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# ON NON-KÄHLERIANITY OF NONUNIFORM LATTICES IN $SO(n,1)(n \geq 4)$

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**Abstract**

In this note, we will show that no nonuniform lattice of $SO(n,1)(n \geq 4)$ is the fundamental group of a quasi-compact Kähler manifold.

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# 1 Introduction and statement of the result

It is well known that a general finitely presented group is not necessarily the fundamental group of a compact Kähler manifold [15]. It is very interesting to know if an abstract group can or cannot be the fundamental group of a compact Kähler or quasi-compact Kähler manifold (by a quasi-compact Kähler manifold we mean a manifold obtained from a compact Kähler manifold by deleting a normal crossing divisor). In the paper [3], Carlson and Toledo showed that no cocompact lattice in $SO(n,1)(n \geq 3)$ is the fundamental group of a compact Kähler manifold. In this note, we will consider the nonuniform lattices' case in $SO(n,1)(n \geq 4)$. Especially, we will show the following:

**Theorem 1** *Let $\Gamma$ be a nonuniform lattice of $SO(n,1)(n \geq 4)$, i.e. $\Gamma \setminus SO(n,1)/SO(n)$ is noncompact and of finite volume with respect to the standard symmetric Riemannian metric. Let $\overline{M}$ be any compact Kähler manifold and let $D$ be any normal crossing divisor of $\overline{M}$. Denote $\overline{M} \setminus D$ by $M$. Then $\Gamma$, as an abstract group, is not isomorphic to $\pi_1(M)$.*

**Remark.** If $M$ is a quasi-projective variety, then by Hironaka's theorem [5], topologically, $M$ is just a smooth projective variety minus a normal crossing divisor. So, one has that a nonuniform lattice in $SO(n,1)(n \geq 4)$ cannot be the fundamental group of any quasi-projective variety.

The idea of the proof is to use infinite energy harmonic maps theory due to Jost and Zuo [8, 9]. So, this note can also be considered as an application of Jost-Zuo's theory on the existence of infinite energy harmonic maps. Assuming that $\Gamma$ is isomorphic to $\pi_1(M)$, by [8, 9], one gets a pluriharmonic map $u$ from $M$ (with an appropriate complete Kähler metric with finite volume and bounded curvature, see the next section for details) to $\Gamma \backslash SO(n,1)/SO(n)$ (with the standard symmetric Riemannian metric), which is possibly of infinite energy and induces an isomorphism from $\pi_1(M)$ to $\Gamma$. Then, a deep analysis [6, 7, 9, 11, 12] of this map shows that there exists a holomorphic foliation on $M$. So, one obtains that the map $u$ factors by a holomorphic map from $M$ to a Riemann surface $S$. This leads to a contradiction.

# 2 On Jost-Zuo's harmonic maps of infinte energy

In this section, we recall the existence and some basic facts on Jost-Zuo's infinite energy harmonic maps.

Let $\overline{M}$ be a compact Kähler manifold with a fixed Kähler metric $\omega_0$, $D$ be a fixed divisor with (at worst) normal crossing condition and $D = \bigcup_{i=1}^{p} D_i$. Here, $D_i$ are irreducible components of $D$. One may also assume that each irreducible component $D_i$ is free from self intersections. Thus, at each intersction point, precisely two components of $D$ meet. Denote $\overline{M} \setminus D$ by $M$.

Let $\sigma_i$ $(i = 1, 2, \cdots, p)$ be a defining section of $D_i$ in $\mathcal{O}(M, [D_i])$, which satisfies $|\sigma_i| \leq 1$ for a certain Hermitian metric of $[D_i]$ and defines a holomorphic coordinate system in each small disk transversal to $D_i$. So, one can get a fibration of a small neighborhood, say $|\sigma_i| \leq \delta \leq 1$, of $D_i$ by small holomophic disks which meet $D_i$ transversally. Similarly, for the boundary of such a small neighborhood, denoted by $\Sigma_i^\delta$, one also gets a fibration by circles. The intersection of two such boundaries is fibered by tori.

Corresponding to the above defining sections $\sigma_i$, one can define a complete Kähler metric on $M$ as follows,

$$g := -\frac{\sqrt{-1}}{2}\sum_{i=1}^{p} \partial\overline{\partial}(\phi(|\sigma_i|)\log|\log|\sigma_i|^2|) + c\omega_0|_M,$$

where $\phi$ is a suitable $C^\infty$ cut-off function on $[0,\infty)$, so that $\phi(s)$ is identical to one on $[0,\epsilon)$ and to zero on $[2\epsilon,\infty)$, for sufficiently small $\epsilon \geq 0$ and $c$ is taken sufficiently large, so that $g$ is positive definite. Then $g$ is a Kähler metric. One can show that $(M,g)$ is complete and has finite volume [2]. In fact, when restricted to small holomorphic disks transversal to $D$, this metric looks like the Poincaré metric on the punctured disk $(D^*, z)$

$$-\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log(-\log|z|^2) = \frac{\sqrt{-1}}{2}\frac{1}{|z|^2(\log|z|^2)^2}\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

In the sequel, unless stated otherwise, we always assume that $M$ is endowed with this complete metric $g$ with finite volume.

Let $N$ be a globally symmetric space of noncompact type, its isometry group denoted by $I(N)$. Given a reduced representation (for its definition, see [8, 9])

$$\rho : \pi_1(M) \to I(N),$$

one wants to get a $\rho$-equivariant harmonic map from the universal covering of $M$ to $N$. The difficulty arises since the representation $\rho$ may map some small loops around $D$ to some hyperbolic or quasi-hyperbolic elements (for their definitions, also see [9]) in $I(N)$. This is why a $\rho$-equivariant harmonic map, if it exists, may have infinite energy (here, we use the above metric $g$ to compute the energy).

Now, we can state Jost-Zuo's theorem on the existence of a $\rho$-equivariant pluriharmonic map, which may be of infinite energy, as follows:

**Theorem 2** *Let $M$, $N$, $I(N)$ and $\rho$ as above. Then there exists a $\rho$-equivariant pluriharmonic map $u$ from the universal covering $M'$ of $M$ with the above metric $g$ to $N$ with the standard symmetric metric.*

For simplicity of notation, we shall consider $u$ in the sequel as a map from $M$ to $Y := N/\rho(\pi_1(M))$, although $Y$, in general, may be singular (but in which we want to apply it, $Y$ is smooth ). Let $w \in D_i$ be a regular point of $D$. Near $w$, one can choose a coordinate system $(z^1, z^2)$ on $\overline{M}$ such that $z^1$ parametrizes small holomorphic discs, which meet $D_i$ transversally near $w$, $z^2$ parametrizes $D_i$ (of course, $z^2$ will have more than one component if the complex dimension of $M$ is greater than 2. In the following, the index 2 will stand for all those $z^2$-directions together), and $z^1 = 0$ on a small neighborhood of $w$ in $D_i$ and $z^2(w) = 0$. Then, one has some derivative estimates of $u$ (see p.481 of [9]):

$$|\frac{\partial u}{\partial z^1}(z^1, z^2)|_g \leq \frac{c}{|z^1|}, \quad |\frac{\partial u}{\partial z^2}(z^1, z^2)|_g \leq c,$$

where $c$ is some positive constant. If $w$ is a singular point of $D$, i.e.,a point at which two irreducible components of $D$ meet, similar estimates can be obtained. One may use $\sigma := \prod_{i=1}^{p} \sigma_i$ to replace the above coordinate component $z^1$. Then, one can get that in the $\sigma$-direction, the derivative of $u$ behaves like $\frac{1}{|\sigma|}$, whereas in directions normal to $\sigma$, it is bounded.

# 3 Some properties of the infinite energy harmonic map

In this section, we shall show that the rank of the harmonic map $u$ in the previous section has a serious restriction if $N = SO(n,1)/SO(n)$. In the following, our exposition is slightly general, which is not restricted to the case of $SO(n,1)$ only.

Let $M$ be as in the previous section with the constructed Kähler metric $g$, the corresponding Kähler form of which is denoted by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=1}^{m} g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$$

where $m = \dim_C M$ and $(z^1, z^2, \cdots, z^m)$ is a local coordinate system of $M$. Let $Y = \rho(\pi_1(M)) \setminus G/K$ a locally symmetric space of noncompact type (it may be singular, but in the case where we want to apply the result, it is smooth). Here, $G$ is a semisimple Lie group, $K$ is a maximal compact subgroup of $G$ (for standard references see [4]). Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{k}$ the Lie algebra of $K$, then one has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Denote by $\mathfrak{p}^C$ the complexification of $\mathfrak{p}$. One can then identify the complexification of the tangent space at any point of $Y$ with $\mathfrak{p}^C$. This is unique up to the right action of $K$ and the left action of $\rho(\pi_1(M))$. Since these actions preserve all relevant structures, we may regard $df(T_x^{1,0}M)$ as a subspace of $\mathfrak{p}^C$, for any map $f : M \to Y$ and any point $x \in M$. Introduce a local coordinate system $(u^1, u^2, \cdots, u^n)$ on $Y$.

As in [13], we introduce a symmetric $(2,0)$-tensor $\phi$ related to the map $u$

$$\phi(X, Y) =< \partial u(X), \partial u(Y) >, \quad X, Y \in T_x^{1,0}M,$$

which can be locally written as $\sum_{\alpha,\beta=1}^{m} \phi_{\alpha\beta} \mathrm{d}z^\alpha \otimes \mathrm{d}z^\beta$. Now, we compute its iterated divergence. By the divergence formula, one has a $(1,0)$-form $\xi$

$$\xi_\alpha = g^{\beta\overline{\gamma}} \phi_{\alpha\beta,\overline{\gamma}}$$

where $(g^{\beta\overline{\gamma}})$ represents the inverse of $(g_{\beta\overline{\gamma}})$ and "," denotes the covariant derivative. Then, taking the divergence of $\xi$ again, one obtains by a direct computation [13]

$$\delta\xi = (|D''\partial u|^2 - g^{\alpha\overline{\beta}} g^{\gamma\overline{\delta}} R_{iklm} u_\alpha^i u_\gamma^k u_{\overline{\beta}}^l u_{\overline{\delta}}^m)$$

where $\delta$ is the codifferential, $R_{iklm}$ is the curvature tensor of $Y$, and $D''\partial u$ is the (0, 1)-type covariant derivative of $\partial u$, which is locally written as

$$(D''\partial u)^i_{\alpha\overline{\beta}} = u^i_{\alpha\overline{\beta}} + \Gamma^i_{jk} u^j_\alpha u^k_{\overline{\beta}}.$$

Here $\Gamma^i_{jk}$ are the Christoffel symbols of $Y$. Then, Jost-Zuo's argument (see p.482 of [9]) shows that the two sides of the above formula are zero pointwise. It should be pointed out that the estimates of the derivatives of $u$ near the divisor $D$ given in the previous section are essential in this reasoning. In particular, using the curvature conditions of $Y$, one obtains that

$$D''\partial u = 0, \quad g^{\alpha\overline{\beta}} g^{\gamma\overline{\delta}} R_{iklm} u^i_\alpha u^k_\gamma u^l_{\overline{\beta}} u^m_{\overline{\delta}} = 0.$$

Note that the above first formula just represents the pluriharmonicity of $u$. Taking the orthogonal frame $e_1, e_2, \cdots, e_m, e_{\overline{1}}, e_{\overline{2}}, \cdots, e_{\overline{m}}$ on $M$, one has

$$\begin{aligned} g^{\alpha\overline{\beta}} g^{\gamma\overline{\delta}} R_{iklm} u^i_\alpha u^k_\gamma u^l_{\overline{\beta}} u^m_{\overline{\delta}} &= < R(\partial u(e_\alpha), \partial u(e_\beta))\overline{\partial} u(e_{\overline{\alpha}}), \overline{\partial} u(e_{\overline{\beta}}) > \\ &= - < [\partial u(e_\alpha), \partial u(e_\beta)], [\overline{\partial} u(e_{\overline{\alpha}}), \overline{\partial} u(e_{\overline{\beta}})] > \end{aligned}$$

So, $[\partial u(e_\alpha), \partial u(e_\beta)] = 0$ for all $\alpha$ and $\beta$. Thus, one has that if one identifies $\partial u(T^{1,0}_x M)$ with a subspace of $\mathfrak{p}^C$, then $\partial u(T^{1,0}_x M)$ is an abelian subspace of $\mathfrak{p}^C$. Therefore, $\dim_C \partial u(T^{1,0}_x M)$ should not be greater than the rank of $Y$. If one applies this assertion to the present case, i.e., $G = SO(n, 1)$, one obtains

**Lemma 1** *Let $u : M \to Y$ be the pluriharmonic map as in the previous section. Assume $Y = \rho(\pi_1(M)) \setminus SO(n,1)/SO(n)$. Then $u$ has real rank at most 2.*

## 4 The proof of Theorem 1

In this section, we will give the proof of Theorem 1. Let $\Gamma$ be a nonuniform lattice of $SO(n,1) (n \geq 3)$, *i.e.*, $\Gamma \setminus SO(n,1)/SO(n)$ is noncompact and of finite volume with respect to the standard symmetric Riemannian metric. Let $\overline{M}$ be any compact Kähler manifold, $D$ be any normal crossing divisor of $\overline{M}$. Denote $\overline{M} \setminus D$ by $M$. *Assume that as abstract groups, $\pi_1(M)$ is isomorphic to* $\Gamma$. We will derive a contradiction. Therefore, the proof of Theorem 1 is completed.

By Jost-Zuo's theorem, we get a pluriharmonic map $u : (M, g) \to \Gamma \setminus SO(n,1)/SO(n) = \Gamma \setminus \mathbf{H}^n_R$, which, by the above assumption, induces an isomorphism from $\pi_1(M)$ to $\Gamma$. The lemma in the previous section tells us that this map is of real rank at most 2. So, one has two cases: 1) the real codimension of the fibres of $u$ is generically equal to 1; 2) the real codimension of the fibres of $u$ is generically equal to 2. In the first case, one gets a map from $M$ to $S^1$, which gives an isomorphism between the fundamental groups of $M$ and $S^1$. However, this is impossible, since a lattice of $SO(n,1)$ can never be $\mathbf{Z}$. So, the remaining is to show that another case is also impossible. We will adopt the argument of Jost-Yau [6, 7]. To this end, we embed $SO(n,1)/SO(n)$ isometrically into some complex hyperbolic space $\mathbf{H}^n_C$. From now on, we assume that $u$ is a pluriharmonic map into some complex hyperbolic space $\mathbf{H}^n_C$ and its real rank is 2 generically. Introduce local complex coordinates $(z^1, z^2, \cdots, z^m)(m = \dim_C M)$ on $M$ and $(u^1, u^2, \cdots, u^n)$ on $\Gamma \setminus \mathbf{H}^n_C$ and denote the Christoffel symbols of $\Gamma \setminus \mathbf{H}^n_C$ by $\Gamma^\alpha_{\beta\gamma}$, $\alpha, \beta, \gamma = 1, 2, \cdots, n$. Similar to the previous section, as a consequence of Siu's Bochner type identity [14] and the argument of [9] (using the strong negativity of the curvature tensor of $\mathbf{H}^n_C$), we obtain

$$D_{\bar{j}}\partial_i u^\alpha = u^\alpha_{z^i z^{\bar{j}}} + \Gamma^\alpha_{\beta\gamma} u^\beta_{z^i} u^\gamma_{z^{\bar{j}}} = 0, \qquad \text{for all} \qquad \alpha, i, j$$

and

$$u^\alpha_{z^{\bar{i}}} u^{\overline{\beta}}_{z^{\bar{j}}} - u^\alpha_{z^{\bar{j}}} u^{\overline{\beta}}_{z^{\bar{i}}} = 0, \qquad \text{for all} \qquad \alpha, \beta, i, j.$$

Then, the argument of Jost-Yau [6, 7] shows that there exists a holomorphic foliation $\mathcal{F}$ on a Zariski open subset of $M$, on the leaves of which $u$ is constant. Arguments of Mok (see Proposition (2.2.1) of [11]) imply that $\mathcal{F}$ can be extended as a holomorphic foliation to $M \setminus V$ for some complex analytic variety $V$ of complex codimension at least 2. Then, the study of [12] (see Proposition (2.2) of [12]) shows that this extended foliation actually defines an open analytic equivalence relation, still denoted by $\mathcal{F}$, on $M$, and the quotient of $M$ by $\mathcal{F}$, denoted by $S$, is an irreducible complex space of complex dimension 1, by a result of Kaup [10]. Therefore, one obtains a factorisaton of $u$ as follows:

$$\begin{array}{ccc} M & \xrightarrow{u} & \Gamma \setminus \mathbf{H}^n_R (\subset \Gamma \setminus \mathbf{H}^n_C) \\ \downarrow \pi & & \updownarrow \mathbf{1}_{\mathbf{H}^n_R} \\ S & \xrightarrow{h} & \Gamma \setminus \mathbf{H}^n_R (\subset \Gamma \setminus \mathbf{H}^n_C) \end{array}$$

where $\pi$ is holomorphic by the construction of $S$ and $h$ is harmonic, since $u$ is pluriharmonic.

By the above assumption, $u_* : \pi_1(M) \to \Gamma$ is an isomorphism, so $\pi_* : \pi_1(M) \to \pi_1(S)$ is injective. Therefore, $\Gamma$, as a subgroup of $\pi_1(S)$ acts freely on the universal covering of $S$, which is either complex plane or unit disk. Thus, the cohomological dimension of $\Gamma$ is at most 2 (see [1]). However, the cohomological dimension of $\Gamma$ is in fact $n-1$, which is at least 3 by the assumption. Thus, we get a contradiction. The proof is completed.

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