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ANALYTICAL CALCULATION OF THE LONGITUDINAL IMPEDANCE OF A SEMI-INFINITE CIRCULAR WAVEGUIDE

S. Kheifets* and L. Palumbo**

Abstract

The longitudinal coupling impedance of a semi-infinite circular waveguide has been analytically calculated. The Wiener-Hopf factorization method has been applied to obtain an expression of the radiation spectrum at any frequency and particle energy.

1. INTRODUCTION

The longitudinal impedance of a semi-infinite circular waveguide has been recently calculated [1] in the low frequency approximation. The problem was expressed in terms of a pair of integral equations whose solution was found by applying the Wiener-Hopf technique. The impedance was expressed as a run of two terms = $Z = Z_1 + Z_2$, where Z_1 is a purely imaginary contribution proportional to $1/\gamma^2$ due to the space charge and to the currents induced on the uniform wall, while Z_2 is a complex contribution accounting for the radiated fields and is given by :

$$Z_2(\omega) = Z_0 \frac{x}{(\beta\gamma)^2} f_I(\xi a) \frac{K_0(\xi b)}{I_0(\xi b)} \left[\gamma \frac{I_1(\xi b)}{I_0(\xi b)} - \frac{\Gamma'_+(x/\beta)}{\Gamma^{0}_+(x/\beta)} - \frac{\beta}{2x(1+\beta)} \right] \tag{1}$$

where $k = \omega/\beta c$, $x = kb$, $\xi = x/|\beta|\gamma$, $f_I(\xi a) = I_1(\xi a)/(\pi\xi a)$, K_0 and $I_{0,1}$ are modified Bessel functions, Z_0 is the free space impedance ($Z_0 = 377\ \Omega$), Γ^+ is obtained by factorization of the function :

$$L(u) = \pi\Omega J_0(\Omega)H_0(\Omega) = \Gamma^+(u)\Gamma^-(u) \tag{2}$$

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* SLAC, Stanford University, Stanford, U.S.A.
** Università La Sapienza, Dipartimento di Energetica, Roma, Italy.

where

$$\Omega = \sqrt{x^2 - u^2} ,$$

The factorized functions, required to construct the solution, were numerically evaluated only at frequencies below the cut-off of the circular pipe.

In this note we will show how to obtain an analytical expression of the factorized functions which is valid at any frequency.

2. EVALUATION OF THE QUANTITY Γ_+'/Γ_+

The application of the factorization method [2] to the function L(u) requires it to be analytic within a strip in the complex u-plane. Assuming the frequency x having a small positive imaginary part ε, the required analytic properties are fulfilled within the strip $-\varepsilon<\mathrm{Im}(u) < \varepsilon$. Noting that also $\ell n[L(u)]$ is analytic within the same strip, it is easy to see that this function may be represented on the whole complex plane by a Cauchy integral; in particular at the point x/β it is given by :

$$\ell n[L(x/\beta)] = \ell n\Gamma^+(x/\beta) + \ell n\Gamma^-(x/\beta) = \frac{1}{2\pi i}\int_{\eta^+}\frac{\ell n[L(u)]du}{u-x/\beta} + \frac{1}{2\pi i}\int_{\eta^-}\frac{\ell n[L(u)]du}{u-x/\beta} \quad (3)$$

where the integration paths are shown in Fig. 1.

Furthermore the same considerations apply also to the logarithmic derivative of (Eq. (2)),which is the quantity we need in Eq.(1). We get :

$$\frac{\Gamma_+'(x/\beta)}{\Gamma_+(x/\beta)} = \frac{-1}{2\pi i}\int_{\eta^+}\frac{u\,du}{\Omega(u-x/\beta)}\left[\frac{1}{\Omega} + \frac{J_1(\Omega)}{J_0(\Omega)} + \frac{H_1(\Omega)}{H_0(\Omega)}\right] \quad (4)$$

The integrand is singular at the points $u = x/\beta$, $u = \pm x$ and at the zeros w_n of the Bessel function $J_0(\Omega)$. Due to the square root $\Omega = \sqrt{x^2-u^2}$, the Hankel function $H_{0,1}(\Omega)$ are multivalued and branch cuts, starting from the points $u = \pm x$, are required to obtain a single-valued Riemann plane. The singularities and the cuts are shown in Fig.2. For our convenience we chose to cut on the curve where $\mathrm{Im}(\Omega) = 0$. The integral (4) can be split into two parts : one integral without Bessel functions that we call I_1 and a second one containing the remaining part of the integrand that we call I_2.

The rational function in I_1 vanishes as u^{-2} for $|u| \to \infty$ therefore we can add to the integration path a circle of radius $R \to \infty$ without changing the integral value (Jordan's Lemmma). We can close the path in the upper complex u-plane where the integrand function has two singular points at $u = x$ and $u = x/\beta$; the Cauchy theorem tells us that the integral is equal to $2\pi i$ times the residua at the singular points, or :

$$I_1 = \frac{\beta}{2x(1+\beta)} \tag{5}$$

Now we want to evaluate I_2. To this end we add and subtract the contour C (shown in Fig. 3) to the original integration path. We obtain :

$$I_2 = I_{2A} + I_{2B} = \frac{1}{2\pi i} \int_{\eta+C} \frac{u\,du}{\Omega(u-k/\beta)} [\cdot] - \frac{1}{2\pi i} \int_{C} \frac{u\,du}{\Omega(u-x/\beta)} [\cdot] \tag{6}$$

where $[\cdot]$ is the expression in square brackets in Eq. (4).

The first integral of Eq. (6) can be evaluated again with the residuum theorem yielding :

$$I_{2A} = \frac{x}{\beta} \left[\frac{J_1(\Omega)}{\Omega\, J_0(\Omega)} - \frac{H_1(\Omega)}{\Omega\, H_0(\Omega)} \right]_{u = x/\beta} \tag{7}$$

For the remaining term I_{2B} one can prove the following results :

1) The integral over the large circle is zero. In fact the J and H term give a constant contribution of opposite sign.

2) The integral over the small circle around the point $u = k$ is zero, (apply again the Jordan Lemma).

Accordingly we will get only the contribution over the two sides of the branch cut. Consider now the J and H terms separately; the former is not multivalued, and its singularities lie just on the curve $\mathrm{Im}(\Omega) = 0$, therefore the contributions on the left and right side of the cut give exactly $2\pi i$ times the sum of the residua at the singular points. The H term, on the other hand, has no poles but is multivalued. Making a change of variable $u \to \Omega$ we get the final expression :

$$I_{2B} = \sum_{n=0}^{\infty}{}' \frac{1}{w_n - x/\beta} + \frac{1}{\pi} \int_0^{\infty} \frac{d\Omega}{x/\beta - \sqrt{x^2-\Omega^2}} \left[\frac{J_0(\Omega)Y_1(\Omega)-J_1(\Omega)Y_0(\Omega)}{J_0^2(\Omega)+Y_0^2(\Omega)} \right] \quad (8)$$

3. LONGITUDINAL IMPEDANCE AND RADIATION SPECTRUM

Exploiting the results of Section 2 we obtain the following expression for the longitudinal impedance :

$$Z = Z_0 \frac{x}{\beta^2\gamma^2} f_I(\xi a) \frac{K_0(\xi b)}{I_0(\xi b)} \left[\gamma \frac{K_1(\xi b)}{K_0(\xi b)} - \frac{\beta}{x(1+\beta)} - I_{2B} \right] \quad (9)$$

The sum over n accounts for the modes excited in the pipe and the remaining terms account for the radiation into free space. In Fig. 4 the real part of the impedance is shown for several γ-values over a wide range of frequencies. It is worth noting that the spectrum falls off for high enough frequencies after a peak. The half maximum value corresponds to a frequency $kb = \gamma$ (where b is the pipe radius). This behaviour of the curves is easily explained : the real part of the impedance is the radiation spectrum and, for a point charge Q, the energy loss U may be expressed as [3] :

$$U = 2Q^2 \int_0^{\infty} Z_r(\omega)\, d\omega \quad (10)$$

For a relativistic particle it is reasonable to think that the radiation from the induced charges on the pipe wall will occur mainly when the edge of the pipe is seen by the self-field which is confined within an angle $\theta = 2\pi/\gamma$. Therefore one can expect the radiation pulse lasting a time $2\pi b/\gamma c$, and its spectrum having a bandwith $kb = \gamma$.

The behaviour of the impedance at $kb = 0.1$ is plotted in Fig. 5 versus the energy. For large energies, the curve seems to tend toward a constant value, but the limit value is not apparent from the results, and there is a non zero slope even at $\gamma = 10^6$.

The above calculations refer to the case of a charge entering the semi-infinite pipe. The case of a charge leaving the waveguide has also been investigated : the expression for the impedance is obtained by replacing β with $-\beta$ in Eq. (9) yielding :

$$Z_2(\omega) = Z_0 \frac{x}{\beta^2\gamma^2} f_I(\xi a) \frac{K_0(\xi b)}{I_0(\xi b)} \left[\frac{\beta(1+\beta)\gamma^2}{x} - \gamma \frac{K_1(\xi b)}{K_0(\xi b)} - I_{2B}(-\beta) \right] \qquad (11)$$

Acknowledgements

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3. A.W. Chao, SLAC-PUB-2946, 1982.

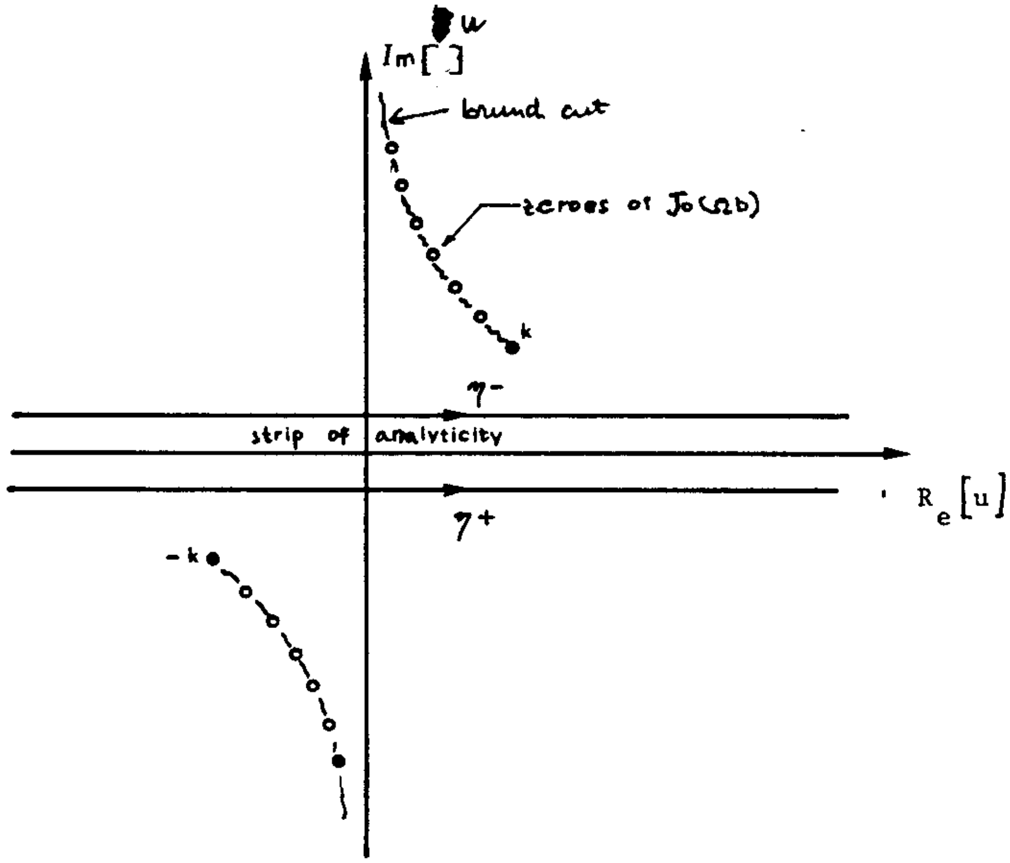

Fig. 1 - Strip of analyticity of L(u) in the complex plane and the integration paths $\eta^{\pm}$

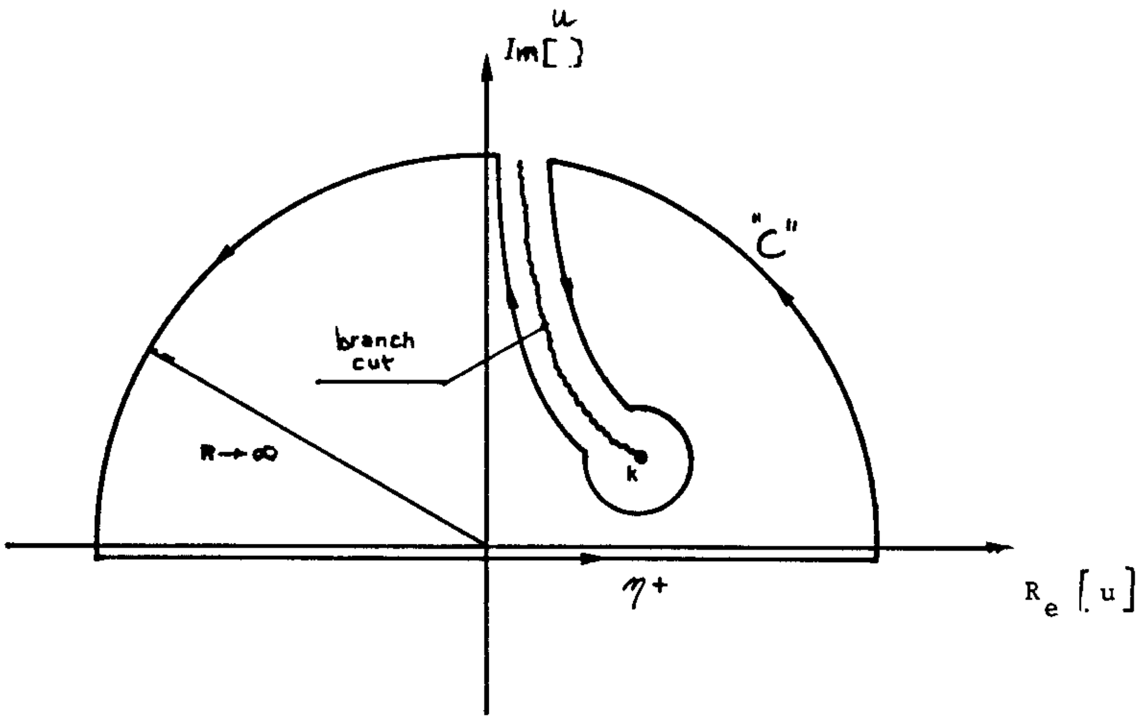

Fig. 2 - The integration contour "C" encompassing the branch cut in the complex U-plane

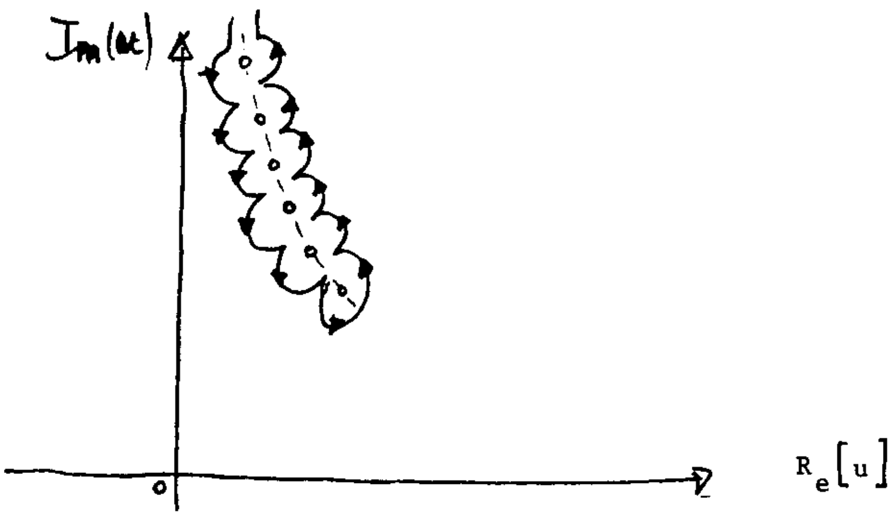

Fig. 3 - Residue contributions at the points $u = w_n$

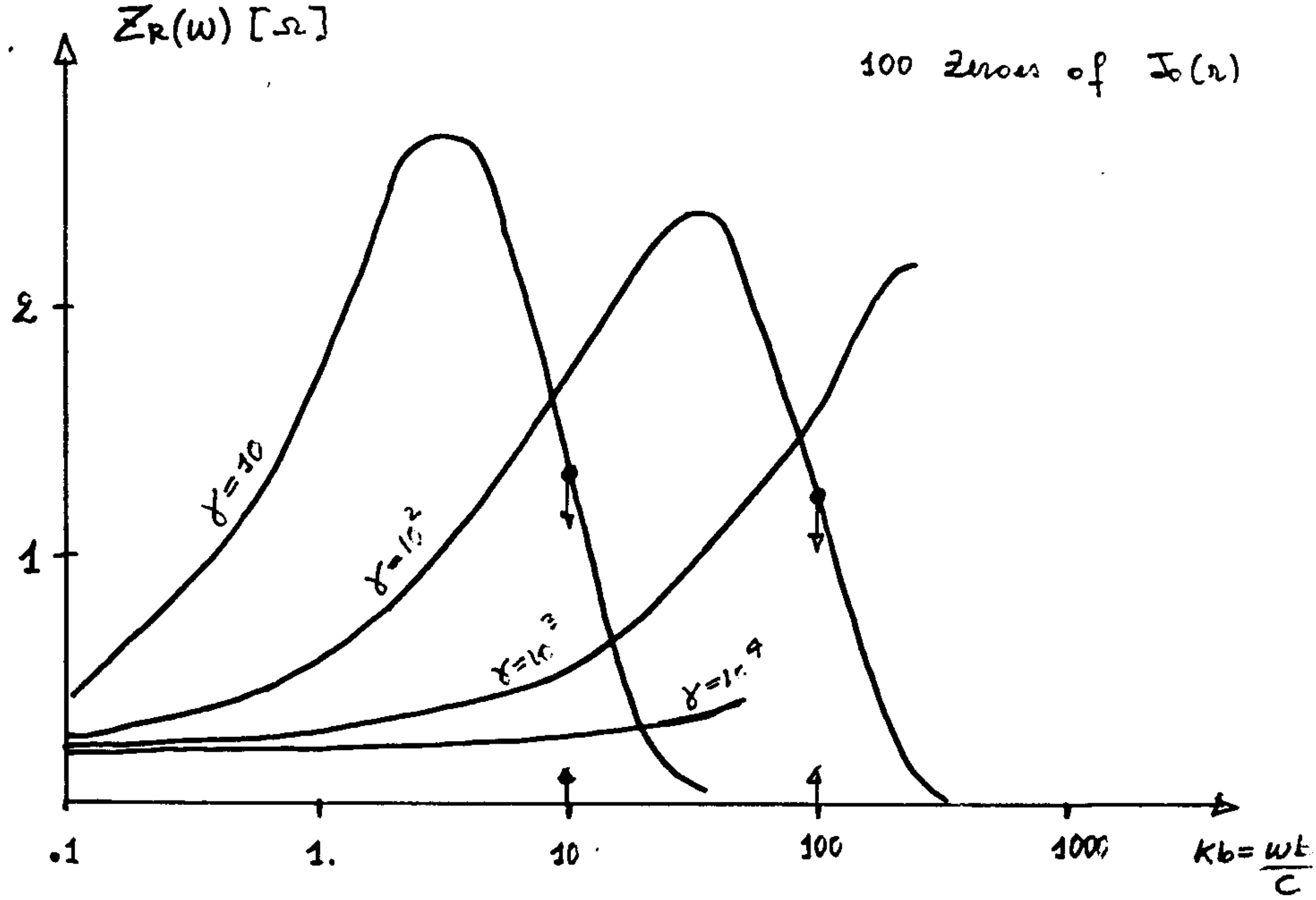

Fig. 4 - Real part of the impedance versus frequency for several values of gamma.

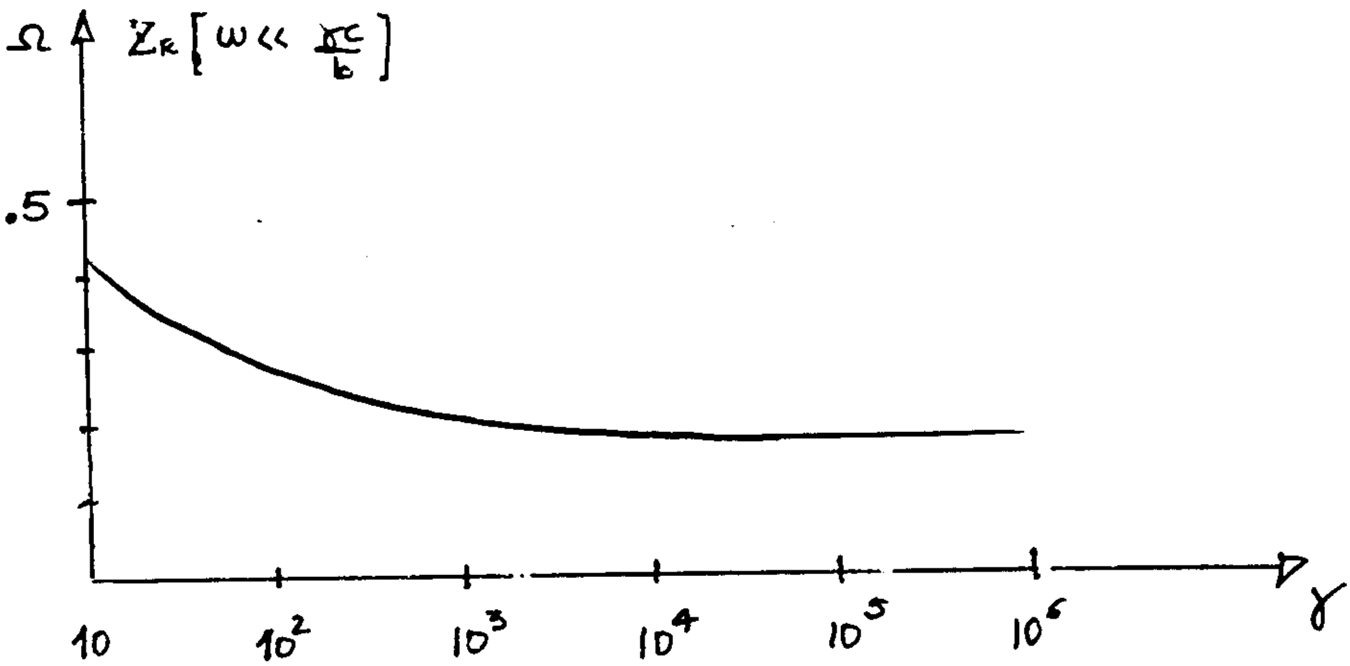

Fig. 5 - Low frequency impedance versus the energy.