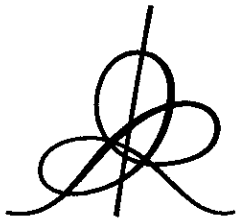
POLYLOGARITHMS, REGULATORS, AND ARAKELOV MOTIVIC COMPLEXES

A.B. GONCHAROV



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Contents

1 Introduction

Abstract. We construct an explicit regulator ma_ρ from Bloch's Higher Chow group complex to the Deligne complex of a complex algebraic variety X. We define the Arakelov motivic complex as the cone of this map shifted by one. Its last cohomology group is (a version of) the Arakelov Chow group defined by H. Gillet and C. Soule.

We relate the Grassmannian n–logarithms (defined as in [G5]) to the geometry of symmetric space $SL_n(\mathbb{C})/SU(n)$. For $n = 2$ we recover Lobachevsky's formula expressing the volume of an ideal geodesic simplex in the hyperbolic space via the dilogarithm. Using the relationship with symmetric spaces we construct the Borel regulator on $K_{2n-1}(\mathbb{C})$ via the Grassmannian n–logarithms.

We study the Chow dilogarithm and prove a reciprocity law which strengthens Suslin's reciprocity law for Milnor's group K_3^M on curves.

Our note [G5] can serve as an introduction to this paper.

1. Beilinson's conjectures on special values of L-functions. Let X be a regular scheme. A.A. Beilinson [B1] defined the rational motivic cohomology of X via Quillen's algebraic K-theory of X by the following formula, where γ is the Adams filtration:

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) = \mathrm{Gr}^{\gamma}_n K_{2n-i}(X) \otimes \mathbb{Q} \tag{1}$$

Let X be a regular complex algebraic variety. Beilinson defined the regulator

$$r_B : H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \longrightarrow H^i_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(n))$$

to the weight n Deligne cohomology of $X(\mathbb{C})$.

Let X be a regular projective variety over a number field F. Denote by $L(h^i(X), s)$ the L-function related to its i-dimensional cohomology. Beilinson conjectured that for any integer $n > 1+i/2$ its special value at $s = n$ is described, up to a nonzero rational factor, by the regulator map

$$r_B : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) \longrightarrow H^{i+1}_{\mathcal{D}}(X \otimes_{\mathbf{Q}} \mathbb{R}_{/\mathbf{R}}, \mathbb{R}(n))$$

to the weight n *real* Deligne cohomology of X, see [B1] and [RSS] for details.

This conjecture is fully established only when $X = Spec(F)$ where F is a number field. In this case the regulator coincides with the Borel regulator, and the relation with special values of Dedekind zeta-functions was given by the famous Borel theorem [Bo].

Although Beilinson's conjectures are far from being proved, it is interesting to see what kind of information about the special values of L-functions they suggest. So we come to the problem of *explicit* calculation of Beilinson's regulator. This problem is already very interesting for the Borel regulator.

2. Regulator maps on motivic complexes and Arakelov motivic cohomology. Beilinson [B2] and S. Lichtenbaum [L1] conjectured that the weight n integral motivic cohomology of X should appear as the cohomology of certain complexes $\Gamma(X; n)$, called the weight n motivic complexes:

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\Gamma(X; n))$$

These complexes are well defined as objects of the derived category.

The first motivic complexes satisfying Beilinson's formula (1) were Bloch's Higher Chow group complexes $\mathcal{Z}^*(X; n)$ [Bl1]. Later on A.A. Suslin and V.A. Voevodsky defined several important versions of these complexes. For another candidates for motivic complexes, called the polylogarithmic motivic complexes, see [G1-2]. They are very explicit and the smallest among possible candidates, however Beilinson's formula (1) is far from being established for them.

The real Deligne cohomology are also cohomology of certain complexes. It was suggested in [G5] and [G7] that the regulator map should be *explicitly* defined on the level of complexes.

Let X be a regular variety over $\mathbb{C}$. In chapter 2 we construct canonical homomorphism of complexes

$$\text{Bloch's weight } n \text{ Higher Chow group complex } \mathcal{Z}^\bullet(X;n) \text{ of } X \quad \longrightarrow \quad (2)$$

$$\text{the weight } n \text{ real Deligne complex } \mathcal{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n) \text{ of } X$$

This construction is a version of the one given in [G5]. The complex $\mathcal{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)$ is the truncation $\tau_{\leq 2n}$ of the complex proposed by Deligne more then twenty years ago. The weight n Arakelov motivic complex $\Gamma^\bullet_{\mathcal{A}}(X;n)$ is the cone of the map (2), shifted by -1:

$$\Gamma^\bullet_{\mathcal{A}}(X;n) := \text{Cone}\Big(\mathcal{Z}^\bullet(X;n) \xrightarrow{(2)} \mathcal{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)\Big)[-1] \tag{3}$$

For a regular variety X over $\mathbb{R}$ the image of map (2) lies in the subcomplex

$$\mathcal{C}^\bullet_{\mathcal{D}}(X_{/\mathbb{R}};n) := \mathcal{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)^{\overline{F}_\infty}$$

where $\overline{F}_\infty$ is the De Rham involution provided by the action of complex conjugation. The weight n *real* Arakelov motivic complex is defined as

$$\Gamma^\bullet_{\mathcal{A}}(X_{/\mathbb{R}};n) := \text{Cone}\Big(\mathcal{Z}^\bullet(X;n) \xrightarrow{(2)} \mathcal{C}^\bullet_{\mathcal{D}}(X_{/\mathbb{R}};n)\Big)[-1] \tag{4}$$

For a regular variety X over a number field F we set

$$\Gamma^\bullet_{\mathcal{A}}(X_{/F};n) := \text{Cone}\Big(\mathcal{Z}^\bullet(X;n) \longrightarrow \mathcal{C}^\bullet_{\mathcal{D}}(X \otimes_{\mathbb{Q}} \mathbb{R}_{/\mathbb{R}};n)\Big)[-1] \tag{5}$$

The Arakelov motivic cohomology are cohomology of these complexes.

Our construction works equally well for the Suslin-Voevodsky versions of the motivic complexes.

The regulator map on the polylogarithmic motivic complexes was defined in [G7] explicitly via the classical polylogarithms. The Arakelov motivic complexes constructed using regulator maps on different motivic complexes are supposed to lead to the same object of the derived category. However a precise relationship between the construction given in [G7] and the one in chapter 2 is not clear.

Higher Arakelov Chow groups. The last group of the complex $\mathcal{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)$ consists of closed distributions of certain type on $X(\mathbb{C})$. Replacing it by the quotient modulo smooth closed forms of the same type we get the quotient complex $\widetilde{\mathcal{C}}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)$. Changing $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ in (3) we define the weight n Higher Arakelov Chow group complex. Its last cohomology group is isomorphic to the Arakelov Chow group $\widehat{CH}^n(X(\mathbb{C}))$ as defined by Gillet and Soule.

It would be very interesting to generalize the Arakelov–Riemann–Roch theorem proved by Gillet and Soule to the case of Higher Arakelov Chow groups.

3. The Chow n–logarithm function. Let us describe the regulator map (2) in the simplest case when $X = \text{Spec}(\mathbb{C})$ is a point.

Let us choose in $\mathbb{P}^m$ a homogeneous coordinates $(z_0 : ... : z_m)$. The union of coordinate hyperplanes is a simplex L. Let $\mathbb{A}^m$ be the complement to the hyperplane $z_1 + ... + z_m = z_0$ in $\mathbb{P}^m$. The abelian group $\mathcal{Z}_m(\mathrm{Spec}(\mathbb{C}); n)$ is freely generated by the codimension n irreducible algebraic cycles in $\mathbb{A}^m$ intersecting properly the faces of the simplex L. The intersection with codimension one faces L_j of L provide homomorphisms

$$\partial_j : \mathcal{Z}_m(\mathrm{Spec}(\mathbb{C}); n) \longrightarrow \mathcal{Z}_{m-1}(\mathrm{Spec}(\mathbb{C}); n); \quad \partial := \sum_{j=0}^{m} (-1)^j \partial_j$$

The weight n Higher Chow group complex over $\mathrm{Spec}(\mathbb{C})$, written as a homological complex, looks as follows:

$$... \xrightarrow{\partial} \mathcal{Z}_2(\mathrm{Spec}(\mathbb{C}); n) \xrightarrow{\partial} \mathcal{Z}_1(\mathrm{Spec}(\mathbb{C}); n) \xrightarrow{\partial} \mathcal{Z}_0(\mathrm{Spec}(\mathbb{C}); n)$$

The Deligne complex of a point reduces to the group $\mathbb{R}(n) := (2\pi i)^n \mathbb{R}$. The regulator map (2) boils down to a construction of a homomorphism

$$\mathcal{Z}_{2n-1}(\mathrm{Spec}(\mathbb{C}); n) \xrightarrow{\mathcal{P}_n} \mathbb{R}(n), \quad \text{such that } \mathcal{P}_n \circ \partial = 0$$

It is provided by a function $\mathcal{P}_n$ on the space of codimension n cycles in $\mathbb{CP}^{2n-1}$ intersecting properly faces of a simplex L. This function, called the *Chow n-logarithm function*, was constructed in [G5]. To recall its construction, observe that a codimension n cycle in $\mathbb{P}^{2n-1} - L$ can be thought of as an $(n-1)$-dimensional variety X (a cycle) equipped with $2n-1$ rational functions $f_1, ..., f_{2n-1}$. These functions are obtained by restriction of the coordinate functions x_i/x_0 to the cycle X. We define a natural $(2n-2)$-form $r_{2n-2}(f_1, ..., f_{2n-1})$ on $X(\mathbb{C})$ and set

$$\mathcal{P}_n(X; f_1, ..., f_{2n-1}) := (2\pi i)^{1-n} \int_{X(\mathbb{C})} r_{2n-2}(f_1, ..., f_{2n-1}) \tag{6}$$

4. An example: the Chow dilogarithm. Let f_1, f_2, f_3 be three arbitrary rational functions on a complex curve X. Set

$$r_2(f_1, f_2, f_3) :=$$

$$\mathrm{Alt}_3\Big(\frac{1}{6} \log|f_1| d\log|f_2| \wedge d\log|f_3| - \frac{1}{2} \log|f_1| d\arg f_2 \wedge d\arg f_3\Big)$$

where Alt_3 is the alternation of f_1, f_2, f_3. Consider the space of quadruples $(X; f_1, f_2, f_3)$. It is a union of finite dimensional algebraic varieties. The Chow dilogarithm is a real function on its complex points defined by the formula

$$\mathcal{P}_2(X; f_1, f_2, f_3) := \frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3)$$

The integral converges since the form $r_2(f_1, f_2, f_3)$ has only logarithmic singularities. The Chow dilogarithm provides a homomorphism

$$\Lambda^3\mathbb{C}(X)^* \to \mathbb{R}, \quad f_1 \wedge f_2 \wedge f_3 \longmapsto \mathcal{P}_2(X; f_1, f_2, f_3) \tag{7}$$

Why dilogarithm appears in the name of the function $\mathcal{P}_2$? Recall the classical dilogarithm

$$Li_2(z) := -\int_0^z \log(1-z)d\log z$$

It has single valued cousin, the Bloch-Wigner function:

$$\mathcal{L}_2(z) := \mathrm{Im}Li_2(z) + \arg(1-z)\log|z|$$

The Chow dilogarithm is defined by a two-dimensional integral over $X(\mathbb{C})$, while $\mathcal{L}_2(z)$ is given by integral over a path on $\mathbb{CP}^1$. In chapter 6 we show that nevertheless the Chow dilogarithm can be expressed by the function $\mathcal{L}_2(z)$. Here is how it works when $X = \mathbb{CP}^1$. For $f \in \mathbb{C}(X)$ let $v_x(f)$ be the order of zero of f at $x \in X(\mathbb{C})$. Choose a point ∞ on $\mathbb{P}^1$. Then

$$\mathcal{P}_2(\mathbb{CP}^1; f_1, f_2, f_3) = \sum_{x_i \in \mathbb{P}^1(\mathbb{C})} v_{x_1}(f_1)v_{x_2}(f_2)v_{x_3}(f_3)\mathcal{L}_2(r(x_1, x_2, x_3, \infty)) \tag{8}$$

A formula for the Chow dilogarithm on elliptic curves see in chapter 6.

The function $\mathcal{L}_2$ satisfies Abel's five term functional equation:

$$\sum_{i=1}^{5}(-1)^i\mathcal{L}_2(x_1, ..., \widehat{x}_i, ..., x_5) = 0 \tag{9}$$

The Chow dilogarithm also satisfies functional equations. They appear as a reformulation of the fact that the composition

$$\mathcal{Z}_4(Spec(\mathbb{C}); 2) \xrightarrow{\partial} \mathcal{Z}_3(Spec(\mathbb{C}); 2) \xrightarrow{\mathcal{P}_2} \mathbb{R}(2)$$

is zero. Namely, let Y be an algebraic surface with four rational functions $g_1, ..., g_4$ on it. This data provides an element of $\mathcal{Z}_4(Spec(\mathbb{C}); 2)$. To evaluate the composition on this element we do the following. Take the divisor $\mathrm{div}(g_i)$ and restrict the other functions g_i on it. Then applying the Chow dilogarithm to the obtained data and taking the alternating sum over $1 \leq i \leq 4$ we get zero. In the special case when $Y = \mathbb{CP}^2$ and $\mathrm{div}g_i = l_i - l_5$, where $l_1, ..., l_5$ are five lines in the plane, this functional equation plus (8) is equivalent to Abel's equation (9).

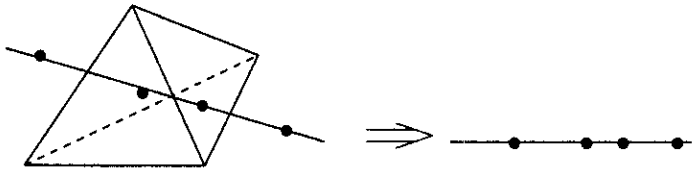
5. The Grassmannian n–logarithm and symmetric space $SL_n(\mathbb{C})/SU(n)$. Restricting the Chow n-logarithm function to the subvariety of $(n-1)$–planes in $\mathbb{CP}^{2n-1}$ in generic position to the simplex L we get the Grassmannian n–logarithm function $\mathcal{L}^G_n$.

There is a natural bijection

$$\{(n-1)\text{–planes in } \mathbb{P}^{2n-1} \text{ in generic position with respect to a simplex } L\}/(\mathbb{G}_m^*)^{2n-1}$$

$$< -- > \quad \{\text{Configurations of } \quad 2n \quad \text{generic hyperplanes in } \mathbb{P}^{n-1}\}$$

given by intersecting of an $(n-1)$–plane h with the codimension one faces of L.

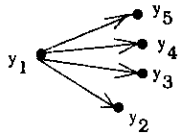


Using it we can view $\mathcal{L}_n^G$ as a function on the configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. Applying the projective duality we can consider it as a function on configurations of $2n$ points in $\mathbb{CP}^{n-1}$.

In fact one can define the Grassmannian n-logarithm $\mathcal{L}_n^G(x_1, ..., x_{2n})$ as a function on configurations of *arbitrary* $2n$ points in $\mathbb{CP}^{n-1}$, see chapter 4. It is a measurable function which is real analytic on generic configurations. It satisfies the two functional equations

$$\sum_{i=0}^{2n}(-1)^i\mathcal{L}_n^G(x_0, ..., \widehat{x}_i, ..., x_{2n}) = 0, \quad \sum_{j=0}^{2n}(-1)^j\mathcal{L}_n^G(y_j|y_0, ..., \widehat{y}_j, ..., y_{2n}) = 0 \tag{10}$$

In the second formula $(y_0, ..., y_{2n})$ is a configuration of $2n+1$ points in $\mathbb{CP}^{n-1}$ and $(y_j|y_0, ..., \widehat{y}_j, ..., y_{2n})$ is a configuration of $2n$ points in $\mathbb{CP}^{n-1}$ obtained by projection from y_j.

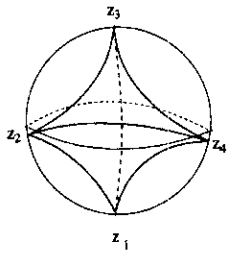


It follows from (8) that the Grassmannian dilogarithm is given by the Bloch-Wigner function:

$$\mathcal{L}_2^G(z_1, ..., z_4) = \mathcal{L}_2(r(z_1, ..., z_4)) \tag{11}$$

Abel's five term equation coincides with (10). (The two functional equations (10) are equivalent when $n = 2$).

Lobachevsky discovered that the dilogarithm appears in the computation of volumes of geodesic simplices in the three dimensional hyperbolic space $\mathcal{H}_3$. Let $I(z_1, ..., z_4)$ be the ideal geodesic simplex with vertices at the points $z_1, ..., z_4$ on the absolute of $\mathcal{H}_3$. The absolute is naturally identified with $\mathbb{CP}^1$.



Lobachevsky's formula relates its volume to the Bloch-Wigner function:

$$\mathrm{vol}\Big(I(z_1,...,z_4)\Big) = \mathcal{L}_2(r(z_1,...,z_4))$$

The volume function $\mathrm{vol}I(z_1,...,z_4)$ is invariant under the group $SL_2(\mathbb{C})$ of isomoetries of $\mathcal{H}_3$. So it depends only on cross ratio of the points $z_1,...,z_4$. It satisfies the five term equation (9). Indeed, $\sum(-1)^i I(z_1,...,\widehat{z}_i,...,z_5) = \emptyset$. By Bloch's theorem [Bl2] any measurable function $f(z)$ on $\mathbb{C}$ satisfying the five term equation is proportional to $\mathcal{L}_2(z)$. So we get the formula up to a constant.

We generalize this picture as follows. $\mathbb{CP}^{n-1}$ is realized as the smallest boundary stratum of the symmetric space $\mathbb{H}_n := SL_n(\mathbb{C})/SU(n)$. We define a function $\psi_n(x_1,...,x_{2n})$ on configurations of $2n$ points of the symmetric space. The function ψ_n is defined by an integral over $\mathbb{CP}^{n-1}$ similar to (6). We show that it can be naturally extended to a function $\overline{\psi}_n$ on configurations of $2n$ points in a compactification $\overline{\mathbb{H}}_n$ of the symmetric space. The Grassmannian n–logarithm function turns out to be the value of the function $\overline{\psi}_n$ on configurations of $2n$ points at the smallest boundary strata, which is identified with $\mathbb{CP}^{n-1}$.

Now let $n = 2$. Then $SL_2(\mathbb{C})/SU(2)$ is identified with the hyperbolic space. We prove that $\psi_2(x_1,x_2,x_3,x_4)$ is the volume of the geodesic simplex with vertices at the points $x_1,...,x_4$. Restricting to the ideal geodesic simplices and using the relation to the Grassmannian dilogarithm plus (11) we get a new proof of Lobachevsky's formula.

6. The Grassmannian n–logarithms and the Borel regulator. For any point $x \in \mathbb{CP}^{n-1}$ the function

$$c^n_{2n-1}(g_1,...,g_{2n}) := \mathcal{L}^G_n(g_1x,...,g_{2n}x) \tag{12}$$

is a measurable $(2n-1)$-cocycle of the Lie group $GL_n(\mathbb{C})$. Indeed, it is invariant under the diagonal action of $GL_n(\mathbb{C})$ and the cocycle condition is just the first functional equation for the function $\mathcal{L}^G_n$. Different points x give canonically cohomologous cocycles. However *a priori* it is not clear that the corresponding cohomology class is non zero.

Let $H^{2n-1}_m(GL_n(\mathbb{C}),\mathbb{R})$ be the space of measurable cohomology of the Lie group $GL_n(\mathbb{C})$. It is known that

$$H^*_m(GL_n(\mathbb{C}),\mathbb{R}) = \Lambda^*_{\mathbb{R}}(b_1,b_3,...,b_{2n-1})$$

where $b_{2k-1} \in H_m^{2k-1}(GL_n(\mathbb{C}), \mathbb{R})$ are certain canonical generators called the Borel classes ([Bo1]).

Theorem 1.1 *The cohomology class of the Grassmannian cocycle (12) is a non zero rational multiple of the Borel class b_{2n-1}.*

For normalization of the Borel classes and precise relationship between the Grassmannian polylogarithms and the Borel regulator see chapter 5, especially sections 5.4 and 5.5.

The essential role in the proof plays the fact that the Grassmannian n–logarithm function $\mathcal{L}_n^G$ is a boundary value of the function $\overline{\psi}_n$. The function $\overline{\psi}_n(x_1, ..., x_{2n})$ is not continuous at certain boundary points, but always satisfies the cocycle condition. So taking any point $x \in \overline{\mathbb{H}}_m$ we get a cocycle

$$c_x(g_1, ..., g_{2n-1}) := \overline{\psi}_n(g_1x, ..., g_{2n}x)$$

of the group $GL_n(\mathbb{C})$. Its cohomology class does not depend on x. If $x \in \mathbb{H}_m$ the corresponding cocycle is smooth. We can differentiate it, getting a cohomology class of the Lie algebra gl_n, and relate it to the Borel class. On the other hand taking x to be a point on the boundary stratum $\mathbb{CP}^{n-1}$ we recover the Grassmannian cocycle (12). So we get the theorem.

Combining it with the technique developed in [G1-2] we get a simple explicit construction of the Borel regulator

$$K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{R}$$

in terms of the Grassmannian n-logarithms. The second functional equation for $\mathcal{L}_n^G$ plays an important role in the proof. Therefore, thanks to the Borel theorem [Bo2], this allows to express the special values of Dedekind ζ–functions at $s = n$ via the Grassmannian n–logarithms.

The definition of the Higher Chow groups of a variety X is much simpler then the definition of algebraic K-groups of X. The situation with the regulator maps is similar. However relating the special values of the Dedekind ζ-functions to motivic cohomology of the corresponding number fields we need to work with the algebraic K-theory (or homology of $GL_n(F)$) of number fields.

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2 Arakelov motivic complexes

1. The Higher Chow groups complex. A (non degenerate) simplex in $\mathbb{P}^m$ is an ordered collection of hyperplanes $L_0, ..., L_m$ in generic position, i.e. with empty intersection. Let us choose in $\mathbb{P}^m$ a simplex L and a generic hyperplane H. We might think about this data as of a simplex in the m-dimensional affine space $\mathbb{A}^m := \mathbb{P}^m - H$. For any two nondegenerate simplices in $\mathbb{A}^m$ there is a unique affine transformation sending one simplex to the other.

Let $I = (i_1, ..., i_k)$ and $L_I := L_{i_1} \cap ... \cap L_{i_k}$. Let $\mathcal{Z}_n(X, n)$ be the free abelian group generated by irreducible codimension n algebraic subvarieties in $X \times \mathbb{A}^m$ which intersect properly (i.e. with the right codimension) all faces $X \times L_I$.

Warning. We use the notation $\mathcal{Z}_m(X; n)$ for the group denoted $\mathcal{Z}^n(X; m)$ by Bloch. This allows us to use upper and lower indices to distinguish between the homological and cohomological notations, see below.

For a given codimension 1 face L_i of a simplex L in $\mathbb{A}^m$ the other faces L_j cut a simplex $\widehat{L}_i := \{L_i \cap L_j\}$, in L_i. So the intersection with codimension 1 faces $X \times L_i$ provides group homomorphisms

$$\partial_i : \mathcal{Z}_m(X; n) \longrightarrow \mathcal{Z}_{m-1}(X; n); \qquad \partial := \sum_{i=0}^{m} (-1)^i \partial_i$$

Then $\partial^2 = 0$, so $(\mathcal{Z}_\bullet(X; m); \partial)$ is a homological complex. Its homology are Bloch's Higher Chow groups. By the fundamental theorem of Bloch ([Bl1-2])

$$H_i(\mathcal{Z}_\bullet(X; n) \otimes \mathbb{Q}) = gr_n^\gamma K_i(X) \otimes \mathbb{Q}$$

Let us cook up a cohomological complex setting

$$\mathcal{Z}^\bullet(X; n) := \mathcal{Z}_{2n-\bullet}(X; n)$$

Its cohomology provide a definition of the integral motivic cohomology of X:

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\mathcal{Z}^\bullet(X; n))$$

Bloch's theorem guarantees Beilinson's formula (1) for the rational motivic cohomology.

2. The Beilinson–Deligne complex. Recall that an n-distribution on a manifold X is a continuous linear functional on the space of $(\dim_{\mathbb{R}} X - n)$-forms with compact support. Denote by $\mathcal{D}^n_X$ the space of all *real* n-distributions on X. The De Rham complex of distributions $(\mathcal{D}^\bullet_X, d)$ is a resolution of the constant sheaf $\mathbb{R}$. The space $\mathcal{A}^p_X$ of all smooth n-forms on X is a subspace of $\mathcal{D}^n_X$.

Let X be a regular variety over $\mathbb{C}$. The n-th Beilinson-Deligne complex

$\mathbb{R}(n)_{\mathcal{D}}$ can be defined as a total complex associated with the following bicomplex:

$$\begin{array}{ccccccccc}
\left(\mathcal{D}^0_X & \xrightarrow{d} & \mathcal{D}^1_X & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{D}^n_X & \xrightarrow{d} & \mathcal{D}^{n+1}_X \xrightarrow{d} \dots\right) \otimes \mathbb{R}(n-1) \\
 & & & & & & \uparrow \pi_n & & \uparrow \pi_n \\
 & & & & & & \Omega^n_X & \xrightarrow{\partial} & \Omega^{n+1}_X \xrightarrow{\partial}
\end{array}$$

Here $\mathbb{R}(n) := (2\pi i)^n \mathbb{R}$ and

$$\pi_n : \mathcal{D}^p_X \otimes \mathbb{C} \longrightarrow \mathcal{D}^p_X \otimes \mathbb{R}(n-1)$$

is the projection induced by the one $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n) \longrightarrow \mathbb{R}(n-1)$. Further, $\mathcal{D}^0_X$ placed in degree 1 and $(\Omega^\bullet_X, \partial)$ is the De Rham complex of holomorphic forms *with logarithmic singularities* at infinity.

The Beilinson-Deligne complex $\mathbb{R}(n)_{\mathcal{D}}$ is quasiisomorphic to the complex

$$\mathbb{R}(n) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow \dots \longrightarrow \Omega^{n-1}_X$$

3. The truncated Deligne complex. Let $\mathcal{D}^{p,q}_X = \mathcal{D}^{p,q}$ be the abelian group of complex valued distributions of type (p,q) on $X(\mathbb{C})$. Consider the following cohomological bicomplex, where $\mathcal{D}^{n,n}_{\mathrm{cl}}$ is the subspace of the space $\mathcal{D}^{n,n}$ of closed currents, and $\mathcal{D}^{0,0}$ is in degree 1:

$$\begin{array}{ccccccccc}
 & & & & & & & & \mathcal{D}^{n,n}_{cl} \\
 & & & & & & & & 2\overline{\partial}\partial \nearrow \\
\mathcal{D}^{0,n-1} & \xrightarrow{\partial} & \mathcal{D}^{1,n-1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,n-1} & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \partial\uparrow & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \overline{\partial}\uparrow & & \\
\mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,1} & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \overline{\partial}\uparrow & & \\
\mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,0} & &
\end{array}$$

Let $Tot^\bullet$ be the total complex of this bicomplex. It is concentrated in degrees $[1, 2n]$. The complex $C^\bullet_{\mathcal{D}}(X(\mathbb{C}); n) = C^\bullet_{\mathcal{D}}(n)$ is a subcomplex of the complex $Tot^\bullet$ defined as follows. Take the intersection of the part of the complex $Tot^\bullet$ coming from the $n \times n$ square in the diagram (and concentrated in degrees

$[1, 2n-1]$) with the complex of distributions with values in $\mathbb{R}(n-1)$. Consider the subgroup $\mathcal{D}^{n,n}_{\mathbb{R},cl}(n) \subset \mathcal{D}^{n,n}_{cl}$ of the $\mathbb{R}(n)$-valued distributions of type (n,n). They form a subcomplex in $Tot^\bullet$ because $\overline{\partial}\partial$ sends $\mathbb{R}(n-1)$-valued distributions to $\mathbb{R}(n)$-valued distributions. This is the complex $C^\bullet_{\mathcal{D}}(n)$. It is truncation of the complex considered by Deligne.

Proposition 2.1 *The complex $C^\bullet_{\mathcal{D}}(n)$ is quasiisomorphic to the truncated Beilinson-Deligne complex $\tau_{\leq 2n}\mathbb{R}(n)_{\mathcal{D}}$.*

Proof. We need the following general construction. Let $f^* : X^* \longrightarrow Y^*$ be a morphism of complexes such that the map f^i is injective for $i \leq p$ and surjective for $i \geq p$ (and hence is an isomorphism for $i = p$). Consider a complex

$$Z^* := \quad \mathrm{Coker} f^{<p}[-1] \xrightarrow{D} \mathrm{Ker} f^{>p}$$

where the differential $D : \mathrm{Coker} f^{p-1} \longrightarrow \mathrm{Ker} f^{p+1}[1]$ is defined via the following diagram (the vertical sequences are exact):

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
0 & & 0 & & & & \mathrm{Ker} f^{p+1} & \longrightarrow & \mathrm{Ker} f^{p+2} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X^{p-2} & \longrightarrow & X^{p-1} & \longrightarrow & X^p & \longrightarrow & X^{p+1} & \longrightarrow & X^{p+2} \\
\downarrow & & \downarrow & & f^p \downarrow = & & \downarrow & & \downarrow \\
Y^{p-2} & \longrightarrow & Y^{p-1} & \longrightarrow & Y^p & \longrightarrow & Y^{p+1} & \longrightarrow & Y^{p+2} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathrm{Coker} f^{p-2} & \longrightarrow & \mathrm{Coker} f^{p-1} & & & & 0 & & \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}$$

Lemma 2.2 *The complex Z^* is canonically quasiisomorphic to* $\mathrm{Cone}(X^* \xrightarrow{f^*} Y^*)$.

Proof. Let

$$\widetilde{\tau}_{<p}X^\bullet := \quad \ldots \xrightarrow{d_X} X^{p-2} \xrightarrow{d_X} X^{p-1} \xrightarrow{d_X} \mathrm{Im} d_X$$

$$\widetilde{\tau}_{\geq p}Y^\bullet := \quad Y^p/\mathrm{Im} d_Y \xrightarrow{d_Y} Y^{p+1} \xrightarrow{d_Y} Y^{p+2} \xrightarrow{d_Y} \ldots$$

Then there is an exact sequence of complexes $0 \longrightarrow \widetilde{\tau}_{<p}X^\bullet \longrightarrow X^\bullet \longrightarrow \widetilde{\tau}_{\geq p}X^\bullet \longrightarrow 0$. The conditions on the maps f^* imply that

$$\widetilde{\tau}_{<p}f^* : \quad \widetilde{\tau}_{<p}X^\bullet \longrightarrow \widetilde{\tau}_{<p}Y^\bullet \quad \text{is injective}$$

$$\widetilde{\tau}_{\geq p}f^* : \quad \widetilde{\tau}_{\geq p}Y^\bullet \longrightarrow \widetilde{\tau}_{\geq p}Y^\bullet \quad \text{is surjective}$$

We get maps of complexes

$$\mathrm{Cone}(\widetilde{\tau}_{<p}X^\bullet \longrightarrow f^*(\widetilde{\tau}_{<p}X^\bullet)) \xrightarrow{\alpha} \mathrm{Cone}(X^\bullet \longrightarrow Y^\bullet) \xrightarrow{\beta} \mathrm{Cone}(\widetilde{\tau}_{\geq p}Y^\bullet \longrightarrow \widetilde{\tau}_{\geq p}Y^\bullet)$$

where α is injective and β is surjective. The complex $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ looks as follows:

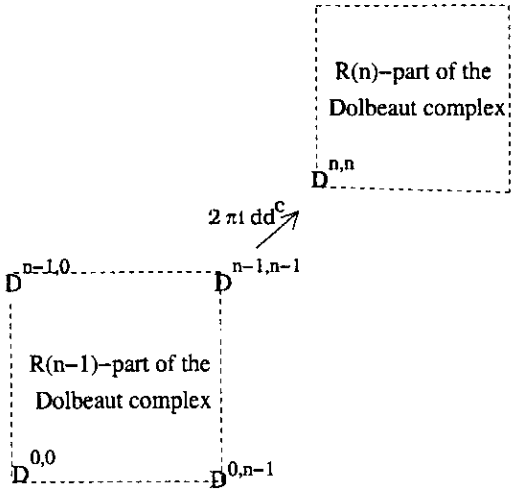
$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & (f^p)^{-1}\mathrm{Im}(d_Y)/\mathrm{Im}(d_X) & \longrightarrow & \mathrm{Ker}f^{p+1} & \longrightarrow & \mathrm{Ker}f^{p+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Coker}f^{p-2} & \longrightarrow & \mathrm{Coker}f^{p-1} & \longrightarrow & \mathrm{Im}(d_Y)/f^p\mathrm{Im}(d_X) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Since the map $f^p : \mathrm{Im}(d_Y)/\mathrm{Im}(d_X) \to \mathrm{Im}(d_Y)/f^p\mathrm{Im}(d_X)$ is an isomorphism it is quasiisomorphic to Z^*. The lemma is proved.

Applying the lemma to the morphism of complexes

$$\mathrm{Tot}(\mathcal{D}^{\geq n,\bullet}) \xrightarrow{\pi_n} \mathcal{D}^{n+\bullet} \otimes_{\mathbb{R}} \mathbb{R}(n-1)$$

we see that the complex $\mathbb{R}(n)_{\mathcal{D}}$ is canonically quasiisomorphic to the following complex:



To compute the differential $\mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1) \longrightarrow \mathcal{D}_{\mathbb{R}}^{n,n}(n)$ we proceed as follows. Take $\alpha \in \mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1)$, so $\alpha = (-1)^{n-1}\overline{\alpha}$. Then

$$d\alpha \quad = \quad \partial\alpha + \overline{\partial}\alpha \quad = \quad \partial\alpha + (-1)^{n-1}\overline{\partial\alpha} \quad = \quad 2\pi_n(\partial\alpha)$$

Applying $d = \partial + \overline{\partial}$ again and taking the (n,n)-component we get $2\overline{\partial}\partial(\alpha)$. Truncating this complex we obtain the proof of the proposition. (Note that $d^{\mathbb{C}} := (4\pi i)^{-1}(\partial - \overline{\partial})$, so $dd^{\mathbb{C}} = (2\pi i)^{-1}\overline{\partial}\partial$.)

Now if X is a variety over $\mathbb{R}$, then we set

$$C_{\mathcal{D}}^\bullet(X_{/\mathbb{R}}; n) := C_{\mathcal{D}}^\bullet(X; n)^{\overline{F}\infty}; \qquad H_{\mathcal{D}}^i(X_{/\mathbb{R}}; \mathbb{R}(n)) := H^i\Big(C_{\mathcal{D}}^\bullet(X_{/\mathbb{R}}; n)\Big)$$

where $\overline{F}_\infty$ is the De Rham involution, i.e. the composition of the involution F_∞ on $X(\mathbb{C})$ induced by the complex conjugation with the complex conjugation of coefficients.

Theorem–Construction 2.3 *a) There exists* canonical *homomorphism of complexes*

$$\mathcal{P}^\bullet(n) : \mathcal{Z}^\bullet(X;n) \longrightarrow C^\bullet_{\mathcal{D}}(X;n)$$

b) If X is defined over $\mathbb{R}$ then the image of the map $\mathcal{P}^\bullet(n)$ lies in the subcomplex $C^\bullet_{\mathcal{D}}(X_{/\mathbb{R}};n)$

To construct this homomorphism let us recall a remarkable differential $(m-1)$-form $r_{m-1}(L;H)$ canonically attached to the pair $(\mathbb{A}^m;L) = (\mathbb{P}^m - H, L)$ ([G5]).

4. The form $r_{m-1}(L;H)$. Let Ω_L be the canonical m-form in $\mathbb{P}^m - L$ with logarithmic singularities at L. It represents a generator of $H^m_{\mathrm{DR}}(\mathbb{P}^m - L)$ defined over $\mathbb{Z}$. Let us give its coordinate description. Choose a homogeneous coordinates $(z_0 : ... : z_m)$ in $\mathbb{P}^m$ such that L_i is given by equation $\{z_i = 0\}$. Then

$$\Omega_L = d\log z_1/z_0 \wedge ... \wedge d\log z_m/z_0$$

The form Ω_L has periods in $\mathbb{Z}(m)$. So $\pi_m(\Omega_L)$ is exact. However there is no canonical choice of a primitive $(m-1)$-form for it: the group $(\mathbb{C}^*)^m$ acting on $\mathbb{CP}^m - L$ leaves the form invariant and acts non trivially on the primitives. But if we consider a simplex L in the affine complex space $\mathbb{A}^m$ (or, what is the same, choose an additional hyperplane H in $\mathbb{CP}^m$, which should be thought of as the infinite hyperplane) then there is a *canonical* primitive.

To define the primitive the following construction is handy.

The homomorphism r_m. Let X be a variety over $\mathbb{C}$ and $f_1, ..., f_m$ are m complex-valued functions on $X(\mathbb{C})$. We attach to them the $(m-1)$-form

$$r_{m-1}(f_1, ..., f_m) := \tag{13}$$

$$\mathrm{Alt}_m \sum_{j\geq 0} c_{j,m} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2j+1}| \wedge di\arg f_{2j+2} \wedge ... \wedge di\arg f_m$$

Here $c_{j,m} := \frac{1}{(2j+1)!(m-2j-1)!}$ and Alt_m is the operation of alternation:

$$\mathrm{Alt}_m F(x_1, ..., x_n) := \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(x_{\sigma(1)}, ..., x_{\sigma(m)})$$

Remark. The form (68) is a part of the cocycle representing the product in real Deligne cohomology of 1-cocycles $(\log|f_i|, d\log f_i)$.

So $r_{m-1}(f_1, ..., f_m)$ is an $\mathbb{R}(m-1)$-valued $(m-1)$-form and it is easy to check that

$$dr_{m-1}(f_1, ..., f_m) = \pi_m\Big(d\log f_1 \wedge ... \wedge d\log f_m\Big)$$

Theorem 2.4 *The form $r_{m-1}(f_1,...,f_m)$ defines a distribution on $X(\mathbb{C})$, and provides a group homomorphism*

$$r_{m-1} : \Lambda^m \mathbb{C}(X)^* \longrightarrow \mathcal{D}^{m-1}_{X(\mathbb{C})}(m-1) \tag{14}$$

Proof. The fact that r_{m-1} is a homomorphism from $\Lambda^m \mathbb{C}(X)^*$ to differential $(m-1)$-forms is clear from the definitions.

Lemma 2.5 *For any rational functions $f_1,...,f_m$ on X the $(m-1)$-form $r_{m-1}(f_1 \wedge ... \wedge f_m)$ defines a distribution on $X(\mathbb{C})$, i.e. the integral*

$$\int_{X(\mathbb{C})} r_{m-1}(f_1,...,f_m) \wedge \omega$$

converges for any smooth $(2\dim X - m + 1)$-form ω with compact support on $X(\mathbb{C})$.

Basic example. The integral $\int_{\mathbb{C}} \log|z| d\log(z-a) \wedge d\log\overline{(z-b)}$ is divergent at infinity, where all the functions $z, z-a, z-b$ have a simple pole, since $\int_{\mathbb{C}} \log|z| \frac{dz \wedge d\bar{z}}{|z|^2}$ is divergent (both near zero and infinity). However

$$4 \cdot \int_{\mathbb{C}} \log|z| d\log|z-a| \wedge d\log|z-b| =$$

$$\int_{\mathbb{C}} \log|z| \Big(d\log(z-a) \wedge d\log\overline{(z-b)} \quad + \quad d\log\overline{(z-a)} \wedge d\log(z-b) \Big)$$

is convergent: the divergent parts cancelled each other. In $r_{m-1}(f_1,...,f_m)$ such divergences cancelled because of multiplicativity and skew-symmetry of r_{m-1}.

Proof. Resolving singularities of X we may assume that X is nonsingular. We may suppose that the divisors of functions f_i are disjoint because r_{m-1} is a homomorphism. Resolving singularities we reduce lemma to the case when these divisors have normal crossing. Our statement is local, so we can assume that in local coordinates $z_1,...,z_m$ one has $f_1 = z_1, ..., f_k = z_k$ and $div f_j$ for $j > k$ does not intersect the origin. After this the statement of lemma is obvious: each term in (68) defines a distribution near the origin. For instance the worse possible singularities has the term $\log|z_1| d\log|z_2| \wedge ... \wedge d\log|z_k| \wedge \omega$ where ω is smooth near the origin. It is clearly integrable with a smooth test form. The lemma is proved.

Remark. In particular if $\dim_{\mathbb{C}} X = n$ the integral

$$\int_{X(\mathbb{C})} r_{2n}(f_1,...,f_{2n+1}) \tag{15}$$

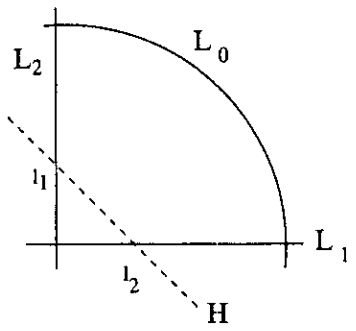
is convergent.

Choose a coordinate system in $\mathbb{P}^m$ as above such that H is given by by $\{\sum_{i=1}^m z_i = z_0\}$. Set

$$r_{m-1}(L; H) := r_{m-1}(z_1/z_0, ..., z_m/z_0)$$

Here is a more invariant definition. Choose one of the faces of the simplex L, say L_0. Consider simplex $(H, L_1, ..., L_m)$. Let t_i be the rational function on $\mathbb{CP}^n$ such that $(f_i) = (L_i) - (L_0)$ normalized by $f_i(l_i) = 1$, where l_i is the vertex of the simplex $(H, L_1, ..., L_m)$ opposite to the face L_i. Then $f_i = \frac{z_i}{z_0}$ and

$$r_{m-1}(L; H) := r_{m-1}(f_1, ..., f_m)$$



This form is skewsymmetric with respect to the permutation of the hyperplane faces of the simplex L. One has

$$dr_{m-1}(L; H) = \pi_m(\Omega_L) \quad \text{in} \quad \mathbb{CP}^m - L$$

So a choice of an "infinite" hyperplane $H \subset \mathbb{CP}^m$ provides the form $r_{m-1}(L; H)$.

Example. If $m = 1$ then

$$\mathbb{A}^1 = \mathbb{P}^1 - \{1\}, \quad L = \{0\} \cup \{\infty\}, \quad \Omega_L = d\log z, \quad \pi_1(d\log z) = d\log|z|,$$

$$r_0(\{0\} \cup \{\infty\}; \{1\}) = \log|z|$$

A coordinate free description of the form $r_{m-1}(L; H)$. Choose a volume form $\mathrm{vol}_m \in \det V_m^*$. Set $\Delta(v_1, ..., v_m) := < \mathrm{vol}_m, v_1 \wedge ... \wedge v_m > \in F^*$.

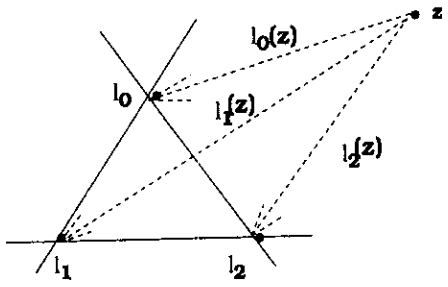
Lemma 2.6 *For a configuration* $(l_0, ..., l_m)$ *of* $m+1$ *vectors in generic position*

$$f_m(l_0, ..., l_m) := \sum_{i=0}^m (-1)^i \Lambda_{i \neq j} \Delta(l_0, ..., \widehat{l_j}, ..., l_i, ..., l_m) \in \Lambda^m F^* \qquad (16)$$

does not depend on the choice of the volume form vol_m.

Proof. See proof of lemma 3.1 in [G3].

For a point $z \in \mathbb{A}^m - L$ let $l_i(z)$ be the vecto. from z to the vertex l_i, see the picture.



We get a canonical element

$$f_m(l_0(z),...,l_m(z)) \in \Lambda^m\mathbb{Q}(\mathbb{A}^m - L)^*$$

If $F = \mathbb{C}$ applying the homomorphism r_m to this element we get a canonical $(m-1)$-form in $\mathbb{CP}^m - L$. It coincides with $r_{m-1}(L; H)$.

Example. If $m = 1$ and $(L_0, L_1, H) = (0, \infty, 1)$ then $t = \frac{z}{z-1}$ is an affine coordinate on $\mathbb{P}^1 - \{1\}$ and

$$l_0(t) = \frac{z}{z-1}, l_1(t) = \frac{1}{z-1}, \quad \text{so} \quad f_1(l_0(t), l_1(t)) = \frac{l_0(t)}{l_1(t)} = z$$

Remark. The map $z \in \mathbb{A}^m - L \longmapsto f_n(l_0(z),...,l_m(z))$ provides an isomorphism $F_m : CH^m(\mathrm{Spec}(F), 0) \longrightarrow K_m^M(F)$; see [NS] where the isomorphism F_m was presented in a bit different way.

For a subvariety $Y \subset X$ we define the δ-distribution δ_Y by setting

$$< \delta_Y, \omega >:= \int_{Y(\mathbb{C})} \omega$$

If f is a rational function on X we set $\delta(f) := \delta_{\mathrm{div}(f)}$.

Let Y be a divisor in a complex variety X. Let $g_1,...,g_{n-1}$ be rational functions on Y. They define an $(n-2)$-form $r_{n-2}(g_1 \wedge ... \wedge g_{n-1})$ on $Y(\mathbb{C})$, which, thanks to the lemma 2.5, defines an n-current $r_{n-2}(g_1 \wedge ... \wedge g_{n-1})\delta_Y$ on $X(\mathbb{C})$ denoted $\varphi(g_1 \wedge ... \wedge g_{n-1})$.

There is the residue homomorphism

$$\mathrm{Res} : \Lambda^n\mathbb{C}(X)^* \longrightarrow \oplus_{Y \subset X^{(1)}} \Lambda^{n-1}\mathbb{C}(Y)^*,$$

given by the formula

$$f_1 \wedge ... \wedge f_n \longmapsto \sum_{i=1}^{n} (-1)^{i-1} v_Y(f_i) \cdot f_{1|_Y} \wedge ... \wedge \widehat{f_{i|_Y}} \wedge ... \wedge f_{n|_Y}$$

Here is a more rigorous definition. Let K be a field with a discrete valuation v and the residue field k_v. The group of units U has a natural homomorphism

$U \longrightarrow k_v^*$, $u \mapsto \overline{u}$. An element $\pi \in K^*$ is prime if $\mathrm{ord}_v \pi = 1$. There is a homomorphism $\mathrm{res}_v : \Lambda^n K^* \longrightarrow \Lambda^{n-1} k_v^*$ uniquely defined by the properties ($u_i \in U$):

$$\mathrm{res}_v(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \overline{u}_1 \wedge \cdots \wedge \overline{u}_{n-1} \quad \text{and} \quad \mathrm{res}_v(u_1 \wedge \cdots \wedge u_n) = 0$$

It does not depend on the choice of π.

We set $\mathrm{Res} := \sum \mathrm{res}_v$ where the sum is over all valuations of the field $\mathbb{C}(X)$ corresponding to the codimension one points of X.

The $(n-1)$-form $r_{n-1}(L; H)$ provides an $(n-1)$-distribution on $\mathbb{CP}^n$. Recall the simplex $\widehat{L}_i$ cut by L in the hyperplane L_i, and $H_i = L_i \cap H$. Consider the $(n-2)$-form $r_{n-2}(\widehat{L}_i; H_i)$ on the hyperplane L_i as n-distribution in $\mathbb{CP}^n$. We denote it as $r_{n-2}(\widehat{L}_i; H_i) \cdot \delta_{L_i}$.

More generally, a current α on a divisor in X provides a current $\varphi(\alpha)$ on X.

Lemma 2.7 *a) Let $f_1, ..., f_n$ be rational functions on a variety X over $\mathbb{C}$. Then*

$$dr_{n-1}(f_1 \wedge ... \wedge f_n) =$$

$$\pi_n\Big(d\log f_1 \wedge ... \wedge d\log f_n\Big) + 2\pi i \cdot \varphi\Big(r_{n-2}\mathrm{Res}(f_1 \wedge ... \wedge f_n)\Big) \qquad (17)$$

b) In particular one has

$$dr_{n-1}(L; H) = \pi_n(\Omega_L) + 2\pi i \cdot \sum_{i=0}^{n} (-1)^i r_{n-2}(\widehat{L}_i; H_i)\delta_{L_i} \qquad (18)$$

Proof. The part b) follows immediately from a). Using the formula

$$d(di \arg z) = 2\pi i \delta(z) := 2\pi i(\delta_0 - \delta_\infty)$$

we get $d(di \arg f) = 2\pi i \delta(f)$. The part a) follows from this.

5. The main construction. We have to construct a morphism of complexes

$$\begin{array}{ccccccccc}
... & \longrightarrow & \mathcal{Z}^1(X; n) & \longrightarrow & ... & \longrightarrow & \mathcal{Z}^{2n-1}(X; n) & \longrightarrow & \mathcal{Z}^{2n}(X; n) \\
 & & \downarrow \mathcal{P}^1(n) & & ... & & \downarrow \mathcal{P}^{2n-1}(n) & & \downarrow \mathcal{P}^{2n}(n) \\
0 & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{0,0}(n-1) & \longrightarrow & ... & \longrightarrow & \mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1) & \stackrel{2\overline{\partial}\partial}{\longrightarrow} & \mathcal{D}_{\mathbb{R}}^{n,n}(n)
\end{array}$$

Let $Y \in \mathcal{Z}^{2n}(X; n)$ be a codimension n cycle in X. By definition

$$\mathcal{P}^{2n}(n)(Y) := (2\pi i)^n \delta_Y$$

Let us construct homomorphisms

$$\mathcal{P}^{(2n-i)}(n) : \mathcal{Z}^{2n-i}(X; n) \longrightarrow \mathcal{D}_{X(\mathbb{C})}^{2n-i-1}(n-1), \quad i > 0$$

Denote by $\pi_{\mathbb{A}}$ (resp. π_X) the projection of $X \times \mathbb{A}^i$ to $\mathbb{A}^i$ (resp. X). Let $k = 2\mathrm{dim}Y - i + 1$. Notice that

$$2\mathrm{dim}X - k = 2(\mathrm{dim}X + i - \mathrm{dim}Y) - i - 1 = 2n - i - 1$$

So $(2n-i-1)$-distributions are functionals on smooth k-forms with compact support on $X(\mathbb{C})$. Let ω be such a k-form and $Y \in \mathcal{Z}^{2n-i}(X;n)$. Then we set

$$< \mathcal{P}^{2n-i}(n)(Y), \omega >:= \quad (2\pi i)^{n-i} \int_{Y(\mathbb{C})} \pi_X^* \omega \wedge \pi_{\mathbb{A}}^* r_{i-1}(L;H) \qquad (19)$$

Theorem 2.8 *The integral (19) is convergent.*

Proof. Let $\overline{Y} \subset X \times \mathbb{P}^i$ be the closure of the cycle Y and $\overline{\pi}_X, \overline{\pi}_{\mathbb{A}}$ the corresponding projections from $\overline{Y}$ to X and $\mathbb{P}^i$. Then the form $\overline{\pi}_X^* \omega$ has compact support on $\overline{Y}$, so, we can apply lemma 2.5. The theorem is proved.

Remark. We just proved that the product of distributions $\delta_Y \wedge \pi_{\mathbb{A}}^* r_{i-1}(L;H)$ makes sense and

$$\mathcal{P}^{2n-i}(n)(Y) = \quad (2\pi i)^{n-i} \pi_{X*}(\delta_Y \wedge \pi_{\mathbb{A}}^* r_{i-1}(L;H))$$

Since the form $r_{i-1}(L;H)$ is $\mathbb{R}(i-1)$-valued, for $i > 0$ the distribution $\mathcal{P}^{2n-i}(n)(Y)$ takes values in $\mathbb{R}(n-1)$. Further, $\mathcal{P}_n^{(2n)}(Y)$ is obviously an $\mathbb{R}(n)$-valued distribution.

Let us show that for $i > 0$ the distribution $\mathcal{P}^{2n-i}(n)(Y)$ lies precisely in the left bottom $(n-1) \times (n-1)$ square of the Dolbeault bicomplex. The integral (19) is non zero only if $\pi_X^* \omega \wedge \pi_{\mathbb{A}}^* r_{i-1}(L;H)$ is of type

$$(\mathrm{dim}Y, \mathrm{dim}Y) = (\mathrm{dim}X + i - n, \mathrm{dim}X + i - n)$$

Since $r_{i-1}(L;H)$ is an $(i-1)$-form we see that the integral vanish if ω is a form of type (p,q) where p or q bigger then $\mathrm{dim}X + 1 - n$. The statement follows.

Therefore we have constructed the maps $\mathcal{P}^i(n)$.

Theorem 2.9 $\mathcal{P}^\bullet(n)$ *is a homomorphism of complexes.*

Proof. One has

$$< d\mathcal{P}^{2n-i}(n), \omega >= - < \mathcal{P}^{2n-i}(n), d\omega >= - \int_{\overline{Y}(\mathbb{C})} \overline{\pi}_X^* d\omega \wedge \overline{\pi}_{\mathbb{A}}^* r_{i-1}(L;H) =$$

$$\int_{\overline{Y}(\mathbb{C})} \overline{\pi}_X^* \omega \wedge d\overline{\pi}_{\mathbb{A}}^* r_{i-1}(L;H) \qquad (20)$$

We need the following lemma

Lemma 2.10 $\int_{Y(\mathbb{C})} \pi_X^* \omega \wedge \pi_{\mathbb{A}}^* \pi_i(\Omega_L) = 0.$

Proof. We need the following obvious fact. Suppose A is a complex manifold and B is a manifold. Let π_A (resp. π_B) be the projection of $A \times B$ onto A (resp. B). Let α (resp. β) be a differential form on $A \times B$ (resp. B). Then $(\pi_B)_*(\alpha \wedge \pi_B^*\beta) = (\pi_B)_*(\alpha) \wedge \beta \neq 0$ only if α is a form of type $(\dim A, \dim A)$.

The form $\pi_i(\Omega_L) = \Omega_L \pm \overline{\Omega}_L$ is a sum of forms of type $(i,0)$ and $(0,i)$ where $i > 0$ by our assumption. Restricting the projection π_X to the generic part Y^0 of Y we get a fibration $Y^0 \to \pi_X(Y^0)$. If the dimension of the fiber is zero, the integral vanish since $i > 0$. If the dimension of the fiber is bigger then zero, it vanish thanks to the argument above, since $\pi_i(\Omega_L)$ of type $(i,0)$ and $(0,i)$.

Applying lemma 2.7 to (20) we get the theorem for $i > 0$.

Since $r_0(f) = \log|f|$, the commutativity of the last square follows from the formula

$$2\overline{\partial}\partial \log|f| = 2\pi i(\delta_{D_0} - \delta_{D_\infty}) \quad \text{where} \quad (f) = D_0 - D_\infty$$

Theorem 2.9 is proved. Therefore we finished the proof of theorem–construction 2.3.

6. The Higher Arakelov Chow groups. Let X be a regular complex variety. Denote by $\widetilde{C}^\bullet_{\mathcal{D}}(n)$ the quotient of the complex $C^\bullet_{\mathcal{D}}(n)$ along the subgroup $\mathcal{A}^{n,n}_{cl}(n) \subset \mathcal{D}^{n,n}_{cl}(n)$ of closed smooth form of type (n,n) with values at $\mathbb{R}(n)$.

Consider the cone of the homomorphism $\mathcal{P}^\bullet(n)$ shifted by -1:

$$\widehat{\mathcal{Z}}^\bullet(X;n) := Cone\Big(\mathcal{Z}^\bullet(X;n) \longrightarrow \widetilde{C}^\bullet_{\mathcal{D}}(X(\mathbb{C});n)\Big)[-1]$$

Definition 2.11 *The Higher Arakelov Chow groups are*

$$\widehat{CH}^n(X;i) := H^{2n-i}(\widehat{\mathcal{Z}}^\bullet(X;n)) \tag{21}$$

Recall the Arakelov Chow groups defined by Gillet-Soule as follows:

$$\widehat{CH}^n(X) :=$$

$$\frac{\{(Z,g); \frac{\overline{\partial}\partial}{\pi i}g + \delta_Z \in \mathcal{A}^{n,n}\}}{\{(0, \partial u + \overline{\partial}v); (\mathrm{div} f, -\log|f|), f \in \mathbb{C}(Y), \mathrm{codim}(Y) = n-1\}} \tag{22}$$

Here Z is a divisor in X, f is a rational function on a divisor Y in X,

$$g \in \mathcal{D}^{n-1,n-1}_{\mathbb{R}}(n-1), \quad (u,v) \in (\mathcal{D}^{n-2,n-1} \oplus \mathcal{D}^{n-1,n-2})_{\mathbb{R}}(n-1)$$

Proposition 2.12 $\widehat{CH}^n(X;0) = \widehat{CH}^n(X)$.

Proof. Let us look at the very right part of the complex $\widehat{\mathcal{Z}}^\bullet(X;n)$:

$$\begin{array}{ccccc} \ldots & \longrightarrow & \mathcal{Z}^{2n-1}(X;n) & \longrightarrow & \mathcal{Z}^{2n}(X;n) \\ & & \downarrow r_n^{(2n-1)} & & \downarrow \mathcal{P}_n^{(2n)} \\ (\mathcal{D}^{n-2,n-1} \oplus \mathcal{D}^{n-1,n-2})_{\mathbb{R}}(n-1)) & \xrightarrow{(\partial,\overline{\partial})} & \mathcal{D}^{n-1,n-1}_{\mathbb{R}}(n-1) & \xrightarrow{\overline{\partial}\partial} & \mathcal{D}^{n,n}_{\mathbb{R}}(n)/\mathcal{A}^{n,n}_{\mathbb{R}}(n) \end{array}$$

Consider the very end of the Gersten complex on X:

$$\prod_{Y \in X_{n-2}} \Lambda^2 \mathbb{C}(Y)^* \xrightarrow{\partial} \prod_{Y \in X_{n-1}} \mathbb{C}(Y)^* \longrightarrow \mathcal{Z}_0(X;n)$$

where ∂ is the tame symbol. It maps to the complex $\widehat{\mathcal{Z}}^\bullet(X;n)$, i.e. to the top row of the bicomplex above, as follows. Recall that $\mathcal{Z}_0(X;n) = \mathcal{Z}^{2n}(X;n)$, so the very right component of our map is provided by this identification. Further, a pair $(Y;f)$ where Y is an irreducible codimension $n-1$ subvariety of X maps to the cycle $(y, f(y)) \subset X \times \mathbb{A}^1$. Similarly any element in $\Lambda^2\mathbb{C}(Y)^*$ can be represented as a linear combinations of elements $\sum_i (Y; f_i \wedge g_i)$ where Y is an irreducible codimension $n-2$ subvariety of X and f_i, g_i are rational functions on Y such that $\mathrm{div} f_i$ and $\mathrm{div} g_i$ share no irreducible divisors. Then we send $(Y; f_i \wedge g_i)$ to the cycle $(y, f_i(y), g_i(y)) \subset X \times \mathbb{A}^2$. It is well known that this way we get an isomorphism on the last two cohomology groups. Computing the composition of this map with the homomorphism $\mathcal{P}^\bullet(n)$ we end up precisely with the denominator in (22). The proposition is proved.

3 The Chow polylogarithms

Suppose $X = \mathrm{Spec}(\mathbb{C})$. Then $\mathcal{P}_n(Y) := \mathcal{P}_n^{(1)}(Y)$ is a function on the space of all codimension n cycles in $\mathbb{P}^{2n-1}$ intersecting properly faces of the simplex L. It is called the *Chow polylogarithm function*. For $i > 0$ all the distributions $\mathcal{P}_n^{(i)}(Y)$ are zero. However modifying the construction of the previous chapter we get a very interesting object, the Chow polylogarithm, even when X is a point. The Chow polylogarithm function is the first component of the Chow polylogarithm. One can define the Chow polylogarithm for an arbitrary variety X, but we spell the details in the most interesting case when X is a point.

1 Chow polylogarithms [G5]. Recall that L is a simplex in $\mathbb{P}^{p+q}$, H is a hyperplane in generic position to L, and $H_i := H \cap L_i$.

Let $\mathcal{Z}_p^q(L)$ be the variety of all codimension q effective algebraic cycles in $\mathbb{CP}^{p+q}$ which intersect properly, i.e. each irreducible component in the right codimension, all faces of the simplex L. It is a union of infinite number of finite dimensional complex algebraic varieties.

Example. $\mathbb{CP}^n - L = (\mathbb{C}^*)^n$ is an irreducible component of $\mathcal{Z}_0^n(L)$ parametrizing the irreducible subvarieties, i.e. points.

The intersection of a cycle with a codimension 1 face L_i of the simplex L provides a map

$$a_i : \mathcal{Z}_p^q(L) \longrightarrow \mathcal{Z}_{p-1}^q(L_i)$$

Further, projection with the center at the vertex l_j of L defines a map

$$b_j : \mathcal{Z}_p^q(L) \longrightarrow \mathcal{Z}_p^{q-1}(L_j)$$

Theorem–Construction 3.1 *For given $q \geq 0$ there is an explicitly constructed chain of $(q-p-1)$-distributions $\omega_p^q = \omega_p^q(L;H)$ on $\mathcal{Z}_p^q(L)$ such that*

$$i) \qquad d\omega_0^q(L,H) = \pi_q(\Omega_L) \tag{23}$$

$$ii) \quad d\omega_p^q(L;H) = \sum_{i=0}^{p+q}(-1)^i a_i^* \omega_{p-1}^q(L;H_i) \tag{24}$$

$$iii) \qquad \sum_{j=0}^{p+q+1}(-1)^j b_j^* \omega_p^q(L;H) = 0 \tag{25}$$

The restriction of ω_p^q to the subvariety $\widehat{\mathcal{Z}}_p^q(L)$ of smooth cycles in generic position with respect to the simplex L is a real-analytic differential $(q-p-1)$ form.

For a given positive integer q the collection $\{\omega_p^q\}$ is called the *q-th Chow polylogarithm.*

The varieties $\mathcal{Z}_p^q(L)$ for $p \geq 0$ form a truncated simplicial variety $\mathcal{Z}_\bullet^q$. The conditions i) and ii) just mean that the sequence of forms ω_p^q is a $2q$-cocycle in the bicomplex computing the Deligne cohomology $H^{2q}(Z_\bullet^q, \mathbb{R}_{\mathcal{D}}(q))$.

Proof. We define ω_p^q as the Radon transform of the distribution $r_{p+q-1}(L;H)$ in $\mathbb{CP}^{p+q}$ over the family of cycles Y_ξ parametrized by $\mathcal{Z}_p^q(L)$. This means the following. Consider the incidence variety:

$$\Gamma_p := \{(x,\xi) \in \mathbb{CP}^{p+q} \times \mathcal{Z}_p^q(L)(\mathbb{C}) \quad \text{such that} \quad x \in Y_\xi\}$$

where Y_ξ is the cycle in $\mathbb{CP}^{p+q}$ corresponding to $\xi \in \mathcal{Z}_p^q(L)$. We get a double bundle

$$\Gamma_p \subset \mathbb{CP}^{p+q} \times \mathcal{Z}_p^q(L)(\mathbb{C})$$

$$\pi_1 \swarrow \qquad\qquad \searrow \pi_2$$

$$\mathbb{CP}^{p+q} \qquad\qquad \mathcal{Z}_p^q(L)(\mathbb{C})$$

Then

$$\omega_p^q := \pi_{2*}\mathrm{Res}_{\Gamma_p}\pi_1^*(2\pi i)^{-q} r_{p+q}(L;H)$$

Observe that since $r_{p+q}(L;H)$ is a distribution on $\mathbb{CP}^{p+q}$, and hence $\pi_1^* r_{p+q}(L;H)$ is a distribution on $\mathbb{CP}^{p+q} \times \mathcal{Z}_p^q(L)(\mathbb{C})$. The fact that this distribution can be restricted to Γ_p is a version of theorem 2.4, and is proved the same way. The push forward π_{2*} of this distribution is well defined since π_2 is a proper map.

The property i) is true by the very definition.

Lemma 3.2 $\sum_{j=0}^{n+1}(-1)^j b_j^* \omega_0^n = 0$.

Proof. Let $s(z_0,...,z_n) := z_1/z_0 \wedge ... \wedge z_n/z_0$. Lemma follows from the identity

$$\sum_{j=0}^{n+1}(-1)^j s(z_0,...,\widehat{z_j},...,z_{n+1}) = 0$$

So we have iii). To check ii) observe that the push forward π_{2*} of distributions commutes with the De Rham differential. The theorem is proved.

2. Properties of the Chow polylogarithm function. The function $\mathcal{P}_q := \omega^q_{q-1}$ on $\mathcal{Z}^q_{q-1}(L)$ is called the *Chow q-logarithm function.* It satisfies two functional equations:

$$\sum_{i=0}^{2q}(-1)^i a_i^* \mathcal{P}_q = 0 \qquad \sum_{j=0}^{2q}(-1)^j b_j^* \mathcal{P}_q = 0$$

Theorem 3.3 *Suppose that $dim X = n$ and $f_1,...,f_{2n+1}$ are rational functions on X. Then the integral*

$$(2\pi i)^{1-n}\int_{X(\mathbb{C})} r_{2n}(f_1,...,f_{2n+1})$$

does not change if we multiply one of the functions f_i by a non zero constant.

Proof. Multiplying, say, f_1 by λ we see that the difference between the two integrals is

$$\log|\lambda|\sum_k a_k \int_{X(\mathbb{C})} \mathrm{Alt}_{2n} d\log|f_2| \wedge ... \wedge d\log|f_{2k-1}| \wedge d\arg f_{2k} \wedge ... \wedge d\arg f_{2n+1} \tag{26}$$

where a_k are some rational constants (easy computable from (68)). We will prove that for each k the corresponding integral in this sum is already zero. Using the identity

$$(d\log|f_2| + id\arg f_2) \wedge ... \wedge (d\log|f_{2n+1}| + id\arg f_{2n+1}) = 0$$

we can rewrite the integral

$$\int_{X(\mathbb{C})} d\arg|f_2| \wedge ... \wedge d\arg f_{2n+1}$$

as a sum of similar integrals containing $d\log|f_i|$. Our statement follows from

Proposition 3.4 *Suppose that $dim X = n$. Then*

$$d\mathrm{Alt}_{2n}\Big(\log|f_2| d\log|f_3| \wedge ... \wedge d\log|f_{2k-1}| \wedge d\arg f_{2k} \wedge ... \wedge d\arg f_{2n+1}\Big) =$$

$$\mathrm{Alt}_{2n}\Big(d\log|f_2| \wedge ... \wedge d\log|f_{2k-1}| \wedge d\arg f_{2k} \wedge ... \wedge d\arg f_{2n+1}\Big) \tag{27}$$

in the sense of distributions.

Proof. Since $ddi\arg f = 2\pi i\delta(f)$, the left hand side is equal to the right hand side *plus* the following terms concentrated on the divisors $f_{2k+j} = 0$:

$$4\pi\cdot 2(n-k+1)\mathrm{Alt}_{2n}\Big(\delta(f_{2n+1})\log|f_2|d\log|f_3|\wedge...\wedge d\log|f_{2k-1}|\wedge d\arg f_{2k}\wedge...\wedge d\arg f_{2n}\Big)$$

However all these additional terms vanish thanks to the following proposition.

Proposition 3.5 *Suppose that* $dim X = n$. *Then for each* $0 \leq j \leq n-1$ *one has*

$$\mathrm{Alt}_{2n}\Big(d\log|f_1|\wedge...\wedge d\log|f_{2j+1}|\wedge d\arg f_{2j+2}\wedge...\wedge d\arg f_{2n}\Big) = 0 \qquad (28)$$

$$\mathrm{Alt}_{2n}\Big(d\log|f_1|\wedge...\wedge d\log|f_{2j}|\wedge d\arg f_{2j+1}\wedge...\wedge d\arg f_{2n}\Big) = \qquad (29)$$

$$\frac{(2n)!\binom{n}{j}}{\binom{2n}{2j}}\cdot d\log|f_1|\wedge...\wedge d\log|f_{2n}|$$

Proof. The idea is this. One has n equations

$$\begin{gathered} d\log f_1\wedge...\wedge d\log f_{2n} = 0 \\ d\log|f_1|\wedge d\log f_2\wedge...\wedge d\log f_{2n} = 0 \\ ..... \\ d\log|f_1|\wedge...\wedge d\log|f_{n-1}|\wedge d\log f_n\wedge...\wedge d\log f_{2n} = 0 \end{gathered} \qquad (30)$$

Taking imaginary part of each of them and alternating $f_1, ..., f_{2n}$ we get n linear equations. Solving them we get the proposition. See the details in the Appendix in [GZ].

Therefore the Chow polylogarithm function is invariant under the natural action of the torus $(\mathbb{C}^*)^{p+q}$ on $\mathcal{Z}^q_p(\mathbb{C})$. In particular it does not depend on the choice of the hyperplane H. These statements are no longer true for the forms ω^q_p for $p < q-1$.

4 The Grassmannian polylogarithms

1. Configurations of vectors and Grassmannians: a dictionary. Let X be a G-set. We define *configurations of* m *points of* X as G-orbits in X^m.

Example 1. If $X := V$ is a vector space and $G := GL(V)$ we get configurations of vectors in V. A configuration of vectors $(l_1, ..., l_m)$ is in generic position if each $k \leq \dim V$ of the vectors are linearly independent.

Example 2. If $X := \mathbb{P}(V)$ is a projective space and $G := PGL(V)$ we get configurations $(x_1, ..., x_m)$ of m points in $\mathbb{P}(V)$. A configuration of points is in generic position if each $k \leq \dim V$ of them generate a plane of dimension $k-1$.

Let T_{p+q} be the quotient of the torus $\mathbb{G}_m^{p+q+1}$ by the diagonal subgroup $\mathbb{G}_m = (t, ..., t)$.

Lemma–Construction 4.1 *i) There are canonical isomorphisms between the following sets of geometric objects:*

(a) Configurations of $p+q+1$ vectors in generic position in V_q.

(b) Isomorphism classes of triples {a projective space $\mathbb{P}^{p+q}$ together with a simplex L, an "infinite" hyperplane H in generic position to L, and a p-dimensional plane in generic position to L (but not necessarily to H) }.

(c) Isomorphism classes of triples {a vector space V_{p+q+1}, a basis $(e_0, ..., e_{p+q})$ of V_{p+q+1}, and a $p+1$-dimensional subspace of V_{p+q+1} in generic position with respect to the coordinate hyperplanes}.

ii) The torus T_{p+q} acts naturally, and without stable points, on each of the objects a), b), c), and the isomorphisms above are compatible with this action.

Proof. i) (a) $\to$ (c). Take the kernel of the linear map from V_{p+q+1} to V_q sending e_i to l_i.

(c) $\to$ (a). Take the quotient of V_{p+q+1}/h along the given subspace h and consider the images of the vectors $(e_0, ..., e_{p+q})$ there.

(c) $\to$ (b). Let $\mathbb{P}^{p+q} := P(V_{p+q+1})$. Let $\mathbb{A}^{p+q}$ be the affine hyperplane in V_{p+q+1} passing through the ends of the basis vectors e_i. Then $\mathbb{A}^{p+q} \subset \mathbb{P}^{p+q}$. The coordinate hyperplanes in V_{p+q+1} provide a simplex $L_{p+q} \subset \mathbb{P}^{p+q}$. The projectivization of a generic subspace h in V_{p+q+1} gives a p-plane $\overline{h}$ in generic position with respect to this simplex. (Notice that we do not impose any condition on the mutual location of H and $\overline{h}$. For instance $\overline{h}$ may be inside of H.)

(b) $\to$ (c). The triple $(\mathbb{P}^{p+q}, H, L)$ provides a unique up to an isomorphism data $(V_{p+q+1}, (e_0, ..., e_{p+q}))$. Namely, the partial data $(\mathbb{P}^{p+q}, H)$ provides us with $(V_{p+q+1}, \widetilde{H})$ where $\widetilde{H}$ is the subspace of V_{p+q+1} projecting to H. Now the vertices l_i of the simplex L provide coordinate lines $\widetilde{l}_i$ in V_{p+q+1}. Intersecting these coordinate lines with a parallel shift of the subspace $\widetilde{H}$ we are getting a point on each of the coordinate lines. By definition the endpoints of the basis vectors e_i are these points. Taking the subspace $\widetilde{h}$ in V_{p+q+1} projecting to a given plane h in $\mathbb{P}^{p+q}$ we get the desired correspondence.

ii) The torus T_{p+q} acts on the configurations of vectors in a) as

$$(t_1, ..., t_{p+q+1}) : (l_1, ..., l_{p+q+1}) \longmapsto (t_1 l_1, ..., t_{p+q+1} l_{p+q+1})$$

The torus T_{p+q} is identified with $\mathbb{P}^{p+q} - L$ in b), and so acts naturally on the data in b). The action on the data c) is similar. The lemma is proved.

If we use the description c) for the Grassmannians then b_j is obtained by factorization along the coordinate axe $< e_j >$.

Configurations of hyperplanes and torus quotients of Grassmannians. Let $\widehat{G}_p^q$ be the Grassmannian of p-planes in $\mathbb{P}^{p+q}$ in generic position with respect to a given simplex L.

Taking the T_{p+q}–orbits of the objects a) and b) in the lemma we arrive to

Corollary 4.2 *There is a bijective correspondence*

$$\widehat{G}_p^q/T_{p+q} < - > \{\textit{Configurations of } p+q+1 \textit{ generic hyperplanes in } \mathbb{P}^p\}$$

sending a p–plane h to the configuration $(h \cap L_0, ..., h \cap L_{p+q})$ *in h.*

2. The Grassmannian and bi-Grassmannian polylogarithms. Let us fix a positive integer q. The operations a_i and b_j from section 2.1 transforms planes to planes. So we get the following diagram of varieties called the bi-Grassmannian $\widehat{G}(q)$:

$$\begin{array}{ccccccc}
 & & & & \downarrow \ldots \downarrow & & \downarrow \ldots \downarrow \\
\widehat{G}(q) := & & & \stackrel{\rightarrow}{\rightrightarrows} & \widehat{G}_1^{q+1} & \stackrel{\rightarrow}{\rightrightarrows} & \widehat{G}_0^{q+1} \\
 & & \downarrow \ldots \downarrow & & \downarrow \ldots \downarrow & & \downarrow \ldots \downarrow \\
 & \ldots \stackrel{\rightarrow}{\rightrightarrows} & \widehat{G}_2^q & \stackrel{\rightarrow}{\rightrightarrows} & \widehat{G}_1^q & \stackrel{\rightarrow}{\rightrightarrows} & \widehat{G}_0^q
\end{array}$$

Here the horizontal arrows are the arrows a_i and the vertical ones are b_j.

Remark. The bi-Grassmannian $\widehat{G}(n)$ is not a (semi)bisimplicial scheme. (It is a truncated semi*hypersimplicial* scheme. See s.2.6 in [G4]).

Let $\psi_p^q(q)$ be the restriction of the differential form ω_p^q to $\widehat{G}_p^q$. The properties i), ii) from theorem 3.1 are exactly the defining conditions for the single–valued Grassmannian polylogarithm whose existence was conjectured in [HM], [BMS], see also [GGL].

Let us extend these forms by zero to the other rows of the bi-Grassmannian $\widehat{G}(q)$, i.e. set $\psi_p^{q+i}(q) = 0$ if $i > 0$. Then the property iii) from theorem 3.1 guarantees that the forms $\psi_p^{q+i}(q)$ form a $2q$-cocycle in the bicomplex computing the Deligne cohomology $H^{2q}(\widehat{G}(q)_\bullet, \mathbb{R}(q)_\mathcal{D})$. It is called *the bi-Grassmannian n-logarithm.* ([G5]).

A sequence of *multivalued analytic* forms on Grassmannians satisfying conditions similar to i), ii) was defined in [HaM1], [HaM2]. Another construction of the multivalued analytic Grassmannian polylogarithms was suggested in [G5] in the more general setting of the multivalued Chow polylogarithm.

3. The Grassmannian n-logarithm function. By theorem 3.3 the Chow polylogarithm function is invariant under the action of the torus $(\mathbb{C}^*)^{2n-1}$. So restricting it to the open Grassmannian $\widehat{G}_{n-1}^n \subset \widehat{\mathcal{Z}}_{n-1}^n$ and using the bijection

$$\{(n-1)\text{–planes in } \mathbb{P}^{2n-1} \text{ in generic position with respect to a simplex } L\}/(\mathbb{G}_m^*)^{2n-1}$$

$$< -- > \quad \{\text{Configurations of } \ 2n \ \text{ generic hyperplanes in } \mathbb{P}^{n-1}\}$$

we get a function on the configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$, called the Grassmannian polylogarithm function $\mathcal{L}_n^G$.

The Grassmannian polylogarithm function has the following simple description on the language of configurations of hyperplanes. It is intersecting that in this description we can work with *any* configurations of $2n$ hyperplanes, assuming nothing about their mutual location.

Let $h_1, ..., h_{2n}$ be arbitrary $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. Choose an additional hyperplane h_0. Let f_i be a rational function on $\mathbb{CP}^{n-1}$ with divisor $h_i - h_0$. It is defined up to a scalar factor. Set

$$\mathcal{L}^G_n(h_1, ..., h_{2n}) := (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(\sum_{j=1}^{2n}(-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ... \wedge f_{2n})$$

It is skewsymmetric by the definition. Notice that

$$\sum_{j=1}^{2n}(-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ... \wedge f_{2n} = \frac{f_1}{f_{2n}} \wedge \frac{f_2}{f_{2n}} \wedge ... \wedge \frac{f_{2n-1}}{f_{2n}}$$

So we can define $\mathcal{L}^G_n(h_1, ..., h_{2n})$ as follows: choose rational functions $g_1, ..., g_{2n-1}$ such that $\mathrm{div} g_i = h_i - h_{2n}$ and put

$$\mathcal{L}^G_n(h_1, ..., h_{2n}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(g_1, ..., g_{2n-1})$$

Remark. The function $\mathcal{L}^G_n$ is defined on the set of all configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. However it is not even continuous on this set. It is real analytic on the submanifold of generic configurations. Since we put no restrictions on the hyperplanes h_i the following theorem is stronger then theorem 3.1 in the case of linear subvarieties.

Theorem 4.3 *The function $\mathcal{L}^G_n$ has the following properties:*

a) It does not depend on the choice of hyperplane h_0.

b) For any $2n+1$ hyperplanes in $\mathbb{CP}^n$ one has

$$\sum_{j=1}^{2n+1}(-1)^j \mathcal{L}^G_n(h_j \cap h_1, ..., h_j \cap h_{2n+1}) = 0 \tag{31}$$

c) For any $2n+1$ hyperplanes in $\mathbb{CP}^{n-1}$ one has

$$\sum_{j=1}^{2n+1}(-1)^j \mathcal{L}^G_n(h_1, ..., \widehat{h_j}, ..., h_{2n+1}) = 0 \tag{32}$$

Proof. a) Choose another hyperplane h'_0. Take a rational function f_0 with divisor $h'_0 - h_0$. Set $f'_i = \frac{f_i}{f_0}$. Then

$$\sum_{j=1}^{2n+1}(-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ..., \wedge f_{2n} - \sum_{j=1}^{2n+1}(-1)^j f'_1 \wedge ... \wedge \widehat{f'_j} \wedge ..., \wedge f'_{2n} = 0$$

Indeed, substituting $f_i' = \frac{f_i}{f_0}$ in this formula we find that the only possible nontrivial term $f_0 \wedge f_1 \wedge ... \wedge \widehat{f_i} \wedge ... \wedge \widehat{f_j} \wedge ... \wedge f_{2n}$ vanish because it is symmetric in i, j.

b) Let $g_1, ..., g_{2n+1}$ be rational functions on $\mathbb{CP}^n$ with $\mathrm{div} g_i = h_i - h_0$. Then

$$dr_{2n}\Big(\sum_{j=1}^{2n+1}(-1)^j g_1 \wedge ... \wedge \widehat{g_j} \wedge ... g_{2n+1}\Big) = \tag{33}$$

$$2\pi i \delta(f_j) df_j \wedge \widetilde{d} f_j \wedge r_{2n-1}\Big(\sum_{j \neq i}(-1)^j g_1 \wedge ... \widehat{g_i} \wedge ... \wedge \widehat{g_j} \wedge ... g_{2n+1}\Big)$$

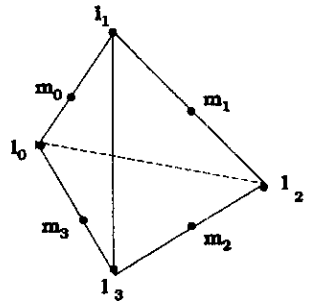
(Notice that $d\log g_1 \wedge ... \wedge \widehat{d\log g_j} \wedge ... \wedge d\log g_{2n+1} = 0$ on $\mathbb{CP}^n$). Integrating (33) over $\mathbb{CP}^n$ we get b).

c) is obvious: we apply r_{2n-1} to zero element. Theorem is proved.

4. $\mathbb{P}^1 - \{0, \infty\}$ as a special stratum in the configuration space of $2n$ points in $\mathbb{P}^n$. A *special configuration* is a configuration of $2n$ points

$$(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) \tag{34}$$

in $\mathbb{P}^{n-1}$ such that $l_0, ..., l_{n-1}$ are vertices of a simplex in $\mathbb{P}^{n-1}$ and m_i is a point on the edge $l_i l_{i+1}$ of the simplex different from l_i and l_{i+1}, as on the picture.



Proposition 4.4 *The set of special configurations of $2n$ points in $\mathbb{P}^{n-1}$ is canonically identified with $\mathbb{P}^1 \setminus \{0, \infty\}$.*

$$\mathbb{P}^1\setminus\{0,\infty\} = \Big\{ \quad \Big\} \Big/ \mathrm{PGL}_n$$

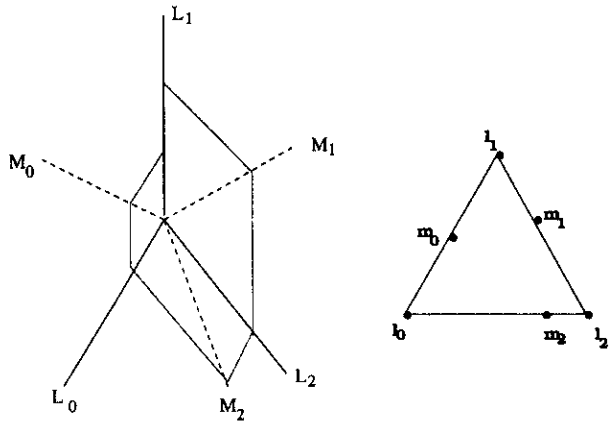
Proof. We define the generalized cross-ratio

$$r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) \in F^*$$

as follows. Consider the one-dimensional subspaces L_i, M_j in the n-dimensional vector space V projecting to the points l_i, m_j in $\mathbb{P}^{n-1}$ respectively. The subspaces L_i, M_i, L_{i+1} generate a two dimensional subspace. Its quotient along M_i can be identified with L_i as well as with L_{i+1}. So we get a canonical linear map $\overline{M}_i : L_i \longrightarrow L_{i+1}$. The composition of these maps (the "linear monodromy")

$$\overline{M}_0 \circ ... \circ \overline{M}_{n-1} : L_0 \longrightarrow L_0$$

is a multiplication by an element of F^* called the generalized cross-ratio of the special configuration (34).



It is clearly invariant under the cyclic permutation

$$l_0 \to l_0 \to ... \to l_{n-1} \to l_0; \quad m_0 \to m_1 \to ... \to m_{n-1} \to m_0$$

Notice that $r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) = 1$ if and only if the points $m_0, ..., m_{n-1}$ belong to a hyperplane.

Let $\widehat{m}_i$ be the point of intersection of the line $l_i l_{i+1}$ with the hyperplane passing through all the points m_j except m_i. Let $r(x_1, ..., x_4)$ be the cross ration of the four points on $\mathbb{P}^1$. Then

$$r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) = r(l_i, l_{i+1}, m_i, \widehat{m}_{i+1})$$

The special configurations and classical polylogarithms. Consider the configuration of $2n$ hyperplanes in $\mathbb{P}^{n-1}$ given by the following equations in homogeneous coordinates $z_0 : ... : z_{n-1}$

$$\begin{aligned} &z_0 = 0, \quad ..., \quad z_{n-1} = 0, \quad z_0 = z_1, \quad z_1 + z_2 = z_0, \\ &z_2 - z_3 = 0, \quad ..., \quad z_{n-2} - z_{n-1} = 0, \quad z_{n-1} = az_0 \end{aligned} \tag{35}$$

It admits the following interpretation. Recall that the classical polylogarithm function $Li_n(z)$ can be defined by an iterated integral:

$$Li_n(a) = \int_0^a \frac{dt}{1-t} \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t} = \int_{\Delta_a} \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_{n-1}}{z_{n-1}}$$

If $a \in (0,1]$ the simplex Δ_a is defined by the equations

$$\Delta_a := \quad \{(z_1,...,z_{n-1}) \in \mathbb{R}^{n-1} | \quad 0 \leq 1 - z_1 \leq z_2 \leq z_3 \leq ... \leq z_{n-1} \leq a\}$$

The faces of the simplex Δ_a can be defined for arbitrary a. Then the codimension one faces $\{z_i = 0\}$ of the coordinate simplex and the codimension one faces of the simplex Δ_a form the configuration (35).

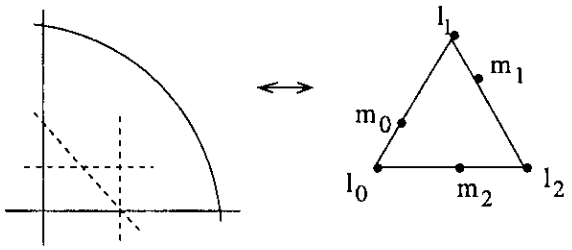
We can reorder hyperplanes of this configuration as follows:

$$z_0 = 0, \quad z_1 = 0, \quad z_1 = z_0, \quad z_1 + z_2 = z_0, \quad z_2 = 0,$$

$$z_2 = z_3, \quad z_3 = 0, \quad ... \quad , \quad z_{n-2} = 0, \quad z_{n-2} = z_{n-1}, \quad z_{n-1} = az_0$$

Applying the projective duality to this configuration of hyperplanes we get the special configuration of $2n$ points in $\mathbb{P}^{n-1}$ with the generalized cross ratio a.

The correspondence between the configuration (35) and the special configuration of points is illustrated in the case $n = 3$ on the picture below.



Remark. It is amusing that the special configuration of $2n$ points in $\mathbb{P}^{n-1}$, which is related to the classical n-logarithm by theorem 4.5 below, is constructed using the geometry of the mixed motive corresponding to $Li_{n-1}(a)$.

5. Restriction of the Grassmannian n–logarithm to the special stratum. The function $Li_n(z)$ has a remarkable single-valued version ([Z1], [BD]):

$$\mathcal{L}_n(z) \quad := \quad \begin{matrix} \text{Re} & (n: \text{ odd}) \\ \text{Im} & (n: \text{ even}) \end{matrix} \left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right) , \quad n \geq 2$$

It is continuous on $\mathbb{CP}^1$. Here $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$, so $\beta_k = \frac{2^k B_k}{k!}$ where B_k are Bernoulli numbers. For example $\mathcal{L}_2(z)$ is the Bloch - Wigner function.

Let us consider the following modification of the function $\mathcal{L}_n(z)$ proposed by A. M. Levin in [Le]:

$$\widetilde{\mathcal{L}}_n(x) :=$$

$$\frac{(2n-3)}{(2n-2)} \sum_{k \text{ even}; \, 0 \leq k \leq n-2} \frac{2^k (n-2)!(2r-k-3)!}{(2n-3)!(k+1)!(n-k-2)!} \mathcal{L}_{n-k}(x) \log^k |x|$$

For example $\widetilde{\mathcal{L}}_n(x) = \mathcal{L}_n(x)$ for $n \leq 3$, but already $\widetilde{\mathcal{L}}_4(x)$ is different from $\mathcal{L}_4(x)$. A direct integration carried out in proposition 4.4.1 of [Le] shows that

$$-(2\pi i)^{n-1}(-1)^{(n-1)(n-2)/2}\widetilde{\mathcal{L}}_n(x) =$$

$$\int_{\mathbb{CP}^{n-1}} \log|1-z_1| \prod_{i=1}^{n-1} d\log|z_i| \wedge \prod_{i=1}^{n-2} d\log|z_i - z_{i+1}| \wedge d\log|z_{n-1} - a|$$

This combined with proposition 5.3 below implies

Theorem 4.5 *The value of the function $\mathcal{L}_n^G$ on the special configuration (34) is equal to*

$$-(-1)^{n(n-1)/2}4^{n-1}\binom{2n-2}{n-1}^{-1}\widetilde{\mathcal{L}}_n(a)$$

where $a = r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1})$.

Another proof in the case $n = 2$ see in section 6.7 below.

Conjecture 4.6 *The Chow n-logarithm function can be expressed by the Grassmannian n-logarithm function.*

Remark. Suppose that an element $\sum_k \{f_1^{(k)}, ..., f_{2n+1}^{(k)}\} \in K_{2n+1}^M(\mathbb{C}(X))$ has zero residues at all the divisors on an n-dimensional variety X over $\mathbb{C}$. Then it defines an element

$$\alpha \in gr_{2n+1}^{\gamma} K_{2n+1}(X) = Ext_{\mathcal{M}}^{2n+1}(\mathbb{Q}(0)_X, \mathbb{Q}(2n+1)_X)$$

Its direct image to the point is an element

$$\pi_*(\alpha) \in gr_n^{\gamma} K_{2n+1}(Spec\mathbb{C}) = Ext_{\mathcal{M}}^1(\mathbb{Q}(0), \mathbb{Q}(n))$$

Applying the regulators we see that the integral $\sum_k \int_{X(\mathbb{C})} r_{2n}(f_1^{(k)}, ..., f_{2n+1}^{(k)})$ coincides with the value of the Borel regulator on $\pi_*(\alpha)$ and so by results of the next chapter is expressible by the Grassmannian n-logarithms. Conjecture 4.6 tells us that this should be true *for any* element in $K_{2n+1}^M(\mathbb{C}(X))$.

5 Grassmannian polylogarithms, symmetric spaces and Borel regulators

1. The function ψ_n. Let

$$\mathbb{H}_n := \{ \text{ positive definite Hermitian forms in } \mathbb{C}^n \} / \mathbb{R}_+^* = SL_n(\mathbb{C})/SU(n)$$

$$= \{ \text{ positive definite Hermitian forms in } \mathbb{C}^n \text{ with determinant } = 1\}$$

It is a symmetric space of rank $n-1$. For example $\mathbb{H}_2 = \mathcal{H}_3$ is the hyperbolic 3-space. Replacing positive definite by non negative definite Hermitian forms we get a compactification $\overline{\mathbb{H}}_n$ of the symmetric space $\mathbb{H}_n$.

Let G_x be the subgroup of $SL_N(\mathbb{C})$ stabilizing the point $x \in \mathbb{H}_n$. A point x defines a one dimensional vector space M_x:

$$x \in \mathbb{H}_n \longmapsto M_x := \{\text{measures on } \mathbb{CP}^{n-1} \text{ invariant under } G_x\}$$

Namely, a point x corresponds to a hermitian metric in V_n. This metric provides the Fubini-Studi metric on $\mathbb{CP}^{n-1} = P(V_n)$. Moreover there is the Fubini-Studi Kahler form on $\mathbb{CP}^{n-1} = P(V_n)$; its imaginary part is a sypmlectic form. Raising it to $(n-1)$-th power we get the Fubini-Studi volume form. The elements of M_x are the multiples of the Fubini-Studi volume form.

So $\mathbb{H}_n$ embeds to the projectivization of the space of all measures in $\mathbb{CP}^{n-1}$. Taking its closure we get a compactification of $\mathbb{H}_n$.

Let us choose for any point $x \in \mathbb{H}_n$ an invariant measure $\mu_x \in M_x$. Then μ_x/μ_y is a real function on $\mathbb{CP}^{n-1}$.

Let $x_0, ..., x_{2n-1}$ be points of the symmetric space $SL_n(\mathbb{C})/SU(n)$. Consider the following function

$$\psi_n(x_0, ..., x_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log|\frac{\mu_{x_1}}{\mu_{x_0}}| d\log|\frac{\mu_{x_2}}{\mu_{x_0}}| \wedge ... \wedge d\log|\frac{\mu_{x_{2n-1}}}{\mu_{x_0}}| \qquad (36)$$

2. General properties of the function ψ_n. Let us study the properties of integral (36) in a more general situation. Let X be an m-dimensional manifold. For any $m+2$ measures $\mu_0, ..., \mu_{m+1}$ on X such that $\frac{\mu_i}{\mu_j}$ are smooth functions we can construct a differential m-form on X:

$$\overline{r}_m(\mu_0 : ... : \mu_{m+1}) := \log|\frac{\mu_1}{\mu_0}| d\log|\frac{\mu_2}{\mu_0}| \wedge ... \wedge d\log|\frac{\mu_{m+1}}{\mu_0}|$$

Proposition 5.1 *The integral*

$$\int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1}) \qquad (37)$$

satisfies the following properties:

1) Skew symmetry with respect to the permutations of μ_i.

2) Homogeneity:

$$\int_X \overline{r}_m(\lambda_0\mu_0 : ... : \lambda_{m+1}\mu_{m+1}) = \int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1})$$

3)Additivity: for any $m+3$ measures μ_i on X one has

$$\sum_{i=0}^{m+2} (-1)^i \int_X \overline{r}_m(\mu_0 : ... : \widehat{\mu}_i : ... : \mu_{m+2}) = 0$$

4) Let g be a diffeomorphism of X. Then

$$\int_X \overline{r}_m(g^*\mu_0 : ... : g^*\mu_{m+1}) = \int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1})$$

Proof. 1). Follows from $\log f \cdot d\log g + \log g \cdot d\log f = d(\log f \cdot \log g)$.
2) Using 1) we may assume $\lambda_i = 1$ for $i > 0$. Then

$$\int_X (\overline{r}_m(\lambda_0\mu_0 : \mu_1 : ... : \mu_{m+1}) - \overline{r}_m(\mu_0 : \mu_1 : ... : \mu_{m+1})) =$$

$$-\log|\lambda| \cdot \int_X d(\log|\frac{\mu_2}{\mu_0}|d\log|\frac{\mu_3}{\mu_0}| \wedge ... \wedge d\log|\frac{\mu_{m+1}}{\mu_0}|) = 0$$

3) Taking into account the skewsymmetry of the integral we have to prove that

$$\mathrm{Alt}_{(0,...,m+2)}\left\{\log|\frac{\mu_2}{\mu_1}|d\log|\frac{\mu_3}{\mu_1}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_1}|\right\} = 0 \qquad (38)$$

Let us write $\frac{\mu_i}{\mu_j} = \frac{\mu_i}{\mu_0}/\frac{\mu_j}{\mu_0}$ and substitute it to (38). Then the terms in (38) where

$$\log|\frac{\mu_2}{\mu_0}|d\log|\frac{\mu_3}{\mu_0}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_0}|$$

will appear looks as follows:

$$\log|\frac{\mu_2}{\mu_1}|d\log|\frac{\mu_3}{\mu_1}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_1}| - \log|\frac{\mu_1}{\mu_2}|d\log|\frac{\mu_3}{\mu_2}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_2}|$$

$$-\log|\frac{\mu_2}{\mu_0}|d\log|\frac{\mu_3}{\mu_0}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_0}| + \log|\frac{\mu_0}{\mu_2}|d\log|\frac{\mu_3}{\mu_2}| \wedge ... \wedge d\log|\frac{\mu_{m+2}}{\mu_2}|$$

(The first two terms comes from $\mathrm{Alt}_{(1,...,m+2)}\overline{r}_m(\mu_1 : ... : \mu_{m+2})$ and the second two from $\mathrm{Alt}_{(0,2,...,m+2)}\overline{r}_m(\mu_0 : \mu_2 : ... : \mu_{m+2})$. The expression $\mathrm{Alt}_{m+2}\overline{r}_m(\mu_0 : ...\widehat{\mu}_i : ... : \mu_{m+2})$ provides no such terms if $i > 1$).

4) Clear. The proposition is proved.

Recall the following general construction. Let X be a G-set and F a function on X^n satisfying

$$\sum_{i=1}^{n}(-1)^i F(x_1, ..., \widehat{x}_i, ..., x_n) = 0$$

Choose a point $x \in X$. Then there is an $(n-1)$-cocycle of the group G:

$$f_x(g_1, ..., g_n) := F(g_1x, ..., g_nx)$$

Lemma 5.2 *The cohomology class of the cocycle f_x does not depend on x.*

Proof. The difference $f_y - f_x$ is the coboundary of the $(n-2)$-chain

$$h_{x,y}(g_1, ..., g_{n-1}) = \sum_{i=1}^{n-1} (-1)^{k-1} f(g_1 x, g_2 x, ..., g_k x, g_k y, g_{k+1} y, ..., g_{n-1} y) \quad (39)$$

Here is the geometric picture leading to this formula. Consider the prism $\Delta^{(n-1)}_{g_1,...,g_{n-1}} \times \Delta^{(1)}_{x,y}$ given by product of the $(n-1)$-simplex with vertices $g_1, ..., g_n$ by the 1-simplex with vertices (x, y). Decomposing its side face $\Delta^{(n-1)}_{g_1,...,g_{n-1}} \times \Delta^{(1)}_{x,y}$ into a sum of simplices we come to the right hand side of (39). Then the terms of the formula $f_y - f_x - \delta h_{x,y}$ correspond to the boundary faces of the prism. Cutting the prism on simplices we see that the sum of the terms corresponding to the prism boundary is zero thanks to the cocycle relation. The lemma is proved.

So for any $x \in \mathbb{H}_n$ $\psi_n(g_0 x, ..., g_{2n-1} x)$ is a smooth $(2n-1)$-cocycle of $GL_n(\mathbb{C})$.

Remark. This cocycle is the restriction to $GL_n(\mathbb{C})$ of the Bott cocycle for the group of diffeomorphisms of $\mathbb{CP}^{n-1}$.

Let $h_0, \ldots, h_{2n-1}$ be any hyperplanes in $\mathbb{CP}^{n-1}$. Recall that the Grassmannian n-logarithm is defined by

$$\mathcal{L}^{\mathrm{G}}_n(h_0, \ldots, h_{2n-1}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-1}(f_1, \ldots, f_{2n-1})$$

where f_i is a rational function on $\mathbb{CP}^{n-1}$ with the divisor $(h_i) - (h_0)$.

Proposition 5.3 *One has*

$$\mathcal{L}^{\mathrm{G}}_n(h_0, \ldots, h_{2n-1}) = -\frac{(-4)^{n-1}((n-1)!)^2}{(2\pi i)^{n-1}(2n-2)!} \int_{\mathbb{CP}^{n-1}} \log|f_1| \bigwedge_{j=2}^{2n-1} d\log|f_j|$$

Proof. See proposition 6.2 in [ZhG].

3. The Grassmannian polylogarithm $\mathcal{L}^G_r$ as the boundary value of the function ψ_n. We start from an explicit formula for the Fubini-Studi form. Let $\widehat{\mathbb{P}}^{n-1}$ be the variety of all hyperplanes in $\mathbb{P}^{n-1}$. Consider the incidence divisor

$$D \subset \widehat{\mathbb{P}}^{n-1} \times \mathbb{P}^{n-1} \qquad D := \{(h, x) | x \in h\}$$

where h is a hyperplane and x is a point in $\mathbb{P}^{n-1}$.

Let $(x_0 : ... : x_{n-1})$ be homogeneous coordinates in $\mathbb{P}^{n-1}$. Let

$$\sigma_n(x, dx) := \sum_{i=0}^{n-1} (-1)^i x_i dx_0 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_{n-1} = i_E \mathrm{vol}_x$$

be the Leray from. Here $\mathrm{vol}_x = dx_0 \wedge ... \wedge dx_{n-1}$ and $E = \sum x_i \partial_{x_i}$.

There is canonical differential $(n-1,n-1)$-form ω_D on $\widehat{\mathbb{P}}^{n-1} \times \mathbb{P}^{n-1} - D$ with a polar singularity at the divisor D. Namely, let $x \in V_n$ and $\xi \in V_n^*$. Then

$$\omega_D := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(\xi, d\xi) \wedge \sigma_n(x, dx)}{< \xi, x >^n}$$

It is PGL_n-invariant. A hermitian metric H in V_n provides an isomorphism $H : V_n \longrightarrow \overline{V}_n^*$, and hence an isomorphism $\mathbb{CP}^{n-1} \longrightarrow \widehat{\overline{\mathbb{CP}}}^{n-1}$. The graph Γ_H of this map does not intersect the incidence divisor D. Thus restricting the form ω_D to Γ_H we get a volume form on $\mathbb{C}P^{n-1}$:

$$\omega_{FS}(H) := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(z, dz) \wedge \sigma_n(\overline{z}, d\overline{z})}{H(z, \overline{z})^n} \tag{40}$$

It is clearly invariant under the group preserving the Hermitian form H. Moreover, it is the Fubini-Studi volume form: a proof can be obtained by using the explicit formula for the Fubini-Studi Kahler form given in [Ar], complement 3.

One can realize $\mathbb{CP}^{n-1}$ as the smallest stratum of the boundary of $\mathbb{H}_n$. Namely, for a hyperplane $h \in V_n$ let

$$F_h := \{ \text{ nonnegative definite hermitian forms in} V_n = \mathbb{C}^n \text{ with kernel } h \} / \mathbb{R}_+^*$$

The set of hermitian forms in V_n with the kernel h is isomorphic to $\mathbb{R}_+^*$, so F_h defines a point on the boundary of $\overline{\mathbb{H}}_n$.

For any nonzero nonnegative definite hermitian form H one can define the corresponding Fubini-Studi form by formula (40). It is a differential form with singularities along the projectivization of kernel of H. In particular if h is a hyperplane then the degenerate hermitian form F_h provides the Lebesgue measure on the affine space $\mathbb{CP}^{n-1} - h$. Indeed, if $h_0 = \{z_0 = 0\}$ then (40) specializes to

$$\frac{1}{(2\pi i)^{n-1}} d\frac{z_1}{z_0} \wedge ... \wedge d\frac{z_{n-1}}{z_0} \wedge d\frac{\overline{z}_1}{\overline{z}_0} \wedge ... \wedge d\frac{\overline{z}_{n-1}}{\overline{z}_0}$$

Denote by M_h the one dimensional real vector space generated by this form. For any hyperplane h in $\mathbb{CP}^{n-1}$ let us choose a measure $\mu_h \in M_h$.

Proposition 5.4 *For any $2n$ hyperplanes $h_0, ..., h_{2n-1}$ in $\mathbb{CP}^{n-1}$ the integral*

$$\psi_n(h_0, ..., h_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log|\frac{\mu_{h_1}}{\mu_{h_0}}| d\log|\frac{\mu_{h_2}}{\mu_{h_0}}| \wedge ... \wedge d\log|\frac{\mu_{h_{2n-1}}}{\mu_{h_0}}| \tag{41}$$

is convergent and equals to

$$(-4)^{-n} \cdot (2\pi i)^{n-1} (2n)^{2n-1} \binom{2n-2}{n-1} \cdot \mathcal{L}_n^G(h_0, ..., h_{2n-1})$$

Proof. Let h_1, h_2 be hyperplanes in $\mathbb{CP}^{n-1}$ and f be a rational function such that $(f) = (h_1) - (h_2)$. From the explicit description of M_h given above we immediately see that

$$\mu_{h_1}/\mu_{h_2} = \lambda \cdot |f|^{2n} \tag{42}$$

Using this and theorem 2.4 we see that integral (41) is convergent. The second statement follows from proposition 5.3 and (42). The proposition is proved.

More generally, take any $2n$ Hermitian forms $H_0, ..., H_{2n-1}$, possibly degenerate. For each of the forms H_i consider the corresponding measure μ_{H_i} (a multiple of the Fubini-Studi form related to H_i). Using convergence of the integral (41) we can deduce that the integral

$$\psi_n(H_0, ..., H_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log|\frac{\mu_{H_1}}{\mu_{H_0}}| d\log|\frac{\mu_{H_2}}{\mu_{H_0}}| \wedge ... \wedge d\log|\frac{\mu_{H_{2n-1}}}{\mu_{H_0}}| = \tag{43}$$

$$-n^{2n-1} \cdot \int_{\mathbb{CP}^{n-1}} \log|\frac{H_1(z,\overline{z})}{H_0(z,\overline{z})}| d\log|\frac{H_2(z,\overline{z})}{H_1(z,\overline{z})}| \wedge ... \wedge d\log|\frac{H_{2n-1}(z,\overline{z})}{H_0(z,\overline{z})}|$$

is also convergent. This enables us to extend ψ_n to the function $\overline{\psi}_n(x_0, ..., x_{2n-1})$ on the configuration space of $2n$ points in $\overline{\mathbb{H}}_{n-1}$. The function $\overline{\psi}_n$ is discontinuous. For instance it is discontinuous at the point $x_1 = ... = x_{2n-1} = F_h$ for a given hyperplane h in $\mathbb{CP}^{n-1}$. It is however a smooth function on an open part of any given strata. We will keep the notation

$$\psi_n(h_0, ..., h_{2n-1}) = \overline{\psi}_n(F_{h_0}, ..., F_{h_{2n-1}})$$

Applying lemma 5.2 to the case when X is $\overline{\mathbb{H}}_n$ and using only the fact that the function $\overline{\psi}_n(x_0, ..., x_{2n-1})$ is well defined for *any* $2n$ points in $\overline{\mathbb{H}}_n$ and satisfies the cocycle condition for any $2n+1$ of them we get

Corollary 5.5 *Let $x \in \mathbb{H}_n$ and h is a hyperplane in $\mathbb{CP}^{n-1}$. Then the cohomology classes of the following cocycles coincide:*

$$\psi_n(g_0 x, ..., g_{2n-1} x) \quad \textit{and} \quad \psi_n(g_0 h, ..., g_{2n-1} h)$$

4. A normalization of the Borel class b_n. Choose a hermitian metric in V_n. Let e be the corresponding point of the symmetric space $\mathbb{H}_n$; its stabilizer is a subgroup $SU(n)$. One has

$$\left(\Lambda^\bullet T_e^* \mathbb{H}_n\right)^{SU(n)} = \mathcal{A}^\bullet(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})}$$

There are well known canonical ring isomorphisms (see [B2] and references there):

$$\left(\Lambda^\bullet T_e^* \mathbb{H}_n\right)^{SU(n)} \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^\bullet(sl_n(\mathbb{C}))^{sl_n(\mathbb{C})} =$$

$$H^\bullet(sl_n(\mathbb{C}), \mathbb{C}) \stackrel{\alpha}{=} H^\bullet_{\text{top}}(SU(n), \mathbb{C}) \stackrel{\beta}{=} H^\bullet_m(SL_n(\mathbb{C}), \mathbb{C}) \tag{44}$$

where $H^\bullet$ is the Lie algebra cohomology, $H^\bullet_{\rm top}$ is the topological cohomology. The first isomorphism is obvious: $T_e\mathbb{H}_n \otimes_{\mathbb{R}} \mathbb{C} = sl_n(\mathbb{C})$. The map

$$\alpha_{\rm DR} : \Lambda^\bullet_{\mathbb{Q}}(sl_n)^{sl_n} \stackrel{\sim}{\longrightarrow} H^\bullet_{\rm DR}(SL_n(\mathbb{C}), \mathbb{Q})$$

sends an sl_n–invariant exterior form on sl_n to the right–invariant, and hence biinvariant differential form on $SL_n(\mathbb{C})$. Let us described the map

$$\beta_{\rm DR} : H^\bullet_{\rm DR}(SL_n(\mathbb{C}), \mathbb{Q}) \longrightarrow H^\bullet_m(SL_n(\mathbb{C}), \mathbb{C})$$

Let C be a biinvariant, and hence closed, differential $(2n-1)$–form on $SL_n(\mathbb{C})$. Let us restrict it first to the Lie algebra, and then to the orthogonal complement $su(n)^\perp$ to the Lie subalgebra $su(n) \subset sl_n(\mathbb{C})$. We identify the $\mathbb{R}$–vector spaces $T_e\mathbb{H}_n$ and $su(n)^\perp$. The obtained exterior form on $T_e\mathbb{H}_n$ is restriction of an invariant differential form, denoted ω_C, on the symmetric space $\mathbb{H}_n$. It is a closed differential form.

For any ordered $2n$ points $x_1, ..., x_{2n}$ in $\mathbb{H}_n$ there is a geodesic simplex $I(x_1, ..., x_{2n})$ in $\mathbb{H}_n$. It is constructed inductively as follows. Let $I(x_1, x_2)$ be the geodesic from x_1 to x_2. The geodesics from x_3 to the points of $I(x_1, x_2)$ form a geodesic triangle $I(x_1, x_2, x_3)$. All the geodesics from x_4 to the points of the geodesic triangle $I(x_1, x_2, x_3)$ form a geodesic simplex $I(x_1, x_2, x_3, x_4)$, and so on. When the rank of the symmetric space is more then 1 (i.e. $n > 2$) the geodesic simplex $I(x_1, ..., x_k)$ depends on the ordering of the vertices $x_1, ..., x_k$.

The differential $(2n-1)$-form ω_C on $SL_n(\mathbb{C})/SU(n)$ provides a volume of the geodesic simplex:

$$\mathrm{vol}_C I(x_1, ..., x_{2n}) := \int_{I(x_1, ..., x_{2n})} \omega_C$$

For every $2n+1$ points $x_1, ..., x_{2n+1}$ the boundary of the simplex $I(x_1, ..., x_{2n+1})$ is the alternated sum of the simplices $I(x_1, ..., \widehat{x}_i, ..., x_{2n+1})$. Since the form ω_C is closed, the Stokes theorem yields

$$\sum_{i=1}^{2n+1} (-1)^i \int_{I(x_1, ..., \widehat{x}_i, ..., x_{2n+1})} \omega_C = \int_{I(x_1, ..., x_{2n+1})} d\omega_C = 0 \qquad (45)$$

This just means that for a given point x the function $\mathrm{vol}_C I(g_1 x, ..., g_{2n} x)$ is a smooth $(2n-1)$-cocycle of the Lie group $SL_n(\mathbb{C})$. It was considered by J.Dupont [D]. By lemma 5.2 cocycles corresponding to different points x are canonically cohomologous. The obtained cohomology class is the class $\beta_{\rm DR}([C])$.

Remark. $\mathrm{vol}_C I(x_0, ..., x_{2n-1})$ is independent of the ordering of its vertices. Indeed, consider $2n+1$ points $(x_0, x_1, x_0, x_2, ..., x_{2n-1})$ and apply relation (45)).

The Betti cohomology of $SL_n(\mathbb{C})$. Recall that $SU(n)$ is a retract of $SL_n(\mathbb{C})$. It is well known that

$$H^\bullet_{\rm top}(SU(n), \mathbb{Z}) = H^\bullet_{\rm top}(S^3 \times S^5 \times ... \times S^{2n-1}, \mathbb{Z}) = \Lambda^*(B_3, B_5, ..., B_{2n-1})$$

The restriction from $SU(n)$ to $SU(m)$ kills the classes B_{2k-1} for $k > m$. If $k \leq m$ it identifies the class B_{2k-1} for $SU(n)$ with the one for $SU(m)$. The class B_{2n-1} for $SU(n)$ is provided by the fundamental class of the sphere $S^{2n-1} \subset \mathbb{C}^n$. Namely, it is the pull back of the fundamental class under the map $SU(n) \longrightarrow S^{2n-1}$ provided by a choice of point on S^{2n-1}. This sphere has the orientation induced by the one of $\mathbb{C}^n$. Thus

$$\mathbb{Z} \cdot B_n = \mathrm{Ker}\Big(H^{2n-1}_{\mathrm{top}}(SU(n), \mathbb{Z}) \longrightarrow H^{2n-1}_{\mathrm{top}}(SU(n-1), \mathbb{Z})\Big) \qquad (46)$$

The transgression in the Leray spectral sequence for the universal $SU(n)$-bundle $EU(n) \longrightarrow BU(n)$ provides an isomorphism

$$\mathbb{Z} \cdot B_n \longrightarrow \frac{H^{2n}(BSU(n), \mathbb{Z})}{\oplus_{0<i<2n} H^i \cdot H^{2n-i}}$$

and identifies B_n with the Chern class $c_n \in H^{2n}(BSU(n), \mathbb{Z})$ of the associated vector bundle.

The De Rham cohomology of $SL_n(\mathbb{C})$. Consider the differential form

$$D_n := \mathrm{tr}(g^{-1}dg)^{2n-1} \in \Omega^{2n-1}(SL) \qquad (47)$$

Its restriction to the subgroup SL_m is zero for $m < n$. It follows that the cohomology class

$$[D_n] \in H^{2n-1}_{\mathrm{DR}}(SL_n, \mathbb{C})$$

is a multiple of B_n. The Hodge considerations shows that $[D_n] \in (2\pi i)^n \mathbb{Q} \cdot B_n$.

Lemma 5.6 *The differential form ω_{D_n} is an $\mathbb{R}(n-1)$-valued form. In particular it provides a cohomology class*

$$b_n := \beta_{\mathrm{DR}}(D_n) \in H^{2n-1}_m(SL_n(\mathbb{C}), \mathbb{R}(n-1))$$

Proof. An easy calculation shows that the value of the exterior form $\omega_{D_n|T_e\mathbb{H}_n}$ on

$$(e_{1,n} + e_{n,1}) \wedge i(e_{1,n} - e_{n,1}) \wedge \ldots \wedge (e_{n-1,n} + e_{n,n-1}) \wedge i(e_{n-1,n} - e_{n,n-1}) \wedge e_{n,n}$$

is non zero, and obviously lies $\mathbb{Q}(n-1)$.

On the other hand the values of the form ω_{D_n} lie in a one dimensional $\mathbb{R}$-vector space. Indeed, the space of $su(n)$-invariant real exterior $(2n-1)$-forms on the space of all hermitian $n \times n$ matrices, which have zero restriction to the subspace of hermitian $(n-1) \times (n-1)$ matrices is one dimensional. The exterior form $\omega_{D_n|T_e\mathbb{H}_n}$ belongs to the complexification of this space. The lemma follows from this.

We call the cohomology class provided by this lemma the Borel class, and use it below to construct the Borel regulator.

5. Comparison of the Grassmannian and Borel cohomology classes of $GL_n(\mathbb{C})$. Let $[C_n^G]$ be the cohomology class of the $(2n-1)$-cocycle of $GL_n(\mathbb{C})$ provided by the Grassmannian n-logarithm. We want to compare it with the Borel class.

Let us consider the following integral

$$\widetilde{C}_n(H_1, ..., H_{2n-1}) :=$$

$$-n^{2n-1} \cdot \int_{\mathbb{CP}^{n-1}} \frac{H_1(z,\overline{z})}{(z,\overline{z})} d\frac{H_2(z,\overline{z})}{(z,\overline{z})} \wedge ... \wedge d\frac{H_{2n-1}(z,\overline{z})}{(z,\overline{z})} \tag{48}$$

where H_i are arbitrary complex matrices and $H_i(z,\overline{z})$ as the bilinear form in $z, \overline{z}$ given by the matrix H_i. We claim that it is a $(2n-1)$-cocycle of the Lie algebra $gl_n(\mathbb{C})$, and it is obtained by differentiating of the group cocycle provided by the function (43). We put these facts in the following framework.

If we restrict to the case when h_i are hermitian matrices, integral (48) admits the following interpretation. Let us construct a map

$$M_e : \mathbb{CP}^{n-1} \longrightarrow T_e^*\mathbb{H}_n$$

which is a version of the moment map. For a point $z \in \mathbb{CP}^{n-1}$ the value of the $< M_e(z), v >$ of the functional $M(z)$ on a vector $v \in T_e\mathbb{H}_n$ is defined as follows. Let $e(t)$ be a path in $\mathbb{H}_n$ such that $e(0) = e$ and $\dot{e}(0) = v$. Then

$$< M_e(z), v >:= \frac{d}{dt} \log \frac{\mu_{e(t)}(z)}{\mu_e(z)}|_{t=0}$$

Choose coordinates $z_1, ..., z_n$ in V_n such that $(z,\overline{z}) := |z_1|^2 + ... + |z_n|^2$ corresponds to the point e. Then $T_e\mathbb{H}_n$ is identified with the space of hermitian matrices H. It follows from (40) that

$$< M_e(z), H >:= n\frac{H(z,\overline{z})}{(z,\overline{z})} \tag{49}$$

The map M_e is clearly $SU(n)$-invariant. Its image is an $SU(n)$-orbit in $T_e^*\mathbb{H}_n$ isomorphic to $\mathbb{CP}^{n-1}$.

We need the following general construction. Let V be a real vector space and M a compact subset of V which is a closure of a k-dimensional submanifold. Any element $\omega \in \Lambda^k V$ can be viewed as a k-form ω on V^*. Integrating it over M we get an exterior form $C_M \in \Lambda^k V^*$. If M is a cone over M' with the vertex at the origin then $\int_M \omega = \int_{M'} i_E\omega$ where E is the Euler vector field on V.

Applying this construction to the cone over the orbit M_e based at the origin we get a $SU(n)$-invariant element $\widetilde{C}_n \in \Lambda^{2n-1}T_e^*\mathbb{H}_n$. It follows from (49) that it is given by formula (48) multiplied by $2n$.

Another invariant $(2n-1)$-cocycle C_n of the Lie algebra gl_n, considered by Dynkin [Dy], is given by

$$C_n(X_1, ..., X_{2n-1}) = \frac{1}{n!}\mathrm{Alt}_{2n-1}\mathrm{Tr}(X_1X_2...X_{2n-1}) \tag{50}$$

Let $[C_n]$ be the cohomology class of $GL_n(\mathbb{C})$ corresponding to the cocycle C_n.

Theorem 5.7 *One has*

$$\widetilde{C}_n \quad = -(-1)^{\frac{(n-1)n}{2}} \frac{(2\pi i)^{n-1} n^{2n-1}(n-1)!}{(2n-1)!} \cdot C_n$$

and the class $[C_n^G]$ is a non zero rational multiple of $[C_n]$:

$$[C_n^G] = -(-1)^{\frac{n(n+1)}{2}} \frac{(n-1)!^2(n-1)!}{(2n-2)!(2n-1)!}[C_n]$$

Proof. The second claim follows from the first using proposition 5.4.

Let us prove the first claim. The restriction of the cocycle C_n to the Lie subalgebra $gl_{n-1}(\mathbb{C})$ equals to zero. This follows, for instance, from the Amitsur-Levitsky theorem: for any $n \times n$ matrices $A_1, ..., A_{2n}$ one has $\mathrm{Alt}_{2n}(A_1, ..., A_{2n}) = 0$.

On the other hand the restriction of the cocycle $\widetilde{C}_n$ to the Lie subalgebra of matrices (a_{ij}) where $a_{1j} = a_{j1} = 0$ is zero. Indeed, in this case the form we integrate in (48) is a differential $(2n-2)$-form in $dz_2, ..., dz_{n-1}, d\overline{z}_2, ..., d\overline{z}_{n-1}$ and thus it is zero. So thanks to (44) and (46) we conclude that the cocycle C_n is proportional to $\widetilde{C}_n$. To determine the proportionality coefficient we compute the values of the both cocycle on a special element $E_n \in \Lambda^{2n-1} gl_n$. To write it down denote by $e_{i,j}$ the elementary $n \times n$ matrix whose only nonzero entry is 1 on the (i,j) place. Then

$$E_n := \bigwedge_{j=1}^{n-1} (e_{j,n} \wedge e_{n,j}) \wedge e_{n,n} \tag{51}$$

A direct computation shows that

$$< C_n, E_n >= 1$$

Indeed, to get a non zero trace we have to multiply $(n-1)$ blocks $e_{n,j}e_{j,n}$, as well as $e_{n,n}$, which can be inserted anywhere between these blocks. So there are $(n-1)!n = n!$ possibilities.

Let us compute the value of the cocycle $\widetilde{C}_n$ on E_n.

Lemma 5.8 *Integral (48) equals to*

$$\frac{-n^{2n-1}}{(2n-1)!} \cdot \mathrm{Alt}_{2n-1} \int_{\mathbb{CP}^{n-1}} \frac{H_1(z,\overline{z}) dH_2(z,\overline{z}) \wedge ... \wedge dH_{2n-1}(z,\overline{z})}{(z,\overline{z})^{2n-1}} \tag{52}$$

Proof. By proposition 5.1 integral (48) equals to

$$\frac{-n^{2n-1}}{(2n-1)!} \mathrm{Alt}_{2n-1} \cdot \int_{\mathbb{CP}^{n-1}} \frac{H_1(z,\overline{z})}{(z,\overline{z})} d\frac{H_2(z,\overline{z})}{(z,\overline{z})} \wedge ... \wedge d\frac{H_{2n-1}(z,\overline{z})}{(z,\overline{z})} \tag{53}$$

One has

$$d\frac{H_i(z,\overline{z})}{(z,\overline{z})} = \frac{(z,\overline{z})dH_i(z,\overline{z}) - H_i(z,\overline{z})d(z,\overline{z})}{(z,\overline{z})^2}$$

Substituting

$$\frac{-H_i(z,\overline{z})d(z,\overline{z})}{(z,\overline{z})^2} \quad \text{instead of} \quad d\frac{H_i(z,\overline{z})}{(z,\overline{z})} \quad \text{in (53)}$$

we get zero since H_1 and H_i appear in a symmetric way and thus disappear after the alternation. The lemma follows.

Let us calculate integral (53) in the special case

$$H_{2n-1}(z,\overline{z}) = |z_n|^2, \quad H_{2k-1}(z,\overline{z}) = z_k\overline{z}_n, \quad H_{2k}(z,\overline{z}) = z_n\overline{z}_k$$

so that $H_1 \wedge ... \wedge H_{2n-1} = E_n$. We will restrict the integrand to the affine part $\{z_n = 1\}$ and then perform the integration. Since $dH_{2n-1}(z,\overline{z}) = 0$ on $\{z_n = 1\}$ and $dz_k \wedge d\overline{z}_k = -2idx_k \wedge dy_k$ we get

$$-(-1)^{\frac{(n-1)(n-2)}{2}}\frac{(-2i)^{n-1}n^{2n-1}}{2n-1}\int_{\mathbb{C}^{n-1}}\frac{d^{n-1}xd^{n-1}y}{(1+|z_1|^2+...+|z_{n-1}|^2)^{2n-1}} =$$

$$-(-1)^{\frac{(n-1)n}{2}}\frac{(2i)^{n-1}n^{2n-1}}{2n-1}\mathrm{vol}(S^{2n-3})\int_0^\infty\frac{r^{2n-3}dr}{(1+r^2)^{2n-1}} =$$

$$-(-1)^{\frac{(n-1)n}{2}}\frac{(2\pi i)^{n-1}n^{2n-1}}{(n-1)!(2n-1)}\binom{2n-2}{n-1}^{-1}$$

since the volume of the sphere S^{2n-3} is $\frac{2\pi^{n-1}}{(n-2)!}$ and

$$\int_0^\infty\frac{r^{2n-3}dr}{(1+r^2)^{2n-1}} = \frac{1}{2}\int_0^\infty\frac{r^{n-2}dr}{(1+r)^{2n-1}} = \frac{1}{2n-2}\binom{2n-2}{n-1}^{-1} \quad (54)$$

To get the last equality we integrate by parts:

$$\int_0^\infty\frac{r^{n-2}dr}{(1+r)^{2n-1}} = -\frac{1}{2n-2}\int_0^\infty r^{n-2}\left(\frac{1}{(1+r)^{2n-2}}\right)'dr =$$

$$\frac{n-2}{2n-2}\int_0^\infty r^{n-3}\frac{dr}{(1+r)^{2n-2}} = ... = \frac{(n-2)!n!}{(2n-2)!}\int_0^\infty\frac{dr}{(1+r)^{n+1}} = \frac{1}{n-1}\binom{2n-2}{n-1}^{-1}$$

Theorem 5.7 is proved.

6. Construction of the Borel regulator via Grassmannian polylogarithms. Let G be a group. The diagonal map $\Delta : G \longrightarrow G \times G$ provides a homomorphism $\Delta_* : H_n(G) \longrightarrow H_n(G \times G)$. Recall that

$$\mathrm{Prim}H_nG := \{x \in H_n(G) | \Delta_*(x) = x \otimes 1 + 1 \otimes x\}$$

Set $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. One has

$$K_n(F)_{\mathbb{Q}} = \mathrm{Prim} H_n GL(F)_{\mathbb{Q}} = \mathrm{Prim} H_n GL_n(F)_{\mathbb{Q}}$$

where the second isomorphism is provided by Suslin's stabilization theorem.

The Borel regulator is a map

$$r_n^{\mathrm{Bo}} : K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \longrightarrow \mathbb{R}(n-1)$$

provided by pairing of the class $b_n \in H^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$ with the subspace $K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \subset H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q})$.

Recall the Grassmannian complex $C_*(n)$

$$... \xrightarrow{d} C_{2n-1}(n) \xrightarrow{d} C_{2n-2}(n) \xrightarrow{d} ... \xrightarrow{d} C_0(n)$$

where $C_k(n)$ is the free abelian group generated by configurations of $k+1$ vectors $(l_0, ..., l_k)$ in generic position in an n–dimensional vector space over a field F, and d is given by the standard formula (see s. 3.1 in [G2]). The group $C_k(n)$ is in degree k. Since it is a homological resolution of the trivial $GL_n(F)$–module $\mathbb{Z}$ (see lemma 3.1 in [G2]), there is canonical homomorphism

$$\varphi^n_{2n-1} : H_{2n-1}(GL_n(F)) \longrightarrow H_{2n-1}(C_*(n))$$

Thanks to lemma 5.6 the Grassmannian n–logarithm function provides a homomorphism

$$\mathcal{L}^G_n : C_{2n-1}(n) \longrightarrow \mathbb{R}(n-1); \quad (l_0, ..., l_{2n-1}) \longmapsto \mathcal{L}^G_n(l_0, ..., l_{2n-1}) \qquad (55)$$

Thanks to the first $(2n+1)$–term functional equation for $\mathcal{L}^G_n$, see (31), it is zero on the subgroup $dC_{2n}(n)$. So it induces a homomorphism

$$\mathcal{L}^G_n : H_{2n-1}(C_*(n)) \longrightarrow \mathbb{R}(n-1);$$

Lemma 5.9 *The composition $\mathcal{L}^G_n \circ \varphi^n_{2n-1}$ coincides with the class $[C^G_n]$.*

Proof. Standard, see [G4].

To construct the Borel regulator we extend, as in s. 3.10 of [G2], the class $[C^G_n]$ to a class of $GL_{2n-1}(\mathbb{C})$. Let us recall the key steps.

Let $\mathbb{Z}[S]$be the free abelian group generated by a set S. Let F be a field. Applying the covariant functor $\mathbb{Z} \longmapsto \mathbb{Z}[X(F)]$ to the bi-Grassmannian $\widehat{G}(n)$ (see s. 4.2), and taking the alternating sum of the obtained homomorphisms, we get a bicomplex. Using lemma 4.1 we see that it looks as follows ([G2], s. 3.7):

$$\begin{array}{ccccccc}
 & & & & ... & \xrightarrow{d} & C_{2n-1}(2n-1) \\
 & & & & & & \downarrow \\
 & & ... & & ... & & ... \\
 & & \downarrow & & & & \downarrow \\
 & ... & \xrightarrow{d} C_{2n-1}(n+1) & \xrightarrow{d} & ... & \xrightarrow{d} & C_{n+1}(n) \\
 & \downarrow & & & & & \downarrow \\
... & \xrightarrow{d} C_{2n-1}(n) & \xrightarrow{d} C_{2n-2}(n) & \xrightarrow{d} & ... & \xrightarrow{d} & C_n(n)
\end{array}$$

In particular the bottom row is the stupid truncation of the Grassmannian complex at the group $C_n(n)$. The total complex of this bicomplex is a homological complex, called the weight n bi-Grassmannian complex $BC_*(n)$. In particular there is a homomorphism

$$H_{2n-1}(C_*(n)) \longrightarrow H_{2n-1}(BC_*(n)) \tag{56}$$

In [G1-2] we proved that there are homomorphisms

$$\varphi^m_{2n-1} : H_{2n-1}(GL_m(F)) \longrightarrow H_{2n-1}(BC_*(n)), \quad m \geq n$$

whose restriction to the subgroup $GL_n(F)$ coincides with the composition

$$H_{2n-1}(GL_n(F)) \stackrel{\varphi^n_{2n-1}}{\longrightarrow} H_{2n-1}(C_*(n)) \stackrel{(56)}{\longrightarrow} H_{2n-1}(BC_*(n)),$$

Let us extend homomorphism (55) to a homomorphism

$$\mathcal{L}^G_n : BC_{2n-1}(n) \longrightarrow \mathbb{R}(n-1)$$

by setting it zero on the groups $C_{2n-1}(n+i)$ for $i > 0$. The second $(2n-1)$-term functional equation for the Grassmannian n-logarithm function, see (32), just means that the composition

$$C_{2n}(n+1) \longrightarrow C_{2n-1}(n) \stackrel{\mathcal{L}^G_n}{\longrightarrow} \mathbb{R}(n-1),$$

where the first map is a vertical arrow in $BC_*(n)$, is zero. Therefore we get a homomorphism

$$\mathcal{L}^G_n : H_{2n-1}(BC_*(n)) \longrightarrow \mathbb{R}(n-1)$$

Corollary 5.10 *One has*

$$[C^G_n] = -(-1)^{n(n+1)/2}\frac{(n-1)!^2}{(2n-2)!^2} \cdot \frac{b_n}{n}$$

Proof. Indeed,

$$\mathrm{Alt}_{2n-1}\mathrm{Tr}(X_1 \cdot ... \cdot X_{2n-1}) = < \mathrm{tr}(g^{-1}dg)^{2n-1}_{|sl_n}, X_1 \wedge ... \wedge X_{2n-1} >$$

So the claim follows from theorem 5.7 since, as it clear from comparison of formulas (50) and (47), $b_n = n![C_n]$. The corollary is proved.

Theorem 5.11 *The composition*

$$K_{2n-1}(\mathbb{C}) \stackrel{\sim}{\longrightarrow} \mathrm{Prim}H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q}) \stackrel{\varphi^{2n-1}_{2n-1}}{\longrightarrow} \tag{57}$$

$$H_{2n-1}(BC_*(n)_{\mathbb{Q}}) \stackrel{\mathcal{L}^G_n}{\longrightarrow} \mathbb{R}(n-1)$$

equals to

$$-(-1)^{n(n+1)/2} \cdot \frac{(n-1)!^2}{n(2n-2)!^2} r^{\mathrm{Bo}}_n \tag{58}$$

Proof. Recall that restriction to GL_n of the map φ^{2n-1}_{2n-1} coincides with the map φ^n_{2n-1}. Therefore lemma 5.9 guarantees that restriction to $GL_n(\mathbb{C})$ of the composition of the last two arrows coincides with the map given by the class $[C^G_{2n-1}]$. So corollary 5.10 implies the theorem.

7. Comparing $[D_n]$ and B_n. The following result is not used below.

Theorem 5.12

$$[D_n] \quad = (2\pi i)^n(2n-1)B_n$$

Proof. The transgression identifies the class B_n with the Chern class of the universal bundle over BG, where $G = GL_n(\mathbb{C})$. We will compute explicitly transgression of the n-th component of the Chern character of the universal vector bundle $p : E \longrightarrow BG$. Let $\mathcal{A}$ be a connection on E. Then the n-th Chern class is represented by the $2n$-form

$$c_n(A) := \frac{trF^n_A}{(2\pi i)^n}$$

where $F_A := d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature form.

Let $q : EG \longrightarrow BG$ be the principal fibration associated with E. Then the form $q^*c_n(A)$ is exact. If $d\omega = q^*c_n(A)$ and F is a fiber of q then $\omega|_F$ is closed, its cohomology class is transgressive, and goes to $[c_n(A)]$. To do the computation we choose a connection $\mathcal{A}_0$ on BG which is flat in a neighbourhood U of a point $x \in BG$. It provides a trivialization of the bundle E over U as well as a trivialization $\varphi : EG|_U \longrightarrow G \times U$.

The bundle q^*E has canonical trivialization. It provides a connection $\mathcal{A}_1$ on q^*E. So there are two connections, $q^*\mathcal{A}_0$ and $\mathcal{A}_1$ on q^*E. One has $\mathcal{A}_1 = q^*\mathcal{A}_0 + g^{-1}dg$, where $(g, u) = \varphi(x)$. Let

$$\mathcal{A}(t) := t\mathcal{A}_1 + (1-t)q^*\mathcal{A}_0 = tg^{-1}dg + q^*\mathcal{A}_0$$

It can be thought of as a connection on the lifting of the bundle q^*E to $EG \times [0,1]$; here $t \in [0,1]$. The curvature $F(t)$ of this connection is

$$F(t) = g^{-1}dgdt + t^2g^{-1}dg \wedge g^{-1}dg$$

The push forward of the form $trF(t)^n$ down to EG is a primitive for the form $trF^n_{A(1)} - trF^n_{A(0)}$. It is given (in $q^{-1}U$) by

$$\int_0^1 trF(t)^n = \frac{1}{2n-1}tr(g^{-1}dg)^{2n-1}$$

Theorem 5.12 is proved.

8. On the motivic nature of the Grassmannian n–logarithm functions. According to the results of the previous section understanding of the

Borel regulator, and hence special values of the Dedekind ζ–functions, is reduced to study of properties of the Grassmannian n–logarithm function $\mathcal{L}^G_n$.

Recall that a framed mixed Hodge-Tate structure has a natural $\mathbb{R}$-valued invariant, called the Lie period. Thus a variation of Hodge-Tate structures $\mathbb{L}$ over a base X provides a period function $\mathbb{L}^{\mathcal{M}}$ on $X(\mathbb{C})$.

Conjecture 5.13 *There exists a variation $\mathbb{L}^{\mathcal{M}}_n$ of framed mixed Tate motives over $\widehat{G}^n_{n-1}$ such that:*

$$\sum_{i=0}^{2n}(-1)^i a_i^*\mathbb{L}^{\mathcal{M}}_n = 0; \quad \sum_{j=0}^{2n}(-1)^j b_j^*\mathbb{L}^{\mathcal{M}}_n = 0;$$

and the Lie period $\mathcal{L}^{\mathcal{M}}_n$ of its Hodge realization satisfies

$$\mathcal{L}^{\mathcal{M}}_n - \mathcal{L}^G_n = \sum_{i=0}^{2n-1}(-1)^i a_i^* F_n \tag{59}$$

where F_n is a function on $\widehat{G}^n_{n-2}(\mathbb{C})$.

b) The functional equations satisfying by $\mathcal{L}^{\mathcal{M}}_n$ essentially determine it: the space of all smooth/measurable functions satisfying these functional equations is finite dimensional.

Remark. The function F_n is obviously not determined by (59) – add a function coming from $\widehat{G}^n_{n-3}(\mathbb{C})$. Nevertheless we expect that there exists *canonical* explicit choice for F_n. Then formula (59) can be considered as an explicit formula for $\mathcal{L}^{\mathcal{M}}_n$ in terms of $\mathcal{L}^G$'s.

Moreover we expect that there exists a canonical homotopy between the Grassmannian n–logarithm (understood as a cocycle in the Deligne cohomology of the bi–Grassmannian) and its "motivic" bi-Grassmannian counterpart. Observe that the motivic bi–Grassmannian n–logarithm should have non trivial components outside of the bottom line of the bi-Grassmannian, while the defined above (or in [G5]) Grassmannian n–logarithm is concentrated entirely at the bottom line.

A variation of mixed Tate motives over $\widehat{G}^n_{n-1}$ was constructed in [HaM]. However it is not clear how to relate it to the function $\mathcal{L}^G_n$.

Conjecture 5.13 is known for $n = 2$ and $n = 3$.

The $n = 2$ case follows from (11), the well known motivic realization of the dilogarithm, and Bloch's theorem characterizing the Bloch-Wigner function by Abel's 5–term equation it satisfies.

The $n = 3$ case of conjecture 5.13 follows from the results of [G1-2], [GZ] and the motivic realization of the trilogarithm. In particular the part b) is given by theorem 1.10 in [G1].

Examples. 1. $n = 2$. Then $\mathcal{L}^{\mathcal{M}}_2 = \mathcal{L}^G_2$.

2. $n = 3$. The motivic Grassmannian trilogarithm function has been constructed in [G1-2] in terms of the classical trilogarithm function. Namely, one has

$$\mathcal{L}_3^{\mathcal{M}}(l_0, ..., l_5) = \frac{1}{90}\mathrm{Alt}_6\mathcal{L}_3\Big(\frac{\Delta(l_0, l_1, l_3)\Delta(l_1, l_2, l_4)\Delta(l_2, l_0, l_5)}{\Delta(l_0, l_1, l_4)\Delta(l_1, l_2, l_5)\Delta(l_2, l_0, l_3)}\Big)$$

According to theorem 1.3 of [GZ] $\mathcal{L}_3^{\mathcal{M}}$ is different from $\mathcal{L}_3^G$, and

$$F_3(l_0, ..., l_4) = \frac{1}{9}\mathrm{Alt}_5\Big(\log|\Delta(l_0, l_1, l_2)|\log|\Delta(l_1, l_2, l_3)|\Delta(l_2, l_3, l_4)|\Big)$$

3. If $n > 3$ then $\mathcal{L}_n^{\mathcal{M}}$ is different from $\mathcal{L}_n^G$ since it is already so for the restriction to the special configuration, see theorem 4.5.

The space of the functional equations for the function $\mathcal{L}_3^G$ is smaller then the one for $\mathcal{L}_3^{\mathcal{M}}$, see chapter 1.5 of [G3]. A similar situation is expected for all $n > 3$.

The space of the functional equations for the motivic n–logarithm function $\mathcal{L}_n^{\mathcal{M}}$ should provide an explicit construction of the weight n part of the motivic Lie coalgebra of an arbitrary field F, as explained in s. 4.1 in [G6], taking into account the following correction.

Correction. In s. 4.2 of [G6] the subgroup of the functional equations $\mathcal{R}_n^G$ is supposed to be defined as the subgroup of all functional equations for the function $\mathcal{L}_n^{\mathcal{M}}$, not $\mathcal{L}_n^G$.

6 The Chow dilogarithm and a reciprocity law

The Chow dilogarithm provides a homomorphism $\Lambda^3\mathbb{C}(X)^* \to \mathbb{R}$ given by

$$f_1 \wedge f_2 \wedge f_3 \longmapsto \mathcal{P}_2(X; f_1, f_2, f_3) := \frac{1}{2\pi i}\int_{X(\mathbb{C})} r_2(f_1, f_2, f_3) \qquad (60)$$

In this section we show that the Chow dilogarithm can be expressed by the function $\mathcal{L}_2(z)$. The precise versions of this claim are discussed below.

1. The set up ([G1-2]). For any field F we defined in [G1] the groups

$$\mathcal{B}_n(F) := \frac{\mathbb{Z}[F^*]}{\mathcal{R}_n(F)}, \quad n \geq 2$$

and homomorphisms

$$\delta_n : \mathcal{B}_n(F) \longrightarrow \mathcal{B}_{n-1}(F) \otimes F^*; \quad \{x\}_n \longmapsto \{x\}_{n-1} \otimes x, \quad n \geq 3$$

$$\delta_2 : \mathcal{B}_2(F) \longrightarrow \Lambda^2 F^*, \quad \{x\}_2 \longmapsto (1-x) \wedge x$$

There is a complex $\Gamma(F; n)$

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_3} \mathcal{B}_2(F) \subset \Lambda^{n-2}F^* \xrightarrow{\delta_3} \Lambda^n F^*$$

where $\delta_n(\{x\}_k \otimes Y) := \{x\}_{k-1} \otimes x \wedge Y$, called the weight n polylogarithmic complex.

If K is a field with a discrete valuation v and the residue field k_v, then there is a homomorphism of complexes $\mathrm{res}_v : \Gamma(K, n) \longrightarrow \Gamma(k_v, n-1)[-1]$ (see s. 14 of §1 in [G1]). For example for $n = 3$ we have

$$\begin{array}{ccccc} \mathcal{B}_3(K) & \xrightarrow{\delta_3} & \mathcal{B}_2(K) \otimes K^* & \xrightarrow{\delta_3} & \Lambda^3 K^* \\ & & \downarrow \mathrm{res}_v & & \downarrow \mathrm{res}_v \\ & & \mathcal{B}_2(k_v) & \xrightarrow{\delta_2} & \Lambda^2 k_v^* \end{array} \tag{61}$$

Here $\mathrm{res}_v(\{x\}_2 \otimes y)$ is zero unless $v(x) = 0$. In the latter case it is $\mathrm{res}_v(\{x\}_2 \otimes y) = v(y)\{\overline{x}\}_2$, where $\overline{x}$ denotes projection of x to the residue field of K.

Let X be a regular curve over an algebraically closed field k and $F := k(X)^*$. Set $\mathrm{Res} := \sum_x \mathrm{res}_x$ where res_x is the residue homomorphism for the valuation on F corresponding to a point x of X. For instance for $n = 3$ we get a morphism of complexes

$$\begin{array}{ccccc} \mathcal{B}_3(F) & \xrightarrow{\delta_3} & \mathcal{B}_2(F) \otimes F^* & \xrightarrow{\delta_3} & \Lambda^3 F^* \\ & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ & & \mathcal{B}_2(k) & \xrightarrow{\delta_2} & \Lambda^2 k^* \end{array}$$

We will also need a more explicit version $B_2(F)$ of the group $\mathcal{B}_2(F)$. Denote by $R_2(F)$ the subgroup of $\mathbb{Z}[\mathbb{P}^1(F)]$ generated by the elements

$$\{0\}, \{\infty\} \quad \text{and} \quad \sum_{i=1}^{5} (-1)^i \{r(x_1, ..., \widehat{x}_i, ..., x_5)\}$$

when $(x_1, ..., x_5)$ run through all 5-tuple of distinct points in $\mathbb{P}^1(F)$. Define the Bloch group $B_2(F)$ as

$$B_2(F) := \frac{\mathbb{Z}[\mathbb{P}^1(F)]}{R_2(F)}$$

One can show that $R_2(F) \subset \mathcal{R}_2(F)$. There is a map

$$i : B_2(k)_{\mathbb{Q}} \longrightarrow \mathcal{B}_2(k)_{\mathbb{Q}} \tag{62}$$

Proposition 6.1 *Let k be a number field. Then (62) is an isomorphism modulo torsion.*

Proof. The map i is clearly surjective. The diagram

$$\begin{array}{ccc} B_2(k) & \longrightarrow & \mathcal{B}_2(k) \\ \downarrow \delta_2 & & \downarrow \delta_2 \\ \Lambda^2 k^* & = & \Lambda^2 k^* \end{array}$$

is commutative. So we need only to show that if $0 \neq x \in B_2(k)_{\mathbb{Q}}$ and $\delta_2(x) = 0$, then $i(x) \neq 0$. This follows from the injectivity of the regulator map on $K_3^{ind}(k)_{\mathbb{Q}}$. Indeed, by Suslin's theorem for a field F one has $K_3^{ind}(F)_{\mathbb{Q}} = \mathrm{Ker}\delta_2 \otimes \mathbb{Q}$. Let us identify $K_3^{ind}(\mathbb{C})_{\mathbb{Q}}$ with this subgroup of $B_2(\mathbb{C})_{\mathbb{Q}}$. The restriction of the dilogarithm map

$$B_2(k) \longrightarrow \mathcal{B}_2(k) \longrightarrow (\mathbb{Z}[Hom(k,\mathbb{C})] \otimes 2\pi i\mathbb{R})^+, \qquad \{z\}_2 \longmapsto \{2\pi i\mathcal{L}_2(\sigma_i(z))\}$$

to the subgroup $\mathrm{Ker}\delta_2 \otimes \mathbb{Q}$ gives the Borel regulator $K_3^{ind}(k)_{\mathbb{Q}} \longrightarrow \mathbb{R}^{r_2}$ ([G1]) and thus injective by Borel's theorem.

Remark. For any field k the rigidity conjecture for K_3^{ind} implies that the map i should be an isomorphism, see [G1].

2. The strong reciprocity law.

Conjecture 6.2 *Let X be a regular projective curve over an algebraically closed field k and $F := k(X)^*$. Then there exists a canonical homomorphism of groups $h : \Lambda^3 F^* \to \mathcal{B}_2(k)$ satisfying the following two conditions:*

a) $h(k^* \wedge \Lambda^2 F^*) = 0$ and the diagram

$$\begin{array}{ccccc} \mathcal{B}_3(F) & \xrightarrow{\delta_3} & \mathcal{B}_2(F) \otimes F^* & \xrightarrow{\delta_3} & \Lambda^3 F^* \\ & & \mathrm{Res}\downarrow & h\swarrow & \downarrow \mathrm{Res} \\ & & \mathcal{B}_2(k) & \xrightarrow{\partial_2} & \Lambda^2 k^* \end{array} \tag{63}$$

is commutative.

b) If X is a curve over $\mathbb{C}$ then

$$\frac{1}{2\pi i}\int_{X(\mathbb{C})} r_2(f_1 \wedge f_2 \wedge f_3) = \mathcal{L}_2\Big(h(f_1 \wedge f_2 \wedge f_3)\Big) \tag{64}$$

Remarks. 1. b) follows easily if we have a functorial map h such that $\mathrm{Res} = \delta_2 \circ h$, see theorem 6.10 below.

2. According to Suslin's reciprocity law for the Milnor group $K_3(F)$ the projection of $\mathrm{Res}(\Lambda^3 F^*) \subset \Lambda^2 k^*$ to $K_2(k)$ is zero. Since by Matsumoto's theorem $K_2(k) = \mathrm{Coker}(\delta_2)$, one has $\mathrm{Res}(\Lambda^3 F^*) \subset \mathrm{Im}(\delta_2)$. However $\mathrm{Ker}(\delta_2)$ is nontrivial, so it is *a priory* unclear that we can lift naturally the map Res to a map h. One of the reasons why we can do this is provided by (64).

We prove this conjecture in the following cases:

a). $X = \mathbb{P}^1$; we construct *explicitly* a reciprocity homomorphism $h : \Lambda^3 F^* \to B_2(k)$ in theorem 6.5.

b). X is an elliptic curve over an algebraically closed field; we construct *explicitly* a reciprocity homomorphism $h : \Lambda^3 F^* \to \mathcal{B}_2(k)$ in theorem 6.14.

c). $k = \overline{\mathbb{Q}}$, X is any curve; see theorem 6.12.

In the cases a) and b) the homomorphism h satisfy the following additional property. Let $F = k(X)$ and k is not necessarily algebraically closed. Let k' be the field of definition of the divisors $(f_1), (f_2), (f_3)$. Then $h(f_1 \wedge f_2 \wedge f_3) \in \mathcal{B}_2(k')$.

Conjecture 6.3 *Let X be a projective regular curve over an algebraically closed field k and $F := k(X)$. Then the homomorphism*

$$\mathrm{Res} : \Gamma(F; n) \longrightarrow \Gamma(k; n-1)[-1]$$

is homotopic to zero.

Lemma 6.4 *Assume that we have a map h such that $h(k^* \wedge \Lambda^2 F^*) = 0$ and $\mathrm{Res} = \delta_2 \circ h$. Then $h \circ \delta_3 = \mathrm{Res}$.*

Proof. The image of the group $\mathcal{B}_2(F) \otimes F^*$ under the map $h \circ \delta_3$ belongs to the subgroup $\mathrm{Ker}\delta_2$. Since $h(k^* \wedge \Lambda^2 F^*) = 0$ one has $h \circ \delta_3 = \mathrm{Res}$ on $\mathcal{B}_2(F) \otimes k^*$. Any element of $k(X)^*$ can be connected via a curve to a constant. This together with the rigidity of $\mathrm{Ker}\delta_2$ (built in the definition of the group $\mathcal{B}_2(k)$) implies the result.

3. The $X = \mathbb{P}^1$ case. Recall that $v_x(f)$ is the order of zero of $f \in k(X)$ at x. Choose a point ∞ on $\mathbb{P}^1$.

Theorem 6.5 *Assume that $k = \overline{k}$. Then the map $h : \Lambda^3(k(\mathbb{P}^1))^* \to B_2(k)$ given by the formula*

$$h(f_1 \wedge f_2 \wedge f_3) := \sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) \{r(x_1, x_2, x_3, \infty)\}_2$$

satisfies all the conditions of conjecture 6.2 modulo 6-torsion.

Proof. Let us show that h is independent of the choice of ∞, i.e.

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) \{r(x_1, x_2, x_3, a)\}_2 \in B_2(k)$$

does not depend on a. Indeed, the 5-term relation for the 5-tuple of points (x_1, x_2, x_3, a, b) gives

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) (\{r(x_1, x_2, x_3, a)\}_2 - \{r(x_1, x_2, x_3, b)\}_2) =$$

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) \mathcal{L}_2(r(x_1, x_2, a, b)\}_2 - r(x_1, x_3, a, b)\}_2 + \{r(x_2, x_3, a, b)\}_2)$$

Each of these 3 terms vanishes because $\sum_{x \in \mathbb{P}^1(k)} v_x(f) = 0$.

Proposition 6.6 *Let $k = \overline{k}$. Then modulo 6-torsion*

$$h((1-f) \wedge f \wedge g) = \sum_{x \in \mathbb{P}^1(k)} v_x(g) \{f(x)\}_2$$

Proof. Using linearity with respect to g and projective invariance of the cross-ratio we see that it is sufficient to prove the identity for $g = t$. Then it boils down to

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(1-f)v_{x_2}(f)\{r(x_1,x_2,0,\infty)\}_2 = \{f(0)\}_2 - \{f(\infty)\}_2 \qquad (65)$$

Lemma 6.7 *Applying δ_2 to both parts of (65) we get the same result modulo 6-torsion.*

Proof. Choose a coordinate t on $\mathbb{P}^1$ such that $f(\infty) = 1$. Then

$$f(t) = \frac{\prod_i (a_i - t)^{\alpha_i}}{\prod_k (c_k - t)^{\gamma_k}}; \quad 1 - f(t) = \frac{B\prod_j (b_j - t)^{\beta_j}}{\prod_k (c_k - t)^{\gamma_k}} \qquad (66)$$

Observe that $\{f(\infty)\}_2 = 0$ modulo 6-torsion. The left hand side equals

$$\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(1-f)v_{x_2}(f)\{x_1/x_2\}_2 = \qquad (67)$$

$$\sum \beta_j \alpha_i \{b_j/a_i\}_2 - \sum \gamma_k \alpha_i \{c_k/a_i\}_2 - \sum \gamma_k \{b_j/c_k\}_2 =$$

Applying δ_2 to it we get

$$\sum \beta_j \alpha_i \cdot \frac{a_i - b_j}{a_i} \wedge \frac{b_j}{a_i} - \sum \gamma_k \alpha_i \cdot \frac{a_i - c_k}{a_i} \wedge \frac{c_k}{a_i} - \sum \gamma_k \beta_j \cdot \frac{c_k - b_j}{c_k} \wedge \frac{b_j}{c_k} =$$

$$\sum \beta_j \alpha_i \cdot b_j \wedge a_i + \sum \alpha_i \gamma_k \cdot a_i \wedge c_k + \sum \gamma_k \beta_j \cdot c_k \wedge b_j \quad +$$

$$-\sum_i \frac{\prod_j (a_i - b_j)^{\beta_j}}{\prod_k (a_i - c_k)^{\gamma_k}} \wedge a_i^{\alpha_i} + \sum_j \frac{\prod_i (a_i - b_j)^{\alpha_i}}{\prod_k (c_k - b_j)^{\gamma_k}} \wedge b_j^{\beta_j} - \sum_k \frac{\prod (a_i - c_k)^{\alpha_i}}{\prod (c_k - b_j)^{\beta_j}} \wedge c_k^{\gamma_k}$$

Using (66) we see that the second line equals modulo 2-torsion to

$$-\sum_i (1 - f(a_i)) \wedge a_i^{\alpha_i} + \sum_j f(b_j) \wedge b_j^{\beta_i} - \sum_k \frac{\prod (a_i - c_k)^{\alpha_i}}{\prod (b_j - c_k)^{\beta_j}} \wedge c_k^{\gamma_k} + B \wedge \prod a_i^{\alpha_i}$$

The first two terms are zero since $f(a_i) = 0$ and $f(b_j) = 1$. The third term equals to $-B \wedge \prod c_k^{\gamma_k}$ since, as it follows from (66),

$$\frac{\prod (a_i - c_k)^{\alpha_i}}{\prod (b_j - c_k)^{\beta_j}} = B$$

On the other hand to

$$\delta_2(\{f(0)\}_2) = (1 - f(0)) \wedge f(0) = \frac{B\prod_j b_j^{\beta_j}}{\prod_k c_k^{\gamma_k}} \wedge \frac{\prod_i a_i^{\alpha_i}}{\prod_k c_k^{\gamma_k}}$$

which matches the expression we got for the left hand side. The lemma is proved.

To prove the proposition it remains to use the standard rigidity argument. Namely, we need to show that the identity is valid for some particular f, which is easy, or use proposition 6.8 plus injectivity of the regulator on $K_3(\overline{\mathbb{Q}})_{\mathbb{Q}}$. The proposition is proved.

Now let us prove the key fact, Res $= \delta_2 \circ h$. We need to show that for any 3 rational functions f_1, f_2, f_3 on $\mathbb{P}^1$

$$\sum_{x \in \mathbb{P}^1(k)} \mathrm{res}_x(f_1 \wedge f_2 \wedge f_3) =$$

$$\delta\Big(\sum_{x_i \in \mathbb{P}^1(k)} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) \{r(x_1, x_2, x_3, \infty)\}_2\Big) \tag{68}$$

The both parts are obviously homomorphisms from $\Lambda^3 F^*$ to $\Lambda^2 k^*$ which are zero on $k^* \wedge \Lambda^2 F^*$. (The last property for the map $\sum \mathrm{res}_x$ is provided by the Weil reciprocity law). We normalize the cross ratio of four points on the projective line by $r(\infty, 0, 1, z) = z$. So it suffices to check the formula on elements

$$\frac{z - a_2}{z - a_1} \wedge \frac{z - b_2}{z - b_1} \wedge \frac{z - c_2}{z - c_1}$$

In this case it follows from

$$\delta_2(\{r(a_2, b_2, c_2, \infty)\}) = \delta_2\left(\frac{a_2 - c_2}{b_2 - c_2}\right) = \frac{b_2 - a_2}{b_2 - c_2} \wedge \frac{a_2 - c_2}{b_2 - c_2}$$

It remains to prove the following proposition.

Proposition 6.8

$$\mathcal{P}_2(\mathbb{P}^1; f_1, f_2, f_3) = \sum_{x_i \in \mathbb{P}^1(\mathbb{C})} v_{x_1}(f_1) v_{x_2}(f_2) v_{x_3}(f_3) \mathcal{L}_2(r(x_1, x_2, x_3, \infty))$$

Proof. We immediately reduce the statement to the situation when $f_1 = 1 - z, f_2 = z, f_3 = z - a$ which is a particular case of the following lemma

Lemma 6.9 *Let X be an arbitrary curve over $\mathbb{C}$. Then*

$$\int_{X(\mathbb{C})} r_2((1 - f) \wedge f \wedge g) = - \sum_{x \in X(\mathbb{C})} v_x(g) \cdot \mathcal{L}_2(f(x))$$

Proof. For functions $f(z)$ and $g(z)$ on $X(\mathbb{C})$ set

$$\alpha(f, g) := \log|f| d\log|g| - \log|g| d\log|f|$$

Consider the following 1-form on $X(\mathbb{C})$

$$\mathcal{L}_2(f)d\arg g - \frac{1}{3}\alpha(1-f,f)\log|g| \tag{69}$$

It defines a current. We claim that its derivative is equal to:

$$2\pi\cdot\mathcal{L}_2(f)\delta(g) + r_2((1-f)\wedge f\wedge g) \tag{70}$$

Using $d(d\arg g) = 2\pi\cdot\delta(g)$ and

$$d\mathcal{L}_2(z) = -\log|1-z|d\arg z + \log|z|d\arg(1-z) \tag{71}$$

we see that the differential of the current (69) equals to

$$2\pi\mathcal{L}_2(f)\delta(g) \quad + \quad \Big(-\log|1-f|d\arg f + \log|f|d\arg(1-f)\Big)\wedge d\arg g +$$

$$\frac{1}{3}\Big(\log|1-f|d\log|f| - \log|f|d\log|1-f|\Big)\wedge d\log|g| - \frac{2}{3}\log|g|\cdot d\log|1-f|\wedge d\log|f|$$

Since $d\log(1-f)\wedge d\log f = 0$ we have

$$d\log|1-f|\wedge d\log|f| = d\arg(1-f)\wedge d\arg f$$

Using this and writing $r_2(f_1\wedge f_2\wedge f_3)$ as

$$\frac{1}{3}(\log|f_1|d\log|f_2|\wedge d\log|f_3| + \quad \text{cyclic permutations})$$

$$-(\log|f_1|d\arg(f_2)\wedge d\arg f_3 + \quad \text{cyclic permutations})$$

we come to (70). Integrating we get the lemma. The theorem is proved.

4. Expressing the Chow dilogarithms via the classical one. Let $\pi: Y\to S$ be a family of curves over a base S over $\mathbb{C}$ and $f_1, f_2, f_3\in\mathbb{C}(Y)^*$. We get a function at the generic point of S. Its value at $s\in S$ is given by the Chow dilogarithm $\mathcal{P}_2(Y^s; f_1^s, f_2^s, f_3^s)$, where Y^s is the fiber of π at s. Denote it by $\mathcal{P}_2(Y\to S; f_1, f_2, f_3)$.

Theorem 6.10 *a) Let $\pi: Y\to S$ be a family of curves over a base S over $\mathbb{C}$. Then there are rational functions φ_i on S such that*

$$\mathcal{P}_2(Y\to S; f_1, f_2, f_3) = \sum\mathcal{L}_2(\varphi_i(s))$$

b) Let $k=\mathbb{C}(S)$, X is the generic fiber of π, and $F=k(X)$. Suppose that there exists a map $h:\Lambda^3F^*\to\mathcal{B}_2(k)$ such that $\mathrm{Res}=\delta_2\circ h$. *Then*

$$d\mathcal{P}_2(Y\to S; f_1, f_2, f_3) \quad = \quad d\mathcal{L}_2(h(f_1, f_2, f_3)) \tag{72}$$

Proof. a) We use the existence of the transfer map on K_3^M to reduce the statement to the case $X = \mathbb{P}^1$.

Choose a projection $p : X \to \mathbb{P}^1$. We may suppose without loss of generality that p is a (ramified) Galois covering with the Galois group G. Indeed, let $p_1 : Y \to X$ be such a covering that its composition with p is a Galois covering. Indeed,

$$\mathcal{P}_2(Y \longrightarrow S; p_1^* f_1, p_1^* f_2, p_1^* f_3) = \frac{1}{\deg p_1} \mathcal{P}_2(X \longrightarrow S; p_1^* f_1, p_1^* f_2, p_1^* f_3)$$

Then $\sum_{g \in G} g^* \{f_1, f_2, f_3\} \in p^* k K_3^M((\mathbb{P}^1))$. It coincides with p^* of the transfer of the element $\{f_1, f_2, f_3\} \in K_3^M(F)$. This means that there exist $s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, \in k(\mathbb{P}^1)$ and $g_j, h_j \in k(X)$ such that

$$\sum_{g \in G} g^*(f_1 \wedge f_2 \wedge f_3) - p^* \sum_i s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)} = \sum_j (1 - g_j) \wedge g_j \wedge h_j \qquad (73)$$

Therefore

$$\mathcal{P}_2(Y \to S; f_1, f_2, f_3) = \frac{1}{|G|} \sum_i \mathcal{P}_2(\mathbb{P}^1 \times S \to S; s_1^{(i)}, s_2^{(i)}, s_3^{(i)}) +$$

$$\sum_j \mathcal{P}_2(\mathbb{P}^1 \times S \to S; (1 - g_j) \wedge g_j \wedge h_j)$$

It remains to use lemma 6.9 and proposition 6.8. The part a) of the theorem is proved.

b) We need the following lemma.

Lemma 6.11

$$d_s \mathcal{P}_2(Y \to S; f_1, f_2, f_3) = (2\pi)^{-1} \cdot \mathrm{Alt}_3\Big(v_x(f_1) \log |f_2(x)| d_s \arg f_3(x)\Big) \qquad (74)$$

Proof. Using $dd \log f = 2\pi i \delta(f)$ we get an equality of 3-currents on Y

$$dr_2(f_1, f_2, f_3) =$$

$$\pi_3\Big(\frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\Big) + 2\pi \cdot \mathrm{Alt}_3\Big(\delta(f_1) \log |f_2(x)| d \arg f_3(x)\Big) \qquad (75)$$

The second term in (75) is a 1-form on the divisor $D := \cup_{i=1}^3 \mathrm{div}(f_i)$ considered as a 3-current on Y. This 1-form is the composition of the residue map

$$\mathrm{res} : \Lambda^3 \mathbb{C}(Y)^* \longrightarrow \coprod_{X \in Y_1} \Lambda^2 \mathbb{C}(X)^*$$

with the map

$$r_1 : \Lambda^2 \mathbb{C}(X)^* \longrightarrow \mathcal{A}^1(\mathrm{Spec}(\mathbb{C}(X))), \qquad f \wedge g \longmapsto -2\pi(\log |f| d \arg g - \log |g| d \arg f)$$

The push forward of the first term in (75) vanish (since the fibers are complex curves the push down of any $(3,0)$-form to S is zero). Integrating the second 3-current in (75) along the fibers of Y we get (74). The lemma is proved.

According to lemma 6.11 and formula (71) for $d\mathcal{L}_2$, and using $\mathrm{Res} = \delta_2 \circ h$ we get the proof of the part b) of the theorem.

Remark The function $\mathcal{L}_2(z)$ is continuous on $\mathbb{CP}^1$. Therefore part a) of the theorem implies that the function $\mathcal{P}_2(Y \to S; f_1, f_2, f_3)$ can be extended to a continuous function on S.

5. Conjecture 6.2 for $\overline{k} = \overline{\mathbb{Q}}$.

Theorem 6.12 *Let X be a regular projective curve over $\overline{\mathbb{Q}}$ and $F := \overline{\mathbb{Q}}(X)^*$. Then there exists a homomorphism $h : \Lambda^3 F^* \to B_2(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ as in conjecture 6.2 such that for any embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ one has*

$$\frac{1}{2\pi i}\int_{X(\mathbb{C})} r_2(\sigma(f_1 \wedge f_2 \wedge f_3)) = \mathcal{L}_2\Big(\sigma(h(f_1 \wedge f_2 \wedge f_3))\Big) \tag{76}$$

Proof. It is similar to the proof of theorem 6.10. Choose a projection $p : X \to \mathbb{P}^1$. We may suppose that p is a Galois covering with the Galois group G. Indeed, let $p_1 : Y \to X$ be such a covering that its composition with p is a Galois covering. Setting

$$h(f_1 \wedge f_2 \wedge f_3) := h(\frac{1}{\mathrm{deg} p_1} \cdot p_1^*(f_1 \wedge f_2 \wedge f_3))$$

we may suppose that p is Galois.

Then $\sum_{g \in G} g^*\{f_1, f_2, f_3\} \in p^* k K_3^M(\mathbb{P}^1))$. So there exist $s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, \in k(\mathbb{P}^1)$ and $g_j, h_j \in k(X)$ such that (73) holds. Set

$$|G| \cdot h(f_1 \wedge f_2 \wedge f_3) := \sum h(s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)}) + \sum_x \sum_{x \in X(\overline{\mathbb{Q}})} \{g_j(x)\}_2 \cdot v_x(h_j)$$

Lemma 6.13 *Suppose $\sum_i (1 - f_i) \wedge f_i \wedge g_i = 0$ in $\Lambda^3 \overline{\mathbb{Q}}(X)^*$. Then*

$$\sum_i \sum_{x \in X(\overline{\mathbb{Q}})} v_x(g_i) \cdot \{f(x)\}_2 = 0 \quad \textit{in the group } B_2(\overline{\mathbb{Q}}).$$

This lemma implies that h is well defined. Indeed, suppose that we have a different presentation

$$\sum_{g \in G} g^*\{f_1, f_2, f_3\} = p^* \sum_i \widetilde{s}_1^{(i)} \wedge \widetilde{s}_2^{(i)} \wedge \widetilde{s}_3^{(i)} + \sum_j (1 - \widetilde{g}_j) \wedge \widetilde{g}_j \wedge \widetilde{h}_j$$

We need to show that

$$h\Big(\sum \widetilde{s}_1^{(i)} \wedge \widetilde{s}_2^{(i)} \wedge \widetilde{s}_3^{(i)} - \sum s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)}\Big) + \tag{77}$$

$$\sum v_x(h_j)\{g_j(x)\}_2 - \sum v_x(\widetilde{h}_j)\{\widetilde{g}_j(x)\}_2 = 0$$

There exist $a_j, b_j \in k(\mathbb{P}^1)$ such that modulo $k^* \wedge \Lambda^2 k(\mathbb{P}^1)$ one has

$$\sum \widetilde{s}_1^{(i)} \wedge \widetilde{s}_2^{(i)} \wedge \widetilde{s}_3^{(i)} - \sum s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)} - \sum (1-a_j) \wedge a_j \wedge b_j = 0$$

According to theorem 6.5 the homomorphism h for $\mathbb{P}^1$ annihilates the left hand side. On the other hand

$$p^*\Big(\sum \widetilde{s}_1^{(i)} \wedge \widetilde{s}_2^{(i)} \wedge \widetilde{s}_3^{(i)} - \sum s_1^{(i)} \wedge s_2^{(i)} \wedge s_3^{(i)}\Big) -$$

$$\sum_j (1-\widetilde{g}_j) \wedge \widetilde{g}_j \wedge \widetilde{h}_j + \sum_j (1-g_j) \wedge g_j \wedge h_j = 0$$

Using lemma 6.13 we get (77). To get (80) we use theorem 6.10 and notice that

$$\mathcal{P}_2(Y \to S; f_1, f_2, f_3) = 1/m \cdot \mathcal{P}_2(Z \to S; p_1^* f_1, p_1^* f_2, p_1^* f_3) \qquad (78)$$

Proof of lemma 6.13. For a regular curve X over an algebraically closed field k there is a commutative diagram

$$\begin{array}{ccc} B_2(F) \otimes F^* & \xrightarrow{\delta_3} & \Lambda^3 F^* \\ \text{res} \downarrow & & \downarrow \text{res} \\ B_2(k) \otimes \mathbb{Z}[X(k)] & \xrightarrow{\delta_2} & \Lambda^2 k^* \otimes \mathbb{Z}[X(k)] \end{array}$$

Thus for any point x of the curve X the element $\sum_i v_x(g_i) \cdot \{f_i(x)\}_2$ lies in $\mathrm{Ker}\delta_2$. Therefore it defines an element $\gamma_x \in K_3(\overline{\mathbb{Q}})_{\mathbb{Q}}$.

For any embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ the value of the Borel regulator on $\sigma(\gamma_x)$ is equal to $\sum_{x \in X} v_x(g_i) \cdot \mathcal{L}_2(\sigma(f_i(x)))$. So by lemma 6.9 the value of the Borel regulator on $\sum_x \sigma(\gamma_x)$ is equal to $2\pi \cdot \int_{\mathbb{CP}^1} r_2(\sum_i (1-f_i) \wedge f_i \wedge g_i)$ and hence it is zero by our assumption. So Borel's theorem implies that the element is also zero.

A similar argument using lemma 6.13 shows that the homomorphism h does not depend on the choice of the (finite) Galois extension of $\mathbb{Q}(\mathbb{P}^1)$ containing the field $\mathbb{Q}(X)$.

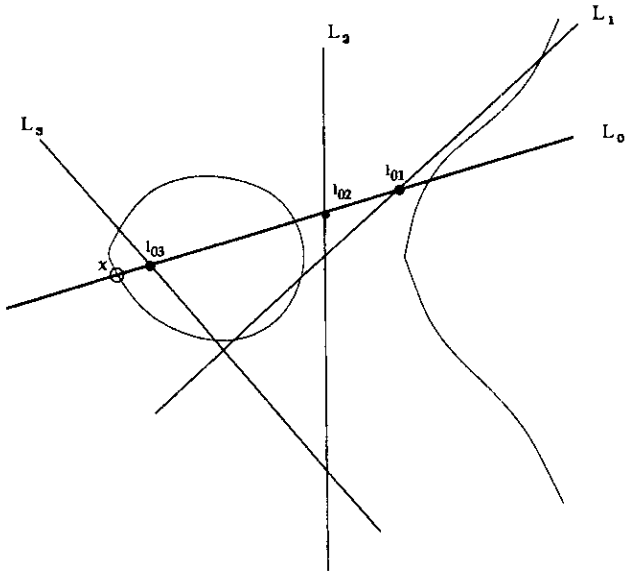
6. Explicit formulas for the reciprocity homomorphism h and the Chow dilogarithm in the case of an elliptic curve. Let E be an elliptic curve. We want to calculate the integral $\int_{E(\mathbb{C})} r_2(f_1 \wedge f_2 \wedge f_3)$. Let us suppose that E is realized as a plane curve. Then any rational function f on E can be written as a ratio of products of *linear* homogeneous functions:

$$f = \frac{l_1 \cdot \ldots \cdot l_k}{l_{k+1} \cdot \ldots \cdot l_{2k}}$$

So it is enough to calculate the integral $\int_{E(\mathbb{C})} r_2(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0)$ where l_i are linear functions in homogeneous coordinates. We will do this in a more general setting.

Notations. Let X be a plane algebraic curve and l_i linear functions in homogeneous coordinates. Denote by L_i the line $l_i = 0$ in $\mathbb{P}^2$. Let D_i be the divisor $L_i \cap X$. Set $l_{ij} := L_i \cap L_j$. For three points a, b, c and a divisor $D = \sum n_i(x_i)$ on a line we will use the following notation (see fig.2):

$$\{r(a,b,c,D)\}_2 := \sum_i n_i \{r(a,b,c,x_i)\}_2$$



Theorem 6.14 *Let E be an elliptic curve over an algebraically closed field k. Then there exists a homomorphism of groups $h : \Lambda^3 F^* \to \mathcal{B}_2(k)$ such that for any linear homogeneous functions $l_0, ..., l_3$ one has*

$$h(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0) = -\sum_{i=0}^{3} (-1)^i \{r(l_{i0}, ..., l_{i3}, D_i)\}_2 \qquad (79)$$

and which satisfy all the properties of conjecture 6.2. In particular, if $k = \mathbb{C}$ then

$$\mathcal{P}_2(E; f_1 \wedge f_2 \wedge f_3) = \mathcal{L}_2\Big(h(f_1 \wedge f_2 \wedge f_3)\Big) \qquad (80)$$

Proof. Suppose we have four generic lines L_0, L_1, L_2, L_3 in $\mathbb{P}^2$. Any two of them, say L_0 and L_1, provide a *canonical* rational function (l_0/l_1) on $\mathbb{P}^2$ with the divisor $(L_0) - (L_1)$ normalized by the condition that its value at the point l_{23} is equal to 1.

Lemma 6.15 *a) On the line L_3 one has $(l_1/l_0) + (l_2/l_0) = 1$.*
b) $\frac{(l_1/l_0)}{(l_2/l_0)} = -(l_1/l_2)$

Proof. Let m be a point on the line L_3. Then

$$(l_1/l_0)(m) = r(l_{03}, l_{13}, l_{23}, m); \qquad (l_2/l_0)(m) = r(l_{03}, l_{23}, l_{13}, m) \tag{81}$$

This gives a). It follows from this that if the point m approaches to the point l_{03} then $\frac{(l_1/l_0)}{(l_2/l_0)}$ tends to -1. This implies b).

Lemma 6.16 *For any plane curve X one has*

$$\sum_{x \in X} \mathrm{res}_x\Big((l_1/l_0) \wedge (l_2/l_0) \wedge (l_3/l_0)\Big) = -\delta_2\Big(\sum_{i=0}^{3}(-1)^i\{r(l_{i0}, ..., l_{i3}, D_i)\}_2\Big)$$

Proof. Let us compute first the residues at the divisors D_1, D_2, D_3 using part a) of lemma 6.15. For example the residue at $x \in D_1$ is equal to

$$v_x((l_1/l_0)) \cdot (l_2/l_0)(x) \wedge (l_3/l_0)(x) = v_x((l_1/l_0)) \cdot (1 - (l_3/l_0)(x)) \wedge (l_3/l_0)(x)$$

$$\stackrel{(81)}{=} v_x((l_1/l_0)) \cdot \{r(l_{13}, l_{10}, l_{12}, x\}_2 = v_x((l_1/l_0)) \cdot \{r(l_{10}, l_{12}, l_{13}, x\}_2$$

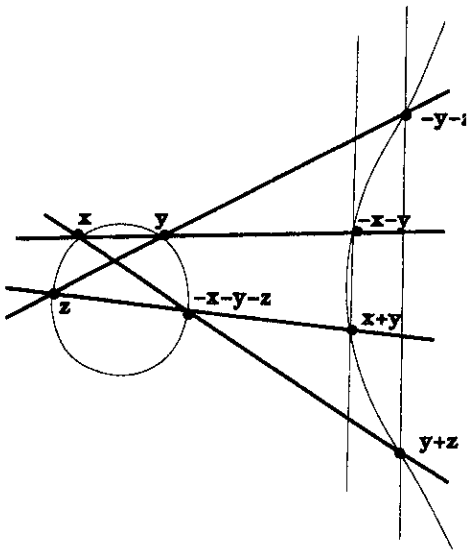
It remains to compute the residues on the line L_0. According to part b) of lemma 6.15 one has

$$(l_1/l_0) \wedge (l_2/l_0) \wedge (l_3/l_0) = -(l_0/l_1) \wedge (l_2/l_1) \wedge (l_3/l_1)$$

Using this we reduce the calculation to the previous case.

Proposition 6.17 *Let E be an elliptic curve over an algebraically closed field k. Then formula (79) provides a well defined homomorphism of groups $h : \Lambda^3 F^* \to \mathcal{B}_2(k)$.*

Proof. Let $D := \sum_i n_i(x_i)$ be a divisor of a rational function f on E. To decompose it into a fraction of products of linear homogeneous functions l_i we proceed as follows. Let $l_{x,y}$ (resp. l_x) be a linear homogeneous equation of the line in $\mathbb{P}^2$ through the points x and y on E (resp. x and $-x$). The divisor of the function $l_{x,y}/l_x$ is $(x) + (y) - (x+y) - (0)$. If $\mathcal{D} = (x) + (y) + D_1$, we write $f = l_{x,y}/l_{x+y} \cdot f'$, so $(f') = (0) + (x+y) + D_1$. After a finite number of such steps we get a desired decomposition.



There are the following relations

$$\frac{l_{x,y}}{l_{x+y}} \cdot \frac{l_{x+y,z}}{l_{x+y+z}} \Big/ \frac{l_{y,z}}{l_{y+z}} \cdot \frac{l_{x,y+z}}{l_{x+y+z}} \quad = \quad \text{a constant}$$

One can prove that they generate all the relations between the functions $l_{x,y}/l_{x+y}$. So h is well defined if it annihilates the following expression:

$$F(x,y,z;l_0,l_2,l_3) := \Big(\frac{l_{x,y}}{l_{x+y}} \cdot \frac{l_{x+y,z}}{l_{x+y+z}} \Big/ \frac{l_{y,z}}{l_{y+z}} \cdot \frac{l_{x,y+z}}{l_{x+y+z}}\Big) \wedge (l_2/l_0) \wedge (l_3/l_0)$$

It follows from lemma 6.16 that $\delta_2(F(x,y,z;l_0,l_2,l_3)) = 0$. Thus according to the definition of the group $\mathcal{B}_2(k)$ it is enough to check that $h(F(x',y',z';l_0,l_2,l_3)) = 0$ for a certain triple of points (x',y',z'). It is easy to see that $h(F(a,a,a;l_0,l_2,l_3)) = 0$ since then the first factor in F is a constant. The proposition is proved.

Proposition 6.18 *Let X be an algebraic curve in $\mathbb{P}^2$ over $\mathbb{C}$ and $l_0,...,l_3$ linear homogeneous functions on $\mathbb{C}^3$. Then one has (using the notations defined above)*

$$\int_{X(\mathbb{C})} r_2(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0) = 2\pi \cdot \sum_{i=0}^{3} (-1)^i \mathcal{L}_2(r(l_{i0},...,l_{i3},D_i) \qquad (82)$$

Proof. It follows from lemma 6.16 and theorem 6.10 that the differentials of the both parts coincide. So its difference is a constant. To show that this constant is zero we deform X to a union of lines in $\mathbb{P}^2$. Using proposition 6.8 one sees that formula (82) is valid when X is a line in $\mathbb{P}^2$.

7 Appendix: on volumes of simplices in symmetric spaces

1. Volumes of hyperbolic geodesic simplices as boundary integrals. A point y of the n-dimensional hyperbolic space $\mathcal{H}_n$ defines a one dimensional space M_y of volume forms on the absolute $\partial\mathcal{H}_n$. It consists of the volume forms invariant under the action of the isotropy group of y. We write them as follows. Let $x_1, ..., x_{n+1}$ be the coordinates in a vector space V_{n+1} and $Q(x) := x_1^2 + ... + x_n^2 - x_{n+1}^2$. Then $\mathcal{H}_n$ can be realized as the projectivization of the cone $Q(x) < 0$, and its boundary is the projectivization of the cone $Q(x) = 0$. Choose a point $y \in \mathcal{H}_n$. Lifting y to a vector $y' \in V_{n+1}$ we have the following volume form on the boundary $\partial\mathcal{H}_n$:

$$\frac{\delta(Q(x))\alpha_{n+1}(x, dx)}{(x, y')^{n-1}}$$

If y belongs to the boundary $\partial\mathcal{H}_n$, this formula provides a space M_y of singular volume forms on the absolute; they are invariant under the isotropy group of y.

Let us choose for any point y such a volume form μ_y. For two points x, y the ratio μ_x/μ_y is a nonzero function on the absolute.

Let $I(y_0, ..., y_n)$ be the geodesic simplex with vertices at $y_0, ..., y_n$ where the points y_i could be on the absolute. Denote by $\mathrm{vol}I(y_0, ..., y_n)$ the volume of this simplex with respect to the invariant volume form in $\mathcal{H}_n$ normalized by the following condition: if we realize the hyperbolic space as the interior of the unit ball $y_1^2 + ... + y_n^2 \leq 1$ then the volume form restricted to the tangent space at the origin $(0, ..., 0)$ is $dy_1 \wedge ... \wedge dy_n$.

Theorem 7.1 *For any hyperbolic geodesic simplex $I(y_0, ..., y_n)$ one has*

$$\frac{(n-1)^n \mathrm{vol}(S^{n-1})}{n} \cdot \mathrm{vol}I(y_0, ..., y_n) = \int_{\partial\mathcal{H}_n} \log|\frac{\mu_{y_1}}{\mu_{y_0}}| d\log|\frac{\mu_{y_2}}{\mu_{y_0}}| \wedge ... \wedge d\log|\frac{\mu_{y_n}}{\mu_{y_0}}| \tag{83}$$

Let $\varphi(y_0, ..., y_n)$ be the function defined by the right hand side of (83). Thanks to property 2) of proposition 5.1 it does not depend on the choice of invariant volume forms μ_y.

Proposition 7.2 *The function $\varphi(y_0, ..., y_n)$ has the following properties:*

1) It is a smooth function on the vertices y_i.

2) Equals to zero if three of the vertices belong to the same geodesic.

3) Additive with respect to cutting of a simplex, i.e. if $y_0, ..., y_n$ are such points that y_0, y_1, y_2 are on the same geodesic, y_1 between y_0 and y_2, then

$$\varphi(y_0, y_2, ..., y_{n+1}) = \varphi(y_0, y_1, y_3, ..., y_{n+1}) + \varphi(y_1, y_2, ..., y_{n+1})$$

4) Invariant under the action of the symmetry group $SO(n, 1)$.

Proof. 1) This is clear.

2) Let us realize the hyperbolic space as the interior part of the unit ball in $\mathbb{R}^n$. Consider the geodesic l passing through the center of the ball in the vertical direction. The subgroup $SO(n-1) \subset SO(n,1)$ preserves pointwise this geodesic. So for any point y on the geodesic the invariant volume form μ_y is invariant under the action of the group $SO(n-1)$. The quotient of $\partial\mathcal{H}_n$ under the action of $SO(n-1)$ is given by the projection $p : \partial\mathcal{H}_n \longrightarrow \mathbb{R}$ onto the vertical axis. Take three points y_1, y_2, y_3 on the geodesic. Then $\frac{\mu_{y_2}}{\mu_{y_1}}$ and $\frac{\mu_{y_3}}{\mu_{y_1}}$ are lifted from the line $\mathbb{R}$. Therefore $d\log|\frac{\mu_{y_2}}{\mu_{y_1}}| \wedge d\log|\frac{\mu_{y_3}}{\mu_{y_1}}| = 0$. So for a degenerate simplex $(y_0, ..., y_n)$ one has

$$\log|\frac{\mu_{y_0}}{\mu_{y_1}}|d\log|\frac{\mu_{y_2}}{\mu_{y_1}}| \wedge d\log|\frac{\mu_{y_3}}{\mu_{y_1}}| \wedge ... \wedge d\log|\frac{\mu_{y_n}}{\mu_{y_1}}| = 0$$

It remains to mention the skewsymmetry of the integral (83). The property 2) is proved.

3) Follows from 2) and the additivity property from proposition (5.1).

4) This is clear from 4) of proposition (5.1). The proposition is proved.

The leading term of the Taylor expansion of the function $\varphi(y_0, y_1, ..., y_n)$ when y_0 is fixed and $y_1, ..., y_n$ are near y_0 provides an exterior n-form in $T_{y_0}\mathcal{H}_n$ denoted $\varphi_{y_0}(Y_1, ..., Y_n)$, $Y_i \in T_{y_0}\mathcal{H}_n$. Let us compare it with the volume form $\mathrm{Vol}_{y_0}(Y_1, ..., Y_n)$ in $T_{y_0}\mathcal{H}_n$ normalized as before theorem ??.

Lemma 7.3 $\varphi_{y_0}(Y_1, ..., Y_n) = \frac{(n-1)^n}{n}\mathrm{vol}(S^{n-1}) \cdot \mathrm{Vol}_{y_0}(Y_1, ..., Y_n)$.

Proof. Below we abuse notations by writing y for y'. One has $\mu_{y_1}/\mu_{y_2} = \frac{(y_2,x)^{n-1}}{(y_1,x)^{n-1}}$. So

$$\varphi(y_0, ..., y_n) = (n-1)^n \int_{\partial\mathcal{H}_n} \log|\frac{(y_1,x)}{(y_0,x)}|d\log|\frac{(y_2,x)}{(y_0,x)}| \wedge ... \wedge d\log|\frac{(y_n,x)}{(y_0,x)}|$$

Thus

$$\varphi_{y_0}(Y_1, ..., Y_n) = (n-1)^n \int_{\partial\mathcal{H}_n} (Y_1, x)d(Y_2, x) \wedge ... \wedge d(Y_n, x)$$

To do the computation of this integral we may suppose that $y_0 = (0, ..., 0, 1)$, $Y_i = \frac{\partial}{\partial y_i}$, so $(Y_i, x) = x_i$. Then the last integral equals to

$$(n-1)^n \int_{S^{n-1}} x_1 dx_2 \wedge ... \wedge dx_n = \frac{(n-1)^n}{n}\mathrm{vol}(S^{n-1})$$

where S^{n-1} is the sphere $x_1^2 + ... + x_n^2 = 1$. The lemma follows.

Proof of theorem 7.1. Let us suppose first that the points x_i are inside of the hyperbolic space.

The function $\varphi(x_0, ..., x_n)$ defines an n-density $\widetilde{\varphi}$ on $\mathcal{H}_n$. Namely, to define the integral $\widetilde{\varphi}$ over a simplex M one has to subdivide it on small simplices and take the sum of the functions φ corresponding to their vertices. When the simplices are getting smaller the limit exists and is by definition $\int_M \widetilde{\varphi}$. Here we used the properties 1) - 3). More precisely 1) and 2) implies that φ defines an additive volume form in $\mathcal{H}_n$, and 3) (together with 1)) guarantee that this volume form is σ-additive.

The skewsymmetry property implies that $\widetilde{\varphi}$ is actually a differential n-form. It is invariant under the action of the group $SO(n,1)$. Therefore it is proportional to the standard volume form.

Now suppose that vertices x_i can be on the absolute. Then it is easy to see that the corresponding integral (37) is still convergent. Moreover, if the vertices of the geodesic simplex are in generic position then it is a continuous function of the vertices. This implies that the volume of an ideal geodesic simplex is finite (which is, of course, an elementary fact) and coincides with the corresponding integral (83).

A completely similar results are valid for the complex n-dimensional hyperbolic space $\mathcal{H}_{n\mathbb{C}} := \{|z_1| + ... + |z_n|^2 < 1\}$, $z_i \in \mathbb{C}$, and the quaternionic hyperbolic space $\mathcal{H}_{n\mathbf{H}} := \{|q_1| + ... + |q_n|^2 < 1\}$ (q_i are quaternions). Indeed, a point x in each of these spaces defines an invariant volume form μ_x on the boundary.

2. Calculation of volume of a three dimensional ideal geodesic simplex. If $n = 3$ the absolute can be identified with $\mathbb{CP}^1$. So

$$f_{\infty,a}(z) = z - a$$

and for the ideal simplex with vertices at the points $\infty, 0, 1, a$ on the absolute we get

$$\mathrm{vol}(I(\infty,0,1,a)) = 3c_3 \cdot \int_{\mathbb{CP}^1} (\log|z| d\log|1-z| - \log|1-z| d\log|z|) \wedge d\log|z-a| =$$

$$3c_3 \cdot \int_{\mathbb{CP}^1} (\log|z| d\arg(1-z) - \log|1-z| d\arg(z)) \wedge d\arg(z-a)$$

because $d\log(z-1) \wedge d\log(z-a) = 0$. Here $\log f = \log|f| + i\arg(f)$. Using

$$d\mathcal{L}_2(z) = \log|z| d\arg(1-z) - \log|1-z| d\arg(z)$$

we rewrite the last integral as

$$3c_3 \cdot \int_{\mathbb{CP}^1} d\mathcal{L}_2(z) \wedge d\arg(z-a) \tag{84}$$

Computing the differential in the sense of distributions we get

$$d(\mathcal{L}_2(z) d\arg(z-a)) = 2\pi \cdot \mathcal{L}_2(z)\delta(z-a)dxdy + d\mathcal{L}_2(z) \wedge d\arg(z-a)$$

So the integral of the right hand side over $\mathbb{CP}^1$ is zero, i.e. the integral (84) is equal to $-6\pi c_3 \cdot \mathcal{L}_2(a)$. ($c_3 = -1/6\pi$).

3. Volumes of geodesic simplices in $SL_n(\mathbb{C})/SU(n)$. Recall the invariant differential $(2n-1)$-form ω_{D_n} in $\mathbb{H}_n$.

Question. Is it true that

$$\mathrm{vol}_{D_n}(I(x_0, ..., x_{2n-1})) = \text{constant} \times \psi_n(x_0, ..., x_{2n-1}) \tag{85}$$

One can show following the lines of s. 7.1 that the positive answer to this question is equivalent to the following statement: if x_0, x_1, x_2 are on the same geodesic then $\psi_n(x_0, ..., x_{2n-1}) = 0$.

4. Another approach to Grassmannian polylogarithms. The following construction was suggested to the author during the Fall of 1989, independently, by M. Kontsevich and by J. Nekovar. A hyperplane h in an n-dimensional complex vector space V determines an arrow in the space of degenerate non negative definite hermitian forms in V consisting of the forms with the kernel h. Let $h_1, ..., h_{2n}$ be hyperplanes in V. Let $C(h_1, ..., h_{2n})$ be projectivization of the convex hull of the arrays corresponding to these hyperplanes. It is a simplex in $\overline{\mathbb{H}}_n$. The idea is to integrate the form ω_{D_n} over this simplex. If $n = 2$ this construction provides an ideal geodesic simplex in the Caley realization of the hyperbolic space, given by the interior part of a ball in $\mathbb{R}P^3$. However the convergence of this integral for $n > 2$ has not been established yet, although it does not seem to be a very difficult problem. If the integral is convergent, we get a function on configurations of $2n$ hyperplanes in $\mathbb{C}P^{n-1}$. It would be very interesting to investigate this construction further and compare it with our construction of the Grassmannian polylogarithms.

5. A $(2n-1)$-cocycle of $GL(\mathbb{C})$. Consider the infinite dimensional vector space with a given basis $e_1, ..., e_m, ...$ The group $GL(\mathbb{C})$ is the group of automorphisms of this space moving only finite number of basis vectors.

Let us describe the restriction of the cocycle to the subgroup $GL_{n+m}(\mathbb{C})$ acting on the subspace generated by first $n+m$ vectors. Take $2n$ elements $g_1, ...g_{2n}$ of this group. Consider the corresponding $2n$ $m+1$-tuples of vectors:

$$g_1(e_n, ..., e_{n+m}), \quad ..., \quad g_{2n}(e_n, ..., e_{n+m})$$

The set of all $(m+1)$-tuples of vectors form a vector space. Let Δ_{2n-1} be the standard simplex $\sum_{j=1}^{2n} \lambda_j = 1$. Consider the set $C_n^m \subset \Delta_{2n-1}$ consisting of $(\lambda_1, ..., \lambda_{2n})$ that

$$\sum_{i=1}^{2n} \lambda_i \cdot g_i(e_n, ..., e_{n+m})$$

is an $(m+1)$-tuples of vectors *not* in generic position. It is a cycle of codimension n. The Chow polylogarithm function evaluated on it provides the desired (measurable) cocycle.

The cocycle property follows from the functional equation for the Chow polylogarithm and the following general fact. The set of those $(\lambda_1, ..., \lambda_{2n+1})$ that $\sum_i \lambda_i \cdot g_i(e_n, ..., e_{n+m})$ is an $(m+1)$-tuples of vectors *not* in generic position is also of codimension n.

The construction is consistent with the restriction just by definition.

Problem. Show that the cohomology class of this cocycle is nontrivial and proportional to the Borel class.

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