QUARKONIA FOR EVERYBODY

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ABSTRACT

The Bohr model of the hydrogen atom is applied to the charmonium and bottonium systems. Simple algebraic equations are derived for the masses of the c and the b quarks, the $c\bar{c}$ and the $b\bar{b}$ bound states and the parameters of the interquark potential. These can be evaluated in a matter of minutes using a pocket calculator.

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I. INTRODUCTION :

In this age of extreme specialisation scientists working in one sub-branch of science are often unable to comprehend the latest developments in other sub-branches of their own specialisation. And the situation gets worse for those working in other branches. It is therefore important to develop simple uncompromising treatments of advanced topics for the elevation of general scientific awareness among the scientists and the science students. In a recent paper we presented such a simple treatment of the gravitational bound states of black holes, called holeums, and other compact astronomical objects using the Bohr method of quantisation(1). In this paper we apply the same method to the bound states of strongly interacting particles like quarks. A bound state of a quark and an antiquark is called a quarkonium. For example, the ψ particles are the bound states of a charm quark c and an anticharm quark $\bar{c}$. This system of $c\bar{c}$ bound states is called the charmonium. Similarly the upsilon particles are the bound states of a bottom (or a beauty) quark b and an antibottom quark $\bar{b}$. This $b\bar{b}$ system is called the bottonium. Finally, it is anticipated that the recently observed top (or truth) quark t will form a system of bound states $t\,\bar{t}$ with its antiquark $\bar{t}$. This system is called the toponium. These bound states are yet to be observed. In all there are six quarks : u (up), d (down), s (strange), c (charm), b (bottom/beauty) and t (top/truth). Each quark comes in three colours red, blue and green. A quark is surrounded by a cloud of massless gluons. In all there are eight gluons carrying different colour combinations. A gluon cloud endows a quark it surrounds with two novel properties : asymptotic freedom and infrared slavery. The former means that the quarks behave like free particles when the distance between them goes to zero. And the latter means that the work required to separate the quarks to infinity is infinite. That is, they can not be separated to an infinite distance from each other. This explains why no free quarks are observed.

II. THE BOHR MODEL :

The u,d,s are light quarks and the c,b,t are heavy. Therefore we may apply the nonrelativistic quantisation for the latter. In fact, we will apply the Bohr quantisation because it is algebraic, transparent and simple.

2.1 The Interquark Potential :

The following potential is found to describe the quarkonia quite well.

$$V(r) = -K/r + ar \qquad ...(1)$$

This is a potential between a quark and an antiquark separated by a distance r, K and a are constants.

2.2 The Balance of Forces :

The centripetal force due to the potential, eq.(1) is given by

Now equating the magnitudes of the centrifugal and the centripetal forces we get,

$$\mu v^2/r = a + K/r^2 \quad ...(3)$$

where $\mu = m/2$ is the reduced mass of the quark-antiquark pair and m that of a quark/antiquark.

2.3 The Quantisation Condition :

According to the de Broglie hypothesis a particle having a momentum $p = \mu v$, where v is its speed, has matter waves associated with it having a wave length,

$$\lambda = h/p \quad (4),$$

where h is the Planck's constant. Under the influence of the centrifugal and the centripetal forces a quark q and an antiquark $\bar{q}$ undergo a circular motion around their centre of mass. This motion is equivalent to that of a fictitious particle of reduced mass μ around their centre of mass which serves as the centre of force. The matter waves associated with the particle of mass μ will form a standing wave pattern if the circumference $2\pi r$ equals $n\lambda$. That is

$$2\pi r = n\lambda \quad ...(5)$$

From Eqs. (4) – (5) we get,

$$\mu v r = n\hbar \quad ...(6)$$

where $\hbar = h/2\pi$. This is the quantisation condition of Bohr.

III. THE ENERGY EIGHEN VALUES :

Eliminating r between Eqs. (3) and (6) we get,

$$v^3 + p' v^2 + q' v + r' = 0 \quad ...(7)$$

where

$$p' = -K/n\hbar,\ q' = 0,\ r' = -an\hbar/\mu^2 \quad ...(8)$$

This equation (7) can be solved as follows(2) :

Let

$$\alpha = (3q' - p'^2)/3 = -K^2/3n^2\hbar^2 \quad ...(9),$$

$$\beta = (2p'^3 - 9p'q' + 27r')/27$$

$$= 2K^3/27n^3\hbar^3 - an\hbar/\mu^2 \quad ...(10),$$

$$\Delta = \beta^2/4 + \alpha^3/27$$

$$= (an\hbar/\mu^2)(K^3/27n^3\hbar^3 + an\hbar/4\mu^2) \quad ...(11).$$

Now if p', q', r' are real and if $\Delta > 0$ then there is one real and two complex conjugate roots(2). To find the real root we define

$$A = (\Delta^{1/2} - \beta/2)^{1/3} \quad ...(12),$$

$$B = (-\Delta^{1/2} - \beta/2)^{1/3} \quad ...(13).$$

Then the real root is given by

$$v = A + B - p'/3$$

$$= A + B + H \quad ...(14),$$

where

$$H = K/3n\hbar \quad ...(15).$$

Let

$$G = an\hbar/\mu^2 \quad ...(16).$$

Then we can rewrite A and B as

$$A = (H^3 + G/2 + F)^{1/3} \quad ...(17),$$

$$B = (H^3 + G/2 - F)^{1/3} \quad ...(18),$$

$$F = G^{1/2}(H^3 + G/4)^{1/2} \quad ...(19).$$

The total energy of the system is given by

$$E_n = (1/2)\mu v^2 + V(r)$$

$$= \mu v^2/2 + ar - K/r$$

$$= 3\mu v^2/2 - 2\mu K v/n\hbar \quad ...(20),$$

where we have used Eqs. (3) and (6) in arriving at Eq. (20)

3.1 The Approximation Scheme :

Let

$$\lambda = G/H^3 = 108\,an^4/m^2K^3 \quad ...(21).$$

Because of the factor $108\,n^4$ in the numerator and the ball-park values a = 0.1 to 0.2 GeV^2, m = 1.5 GeV to 2.0 GeV and $K \le 1$ we expect this quantity λ to be much larger than unity. That is

$$\lambda >> 1 \quad ...(22).$$

Thus, keeping terms upto $\lambda^{-1/3}$ we can show that

$$E_n = (a^2n^2/4m)^{1/3} - (4K/3)(am/2n^2)^{1/3} \quad ...(23).$$

IV. THE EQUATIONS AND THEIR SOLUTIONS :

The mass m_n of a bound state of a quark and an antiquark is given by

$$m_n = 2m + E_n \quad ...(24).$$

Let m_1, m_2, m_3 be the masses of the 1S, 2S and the 3S states, respectively of a quarkonium. Then from Eqs. (23) and (24) we get the following equations for $m = x^3$, $a = y^3$ and K.

$$2x^3 - m_1 + 3y^2/2^{2/3}\,x - 4Kxy/(3 \cdot 2^{1/3}) = 0 \quad ...(25),$$

$$2x^3 - m_2 + 3y^2/x - 2Kxy/3 = 0 \quad ...(26),$$

$$2x^3 - m_3 + 3(3/2)^{2/3}\,y^2/x - 4Kxy/(3^{5/3}\,2^{1/3}) = 0 \quad ...(27).$$

These equations are easy to solve and we get the following solution.

$$m = (N_2/D_2 - N_1/D_1)/(N_3/D_3 - N_4/D_4) \quad ...(28),$$

where

$$N_1 = m_2 - m_1/2^{2/3} \quad ...(29),$$

$$N_2 = (3/2)^{2/3}\,m_3 - m_2 \quad ...(30),$$

$$N_3 = (18)^{1/3} - 2 \quad ...(31),$$

$$N_4 = 2 - 2^{1/3} \quad ...(32),$$
$$D_1 = 1 - 1/2^{4/3} \quad ...(33),$$
$$D_2 = (3/2)^{4/3} - 1 \quad ...(34),$$
$$D_3 = D_2 \quad ...(35),$$
$$D_4 = D_1 \quad ...(36).$$

And

$$a = [\, m_1^{1/3} (N_1 / D_1 - mN_4 / D_4) / 3 \,]^{3/2} \quad ...(37),$$
$$K = (3/2)\, [\, 2m - m_2 + 3\, (a^2/ m)^{1/3} \,] / (ma)^{1/3} \quad ...(38).$$

V. THE CHARMONIUM

The experimental values of the first four S-states (l=0) of charmonium and bottonium are presented in Table-I[3]. Now we have three unknown parameters m, K and a. These can be determined using the above equations and taking m_1, m_2, m_3 from Table-I. We get the following values :

$$m = 1.8058 \text{ GeV} \quad ...(39),$$
$$K = 1.3947 \quad ...(40),$$
$$a = 0.1389 \text{ GeV}^2 \quad ...(41).$$

Now one can show that

$$\hbar c = 0.1974 \text{ GeV Fermi} \cong 0.2 \text{ GeV Fermi} \quad ...(42),$$

where 1 Fermi = 10^{-13} cm and 1 GeV = 10^9eV. Here we use the natural units $\hbar = c = 1$. Then from Eq.(42) we get

$$1 \text{ Fermi} = 5 \text{ GeV}^{-1} \quad ...(43).$$

Thus, in natural units mass and energy are expressed in units of GeV and lengths are expressed in units of GeV^{-1}. Of course, we can use Eq.(43) to express the latter in units of Fermis. The charmonium parameters given in Eqs. (39-41) compare favourably with "Standard" ones[4]:

$$m = 1.65 \text{ GeV},\ K = 0.132,\ a = 0.23 \text{ GeV}^2 \quad ...(44).$$

It must be emphasized that there is no unique set of values for these parameters and that our values are quite in the ball-park. As a test of our approximation scheme we calculate the mass m_4 of the 4S state of the charmonium using Eqs. (23—24) and (39-41), we get

$$m_4 = 4.291 \text{ GeV} \quad ...(45),$$

to be compared with the experimental value 4.414 GeV given in Table-I. This is all right within the spirit of our approximation.

VI. THE BOTTONIUM

Next we turn to the upsilon or bottonium family whose masses are also given in Table-I. Using the first three masses and Eqs. (25-38) we get,

$$m = 4.9597 \text{ GeV} \quad ...(46),$$
$$K = 0.7923 \quad ...(47),$$
$$a = 0.2228 \text{ GeV}^2 \quad ...(48).$$

Here m is the mass of the bottom quark. As a test of our method we calculate the mass m_4 of the 4S state of the upsilon system. We find

$$m_4 = 10.60 \text{ GeV} \quad ...(49),$$

to be compared with m_4 = 10.57 GeV given in Table-I. The consistently good agreement shows that this method is very useful and appropriate for the quarkonium families.

VII. CONCLUSION AND DISCUSSION

The Bohr model ushered us in to the atomic age and led to the development of the complete quantum theory. The latter showed that the Bohr model was deterministic and therefore it violated the uncertainty principle. This led to its fall from grace. However, the taste of pudding is in eating. The Bohr model gives exact energy eigen values for the 1/r (Coulomb) potential and approximate ones for others. The effects of the uncertainty principle are most pronounced in the ground state where the zero point vibrations make a characteristic contribution which is absent in the Bohr model. Therefore the latter is expected to be a good approximation for n >> 1 which is also the region of validity of the WKB method. However, Bohr's is the simplest method to obtain the energy eigen values without solving a differential equation or even without doing an integration. Therefore, it is highly recommended both to a novice and an expert especially in anticipation of the toponium family waiting to be discovered.

The aim of reference (1) and the present paper is two-fold (1). Revival of interest in the Bohr method as an approximation method valid for large n. (2) To provide a completely transparent and simple treatment of advanced topics in Astrophysics and Elementary Particle Physics which any student of science can follow and work through completely on his own.

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4. T. Appelquist et al., Annual Review of Nuclear and Particle Science 28, 412 (1978).

TABLE-I

Charmonium		Bottonium
nl	m_n	m_n
1S	3.097	9.47
2S	3.686	10.02
3S	4.030	10.35
4S	4.414	10.57

TABLE CAPTION

Table-I. The masses in GeV of the bound states of $c\bar{c}$ and bb systems.