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V. Gomes Lima, V. Silva Santos and R. de Lima Rodrigues

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ON THE SCALAR POTENTIAL MODELS FROM THE ISOSPECTRAL POTENTIAL CLASS

V. Gomes Lima[(a)], *V. Silva Santos*[(b)] *and R. de Lima Rodrigues*[(c)]*

[(a,c)] *Centro Brasileiro de Pesquisas Físicas (CBPF)*

Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro-RJ, Brazil

[(a)] *Departamento de Física, Universidade Federal Rural do Rio de Janeiro*

Antiga Rodovia Rio-São Paulo Km 47, BR 465, CEP 23.890-000, Seropédica-RJ - Brazil

[(b)] *Departamento de Engenharia Civil, Universidade Federal da Paraíba*

CEP 58.109-970, Campina Grande - PB - Brazil

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Abstract

The static field classical configuration in (1+1)-dimensions for new non-linear potential models is investigated from an isospectral potential class and the concept of bosonic zero mode solution. One of the models considered here has a static nontopological configuration with a single vacuum state, whose potential in the stability equation corresponds to broken a supersymmetry.

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*Permanent address: Departamento de Ciências Exatas e da Natureza, Universidade Federal da Paraíba, Cajazeiras – PB, 58.900-000 - Brazil, e-mail: rafaelr@cbpf.br ou rafael@cfp.ufpb.br

I. INTRODUCTION

Rencently, static classical configurations that are exact solution of the equation of motion in scalar potential model have drawn much attention mainly because they have found interesting applications as a solitary wave [1]. There are several methods to construct soliton (kink) solutions associated with some nonlinear differential equations [2]. The kink of a scalar field theory is a static, non-singular, *classically stable* and finite localized energy solution of the equation of motion [3]. When $\phi(x,t) = f(x - vt)$, a kink solution is called solitary wave. The classical solutions in quantum field theories have been recently overviewed [4]. There one can see how a quantum field theory has topological and nontopological soliton solutions in higher spatial dimensions. In literature, there is a surprisingly number of scalar potential model in higher spartial dimensions with exactly solvable equation of motion. Also, a variety of classical finite-energy static solutions are known.

The connection between supersymmetry in quantum mechanics (SUSY QM) [5–8] and the topological and nontopological solitons in terms of a scalar potential for the case with only one single field has been approached in literature [9–13]. Recently, the reconstructing 2-dimensional scalar field potential models have been considered starting from the Morse and the Scarf II hyperbolic potential and quantum corrections to the solitonic sectors of both potentials are point out [14].

In the present work, from a nonpolynomial potential in supersymmetric quantum mechanics, we will construct new potencials as a function of one single real scalar field. We consider the interesting program to construct new potential models in 1+1 dimensions, which the essential point is associated with the translational invariance of the static field configuration [9]. There the aim is to construct field potential models via SUSY QM considering an one-dimensional quantum mechanical isospectral potential class such that corresponding fluctuation operator Schrödinger-like eigenvalue problem as a stability equation exactly sovable associated to the ϕ^4 model. However, the corresponding field potential could not be put in a closed form. Indeed, we show that in such a procedure there appears a new nonpolynomial field potential model with infinite-energy static configuration, so that such a new potential may not be considered as a well defined theoretical field potential model.

Actually, we will bring two constructions of classical solutions. In the first case we obtain the well-known topological kink and in the second case we obtain a nontopological static configuration which is not a kink. In the second case the new static field configuration is not a smooth function of position coordinate because it has a divergence for some values of the spatial coordinate.

We also construct some pictures of the field potential models (figures I and III) and the static classical field configurations (figures II and IV).

II. STATIC CLASSICAL CONFIGURATION

Consider the Lagrangian density for a single scalar field $\phi(x,t)$ in (1+1)-dimensions, in natural system ($c = 1 = \hbar$), given by

$$\mathcal{L}\left(\phi, \partial_\mu \phi\right) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V\left(\phi\right), \tag{1}$$

where $V(\phi)$ is any positive semidefinite function of ϕ, which must have at least two zeros for kinks to exist. It represents a well-behaved potential energy. However, we have found a new potential which is solveble exactly in the context of classical theory in (1+1)-D which is not a kink.

The field equation for a static classical configuration $\phi = \phi_c(x)$ becomes

$$-\frac{d^2}{dx^2}\phi_c(x) + \frac{d}{d\phi_c}V\left(\phi_c\right) = 0, \qquad \dot{\phi}_c = 0, \tag{2}$$

with the following boundary condition: $\phi_c(x) \rightarrow \phi_{vacuum}(x)$ as $x \rightarrow \pm\infty$. Since the potential $V(\phi)$ is positive it can be written as

$$V(\phi) = \frac{1}{2}U^2(\phi), \tag{3}$$

giving the well-known Bogomol'nyi condition,

$$\frac{d\phi}{dx} = \pm U(\phi) \tag{4}$$

where the solutions with the plus and minus signs represent two configurations.

III. STABILITY EQUATION AND NEW POTENTIALS

The classical stability of the static solution is investigated by considering small perturbations around it,

$$\phi(x,t) = \phi_c(x) + \eta(x,t), \tag{5}$$

where we expand the fluctuations in terms of the normal modes,

$$\eta(x,t) = \sum_n \epsilon_n \eta_n(x) e^{i\omega_n t}; \tag{6}$$

the ϵ_n's are so chosen that $\eta_n(x)$ are real. A localised classical configuration is said to be dynamically stable if the fluctuation term does not destroy it. The equation of motion becomes a Schrödinger-like equation, viz.,

$$F\eta_n(x) = {\omega_n}^2 \eta_n(x), \quad F = -\frac{d^2}{dx^2} + V''(\phi_c). \tag{7}$$

According to Eq. (3) one then obtains the supersymmetric form [13],

$$V''(\phi_c) = U'^2(\phi_c) + U(\phi_c)U''(\phi_c), \tag{8}$$

where the two primes mean a second derivative with respect to the argument.

If the normal modes of (7) satisfy ${\omega_n}^2 \geq 0$, the stability of the Schrödinger-like equation is ensured. Now, we are able to implement a method that provides a new potential from the potential term that appears in the fluctuation operator.

Next, we consider the following generalized isospectral potential as being the potential term (8) to the fluctuation operator:

$$V(x;\alpha,\beta) = m^2\left[3tanh^2\left(\frac{m}{\sqrt{2}}x\right)-1\right]$$
$$+ 2m^2\beta\left[sech^4\left(\frac{m}{\sqrt{2}}x\right)\left(2tanh\left(\frac{m}{\sqrt{2}}x\right)+\frac{\beta}{2}sech^4\left(\frac{m}{\sqrt{2}}x\right)\chi\right)\chi\right],$$
$$\chi = \chi(x;\alpha,\beta) = \left\{\alpha+\beta\left[tanh\left(\frac{m}{\sqrt{2}}x\right)-\frac{1}{3}tanh^3\left(\frac{m}{\sqrt{2}}x\right)\right]\right\}^{-1}, \tag{9}$$

where α and β are constant parameters. This nonpolynomial potential satisfies the condition ${\omega_n}^2 \geq 0$ and the ground state associated to the zero mode ($\omega_0^2 = 0$) is given by

$$\eta^{(0)}(x;\alpha,\beta) = N\chi(x)sech^2\left(\frac{m}{\sqrt{2}}x\right), \tag{10}$$

where N is the normalization constant. Note that if $|\alpha| > \frac{2\beta}{3}$, the eigenfunction of the ground state is non-singular. This ground state was independently constructed by two authors [9,10] as $\beta = \frac{1}{2}\sqrt{\frac{3m}{\sqrt{2}}}$.

It is well-known that the bosonic zero mode eigenfunction of the stability equation is related with the kink by

$$\eta^{(0)}(x;\alpha,\beta) \propto \frac{d}{dx}\phi_c(x), \tag{11}$$

so that at priori we can find the static classical configuration by a first integration. Thus, from the potential

$$V(\phi;\alpha,\beta) = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2, \tag{12}$$

we can obtain a class of scalar potential $V(\phi) = V(\phi;\alpha,\beta)$ which has exact solutions.

There, we can find various static field configurations, however, let us consider here only two cases.

Caso (i): $\beta = 0$ and $\alpha = 2N\sqrt{\frac{\sqrt{2}}{3m}}$

In this case, $\chi(x) \to \frac{1}{\alpha}$, so that from (9) and (10) we obtain the following bosonic zero mode solution

$$\eta^{(0)}(x) = \frac{1}{2}\sqrt{\frac{3m}{\sqrt{2}}}sech^2\left(\frac{m}{\sqrt{2}}x\right) \tag{13}$$

and the nonpolynomial potential becomes

$$V_{1-}(x) = m^2 \left[3tanh^2\left(\frac{m}{\sqrt{2}}x\right) - 1\right]. \tag{14}$$

Notice that the two Schrödinger-like fluctuation operators associated with both $V(x;\alpha,\beta)$ and $V(x)$ nonpolynomial potentials are positive semi-definite and completly isospectrals. However, their factorization has been implemented from distinct superpotencials. Indeed, while the Riccati equation

$$V_{1-}(x) = W_1^2(x) + W_1'(x) \tag{15}$$

has a particular solution given by

$$W_1(x) = -\sqrt{2}mtgh\left(\frac{m}{\sqrt{2}}x\right), \tag{16}$$

which provides

$$\eta_{1-}^{(0)}(x) = e^{\int W_1(y)dy} = \frac{1}{2}\sqrt{\frac{3m}{\sqrt{2}}}sech^2\left(\frac{m}{\sqrt{2}}x\right) = \eta^{(0)}(x), \tag{17}$$

where $\eta^{(0)}(x)$ is in according to Eq. (13), the Riccati equation

$$V_{0-} = V(x;\alpha,\beta) = W_0^2(x) + W_0'(x)$$

has a particular solution given by $W_0(x) = \frac{d}{dx}\ell n\left(\eta_0(x;\alpha,\beta)\right).$

By substituing Eq. (13) in Eq. (11), we get the well-known kink of the double-well potential

$$\phi_k(x) = \frac{m}{\sqrt{\lambda}}tanh\left(\frac{m}{\sqrt{2}}x\right). \tag{18}$$

When we put the value for the position coordinate in terms of the kink, i.e. $x = x(\phi_k)$, we find the ϕ^4-potential model with spontaneously broken symmetry in scalar field theory, viz.,

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2. \tag{19}$$

The mass kink is finite, but in the next case we obtain an infinite mass. Such a classical configuration can not represents a stable particle. The pictures of the potential (19) and of the kink (18) are in figures I and II. Note that both are smooth functions of field and spatial coordinate, respectivelly.

Case (ii): $\alpha = 0$.

In this case, from (9), (10) and (11) we obtain the following nonpolynomial potential within the singularity region:

$$\tilde{V}(\phi) = \frac{\lambda}{2}\left(1 + 3e^{-\gamma\phi}\right)\left(1 - \frac{2}{3}e^{\gamma\phi}\right)^2, \tag{20}$$

where $\gamma = \frac{2}{\sqrt{3}}$ and λ is adimensionless constant.

The vacuum state, ϕ_v, is given by

$$\phi_v = \frac{\sqrt{3}}{2}\ell n(\frac{3}{2}). \tag{21}$$

This nonpolynomial potential does not have a discrete symmetry as $\phi \to -\phi$ and there exist only one vacuum state so that it is nontopological.

From (3), (4) and (20), for the minus sign of the Bogomol'nyi condition and the coupling constant $\lambda \neq 0$, the static classical configuration has the following explicit form

$$\phi_c(x) = \frac{\sqrt{3}}{2}\ell n\left(\frac{tanh^2\left(\sqrt{\lambda}x\right)}{1 - \frac{1}{3}tanh^2\left(\sqrt{\lambda}x\right)}\right), \tag{22}$$

where the integration constant is taken to be zero. Note that this static configuration satisfies Eq. (11) and the following boundary conditions $\phi_c(x) \to \phi_v$ as $x \to \pm\infty$. The pictures of the nonpolynomial potential and the static solution are in figures II and IV. However, in figure IV we have plotted this static field configuration only for the region in that it is well behaved without no singularity.

The energy density for the static solution for the nonpolynomial potential is given by

$$E(x) \propto \left\{sinh^2\left(\sqrt{2}mx\right)\left(1 - \frac{1}{3}tanh^2\left(\frac{m}{\sqrt{2}}x\right)\right)^2\right\}^{-1} \tag{23}$$

which leads us an undefined total energy or classical mass. Here, we have used the explicit relations between the static configuration and one-dimensional spatial coordinate.

IV. BROKEN SUSY QM

In this section, we consider the fluctuation operator in context of the supersymmetry in quantum mechanics (SUSY QM), whose supersymmetric partners are constructed from the stability equation. In $N = 2-$SUSY, we define the following first order differential operators

$$A_2^{\pm} = \pm\frac{d}{dx} + W_2(x), \qquad A_2^{+} = \left(A_2^{-}\right)^{\dagger}. \tag{24}$$

The bosonic sector fluctuation operator is given by

$$F_{2-} \equiv A_2^{+}A_2^{-} = -\frac{d^2}{dx^2} + V_{2-}(x), \quad V_{2-}(x) = \tilde{V}''_{|\phi=\phi_c}, \tag{25}$$

so that in terms of the superpotential we obtain the following nonlinear first order diferential equation

$$V_{2-}(x) = W_2^2(x) + W_2'(x) \equiv V(x;0,\beta) \tag{26}$$

where the prime means a derivative with respect to x.

The superpotential that solved this Riccati equation for the nonpolynomial potential has the following explicit form

$$W_2(x) = -\frac{m}{\sqrt{2}}\frac{tanh^4\left(\frac{m}{\sqrt{2}}x\right) + 3}{tanh\left(\frac{m}{\sqrt{2}}x\right)\left[3 - tanh^2\left(\frac{m}{\sqrt{2}}x\right)\right]}. \tag{27}$$

Note that this particular solution to the Riccati differential equation has the following asymptotic behaviour: $W_2(x) \to -\sqrt{2}m$ as $x \to \infty$ and $W_2(x) \to \sqrt{2}m$ as $x \to -\infty$. The supersymmetric partner of F_{2-} is given by

$$F_{2+} = A_2^{-}A_2^{+} = -\frac{d^2}{dx^2} + V_{+}(x)$$
$$V_{2+}(x) = W_2^2(x) - W_2'(x) = m^2\left\{1 + tanh^2\left(\frac{m}{\sqrt{2}}x\right)\right\}. \tag{28}$$

These fluctuation operators are isospectral and consist of a pair Schrödinger-like Hamiltonians of Witten's model of broken SUSY [5]. Note that the shape invariance condition

[6,8] is not satisfied for $V_\pm$ given by Eqs. (26) and (28), i.e. $V_{2+}(x;a_2) \neq V_{2-}(x;a_1) + R$, where a_1, a_2 and R are constants.

The eigenvalue equations for the supersymmetric partners $F_{2\pm}$ are given by

$$F_{2\pm}\eta_{2\pm}^{(n)}(x) = \omega_{2\pm}^{(n)}\eta_{2\pm}^{(n)}(x), \quad \omega_{2-}^{(n)} = \omega_n^2, \quad \omega_0^2 = 0 \tag{29}$$

which in general F_{2-} can have as eigenstates the well-known normal modes. However, when $\alpha = 0$ the bosonic zero mode ($\omega_0^2 = 0$) satisfies the annihilation condition

$$A_2^- \eta_{2-}^{(0)} = 0 \Rightarrow \eta_{2-}^{(0)}(x) \propto \frac{1}{sinh\left(\sqrt{2}mx\right)\left(2 + sech^2\left(\frac{m}{\sqrt{2}}x\right)\right)} = \eta^{(0)}(x;0,\beta). \tag{30}$$

This eigensolution is not normalizable, so that the bosonic sector fluctuation operator does not have the zero mode. In this case, the integral $\int_{-\infty}^{+\infty}\left(\eta_{2-}^{(0)}(x)\right)^2 dx$ is undefined. This result is in according to Eq. (10), for $\alpha = 0$. Furthermore, bosonic zero mode satisfies $\eta_{2-}^{(0)}(x) = \frac{d}{dx}\phi_c(x)$. The fermionic sector fluctuation operator F_{2+} does not also have the zero mode because $\eta_{2+}^{(0)}$,

$$A_2^+ \eta_{2+}^{(0)} = 0 \Rightarrow \eta_{2+}^{(0)} \propto sinh\left(\sqrt{2}mx\right)\left[2 + sech^2\left(\frac{m}{\sqrt{2}}x\right)\right] \tag{31}$$

because it is not normalizable. In this case we have broken SUSY. Indeed, it is easy to see that

$$\int_{-\infty}^{+\infty}\left(\eta_{2+}^{(0)}(x)\right)^2 dx \to \infty.$$

The eigenvalues $\omega_{2\pm}$ and eigenfunctions $\eta_{2\pm}^{(n)}$ can be exactly solved in a similar way for a general potential. All eigenvalues of F_{2+} are eigenvalues of F_{2-}, i.e. $\omega_{2-}^{(n)} = \omega_{2+}^{(n)} > 0$, so that the ground state and the excited state of both $F_{2\pm}$ have energy different from zero.

V. CONCLUSION

In this work, we investigate the classical stability of a new isospectral nonpolynomial potential model with static classical configuration which solve exactly the equation of motion.

Indeed, the classical finite-energy static solutions appear in field theoretical models with spontaneously broken symmetry (SUSY), for example, in the double-well potential given by Eq. (19). It is well known that double-well potential model which has two zeros corresponding to the vacuum states ϕ_1 and ϕ_2. In this case the topological kink interpolate smoothly and monotonically between ϕ_1 and ϕ_2, according to figures I and II. But in our nonpolynomial scalar potential one build a static, infinite-energy and classically singular field configuration, which is in a nontopological sector. Indeed, we have found a new field potential model given by Eq. (20) that is solved exactly in the context of classical theory in (1+1)-dimensions.

In conclusion, we found V_{1-} and $V_{1+} = V_{2+}$ is a supersymmetric potential pair with unbroken SUSY, so that F_{1-} refers to the bosonic sector of the SUSY fluctuation operator F_{SUSY}, while F_{1+} is the fermionic sector of F_{SUSY}. In this case, $\eta_{1-}^{(0)}$ then becames the unique normalizable eigenfunction of the F_{SUSY} corresponding to the zero mode of the ground state. On the other hand, the spectra to $F_{2\pm}$ are identical and in this case, there are no zero mode for the ground state i.e. $\omega_- = \omega_+ > 0$, thus one has broken SUSY.

Therefore, our nonpolynomial potential does not have the reflection symmetry $\phi \rightarrow -\phi$, with a stability equation so that it does not lead to either a bosonic zero mode or its supersymmetric partner because both eigenfunctions are non-normalizable with SUSY broken. Thus the above scheme to construct new field potential models is not always physically acceptable, because it may lead to infinit energy configuration.

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FIGURES

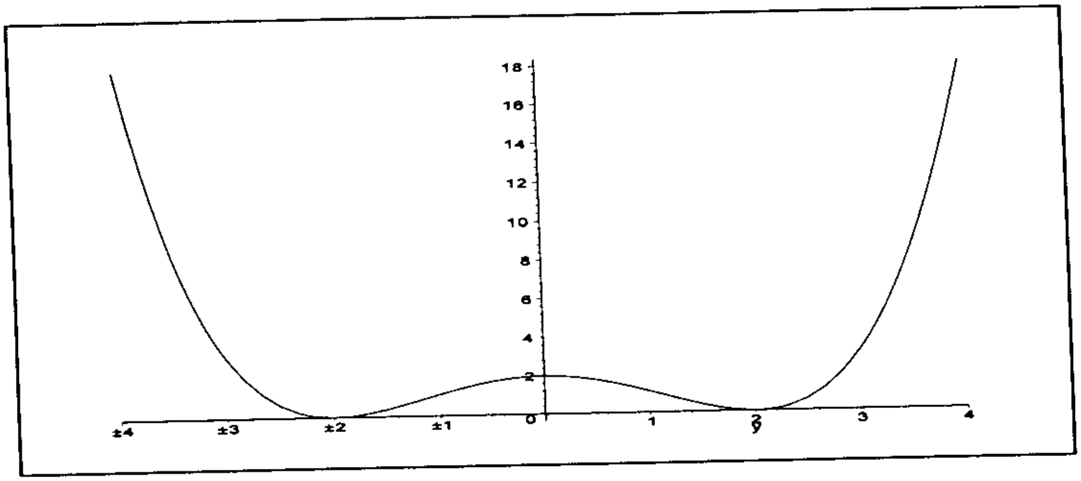

FIG. 1. Double-well potential $V(y) = \frac{1}{8}\left(y^2 - 4\right)^2$, for $\lambda = \frac{1}{4}, \quad m = 1.$

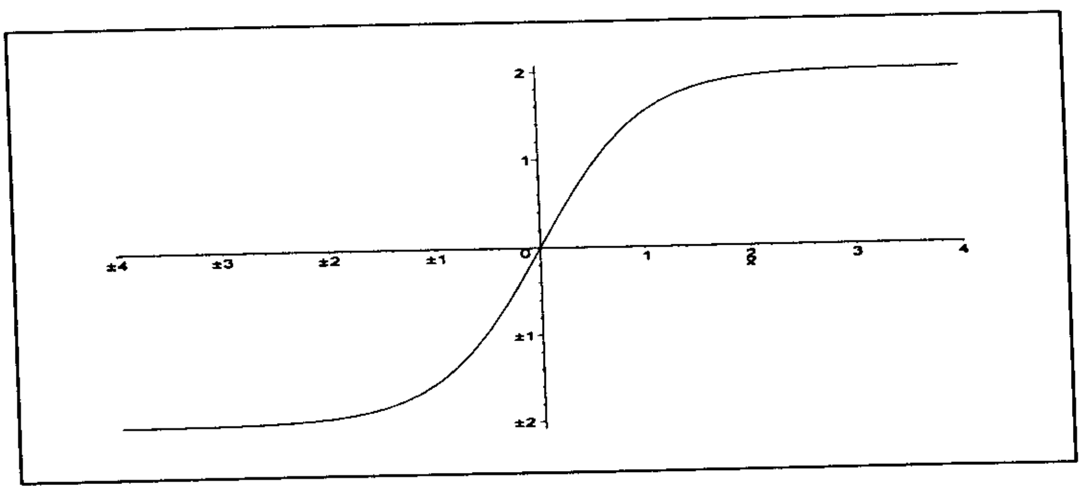

FIG. 2. The Kink $y = \phi(x)$ of the double-well potential, for $\lambda = \frac{1}{4}, \quad m = 1.$

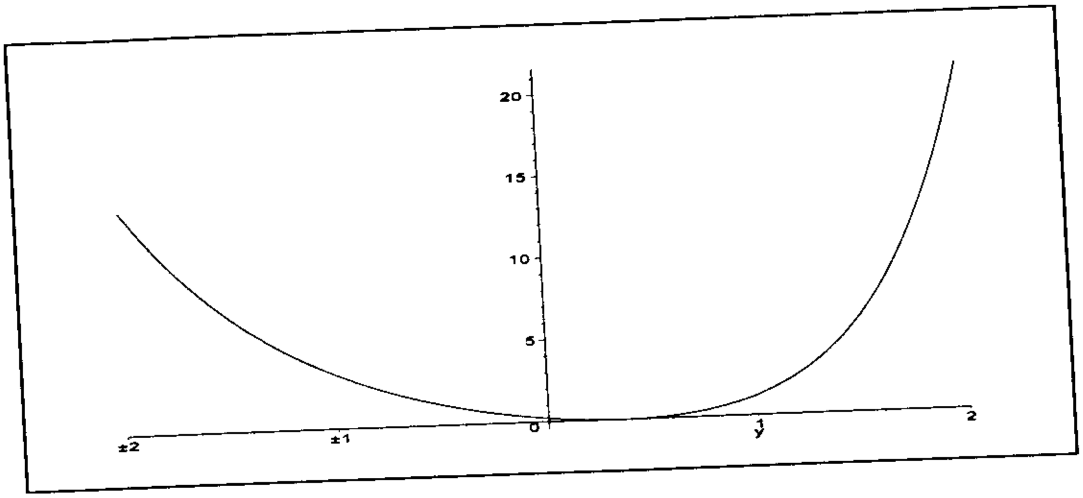

FIG. 3. Nonpolynomial potential, for $\lambda = 1, \quad m = 1$.

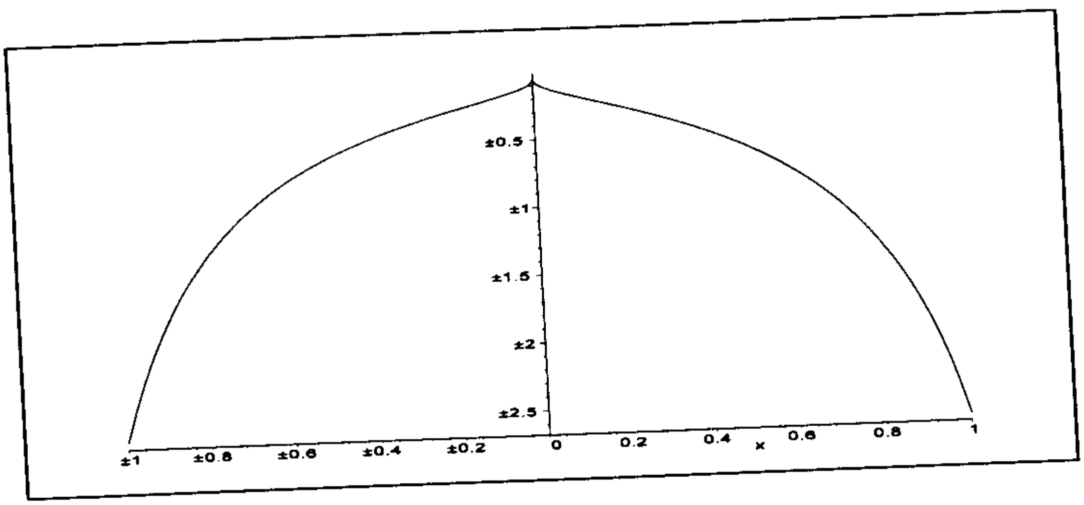

FIG. 4. Static classical configuration associated to the nonpolynomial potential in a well behaved region, for $\lambda = 1, \quad m = 1$.

REFERENCES

[1] R. Rajaraman, *Solitons and Instantons*, (North-Holland, Amsterdam, 1982), p. 20-21; S. Coleman, *Aspects of Symmetry*, (Cambridge University, London, 1985), p. 190-191.

[2] P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, (Cambridge University Press, Cambridge, 1993.)

[3] A. P. Balachandran, G. Marmo, B. S. Skagerstam and A. Stern, *Classical Topology and Quantum States*, (World Scientific, Singapore, 1991), p. 104-106.

[4] E. J. Weinberg, *Annu. Rev. Nucl. Part. Sci.* **42**, 177 (1992).

[5] E. Witten, *Nucl. Phys.* **B185**, 513 (1981); C. V. Sukumar *J. Phys. A: Math. Gen.* **18**, 2917 (1985);D. L. Pursey, *Phys. Rev.* **D33**, 1047 (1986); D. L. Pursey, *Phys. Rev.* **D33**, 2267 (1986).

[6] L. É. Gendenshtein, *JETP Lett.* **38**, 356 (1983); L. É. Gendenshtein and I. V. Krive, *Sov. Phys. Usp.* **28**, 645 (1985); R. Dutt, A. Khare and U. P. Sukhatme, *Am. J. Phys.* **56**, 163 (1988); A. Lahiri, P. K. Roy and B. Bagchi, *Int. J. Mod. Phys.* **A5**, 1383, (1990).

[7] M. N. Nieto, *Phys. Lett.* **B145** 208 (1984); W.-Y. Keung, U. P. Sukhatme, Q. Wang and T. D. Imbo, *J. Phys. A: Math. Gen.* **22**, L987 (1989). *Am. J. Phys.* **56**, 163 (1988).

[8] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* **251**, 267-285, (1995).

[9] C. N. Kumar, *J. Phys. A: Math. Gen.* **20**, 5397 (1987).

[10] R. de Lima Rodrigues *Mod. Phys. Lett.* **A10**, 1309 (1995).

[11] Q. Wang, U. P. Sukhatme and W.-Y. Keung, *Mod. Phys. Lett.* **A5**, 525 (1990); D. S. Kulshreshtha, J.-Q. Liang and H. J. W. Müller-Kirsten, *Ann. Phys. (N. Y.)* **225**, 191 (1993).

[12] J. Casahorran and S. Nam, *Int. J. Mod. Phys.* **A6**, 5467 (1991); C. V. Sukumar, *J. Phys. A: Math. Gen.* **19**, 2297 (1986); J. Hruby, *J. Phys. A: Math. Gen.* **22**, 1807 (1989); L. J. Boya. and J. Casahorran, *Ann. Phys. (N. Y.)* **196**, 361 (1989).

[13] W.-Y. Keung, U. P. Sukhatme, Q. Wang and T. D. Imbo, *J. Phys. A: Math. Gen.* **22**, L987 (1989); G. Junker and P. Roy, *Ann. of Phys.* **256**, 302 (1997); P. Barbosa da Silva Filho, R. de Lima Rodrigues, A. N. Vaidya, *J. Phys. A: Math. Gen.*, **32**, 2395 (1999).

[14] G. H. Flores and N. F. Svaiter, preprint Notas de Física, CBPF-NF-014/01; hep-th/01017043.

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