COMPUTER CALCULATION OF THE DEGREE OF MAPS INTO THE POINCARÉ HOMOLOGY SPHERE

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Computer calculation of the degree of maps into the Poincaré homology sphere

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Abstract

Let M, P be Seifert 3-manifolds. Does there exist a degree one map $f: M \to P$? The problem was completely solved in [HWZ] for all cases except when P is the Poincaré homology sphere. We investigate the remaining case by elaborating and implementing a computer algorithm that calculates the degree. As a result, we get an explicit experimental expression for the degree through numerical invariants of the induced homomorphism $f_*: \pi_1(M) \to \pi_1(P)$.

1. Introduction

Let M, P be closed oriented 3-manifolds such that the order n of the fundamental group $\pi_1(P)$ is finite. Let $\varphi: \pi_1(M) \to \pi_1(P)$ be a homomorphism. Using elementary facts from obstruction theory, one can easily show that

(1) φ is realizable geometrically, i. e. there exists a map $f: M \to P$ such that $f_* = \varphi$;

(2) $\deg(f)$ mod n depends only on φ.

The paper is devoted to the elaboration of a computer algorithm for calculating the degree. We apply the algorithm to maps into the Poincaré homology sphere P and under certain restrictions give an experimental explicit formula for $\deg(f)$ through numerical invariants of φ. The formula reduces the problem of finding out degree one maps onto P to purely number-theoretical questions. For background informations see [HWZ] and [HZ].

The calculation of the degree is usually a difficult procedure, even for concrete examples. It requires vast manipulations with group presentations and calculations in group rings. For instance in [P], a self mapping of the Poincaré homology sphere is constructed which has degree 49 and induces an automorphism of the fundamental group.

2. A little bit of theory

Let $f: M \to P$ be a map between closed 3-manifolds such that the order n of $\pi_1(P)$ is finite. We assume that both manifolds are equipped with CW structures and f is cellular. We also assume that each manifold has exactly one vertex, and that P has only one 3-cell. Let $p: \tilde{P} \to P$, $p_1: \tilde{M} \to M$ be universal coverings and $\tilde{f}: \tilde{M} \to \tilde{P}$ the map induced by f. We describe three items needed for calculating the degree.

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I. THE BOUNDARY CYCLE. Let $B_1, B_2, \ldots, B_k$ be the 3-cells of M and $\beta_M \in C_3(M; Z)$ the corresponding 3-chain composed from the 3-cells taken with the orientation inherited from the orientation of M. Since the 2-cycle $\partial\beta_M \in C_2(M; Z)$ is a sum of spherical cycles, it can be lifted to a 2-cycle in $\tilde{M}$. Let $\partial\tilde{\beta}_M \in C_2(\tilde{M}; Z)$ be a lifting of $\partial\beta_M$. Then the *boundary cycle* $\partial\tilde{\beta}_M$ will be the first item needed for the calculation of the degree. By definition, it is the sum of boundary cycles $\partial\tilde{\beta}_{B_i}$ for the 3-cells B_i.

Remark 2.1. The group $C_2(\tilde{M}; Z)$ considered as a $\pi_1(M)$-module with respect to the covering translations can be identified with the free $\pi_1(M)$-module $C_2(M; Z[\pi_1(M)])$. To specify the identification, one should fix the orientations and the base points for the attaching curves of all 2-cells, as well as a base point for $\tilde{M}$ over the vertex of M.

II. THE CHARACTERISTIC COCHAIN. Choose a point x_0 in the interior of the unique 3-cell B^3 of P. The set $X = p^{-1}(x_0)$ can be considered as a 0-dimensional cycle in P with coefficients in Z_n. Since $H_1(\tilde{P}; Z_n) = 0$, X bounds an 1-dimensional chain Y in $\tilde{P}$ with coefficients in Z_n. Let us point out that X and Y are actually elements of the corresponding chain groups of $\tilde{D}$, where $\tilde{D}$ is the decomposition of $\tilde{P}$ dual to the one induced by the cell decomposition of P. Alternatively, one can imagine X and Y as singular chains with the additional requirement that Y should be transverse to the 2-skeleton of $\tilde{P}$. Hence, for any 2-chain $\sigma \in C_2(\tilde{P}; Z)$, the intersection index $\sigma \cap Y \in Z_n$ is well-defined. Therefore we get a homomorphism $\xi_P: C_2(\tilde{P}; Z) \to Z_n$ that is, a 2-cochain in $C^2(\tilde{P}; Z_n)$; this is the second item needed.

III. THE INDUCED CHAIN MAP. The third item we need for calculation is the module homomorphism $\tilde{f}_*: C_2(M; Z[\pi_1(M)]) \to C_2(P; Z[\pi_1(P)])$. It can be described as the chain map induced by f, and takes $C_2(\tilde{M}; Z) = C_2(M; Z[\pi_1(M)])$ to $C_2(\tilde{P}; Z) = C_2(P; Z[\pi_1(P)])$. It is easy to see that $\tilde{f}_*$ preserves the module structures in the sense that for all $g \in \pi_1(M)$, $\sigma \in C_2(M; Z[\pi_1(M)])$ we have $\tilde{f}_*(g\sigma) = f_*(g)\tilde{f}_*(\sigma)$, where $f_*: \pi_1(M) \to \pi_1(P)$ is the induced homomorphism.

Theorem 2.1. *$deg(f) = \xi_P(\tilde{f}_*(\partial\tilde{\beta}_M))$ mod n .*

Proof. Let $\sigma^1 \in C_1(Q; Z_n)$, $\sigma^3 \in C_3(Q; Z)$ be two (say, singular) chains in an orientable 3-manifold Q in general position; in particular, their boundary cycles are disjoint. Then the linking number $\mathrm{lk}(\partial\sigma^3, \partial\sigma^1) \in Z_n$ is well-defined and can be calculated as the intersection number $\sigma^3 \cap \partial\sigma^1$ as well as the intersection number $\partial\sigma^3 \cap \sigma^1$.

Thus $\sigma^3 \cap \partial\sigma^1 = \partial\sigma^3 \cap \sigma^1$. Taking $Q = \tilde{P}$, $\sigma^1 = Y$ and $\sigma^3 = \tilde{f}_*(\tilde{\beta}_M)$, we get $\tilde{f}_*(\tilde{\beta}_M) \cap X = \partial\tilde{f}_*(\tilde{\beta}_M) \cap Y$. It remains to note that, by the definition of the degree (say, in terms of singular homology theory), the left part of the equality coincides with $\deg(f)$ mod n while the right side is just $\xi_P(\tilde{f}_*(\partial\tilde{\beta}_M))$, by the definition of ξ_P. ∎

Remark 2.2. It is important to note that ξ_P and $\partial\tilde{\beta}_M$ depend only on P and M, respectively, and do not depend on f.

3. How to calculate the boundary cycle

3.1. General case

Let M be a CW 3-manifold such that its 2-skeleton K_M^2 has exactly one vertex. We say that a presentation $\langle a_1, \dots a_r \mid R_1, \dots R_q \rangle$ of $\pi_1(M)$ is *geometric* if:

(1) all edges of K_M^2 are oriented and correspond bijectively to the generators $a_1, \dots, a_r$;

(2) all 2-cells of K_M^2 are oriented and correspond bijectively to the relations $R_1, \dots, R_q$;

(3) the boundary curve of each 2-cell is equipped with a base point such that, starting from this point, the curve follows the edges just so as they are written in the corresponding relation.

Let $\langle a_1, \dots a_r \mid R_1, \dots R_q \rangle$ be a geometric presentation of $\pi_1(M)$, and let B be a 3-cell of M. The simplest way to calculate the contribution $\partial\tilde{\beta}_B$ to the boundary cycle made by the boundary of B is to construct a *spherical diagram* for the attaching map $h_B: S_B^2 \to K_M^2$ of B. Recall that the spherical diagram is a cellular decomposition θ_B of S_B^2 such that: (1) every edge of θ_B is oriented and labeled with a generator a_i; (2) every 2-cell of θ_B is oriented and labeled with a relation R_j; (3) the boundary curve of each 2-cell is equipped with a base point such that, starting from this point, the curve follows the edges just so as they are written in the corresponding relation; (4) the attaching map h_B is cellular and preserves the orientations, labels and base points.

Let us fix a vertex v of S_B^2 as a global base point for S_B^2, and assign to every 2-cell c of S_B^2 the following data:

(1) the sign $\varepsilon(c) = \pm 1$ that shows whether or not the orientation of c agrees with the fixed orientation of S_B^2;

(2) the element $g(c) = [h_B(\gamma(c))]$ of $\pi_1(M)$, where $\gamma(c)$ is a path in S_B^2 joining v to the base point of c and $[h_B(\gamma(c))]$ denotes the element of $\pi_1(M)$ that corresponds to the loop $h_B(\gamma(c))$ in K_M^2;

(3) the relation $R_{i(c)}$ which labels c.

It is convenient to consider the chain group $C_2(\tilde{M}, Z)$ as the free $\pi_1(M)$-module generated by the relations $R_1, \dots, R_q$. The proof of the following statement is evident.

Lemma 3.1.1. *The contribution $\partial\tilde{\beta}_B$ made by a 3-cell B of M to $\partial\tilde{\beta}_M$ equals $\sum_c \varepsilon(c)g(c)R_{i(c)}$, where the sum is taken over all 2-cells in S_B^2.*

3.2. Useful example

Let us consider an informative example. Present the torus $T^2 = S^1 \times S^1$ as a CW complex with one vertex, two edges $a = S^1 \times \{*\}, t = \{*\} \times S^1$, and one

2-cell r_1 that corresponds to the relation $R_1 = at^{-1}a^{-1}t$ of $\pi_1(T^2) = \langle a, t \mid R_1 \rangle$. Choose a pair $\alpha \geq 0, \beta$ of coprime integers. We extend the cell decomposition of T^2 to a cell decomposition of a solid torus M, $\partial M = T^2$, by attaching two new cells: a 2-cell r_2 and a 3-cell B. Note that the boundary curve m of r_2 lies in $a \cup t$ and thus can be written as a word in generators a, t. To make the situation interesting, we require that m wraps totally α times around a and β times around t. In other words, the corresponding word (denote it by $w_{\alpha,\beta}(a,t)$) should determine the element $\alpha a + \beta t$ in the homology group $H_1(a \cup t; Z)$. In general we cannot take $w_{\alpha,\beta}(a,t) = a^\alpha t^\beta$ since the presentation $\langle a, t \mid at^{-1}a^{-1}t, a^\alpha t^\beta \rangle$ may be not geometric.

Let us describe a simple geometric procedure for finding out $w_{\alpha,\beta}(a,t)$. Present a regular neighborhood N of $a \cup t$ in T^2 as a disc with two index one handles. The key observation is that m, being the boundary of a meridional disc of the solid torus, can be shifted in N to a simple closed curve m_1 which is normal with respect to the handle decomposition of N. Therefore, we can reconstruct m_1 as follows:

1. Take α parallel copies of a and $\mid \beta \mid$ parallel copies of t such that the end points of them lie on the boundary of the disc around the vertex;

2. Join the copies inside the disc to get a normal curve without intersections. This can be done in two ways, and the right choice depends on the sign of β.

It remains only to read off $w_{\alpha,\beta}$ by travelling along m_1, see Fig. 1 for the case $\alpha = 5, \beta = 2$ when we get $w_{\alpha,\beta} = a^3ta^2t$.

Lemma 3.2.1. *Let K be a CW complex realizing the presentation $\langle a, t \mid R_1, R_2 \rangle$, where $R_1 = at^{-1}a^{-1}t$ and $R_2 = w_{\alpha,\beta}(a,t)$ for a pair $\alpha \geq 0, \beta$ of coprime integers. Suppose that a solid torus M is obtained from K by attaching a 3-cell B. Then $\partial\tilde{\beta}_M = -R_1 + (1 - a^x t^y)R_2$, where $\alpha y - \beta x = 1$.*

Proof. To calculate the boundary cycle $\partial\tilde{\beta}_M$ (which in our case coincides with the contribution $\partial\tilde{\beta}_B$ made by B), construct a spherical diagram for B. It contains only three 2-cells c_1, c_2, c_3 labeled by R_1, R_2, R_2 and having signs $-1, 1, -1$, respectively. Let us choose the common base point of c_1, c_2 as a global vertex v, see Fig. 2.

By Lemma 3.1.1 we get $\partial\tilde{\beta}_B = -R_1 + R_2 - g(c_3)R_2$, where $g(c_3) \in \pi_1(M)$ corresponds to a path $\gamma(c_3)$ in S^2_B joining v to the base point of c_3. Note that if we push slightly the loop $h_B(\gamma(c_3))$ into the interior of M, we get a circle that intersects the meridional disc r_2 of the solid torus M positively in exactly one point. It follows that $g(c_3)$ can be presented as $a^x t^y \in \pi_1(M)$, where the integers x, y satisfy the equation $\alpha y - \beta x = 1$ and thus serve as the coordinates of a positively oriented longitude of the torus. ∎

Remark 3.2.1. The following simple rules can be used for recursive calculating of the word $w_{\alpha,\beta}$:

$$w_{1,0}(a,t) = a, \; w_{0,\pm 1}(a,t) = t^{\pm 1};$$

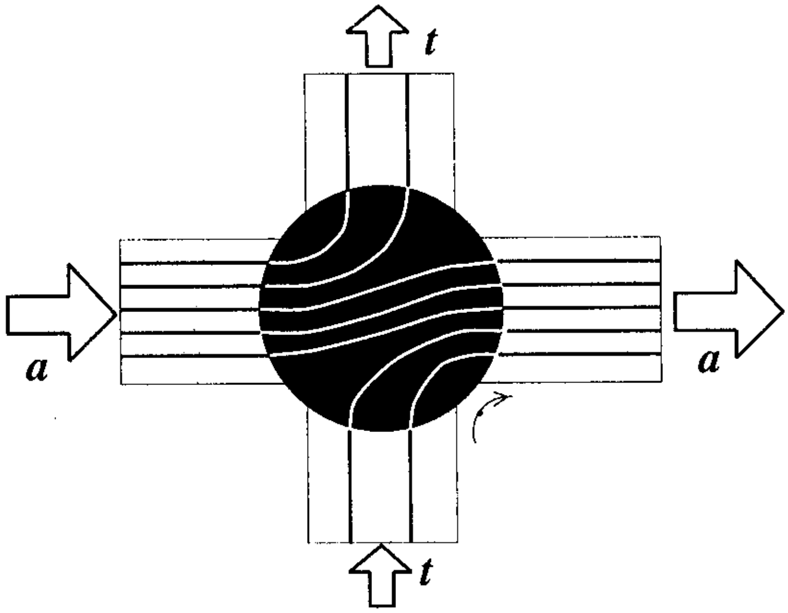

Figure 1: Simple closed curve of the type (α, β)

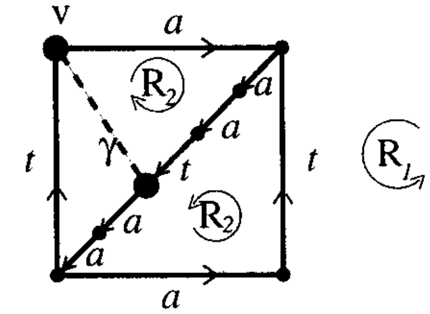

Figure 2: Spherical diagram for the 3-cell of M

$w_{\alpha+\beta,\beta}(a,t) = w_{\alpha,\beta}(a,at);$

$w_{\alpha,\alpha+\beta}(a,t) = w_{\alpha,\beta}(at,t).$

3.3. Boundary cycles of Seifert manifolds

We restrict ourselves to Seifert manifolds fibered over the 2-sphere with three exceptional fibers. Let $M = M((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3))$ be a Seifert manifold, where $\alpha_i \geq 0, \beta_i, 1 \leq i \leq 3$, are non-normalized parameters of the exceptional fibers. Then $\pi_1(M)$ can be presented by $\langle a_1, a_2, a_3, t \mid a_i t a_i^{-1} t^{-1}, a_i^{\alpha_i} t^{\beta_i}, a_1 a_2 a_3, i = 1,2,3\rangle$, but this presentation is not geometric. To improve the shortcoming, we will use another presentation $\langle a_1, a_2, a_3, t \mid R_j, 1 \leq j \leq 7\rangle$ of the same group with the same generators, where $R_{2i-1} = a_i t^{-1} a_i^{-1} t, R_{2i} = w_{\alpha_i\beta_i}(a_i,t)$ for $i = 1,2,3$, $R_7 = a_1 a_2 a_3$, and the words $w_{\alpha_i\beta_i}$ have been described above. The CW complex that realizes this presentation embeds in M such that the complement consists of four 3-balls $B_i, 1 \leq i \leq 4$. First three of them correspond to solid tori containing exceptional fibers, and the last corresponds to a regular fiber.

Theorem 3.3.1. *If* $M = M((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3))$*, then* $\partial\tilde{\beta}_M = \sum_{i=1}^{3}(-R_{2i-1} + (1 - a_i^{x_i} t^{y_i})R_{2i}) + (R_1 + a_1 R_3 + a_1 a_2 R_5 - (1-t)R_7)$*, where* $\alpha_i y_i - \beta_i x_i = 1$.

Proof. The first three summands can be obtained by Lemma 3.2.1. The last summand can be obtained in a similar way. The only difference from the proof of Lemma 3.2.1 is that the corresponding spherical diagram, instead of one 2-cell with one odd R-label, contains three positively oriented 2-cells with labels R_1, R_3, R_5. ∎

4. How to calculate the characteristic cochain?

We assume that a closed 3-manifold P with order n group $\pi_1(P)$ is equipped with a CW structure such that there is only one vertex v, only one 3-cell B, and the 2-dimensional skeleton K_P^2 of P is a geometric realization of a presentation $\langle b_1,\ldots,b_s \mid Q_1,\ldots,Q_s\rangle$ of $\pi_1(P) = \pi_1(P;v)$. We will identify generators and relations of $\pi_1(P)$ with edges and 2-cells of K_P^2, respectively. Fix a base point x_0 in the interior of B. For each $i, 1 \leq i \leq s$, choose a loop u_i in P with end points in x_0 such that $K_P^2 \cap u_i$ is a point in Q_i, and the intersection is transverse and positive. Clearly, the loops $u_i, 1 \leq i \leq s$ generate the group $\pi_1(P;x_0)$ isomorphic to $\pi_1(P;v)$; we will call them *dual generators*.

Let w be a word in the generators u_i. Then the formula $p(w)(c) = w \cap c$, where $c \in C_2(\tilde{P};Z)$ and $w \cap c$ is the intersection number, determines a homomorphism $p(w): C_2(\tilde{P};Z) \to Z$. It means that we have a cochain $p(w) \in C^2(\tilde{P};Z)$. It follows from the construction that:

(1) $p(u_i) = 1;$

(2) $p(w_1 w_2) = p(w_1) + w_1 p(w_2).$

These rules describe actually nothing more than a sort of *Fox calculus*, see [CF]. They are sufficient for finding out $p(w)$ for any word w in the generators u_i.

Proposition 4.1. *Let words $w_0, \dots, w_{n-1}$ in generators u_i present (without repetitions) all elements of $\pi_1(P; x_0)$. Then $\xi_P = \sum_{i=0}^{n-1} p(w_i)$ mod n is a characteristic cochain for P.*

Proof. Evident, since the union of paths given by $w_i, 0 \leq i \leq n-1$ presents the 1-chain Y such that ∂Y is the union of all points in $p^{-1}(x_0)$. ∎

We may conclude that all what we need for calculation is a sort of *normal form* for elements of $\pi_1(P)$, that is, a list of words in dual generators that presents without repetitions all elements of $\pi_1(P)$.

5. How to calculate the induced chain map?

5.1 What are logs?

Recall that M, P are closed oriented CW 3-manifolds such that the order n of $\pi_1(P)$ is finite. Let $\langle a_1, \dots a_r \mid R_1, \dots R_q \rangle$ and $\langle b_1, \dots, b_s \mid Q_1, \dots, Q_s \rangle$ be geometric presentation of their fundamental groups. Suppose that the homomorphism $\varphi = f_*: \pi_1(M) \to \pi_1(P)$ is given by a set of words h_i in the generators b_j that present the elements $\varphi(a_i)$ of $\pi_1(P), 1 \leq i \leq r$. We consider the chain group $C_2(M, Z[\pi_1(M)])$ as a free $Z[\pi_1(M)]$-module generated by the set $R_1, \dots, R_r$. Similarly, $C_2(P, Z[\pi_1(P)])$ is a free module generated by $Q_1, \dots, Q_s$. Denote by K the kernel of the quotient map $F(b_1, \dots, b_s) \to \pi_1(P)$, where $F = F(b_1, \dots, b_s)$ is the free group generated by $b_1, \dots, b_s$.

Let R be one of the relations R_i. In order to calculate the image $f_*(R) \in C_2(P, Z[\pi_1(P)])$ of the corresponding 2-cell, one may do the following:

(1) Replace each generator $a_i^{\pm 1}$ in R by the corresponding word $h_i^{\pm 1}$. We get a word $w \in K$;

(2) Present w as a product of conjugated relations, that is, in the form $w = \prod_k v_k Q_{i_k}^{\varepsilon_k} v_k^{-1}$, where $\varepsilon_k = \pm 1$ and $v_k \in F$;

(3) Then the image $f_*(R)$ is obtained by "taking log": $f_*(R) = \log(w)$, where $\log(w) = \sum_k \varepsilon_k \bar{v}_k Q_{i_k}$ and $\bar{v}_k$ is the image of v_k in $\pi_1(P)$.

We note that $\log(w)$ is a multivalued function since w can be presented as a product of conjugated relations in many different ways. This corresponds to the fact that φ can be realized by many different f; this arbitrariness does not affect the degree. One should point out that finding out $\log(w)$ (actually, step (2) above) is a non-trivial procedure. The problem is to realize it algorithmically. We solve the problem for the case when P is the Poincaré homology sphere.

5.2. On the fundamental group of the Poincaré sphere.

Later on we denote by P the Poincaré sphere. It is homeomorphic to the Seifert manifold $M((2,1),(3,1),(5,-4))$. Its fundamental group $\pi_1(P)$ consists

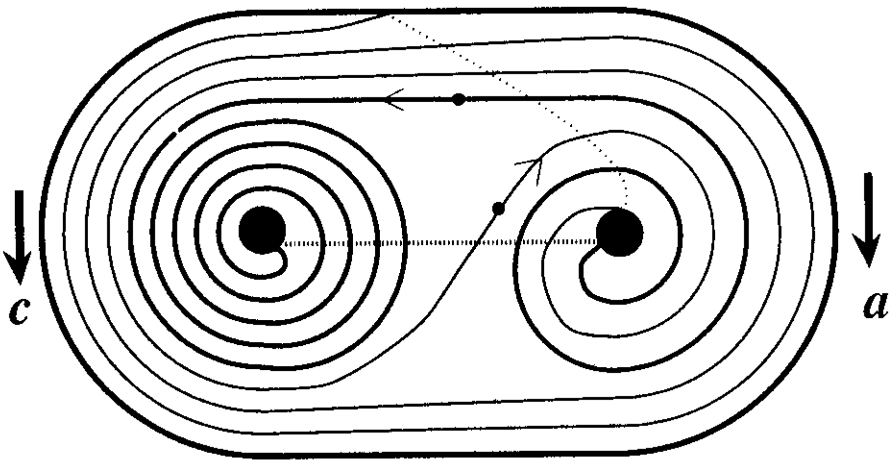

Figure 3: Heegaard diagram of the Poincaré homology sphere

of 120 elements and is isomorphic to $SL_2(Z_5)$ and to the binary icosahedron group I^*. We will denote it by π.

We will use the presentation $\langle a, c \mid Q_1, Q_2 \rangle$, where $Q_1 = c^5 a^{-2}$ and $Q_2 = aca^{-1}ca^{-1}c$. This presentation corresponds to the Heegaard diagram of P shown in Fig. 3. Therefore, the presentation is geometric.

Remark 5.2.1. Looking at Fig. 3, one can easily find out the dual generators (see Section 4): $u_1 = c, u_2 = c^{-1}a$.

Let us play a little with the relations $c^5a^{-2} = 1$, $aca^{-1}ca^{-1}c = 1$. Our goal is to get new relations. We will use essentially one basic transformation:

(*): If $w_1 = w_2$ is a relation, then $uw_1v = uw_2v$ is a relation for any words u, v.

Lemma 5.2.1. *In π, the following relations are true:*

(1) $aca = c^4ac^{-1}, \quad ac^{-1}a = cac;$

(2) $ac^{-2}a = c^4ac^2ac;$

(3) $ac^2ac^2a = c^4ac^2ac^{-1};$

(4) $ac^2ac^{-2}ac^{-1} = cac^2ac^{-2}a;$

(5) $c^{10} = 1.$

Proof. The first three relations are easy, so we concentrate our attention on proving the last two.

a). Multiplying $Q_2 = 1$ by $c^{-1}ac^{-1}$, we get $aca^{-1} = c^{-1}ac^{-1}$;

b). Using a), we get $ac^{-2}a = c\ c^{-1}ac^{-1}\ c^{-1}ac^{-1}\ c = c\ aca^{-1}aca^{-1}c = cac^2a^{-1}c$;

c). Let $w = ac^2ac^{-2}a$. Then from b) and $a^2 = c^5$ we have: $wc^{-1} = ac^2\ ac^{-2}a\ c^{-1} = ac^3ac^2a^{-1} = ac^{-2}a\ c^2a = c\ ac^2a^{-1}c^3a = cw$;

d). It follows from c) that $c^{10} = c^5ww^{-1}c^5 = wc^{-5}w^{-1}c^5$. On the other hand, the relation $a^2 = c^5$ allows one to permute c^{-5} with any other word. Thus we have $c^{10} = wc^{-5}w^{-1}c^5 = ww^{-1}c^{-5}c^5 = 1$. ∎

The following list L presents without repetitions all 120 elements of π: c^i, c^iac^j, $c^iac^2ac^j$, $c^iac^2ac^{-2}a$, where $0 \le i \le 9$, $0 \le j \le 4$. The words from L will be called *normal forms.*

Lemma 5.2.2. *There exists an effective algorithm that transforms any word in the generators a, c to a normal form presenting the same element of π.*

Proof. The normalizing algorithm works with reduced words. Thus, before and after each step, one should reduce the word we are working with. By the *a-size* of a word in generators a, c we mean the total number of occurrences of $a^{\pm 1}$ in it.

Let w be a word in generators a, c. Steps 1 – 5 below are based on the corresponding relations 1 – 5 from Lemma 5.2.1.

STEP 0. Using the relation $a^2 = c^5$, transform w to the form $w = c^{k_1}ac^{k_2}a \dots c^{k_m}ac^{k_{m+1}}$, where $m \ge 0$ and $k_i = \pm 1, \pm 2$ for $1 < i < m+1$ and $|k_{m+1}| \le 2$.

STEP 1. If w contains subwords aca or $ac^{-1}a$, we replace them by c^4ac^{-1} or cac, respectively.

STEP 2. If the initial segment of w has a form $c^{k_1}ac^{-2}a$, that is, $k_2 = -2$ and $m \ge 2$, we replace it by $c^{k_1-4}ac^2ac$.

STEP 3. If the initial segment of w has a form $c^{k_1}ac^2ac^2a$, we replace it by $c^{k_1+4}ac^2ac^{-1}$.

STEP 4. If the initial segment of w has a form $c^{k_1}ac^2ac^{-2}ac^{k_4}a$, we replace it by $c^{k_1-k_4}ac^2ac^{-2}$.

STEP 5. Reduce modulo 10 the power k_1 of the first term c^{k_1} of w.

Now we are ready to describe the algorithm. Let us apply to the given word w Steps 0 – 5 as long as possible. Since Steps 1 – 4 strictly decrease the a-size, the process terminates after a finite number of steps. It is easy to verify that the resulting word is in the normal form. ∎

5.3. Logs in the case of the Poincaré sphere

The algorithm for calculating logs is similar to the one described in Lemma 5.2.2. The only difference is that instead of operating with words in $F(a, c)$ we will operate with their *shadows.*

Denote by K the kernel of the quotient map $F \to \pi$. Let $\mathcal{M}$ be the free π-module generated by Q_1, Q_2. We define the *shadow group* $S(F)$ of F to be the semidirect product $\mathcal{M}\lambda F$, where the action of F on $\mathcal{M}$ is induced by the action of π. In other words, $\mathcal{S}(F)$ consists of pairs (μ, w), where $\mu \in \mathcal{M}$, $w \in F$. Multiplication is given by the rule $(\nu, u)(\mu, w) = (\nu + \bar{u}\mu, uw)$, where $\bar{u}$ is the image of u in π. Note that the unit of the group $\mathcal{S}(F)$ is $(0, 1)$, and the inverse element of (μ, w) is $(-\bar{w}^{-1}\mu, w^{-1})$.

If $w \in F$, then any pair $(\mu, w) \in \mathcal{S}(F)$ is called a *shadow* of w. The shadow $(0, w)$ is called *pure*. Similarly, the pair $(\mu, 1)$ is the *pure shadow* of $\mu \in \mathcal{M}$. Note that any product of shadows can be replaced by a product of pure shadows: $\prod_{i=1}^{k}(\lambda_i, w_i) = (\mu, 1)\prod_{i=1}^{k}(0, w_i) = (\mu, 1)(0, \prod_{i=1}^{k} w_i)$, where $\mu = \lambda_1 + \bar{w}_1\lambda_2 + \ldots \bar{w}_1\bar{w}_2 \ldots \bar{w}_{k-1}\lambda_k$. In other words, one can *purify* the factors.

Let $\mathcal{S}(K)$ denote the normal subgroup of $\mathcal{S}(F)$ generated by elements $(-Q_i, Q_i)$, $i = 1, 2$. Note that the left Q_i in the above expression is considered as a generator of $\mathcal{M}$ while the right Q_i is the word c^5a^{-2} or $aca^{-1}ca^{-1}c$. We define the *shadow group* of π as $\mathcal{S}(\pi) = \mathcal{S}(F)/\mathcal{S}(K)$. Another way to get $\mathcal{S}(\pi)$ is to take the quotient of $\mathcal{S}(F)$ by the relations $(0, Q_i) = (Q_i, 1), i = 1, 2$.

Lemma 5.3.1. $(-\mu, w) \in \mathcal{S}(K) \iff w \in K$ *and* $\mu = \log(w)$.

Proof. Recall that an element of a group lies in the normal subgroup generated by some elements if and only if it is a product of conjugates of the generators and their inverses. Thus for some $(\lambda_k, v_k) \in \mathcal{S}(F)$, $\varepsilon_k = \pm 1$, and $i_k = 1, 2$ we have:

$$(-\mu, w) \in \mathcal{S}(K) \iff (-\mu, w) = \prod_k (\lambda_k, v_k)(-\varepsilon_k Q_{i_k}, Q_{i_k}^{\varepsilon_k})(-\bar{v}_k^{-1}\lambda_k, v_k^{-1}) =$$

$$\prod_k(-\varepsilon_k\bar{v}_k Q_{i_k}, v_k Q_{i_k}^{\varepsilon_k}\bar{v}_k^{-1}) = (-\sum_k \varepsilon_k \bar{v}_k Q_{i_k}, \prod_k v_k Q_{i_k}^{\varepsilon_k} v_k^{-1}) = (-\log w, w)$$

by definition of log (see Section 5.1).

To construct an algorithm for calculating logs, we need a shadow counterpart of Lemma 5.2.1.

Lemma 5.3.2. *One can calculate* $\lambda_i \in \mathcal{M}, 0 \leq i \leq 5$, *such that in* $\mathcal{S}(\pi)$ *the following relations are true:*

$\mathcal{S}$*(1)* $(0, aca) = (\lambda_0, c^4ac^{-1})$, $\quad (0, ac^{-1}a) = (\lambda_1, cac)$;

$\mathcal{S}$*(2)* $(0, ac^{-2}a) = (\lambda_2, c^4ac^2ac)$;

$\mathcal{S}$*(3)* $(0, ac^2ac^2a) = (\lambda_3, c^4ac^2ac^{-1})$;

$\mathcal{S}$*(4)* $(0, ac^2ac^{-2}ac^{-1}) = (\lambda_4, cac^2ac^{-2}a)$;

$\mathcal{S}$*(5)* $(0, c^{10}) = (\lambda_5, 1)$.

Proof. The existence of λ_i is evident, since both sides of each relation are shadows of the same element of π. The problem consists in calculating λ_i. To solve it, we repeat the proof of Lemma 5.2.1 in terms of shadows starting with the relations $(0, Q_i) = (Q_i, 1)$ instead of $Q_i = 1$, $i = 1, 2$. In particular, we apply the following shadow version of the basic transformation $(*)$ (see Section 5.2):

$\mathcal{S}(*)$: If $(0, w_1) = (\lambda, w_2)$ is a relation, then $(0, uw_1v) = (\bar{u}\lambda, uw_2v)$ is a relation for any words u, v.

For example, the shadow versions of items a) and b) in the proof of Lemma 5.2.1 look as follows:

a). Multiplying $(0, Q_2) = (Q_2, 1)$ by $(0, c^{-1}ac^{-1})$, we get $(0, aca^{-1}) = (Q_2, c^{-1}ac^{-1})$ or, equivalently, $(0, c^{-1}ac^{-1}) = (-Q_2, aca^{-1})$;

b). Using a), we get $(0, ac^{-2}a) = (0, c\, c^{-1}ac^{-1}\; c^{-1}ac^{-1}\; c) =$ $(0, c)(-Q_2, aca^{-1})(-Q_2, aca^{-1})(0, c) = (-(\bar{c} + \bar{c}\bar{a}\bar{c}\bar{a}^{-1})Q_2, cac^2a^{-1}c)$.

We do not present here the values of λ_i since they are large (especially λ_5) in the sense that many of 240 integers presenting each of them are not zeros. Nevertheless, the authors calculated them, and in the sequel we will think that they are known. ∎

Proposition 5.3.1. *There exists an effective algorithm that, given $w \in K$, calculates* $\log(w) \in \mathcal{M}$.

Proof. The algorithm is a shadow twin of the one described in Lemma 5.2.2. Starting with the shadow $(0, w)$ of w, we apply shadow versions of Steps 0 - 5 as long as possible. It means that we use the shadow relations $\mathcal{S}(1) - \mathcal{S}(5)$ from Lemma 5.3.2 instead of the relations (1) - (5) from Lemma 5.2.1. After each step we purify the words by taking non-zero lambdas to the beginning of the word. We terminate with a shadow $(\mu, 1)$ of 1. Then $\log(w) = \mu$ by Lemma 5.3.1, since $(0, w) = (\mu, 1)$ in $\mathcal{S}(\pi)$ implies $(-\mu, w) \in \mathcal{S}(K)$. ∎

6. Computer implementation

6.1. Description and verification of the program

Recall that calculation of the degree of a map $f: M \to P$ requires knowledge of the three items: the boundary cycle $\partial\tilde{\beta}_M$, the characteristic cochain ξ_P, and the induced chain map $\tilde{f}$, see Section 2. If $M = M((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3))$, an explicit expression for $\partial\tilde{\beta}_M$ was obtained in Section 3, Theorem 3.3.1. Proposition 4.1 and the information on $\pi = \pi_1(P)$ obtained in Section 5.2 show how one can calculate ξ_P. The authors made this by hand, without computers. It is calculation of $\tilde{f}$ that requires a computer.

We assume that f is given by images $\tau, x_1, x_3 \in F$ of the generators t, a_1, a_3 of $\pi_1(M)$, respectively; the image x_2 of the generator a_2 can be found from the relation $a_1a_2a_3 = 1$. Let us describe the main steps of the computer program.

(1) For each relation R_i of the geometric presentation of $\pi_1(M), 1 \le i \le 7$, computer finds out its image w_i in F. In fact, it substitutes each generator of $\pi_1(M)$ by the given word in F that presents its image.

(2) Then computer works according to the algorithm described in Proposition 5.3.1 and finds out logs of all w_i. This is sufficient for obtaining $\tilde{f}$, since

$\tilde{f}_*(R_i) = \log(w_i)$.

(3) To get $\tilde{f}_*(\partial\tilde{\beta}_M)$, the computer substitutes all relation R_i in the expression for $\partial\beta_M$ by corresponding $\tilde{f}_*(R_i)$.

(4) The computer calculates the degree by evaluating ξ_P on $\tilde{f}_*(\partial\tilde{\beta}_M)$.

An extended version of the program calculates the degree for all possible homomorphisms $\pi_1(M) \to \pi$ by letting τ, x_1, x_3 run over all 120 elements of π each.

The program is written in PASCAL and occupies about 1000 lines (not including commentaries). It works sufficiently fast: the extended version requires a few seconds to run over all 120^3 cases. The maximal range of α_i, β_i is about 1000. The cause of the restriction is that for large α_i, β_i the words $w_{\alpha_i\beta_i}(a_i, t)$ can be too long, especially after substituting the generators by their images.

The program had passed all verification tests, among them:

- It gives correct answers for evident cases, in particular, for the identity homomorphism $\pi \to \pi$;
- It gives the same list of degrees for maps into P for differently presented homeomorphic Seifert manifolds;
- It gives the same degree for maps into P that differ by an inner automorphism of π. Multiplication of a degree d map $M \to P$ by a degree 49 map $P \to P$ inducing the unique non-trivial element of $\mathrm{Out}(\pi)$ produces a map of degree $49d$;
- The results of a vast computer experiment completely agree with all known facts about the degree of maps into P. In particular, computer rediscovered the set of Seifert homology spheres that admit degree one map onto P. We mean homology spheres $M((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3))$ such that $\alpha_1/2, \alpha_2/3, \alpha_3/5$ are integer and $\alpha_1\alpha_2\alpha_3/30 = \pm1, \pm49 \mod 120$. They are the only known Seifert homology spheres that admit degree one maps onto P, see [HWZ].

6.2. Results

Let $M = M((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3))$ be a Seifert manifold and $\langle a_1, a_2, a_3, t \mid a_i t a_i^{-1} t^{-1}, a_i^{\alpha_i} t^{\beta_i}, a_1 a_2 a_3, i = 1, 2, 3\rangle$ the standard presentation of $\pi_1(M)$.

The main goal of the computer experiment was to investigate the following question:

Problem 1. Let $\alpha_1 = 2p_1, \alpha_2 = 3p_2, \alpha_3 = 5p_3$, where p_i are relatively prime with 30 , $1 \le i \le 3$. Does there exist a degree one map of a Seifert manifold $M = M((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3))$ onto P?

Here one should point out that conditions $2 \mid \alpha_1, 3 \mid \alpha_2, 5 \mid \alpha_3$, and $\gcd(\alpha_1, 15) = 1$ are necessary for having a degree one map $M \to P$, see [HWZ, Corollary 9.3]. Also, for the most interesting case when M is a homology sphere we have $\alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_2 + \alpha_2\alpha_3\beta_1 = \pm 1$, which implies that p_i are relatively prime with 30, $1 \le i \le 3$.

Remark 6.2.1. It is known that any homomorphism $\varphi: \pi_1(M) \to \pi$ inducing a degree one map $f: M \to P$ must be surjective. Moreover, any map $f: M \to P$ can be lifted to a map $\hat{f}: M \to \hat{P}$, where $\hat{P}$ is the covering of P corresponding to the subgroup $G = f_*(\pi_1(M)) \subset \pi$. Note that $\deg(f) = [\pi: G] \deg(\hat{f})$, where $[\pi: G]$ is the index of G in π. If $[\pi: G] > 1$, it reduces the calculation of the degree for f to the one for $\hat{f}$, which is simpler.

Let $M = M((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3))$ be a manifold satisfying the conditions $\alpha_1 = 2p_1, \alpha_2 = 3p_2, \alpha_3 = 5p_3$ and $\gcd(p_1p_2p_3, 30) = 1$. We will assume also that all β_i are odd. This can be easily achieved by transformations $\beta_i \to \beta_i + \alpha_i$, $\beta_j \to \beta_j - \alpha_j$ for $i \ne j$. Under that assumption the rule $t \to a^2, a_1 \to a, a_2 \to a^{-1}c^{-1}, a_3 \to c$ determines a surjective homomorphism $\varphi_0: \pi_1(M) \to \pi$ called *standard*. Denote by ext_a the external automorphism of π that takes a to a and c to cac^2ac^{-1}. It is induced by a map $P \to P$ of degree 49.

Lemma 6.2.1. *Let α_i and β_i satisfy the above assumptions. Then for any homomorphism $\varphi: \pi_1(M) \to \pi$ the following conditions are equivalent:*

(1) φ is surjective;

(2) φ has the form $\varphi = \psi\varphi_0$, where $\psi: \pi \to \pi$ is either an inner automorphism of π or the product of ext_a and an inner automorphism;

(3) $\varphi(t) = a^2$.

Proof. Denote $\varphi(t)$ by τ, $\varphi(a_i)$ by x_i, and the order of x_i by k_i, $1 \le i \le 3$. Assume that τ is in the center $\{1, a^2\}$ of π. Since $x_1^{2p_1}\tau^{\beta_1} = 1$ and $\tau^{2\beta_1} = 1$, k_1 divides $4p_1$. Note that all possible orders of elements of π are contained in the following list: 1,2,3,4,5,6,10. Recall that p_1 is not divisible by 3 or 5. It follows that k_1 divides 4. Similar arguments show that k_2 divides 6 and k_3 divides 10.

(1) $\Longrightarrow$ (2). STEP 1. Since φ is surjective, τ is in the center of π and, as it is shown above, k_1 divides 4. It follows that $k_1 = 4$. Indeed, the relation $x_1x_2x_3 = 1$ shows that for $k_1 = 1, 2$ the image of φ would be generated by τ, x_2 and possibly a^2, the unique element of π having order 2. In this case the image would be abelian, which contradicts the surjectivity.

It is well known that all order 4 elements of π are conjugates of a. Thus, up to multiplication of φ by an inner automorphism of π, we may assume that $x_1 = a$. Since p_1 and β_1 are odd, the relation $x_i^{2p_1}\tau^{\beta_1} = 1$ implies that $\tau = a^2$.

STEP 2. Recall that k_2 divides 6. Just as above, we cannot have $k_2 = 1, 2$ because of the surjectivity of φ. Since $x_2^{3p_2}\tau^{\beta_2} = 1$, $\tau = a^2$, and β_2 is odd, we have $k_2 \ne 3$. The only remaining case is $k_2 = 6$. Similarly, $k_3 = 10$.

STEP 3. There are only four elements x of π such that x has order 10 and $a^{-1}x^{-1}$ has order 6: c, aca^{-1}, and their images under ext_a. Certainly, this fact could be obtained theoretically, but the authors got it by letting a simple computer program run over all elements of π. It implies easily (2).

(2) $\Longrightarrow$ (3). Since a^2 is fixed under all automorphisms of π, this implication is evident.

(3) $\Longrightarrow$ (1). Since $\tau = a^2$ is in the center, k_1, k_2, and k_3 divide 4,6, and 10, respectively (see above). The equality $x_1^{2p_1} a^{2\beta_1} = x_1^{2p_1} a^2 = 1$ says us that $k_1 = 1, 2$ is impossible. Thus $k_1 = 4$. We cannot have $k_2 = 1, 2$, since then $x_3 = x_2^{-1} x_1^{-1}$ would have order 4, which is impossible. Thus k_2 is divisible by 3. Similarly, k_3 is divisible by 5. It follows that the order of the subgroup $G \subset \pi$ generated by x_1, x_2, x_3 should be divisible by 4.3 and 5. Since π contains no subgroups of order 60, $G = \pi$. ∎

To a great extent, Lemma 6.2.1 facilitates the computer search for new degree one maps of Seifert manifolds onto P: under above conditions on α_i, β_i, it suffices to check only *standard* maps $M \to P$ that correspond to the standard homomorphisms $\pi_1(M) \to \pi$. The result of the corresponding computer experiment was negative: no new examples of degree one maps. Nevertheless, a manual analysis of the output had shown that the degrees of the standard maps are periodical with respect to any of parameters $\alpha_1 = 2p_1, \alpha_2 = 3p_2, \alpha_3 = 5p_3$, and β_i. Moreover, the periods are different for different i: 8 for $i = 1$, 12 for $i = 2$, and 20 for $i = 3$. This observation allows one to suggest an explicit artificial formula for the degrees of standard maps. Since we do not have a theoretical proof of the periodicity, we present the formula in a form of a conjecture. By $[x]_k$ we denote the residue of x modulo k. In other words, $[x]_k$ is the integer satisfying the conditions $x - [x]_k$ is divisible by k and $0 \le [x]_k < k$.

Conjecture 6.2.2. *Let $f_0: M((2p_1, \beta_1); (3p_2, \beta_2); (5p_3, \beta_3)) \to P$ be the standard map, where $gcd(p_1 p_2 p_3, 30) = 1$ and all β_i are odd. Then $deg(f_0) = A_1 + A_2 + A_3 + 39$ mod 120, where*

$$A_1 = 30\left(\left[\frac{p_1+\beta_1}{2}\right]_2\left[\frac{\beta_1-p_1-2}{2}\right]_4 + \left[\frac{p_1+\beta_1+2}{2}\right]_2\left[\frac{\beta_1+p_1+4}{2}\right]_4\right),$$

$$A_2 = 10\left(\left[\frac{1+\beta_2}{2}\right]_3[\beta_2+p_2-1]_{12} + \left[\frac{1-\beta_2}{2}\right]_3[\beta_2-p_2+1]_{12}\right),$$

$$A_3 = 12\left(\left[\frac{1+\beta_3^2}{2}\right]_5\left[\frac{1+p_3\beta_3}{2}\right]_{10} + \left[\frac{1-\beta_3^2}{2}\right]_5\left[\frac{11-p_3\beta_3}{2}\right]_{10}\right).$$

If the conjecture is true, then the solution of the degree one mapping problem can be reduced to purely number-theoretical questions.

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