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A NOTE ON NONUNIFORM LATTICES OF $SO(n,1)$

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Abstract

In this note, we will prove that any finite volume quotient of real hyperbolic space of dimension at least 3 is not properly homotopical equivalent to a quasi-projective variety.

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1. Introduction

The purpose of this short note is to show the following

Theorem. *Let $\overline{M}$ be a compact Kähler manifold, D be a normal crossing divisor of $\overline{M}$. Denote $\overline{M} \setminus D$ by M. Assume that , is a nonuniform lattice of $SO(n,1)(n \geq 3)$, i.e., , $\setminus SO(n,1)/SO(n)$ is noncompact and of finite volume with respect to standard symmetric Riemannian metric. Then M is not properly homotopical equivalent to , $\setminus SO(n,1)/SO(n)$.*

Remark. a) In the compact case, J. Carlson and D. Toledo [1] proved that any cocompact lattice of $SO(n,1)(n \geq 3)$ cannot be the fundamental group of a compact Kähler manifold. Corresponding to this, one should conjecture that any nonuniform lattice of $SO(n,1)(n \geq 3)$ cannot be the fundamental group of $\overline{M} \setminus D$ for a compact Kähler manifold $\overline{M}$ and its normal crossing divisor D; b) if M is a quasiprojective variety, by Hironaka's theorem [2], topologically, M is just a smooth projective variety deleting a normal crossing divisor; c) the condition of $n \geq 3$ is obviously necessary.

The idea of the proof is to use harmonic map theory. We assume that the theorem is not true, namely, M is properly homotopical equivalent to $N :=$, $\setminus SO(n,1)/SO(n)$. First, we choose a complete Kähler metric of finite volume on M. Second, we construct a harmonic homotopic equivalence of finite energy from M (under the metric constructed) to N using the theory due to Jost-Yau and Jost-Zuo ([3, 4] and [5]). The generalization of the famous argument of Siu and Sampson [6, 7] to the noncompact case then gives a strong restriction to this harmonic map. But, this will be a contradiction to our assumption at the beginning of the proof.

2. The construction of harmonic maps of finite energy

Let $\overline{M}$ be a compact Kähler manifold with a fixed Kähler metric ω, D be a fixed divisor with (at worst) normal crossing condition and $D = \bigcup_{i=1}^{p} D_i$ be a disjoint union of connected components of D. D_i consists of some irreducible components $D_i^1, D_i^2, \cdots, D_i^{m_i} (i = 1, 2, \cdots, p)$. Let $N =$, $\setminus SO(n,1)/SO(n)$ be a noncompact quotient of finite volume of real hyperbolic space of dimension at least 3. *Assume $M := \overline{M} \setminus D$ is properly homotopical equivalent to N.* We will construct a harmonic homotopic equivalence from M to N under an appropriate metric (we will construct it in the following) on M and the standard metric of N.

Let $\sigma_i^j (i=1,2,\cdots,p; j=1,2,\cdots,m_i)$ be a defining section of D_i^j in $\mathcal{O}(M,[D_i^j])$, which satisfies $|\sigma_i^j| \leq 1$ under certain Hermitian metric of $[D_i^j]$ and defines a coordinate system in a small disk transval to D_i^j. Set $\sigma_i = \sigma_i^1 \otimes \cdots \otimes \sigma_i^{m_i}$. One can then take on M

$$g := -\sum_{i=1}^{p} \partial\overline{\partial}(\phi(|\sigma_i|)\log|\log|\sigma_i|^2|) + c\omega|_M,$$

where ϕ is a suitable C^∞ cut-off function on $[0,\infty)$ so that $\phi(s)$ is identical to one on $[0,\epsilon)$ and to zero on $[2\epsilon,\infty)$ for sufficiently small $\epsilon \geq 0$ and c is taken sufficiently large so that g is positive definite. Then g is a Kähler metric. Furthermore, one can show that (M,g) is complete and has finite volume [8], [4], [5].

Since N is a finite volume noncompact quotient of real hyperbolic space under the nonuniform lattice , , it is well known that N can be simply compactified by adding some isolated points to its cusps. In particular, near each cusp, it is a topological product of R^+ and a torus and its metric is of form:

$$d\rho^2 + e^{-2\rho}d\omega^2,$$

where $d\omega^2$ is the flat metric of the torus.

Before going on our construction, we recall a notion due to Jost and Zuo [5] (for the noncompact curves case, also see [9]). A loop γ_∞ in M around D is called small if there exists a homotopy $H : [0,1]\times[0,1] \to \overline{M}$ with $H(\cdot,0) = \gamma_\infty, H([0,1]\times[0,1)) \subset M$ and $H(\cdot,1)$ is a point on D. Thus, γ_∞ is freely homotopic in M to arbitrary short loop under the induced metric or the above constructed complete metric.

Definition. Let G/K be a symmetric space of noncompact type. A homomorphism $\rho : \pi_1(M) \to G$ is called stabilizing at infinity if for every small loop γ_∞ around a connected component of D,

$$\inf_{z\in G/K} \mathrm{dist}(z,\rho(\gamma_\infty)z) = 0$$

Remark. $\rho(\gamma_\infty)$, if nontrivial, is strictly parabolic or elliptic. In our present case, it will be strictly parabolic, as we will see.

In [4], Jost and Zuo obtained

Jost-Zuo's theorem. *Let M be as above with the above constructed metric. Assume that the representation $\rho : \pi_1(M) \to G$ is stabilizing at infinity and reductive. Then there exists a ρ-equivariant harmonic map $u : \tilde{M} \to G/K$ from the universal covering of M to G/K with finite energy on a fundamental domain of M.*

Now, we turn back to our construction. Since M is properly homotopical equivalent to N, one can get a special homomorphism $\rho : \pi_1(M) \cong \Gamma \subset SO(n,1)$, which certainly satisfies the conditions of Jost-Zuo's theorem except stabilizing property at infinity, which we will show in the following. Take a splitting $M = M_b \bigcup \cup_{i=1}^{p} M_i$, here M_b is bounded in M, M_i is a neighborhood of D_i. Obviously, any small loop γ_∞ around D can be assumed to lie in one of $M_i, i = 1, 2, \cdots, p$. Since M is properly homotopic to N, corresponding to this splitting, we also have a splitting of $N : N = N_b \bigcup \cup_{i=1}^{p} N_i$, here N_b is bounded in N, N_i is a neigborhood of a cusp of N. The homotopy maps M_b into N_b, M_i into $N_i, i = 1, 2, \cdots, p$. So, if a small loop γ_∞ lies in M_i and represents a nontrivial element of $\pi_1(M)$, $\rho(\gamma_\infty)$ also represents a nontrivial element of Γ, and can be represented by a loop lying in N_i. Since $\rho(\gamma_\infty)$ is nontrivial and lies in N_i, by a point compactification of N_i, one has that $\rho(\gamma_\infty)$ is strictly parabolic. So, one obtains that ρ is stabilizing at infinity. By Jost-Zuo's theorem, we get a harmonic map of finite energy from M (under the constructed metric) to N, which induces ρ, and hence is a homotopic equivalence.

Remark. In [5], Jost and Zuo first constructed a finite energy map, then they used a variational technique to get the above harmonic map. In fact, if applying the construction of this finite energy map to our present setting, it is easy to see that this map is actually proper and one can also get the above harmonic map using Schoen-Yau's technique [10]. Then the same reasoning as in section 3c of [3] derives that this harmonic map is essentially proper.

Summing up the above arguments, we have the following

Lemma 1. *Let M and N be as above. Then there exists a properly harmonic homotopic equivalence of finite energy from M to N.*

3. Some properties of the harmonic maps

The argument in this section is the generalization of that of Siu and Sampson in the compact case to the noncompact case. Our exposition is general, which obviously includes our present case.

Let M be a complete Kähler manifold with a fixed Kähler metric ω, $N = \, , \backslash G/K$ a locally symmetric space of noncompact type. Here G is a semisimple Lie group, K is a maximal compact subgroup of G, and , is a lattice of G (for standard references see [11] and [12]). Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K, then one has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Denote by $\mathfrak{p}^C$ the complexification of $\mathfrak{p}$. One can then identify the complexification of the tangent space at any point of N with $\mathfrak{p}^C$. This is unique up to the right action of K and the left action of , . Since these actions preserve all relevant structures, we may regard $df(T_x^{1,0}M)$ as a subspace of $\mathfrak{p}^C$, for any map $f : M \to N$ and any point $x \in M$.

On the other hand, in general, let $h : M \to N$ be a smooth map from a Kähler manifold to a Riemannian manifold. If we denote $\{\omega^1, \omega^2, \cdots, \omega^m, \omega^{\overline{1}}, \omega^{\overline{2}}, \cdots, \omega^{\overline{m}}\}$ to be local orthonormal coframes of M and $\{\theta^1, \theta^2, \cdots, \theta^n\}$ to be local orthonormal coframes of N, then one can define h^i_α and $h^i_{\overline{\alpha}}$ by the equation

$$h^*(\theta^i) = h^i_\alpha \omega^\alpha + h^i_{\overline{\alpha}} \omega^{\overline{\alpha}},$$

for all $1 \leq i \leq n, 1 \leq \alpha \leq m$, and $\overline{1} \leq \overline{\alpha} \leq \overline{m}$. As usual, one can also define the higher order covariant derivatives $h^i_{\alpha\beta}, h^i_{\alpha\overline{\beta}}, h^i_{\overline{\alpha}\beta}, h^i_{\overline{\alpha}\overline{\beta}}, h^i_{\alpha\beta\gamma}, h^i_{\overline{\alpha}\beta\gamma}, h^i_{\alpha\overline{\beta}\overline{\gamma}}$, etc. And one has the commutation formulae concerning higher order covariant derivatives, for example,

$$h^i_{\alpha\overline{\beta}} = h^i_{\overline{\beta}\alpha};$$

$$h^i_{\alpha\overline{\beta}\overline{\gamma}} - h^i_{\alpha\overline{\gamma}\overline{\beta}} = -R^N_{iklm} h^k_\alpha h^l_{\overline{\beta}} h^m_{\overline{\gamma}},$$

in the second formula, we used Kähler condition.

Using the above notations, the harmonic map equation is then given by

$$h^i_{\alpha\overline{\alpha}} = 0.$$

We say that h is pluriharmonic if it satisfies

$$h^i_{\alpha\overline{\beta}} = 0$$

for all $i, \alpha, \overline{\beta}$.

From now on, we assume that $h : M \to N$ is a harmonic map with finite energy, here M is a complete Kähler manifold with a fixed Kähler metric ω and N is a locally symmetric space of noncompact type. Fix a point $O \in M$. For any positive integral number, take a C^∞cut-off function $\eta_k : M \to R^+$ as follows:

$$\begin{aligned}
&\eta_k \equiv 1, \text{ on } B_0(R_k);\\
&\eta_k \equiv 0, \text{ on } M \setminus B_0(2R_k);\\
&|\nabla \eta_k|_\omega \le \frac{2}{R_k}, \text{ on } M,
\end{aligned}$$

here, $\{R_k\}$ is a sequence of positive numbers, which is strictly monotone and divergent. One then has

$$\begin{aligned}
\eta_k^2 |h^i_{\alpha\overline{\beta}}|^2 &= \eta_k^2 h^i_{\alpha\overline{\beta}} h^i_{\overline{\alpha}\beta}\\
&= (\eta_k^2 h^i_{\overline{\alpha}\beta} h^i_\alpha)_{\overline{\beta}} - (\eta_k^2 h^i_{\overline{\alpha}\beta})_{\overline{\beta}} h^i_\alpha\\
&= (\eta_k^2 h^i_{\overline{\alpha}\beta} h^i_\alpha)_{\overline{\beta}} - 2\eta_k \eta_{k,\overline{\beta}} h^i_{\overline{\alpha}\beta} h^i_\alpha - \eta_k^2 h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha.
\end{aligned}$$

Integrating both sides on M and using Hölder inequality, one has

$$\begin{aligned}
\int_M \eta_k^2 |h^i_{\alpha\overline{\beta}}|^2 &= -2\int_M \eta_k \eta_{k,\overline{\beta}} h^i_{\overline{\alpha}\beta} h^i_\alpha - \int_M \eta_k^2 h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha\\
&\le 2\epsilon \int_M \eta_k^2 |h^i_{\alpha\overline{\beta}}|^2 + \frac{2}{\epsilon}\int_M |\eta_{k,\overline{\beta}}|^2 |h^i_\alpha|^2 - \int_M \eta_k^2 h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha.
\end{aligned}$$

As ϵ is sufficiently small, say $\epsilon \le \frac{1}{3}$, one has

$$0 \le (1-2\epsilon)\int_M \eta_k^2 |h^i_{\alpha\overline{\beta}}|^2 \le \frac{2}{\epsilon}\int_M |\eta_{k,\overline{\beta}}|^2 |h^i_\alpha|^2 - \int_M \eta_k^2 h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha.$$

On the other hand, using the commutation formulae and the Hermitian negative curvature condition of N under Sampson's sense, one has

$$\begin{aligned}
h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha &= h^i_{\beta\overline{\alpha}\overline{\beta}} h^i_\alpha\\
&= (h^i_{\beta\overline{\beta}\overline{\alpha}} - R_{iklm} h^k_\beta h^l_{\overline{\alpha}} h^m_{\overline{\beta}}) h^i_\alpha\\
&= -R_{iklm} h^i_\alpha h^k_\beta h^l_{\overline{\alpha}} h^m_{\overline{\beta}}\\
&\ge 0.
\end{aligned}$$

Thus, one has

$$0 \leq (1-2\epsilon)\int_M \eta_k^2 |h^i_{\alpha\overline{\beta}}|^2 \leq \frac{2}{\epsilon}\int_M |\eta_{k,\overline{\beta}}|^2 |h^i_\alpha|^2$$
$$\leq \frac{c(\epsilon)}{R_k^2}\int_{B_O(2R_k)} |h^i_\alpha|^2,$$

where $c(\epsilon)$ is a positive constant depending only on ϵ. As $k \to \infty$, by the choice of η_k and the finiteness of energy of h, one has $h^i_{\alpha\overline{\beta}} = 0$, i.e., h is pluriharmonic. By the above argument, one also has for any k

$$0 = \int_M \eta_k^2 h^i_{\overline{\alpha}\beta\overline{\beta}} h^i_\alpha = -\int_M \eta_k^2 R_{iklm} h^i_\alpha h^k_\beta h^l_{\overline{\alpha}} h^m_{\overline{\beta}}$$

Since the integrand of the right-hand side integral is nonnegative, so, one has on M

$$R_{iklm} h^i_\alpha h^k_\beta h^l_{\overline{\alpha}} h^m_{\overline{\beta}} \equiv 0.$$

On the other hand,

$$\begin{aligned}
&R_{iklm} h^i_\alpha h^k_\beta h^l_{\overline{\alpha}} h^m_{\overline{\beta}} \\
&=< R(\partial h(e_\alpha), \partial h(e_\beta))\overline{\partial} h(e_{\overline{\alpha}}), \overline{\partial} h(e_{\overline{\beta}}) > \\
&= -< [\partial h(e_\alpha), \partial h(e_\beta)], [\overline{\partial} h(e_{\overline{\alpha}}), \overline{\partial} h(e_{\overline{\beta}})] >,
\end{aligned}$$

here $e_1, e_2, \cdots, e_m, e_{\overline{1}}, e_{\overline{2}}, \cdots, e_{\overline{m}}$ are the dual frames of $\omega^1, \omega^2, \cdots, \omega^m, \omega^{\overline{1}}, \omega^{\overline{2}}, \cdots, \omega^{\overline{m}}$. So, $[\partial h(e_\alpha), \partial h(e_\beta)] = 0$ for all α and β. Thus, one has that if one identifies $\partial h(T^{1,0}_x M)$ with a subspace of $\mathfrak{p}^C$, then $\partial h(T^{1,0}_x M)$ is an abelian subspace of $\mathfrak{p}^C$. Therefore, $\dim_C \partial h(T^{1,0}_x M)$ should not be greater than the rank of N. If one applies this assertion to our case, one obtains

Lemma 2. *Let $u : M \to N$ be the harmonic homotopic equivalence of finite energy in the last section. Then u has rank at most* 2.

4. The proof of the theorem

Assume that M is properly homotopical equivalent to $N = \, , \setminus SO(n,1)/SO(n), n \geq 3$. Therefore, by section 2, one has a harmonic homotopic equivalence of finite energy h from M (with the constructed metric) to N (with standard symmetric metric); by section 3, this harmonic homotopic equivalence has rank at most 2. Now, we will proceed to derive a contradiction.

Assume that N has at least two cusps. It is easy to see that the boundary of a suitable neighborhood of each cusp represents a nontrivial homology element in $H^{n-1}(N, R)$; by lemma 1, one also sees that the above harmonic map is proper. So, there exists at least some point at which this harmonic map is of maximal rank n. This is a contradiction (note that if one assumes $n \geq 4$, we need not consider the properness of the harmonic map). Therefore, the proof is reduced to the case that N has only one cusp. We will use a result of Selberg [13, p.39] to reduce this case to the case of at least two cusps. More precisely, assuming that N has only one cusp, we will make a finite covering N' of N, which is of at least two cusps, using Selberg's result. Then, we make the corresponding covering M' of M and lift the above harmonic map to another harmonic map $h' : M' \to N'$ with respect to the corresponding lifting metrics. It is not difficult to see that h' satisfies all properties of h, namely, h' is a properly harmonic homotopic equivalence of finite energy from M' to N'. So, the above argument also implies that h' is of maximal rank at some point of M'. Since h' is the lifting of h, so, h also has maximal rank at some point of M. The contradiction is obtained.

Assume that N has only one cusp. Since N is of finite volume, , is not a parabolic subgroup of $SO(n, 1)$, namely, there exists an element γ of , , which does not fix any lifting of the unique cusp of N to the boundary at infinity of $SO(n, 1)/SO(n)$. On the other hand, a result of Selberg [13, p.39] says that any lattice of $SO(n, 1)$ is residually finite, that is to say, the intersection of all subgroups of finite index of a lattice is its unit. In particular, so is , . Therefore, we can choose a subgroup of finite index of , which does not contain the above element γ. A lifting argument shows that one can actually choose a normal subgroup of finite index. Corresponding to this normal subgroup, we have a smooth finite covering of N, which has at least two cusps thanks to the properties of γ. This completes the proof of the theorem.

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