Confidence intervals for the parameter of Poisson distribution in presence of background

S.I. Bityukov[a,*] N.V. Krasnikov[b]

[a]Division of Experimental Physics, Institute for High Energy Physics, Protvino, Moscow Region, Russia

[b]Division of Quantum Field Theory, Institute for Nuclear Research RAS, Moscow, Russia

Abstract

A numerical procedure is developed for construction of confidence intervals for parameter of Poisson distribution for signal in the presence of background which has Poisson distribution with known value of parameter.

Keywords: statistics, confidence intervals, Poisson distribution, Gamma distribution, sample.

I. INTRODUCTION

In paper [1] the unified approach to the construction of confidence intervals and confidence limits for a signal with a background presence, in particular for Poisson distributions, has been proposed. The method is widely used for the presentation of physical results [2] though a number of investigators criticize this approach [3]

In present paper we propose a simple method for construction of confidence intervals for parameter of Poisson distribution for signal in the presence of background which has Poisson distribution with known value of parameter. This method is based on the statement [4] that the true value of parameter of the Poisson distribution in the case of observed number of

*Corresponding author

Email addresses: bityukov@mx.ihep.su, Serguei.Bitioukov@cern.ch

events $\hat{x}$ has a Gamma distribution. In contrast to the approach proposed in [1], the width of confidence intervals in the case of $\hat{x} = 0$ is independent on the value of the parameter of the background distribution.

In Section 2 the method of construction of confidence intervals for parameter of Poisson distribution for signal in the presence of background which has Poisson distribution with known value of parameter is described. The results of confidence intervals construction and their comparison with the results of unified approach are also given in the Section 2. The main results of this note are formulated in the Conclusion.

II. THE METHOD OF CONSTRUCTION OF CONFIDENCE INTERVALS

Assume that in the experiment with the fixed integral luminosity (i.e. a process under study may be considered as a homogeneous process during given time) the $\hat{x}$ events of some Poisson process were observed. It means that we have an experimental estimation $\hat{\lambda}(\hat{x})$ of the parameter λ of Poisson distribution. We have to construct a confidence interval $(\hat{\lambda}_1(\hat{x}), \hat{\lambda}_2(\hat{x}))$, covering the true value of the parameter λ of the distribution under study with confidence level $1 - \alpha$, where α is a significance level. It is known from the theory of statistics [5], that the mean value of a sample of data is an unbiased estimation of mean of distribution under study. In our case the sample consists of one observation $\hat{x}$. For the discrete Poisson distribution the mean coincides with the estimation of parameter value, i.e. $\hat{\lambda} = \hat{x}$ in our case. As it is shown in ref [4] the true value of parameter λ has Gamma distribution $\Gamma_{1,\hat{x}+1}$, where the scale parameter is equal to 1 and the shape parameter is equal to $\hat{x} + 1$, i.e.

$$P(\lambda|\hat{x}) = \frac{\lambda^{\hat{x}}}{\hat{x}!} e^{-\lambda}. \tag{2.1}$$

Let us consider the Poisson distribution with two components: signal component with a parameter λ_s and background component with a parameter λ_b, where λ_b is known. To construct confidence intervals for parameter λ_s of signal in the case of observed value $\hat{x}$ we must find the distribution $P(\lambda_s|\hat{x})$.

At first let us consider the simplest case $\hat{x} = \hat{s} + \hat{b} = 1$. Here $\hat{s}$ is a number of signal events and $\hat{b}$ is a number of background events among observed $\hat{x}$ events.

The $\hat{b}$ can be equal to 0 and to 1. We know that the $\hat{b}$ is equal to 0 with probability

$$p_0 = P(\hat{b} = 0) = \frac{\lambda_b^0}{0!} e^{-\lambda_b} = e^{-\lambda_b} \tag{2.2}$$

and the $\hat{b}$ is equal to 1 with probability

$$p_1 = P(\hat{b} = 1) = \frac{\lambda_b^1}{1!} e^{-\lambda_b} = \lambda_b e^{-\lambda_b}. \tag{2.3}$$

Correspondingly, $P(\hat{b} = 0|\hat{x} = 1) = P(\hat{s} = 1|\hat{x} = 1) = \dfrac{p_0}{p_0 + p_1}$ and $P(\hat{b} = 1|\hat{x} = 1) = P(\hat{s} = 0|\hat{x} = 1) = \dfrac{p_1}{p_0 + p_1}$.

It means that distribution of $P(\lambda_s|\hat{x} = 1)$ is equal to sum of distributions

$$P(\hat{s} = 1|\hat{x} = 1)\Gamma_{1,2} + P(\hat{s} = 0|\hat{x} = 1)\Gamma_{1,1} = \frac{p_0}{p_0 + p_1}\Gamma_{1,2} + \frac{p_1}{p_0 + p_1}\Gamma_{1,1}, \tag{2.4}$$

where $\Gamma_{1,1}$ is Gamma distribution with probability density $P(\lambda_s|\hat{s} = 0) = e^{-\lambda_s}$ and $\Gamma_{1,2}$ is Gamma distribution with probability density $P(\lambda_s|\hat{s} = 1) = \lambda_s e^{-\lambda_s}$. As a result we have

$$P(\lambda_s|\hat{x} = 1) = \frac{\lambda_s + \lambda_b}{1 + \lambda_b} e^{-\lambda_s}. \tag{2.5}$$

Using formula (2.5) for $P(\lambda_s|\hat{x} = 1)$ we construct the shortest confidence interval of any confidence level in a trivial way [4].

In this manner we can construct the distribution of $P(\lambda_s|\hat{x})$ for any values of $\hat{x}$ and λ_b. We have obtained the formula

$$P(\lambda_s|\hat{x}) = \frac{(\lambda_s + \lambda_b)^{\hat{x}}}{\hat{x}! \displaystyle\sum_{i=0}^{\hat{x}} \frac{\lambda_b^i}{i!}} e^{-\lambda_s}. \tag{2.6}$$

The numerical results for the confidence intervals and for comparison the results of paper [1] are presented in Table 1 and Table 2.

It should be noted that in our approach the dependence of the width of confidence intervals for parameter λ_s on the value of λ_b in the case $\hat{x} = 0$ is absent. For $\hat{x} = 0$ the method proposed in ref. [6] also gives a 90% upper limit independent of λ_b.

III. CONCLUSION

In this note the construction of classical confidence intervals for the parameter λ_s of Poisson distribution for the signal in the presence of background with known value of parameter λ_b is proposed. The results of numerical construction are presented.

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REFERENCES

[1] Feldman, G.J. and R.D. Cousins, Unified approach to the classical statistical analysis of small signal, *Phys.Rev.* **D57** 3873-3889 (1998).

[2] Groom, D.E. et al., Review of particle physics, *Eur.Phys.J.* **C 15** (2000) 198-199.

[3] as an example, Zech, G., Classical and Bayesian Confidence Limits, in: F. James, L. Lyons, and Y. Perrin (Eds.), Proc. of 1st Workshop on Confidence Limits, CERN 2000-005, Geneva, Switzerland, (2000) 141-154.

[4] Bityukov S.I, N.V. Krasnikov and V.A. Taperechkina, On the confidence interval for the parameter of Poisson distribution. e-print arXiv: physics/0008082, August 2000.

[5] as an example, *Handbook of Probability Theory and Mathematical Statistics (in Russian),* V.S. Korolyuk (Ed.), (Kiev, "Naukova Dumka", 1978)

[6] Roy B.P. and M.B. Woodroofe, *Phys.Rev.* **D59** (1999) 053009.

TABLE I. 90% C.L. intervals for the Poisson signal mean λ_s, for total events observed $\hat{x}$, for known mean background λ_b ranging from 0 to 4. A comparison between results of ref.[1] and results from present note.

$\hat{x}\backslash\lambda_b$	0.0 ref.[1]	0.0	1.0 ref.[1]	1.0	2.0 ref.[1]	2.0	3.0 ref.[1]	3.0	4.0 ref.[1]	4.0
0	0.00, 2.44	0.00, 2.30	0.00, 1.61	0.00, 2.30	0.00, 1.26	0.00, 2.30	0.00, 1.08	0.00, 2.30	0.00, 1.01	0.00, 2.30
1	0.11, 4.36	0.09, 3.93	0.00, 3.36	0.00, 3.27	0.00, 2.53	0.00, 3.00	0.00, 1.88	0.00, 2.84	0.00, 1.39	0.00, 2.74
2	0.53, 5.91	0.44, 5.48	0.00, 4.91	0.00, 4.44	0.00, 3.91	0.00, 3.88	0.00, 3.04	0.00, 3.53	0.00, 2.33	0.00, 3.29
3	1.10, 7.42	0.93, 6.94	0.10, 6.42	0.00, 5.71	0.00, 5.42	0.00, 4.93	0.00, 4.42	0.00, 4.36	0.00, 3.53	0.00, 3.97
4	1.47, 8.60	1.51, 8.36	0.74, 7.60	0.51, 7.29	0.00, 6.60	0.00, 6.09	0.00, 5.60	0.00, 5.35	0.00, 4.60	0.00, 4.78
5	1.84, 9.99	2.12, 9.71	1.25, 8.99	1.15, 8.73	0.43, 7.99	0.20, 7.47	0.00, 6.99	0.00, 6.44	0.00, 5.99	0.00, 5.72
6	2.21,11.47	2.78,11.05	1.61,10.47	1.79,10.07	1.08, 9.47	0.83, 9.01	0.15, 8.47	0.00, 7.60	0.00, 7.47	0.00, 6.76
7	3.56,12.53	3.47,12.38	2.56,11.53	2.47,11.38	1.59,10.53	1.49,10.37	0.89, 9.53	0.57, 9.20	0.00, 8.53	0.00, 7.88
8	3.96,13.99	4.16,13.65	2.96,12.99	3.18,12.68	2.14,11.99	2.20,11.69	1.51,10.99	1.21,10.60	0.66, 9.99	0.34, 9.33
9	4.36,15.30	4.91,14.95	3.36,14.30	3.91,13.96	2.53,13.30	2.90,12.94	1.88,12.30	1.92,11.94	1.33,11.30	0.97,10.81
10	5.50,16.50	5.64,16.21	4.50,15.50	4.66,15.22	3.50,14.50	3.66,14.22	2.63,13.50	2.64,13.21	1.94,12.50	1.67,12.16
20	13.55,28.52	13.50,28.33	12.55,27.52	12.53,27.34	11.55,26.52	11.53,26.34	10.55,25.52	10.53,25.34	9.55,24.52	9.53,24.34

TABLE II. 90% C.L. intervals for the Poisson signal mean λ_s, for total events observed $\hat{x}$, for known mean background λ_b ranging from 6 to 15. A comparison between results of ref.[1] and results from present note.

$\hat{x}\backslash\lambda_b$	6.0 ref.[1]	6.0	8.0 ref.[1]	8.0	10.0 ref.[1]	10.0	12.0 ref.[1]	12.0	15.0 ref.[1]	15.0
0	0.00, 0.97	0.00, 2.30	0.00, 0.94	0.00, 2.30	0.00, 0.93	0.00, 2.30	0.00, 0.92	0.00, 2.30	0.00, 0.92	0.00, 2.30
1	0.00, 1.14	0.00, 2.63	0.00, 1.07	0.00, 2.56	0.00, 1.03	0.00, 2.51	0.00, 1.00	0.00, 2.48	0.00, 0.98	0.00, 2.45
2	0.00, 1.57	0.00, 3.01	0.00, 1.27	0.00, 2.85	0.00, 1.15	0.00, 2.75	0.00, 1.09	0.00, 2.68	0.00, 1.05	0.00, 2.61
3	0.00, 2.14	0.00, 3.48	0.00, 1.49	0.00, 3.20	0.00, 1.29	0.00, 3.02	0.00, 1.21	0.00, 2.91	0.00, 1.14	0.00, 2.78
4	0.00, 2.83	0.00, 4.04	0.00, 1.98	0.00, 3.61	0.00, 1.57	0.00, 3.34	0.00, 1.37	0.00, 3.16	0.00, 1.24	0.00, 2.98
5	0.00, 4.07	0.00, 4.71	0.00, 2.60	0.00, 4.10	0.00, 1.85	0.00, 3.72	0.00, 1.58	0.00, 3.46	0.00, 1.32	0.00, 3.20
6	0.00, 5.47	0.00, 5.49	0.00, 3.73	0.00, 4.67	0.00, 2.40	0.00, 4.15	0.00, 1.86	0.00, 3.80	0.00, 1.47	0.00, 3.46
7	0.00, 6.53	0.00, 6.38	0.00, 4.58	0.00, 5.34	0.00, 3.26	0.00, 4.65	0.00, 2.23	0.00, 4.19	0.00, 1.69	0.00, 3.74
8	0.00, 7.99	0.00, 7.35	0.00, 5.99	0.00, 6.10	0.00, 4.22	0.00, 5.23	0.00, 2.83	0.00, 4.64	0.00, 1.95	0.00, 4.06
9	0.00, 9.30	0.00, 8.41	0.00, 7.30	0.00, 6.95	0.00, 5.30	0.00, 5.89	0.00, 3.93	0.00, 5.15	0.00, 2.45	0.00, 4.42
10	0.22,10.50	0.02, 9.53	0.00, 8.50	0.00, 7.88	0.00, 6.50	0.00, 6.63	0.00, 4.71	0.00, 5.73	0.00, 3.00	0.00, 4.83
20	7.55,22.52	7.53,22.34	5.55,20.52	5.53,20.34	3.55,18.52	3.55,18.30	2.23,16.52	1.70,16.08	0.00,13.52	0.00,12.31