$$

Explicitly, on objects we have $Jn=n$, and on morphisms $f:k\bullet G\longrightarrow n\bullet G$ we let $J(f)$ denote the Yoneda preimage of

$$\mathcal{T}(k,-)\xrightarrow{\delta_k^{-1}}\mathcal{T}_G(k,-)\xrightarrow{(-)\cdot f}\mathcal{T}_G(n,-)\xrightarrow{\delta_n}\mathcal{T}(n,-).$$

This isomorphism J fulfils

$$\mathcal{T}(1,J-)\cong\mathcal{T}_G(1,-)$$

because we have natural isomorphisms

$$\varphi_n:\mathcal{T}(1,n)\longrightarrow\mathcal{T}_G(1,n)\qquad(n\in\omega)$$

given by

$$\begin{aligned}\mathcal{T}(1,n)&\cong Mod\,\mathcal{T}(\mathcal{T}(n,-),\mathcal{T}(1,-))\\&\cong Mod\,\mathcal{T}(E(n\bullet G),EG)\\&\cong\mathcal{K}(n\bullet G,G)\\&=\mathcal{T}_G(1,n).\end{aligned}$$

This proves that the objects

$$((Mod\,J)\cdot E)G=E(G)\cdot J\cong\mathcal{T}(1,J-)$$

and

$$E_GG\cong\mathcal{T}_G(1,-)$$

are isomorphic in $Mod\,\mathcal{T}_G$. Extend this to isomorphisms of $((Mod\,J)\cdot E)(n\bullet G)\cong E_G(n\bullet G)$ naturally for all $n\in\omega$. Since the objects $n\bullet G$, $n\in\omega$, are dense in $\mathcal{K}$, see II.17, it follows that $(Mod\,J)\cdot E$ and E_G are naturally isomorphic. This proves (2).

(3) Suppose that

$$J : \mathcal{T}_G \longrightarrow \mathcal{T}_{G'}$$

is an isomorphism of categories with a natural isomorphism

$$\varphi : E_G \longrightarrow (Mod\ J)E_{G'}.$$

If we identify $\mathcal{T}_G$ and $\mathcal{T}_{G'}$ with the full subcategories of $\mathcal{K}^{op}$ formed by the finite copowers of G and G', respectively, then the components of φ are natural isomorphisms

$$\varphi_K : \mathcal{K}(-,K)/\mathcal{T}_{G'} \longrightarrow \mathcal{K}(J-,K)/\mathcal{T}_G.$$

In particular, φ_G assigns to $id_G \in \mathcal{K}(G,G)$ the value

$$\varphi_G(id_G) = f \in \mathcal{K}(G',G).$$

(Recall that J is the identity map on objects, more precisely, it sends $n \bullet G$ to $n \bullet G'$!) The morphism $f : G' \longrightarrow G$ is an isomorphism. In fact, let g denote the preimage of $id_{G'}$ under $\varphi_{G'} : \mathcal{K}(-,G')/\mathcal{T}_{G'} \longrightarrow \mathcal{K}(J-,G')/\mathcal{T}_G$. The naturality of φ implies that the following square

$$\begin{array}{ccc} \mathcal{K}(-,G)/\mathcal{T}_{G'} & \xrightarrow{\ \varphi_G\ } & \mathcal{K}(J-,G)/\mathcal{T}_G \\ {\scriptstyle g\cdot(-)}\downarrow & & \downarrow{\scriptstyle g\cdot(-)} \\ \mathcal{K}(-,G')/\mathcal{T}_{G'} & \xrightarrow[\ \varphi_{G'}\ ]{} & \mathcal{K}(J-,G')/\mathcal{T}_G \end{array}$$

commutes. Applied to id_G, we obtain

$$\varphi_{G'}(g) = g \cdot f$$

but since $\varphi_{G'}(g) = id_{G'}$, this proves $g \cdot f = id_{G'}$. Moreover, g is a monomorphism because G is a generator and we have

$$gu = gv \text{ implies } u = v \quad \text{for all } u, v : G \longrightarrow G$$

(since $\varphi_G(gu) = \varphi_G(g)u = u$, analogously for v). Thus, $gf = id$ implies that g is an isomorphism.

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