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A COUNTEREXAMPLE TO A GENERALIZATION OF THE LEVI PROBLEM

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Abstract

We prove by means of a 5-dimensional counterexample that an open subset $X \subset \mathbb{C}^n$ which is exhaustable by an increasing sequence $(X_j)_{j\geq 1}$ of q-complete open sets in $\mathbb{C}$ is not necessarily q-complete.

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1. Introduction

Let X be a Stein manifold and $D \subset X$ an open subset which is exhaustable by an increasing sequence $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ of q-complete domains in X. Does it follow that D itself is q-complete?

The answer to this problem is yes if $q = 1$ where the special case when $(D_j)_{j\geq 1}$ is a sequence of Stein domains in $\mathbb{C}^n$ had been solved a long time ago by Behnke and Stein [2].

If X is arbitrary, the space D is not necessarily q-complete. Vajaitu [5], gave an example of such situation.

But it is unknown if an open subset $D \subset \mathbb{C}^n$ which is the union of an increasing sequence of q-complete domains is itself q-complete [3].

In this paper, we give an example of an open subset $D \subset \mathbb{C}^5$ which is an increasing sequence $(D_j)_{j\geq 1}$ of 3-complete open subsets, but such that D is not 3-complete.

For this, we modify a counterexample of Fornaess [4] concerning a generalization of the Levi problem.

2. 5-dimensional counterexample to a generalization of the union problem

We consider pairwise disjoint closures discs

$$\Delta_n = \{z \in \mathbb{C} : |z - \frac{1}{n}| < r_n\}, n \geq 2.$$

There exist numbers $\varepsilon_n > 0$ such that

$$\frac{-1}{2^n} < \varepsilon_n log(\frac{1}{2}|z - \frac{1}{n}|) < 0$$

in the set $\Delta - \Delta_n$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

The function $h : \Delta \longrightarrow \mathbb{R}^- \cup \{-\infty\}$ defined by, $h(z) = \sum_{n\geq 2} \varepsilon_n log(\frac{1}{2}|z - \frac{1}{n}|)$, is subharmonic, and $h(z) > \frac{-1}{2}$ on $\Delta - \bigcup_{n\geq 2} \Delta_n$.

Then, we modify the function h as follows:

Choose $k_n \in \mathbb{N}, k_n \geq 2$, such that $\frac{1}{k_n} < \varepsilon_n$, and put:

$$H(z) = \begin{cases} h(z) \;\; on \;\; \Delta - \bigcup_{n\geq 2} \Delta_n \\ Max(h(z), \frac{1}{k_n} log(\frac{1}{2}|z - \frac{1}{n}|) - 1 \;\; on \;\; \Delta_n \end{cases}$$

In [4] it is proved that J is subharmonic, and there exist pairwise disjoint discs $D_n = \{z \in \mathbb{C} : |z - \frac{1}{n}| < r'_n\}, n \geq 2$, relatively compact in Δ,
such that H is bounded from below on $\Delta - \bigcup_{n\geq 2} D_n$, and in some neighborhood of $\partial\Delta_n$, $H = h$ while on D_n, we have
$H(z) = \frac{1}{k_n} log(\frac{1}{2}|z - \frac{1}{n}|) - 1$.

Clearly the domain $D \subset \mathbb{C}^5$ defined by

$$D = \{(z, w) \in \Delta \times \mathbb{C}^4 - \{w_1 = w_2 = 0\} : H(z) - log|w| < 0\},$$

satisfies the following conditions:

(i) If $(z, w) \in D$ *and* $z \in \Delta - \bigcup_{n\geq 2} \Delta_n$, *then* $|w| > \frac{1}{2}$

(ii) If $z \in D_n$ *then* $(z, w) \in D$ *if and only if* $z \in D_n$ *and* $|(z - \frac{1}{n})||w|^{-k_n} < 2e^{k_n}$

Consider now the homeomorphic map Γ_n *defined, on* $\mathbb{C} \times (\mathbb{C}^4 - \{w_1 = w_2 = 0\})$, *by* $\Gamma_n(z, w) = ((z - \frac{1}{n})\frac{1}{(|w_1|^2+|w_2|^2)^{k_n}}, w)$;
and put

$$B_n = \{(\eta, w) \in \mathbb{C} \times (\mathbb{C}^4 - \{w_1 = w_2 = w_4 = 0\}) : |\eta| < 2e^{k_n}, (|w_1|^2 + |w_2|^2) < l_n\}$$

We may choose, by (ii), $l_n > 0$ sufficiently small such that if $(\eta, w) \in \mathbb{C} \times (\mathbb{C}^4 - \{w_1 = w_2 = 0\})$ *with* $|\eta| < 2e^{k_n}, |w_1|^2 + |w_2|^2 < l_n$, then:

$$(z, w) = \Gamma_n^{-1}(\eta, w) = (\frac{1}{n} + \eta(|w_1|^2 + |w_2|^2)^{k_n}, w) \in D_n \times (\mathbb{C}^4 - \{w_1 = w_2 = 0\}) \cap D$$

Consider now the open set X in $\mathbb{C}^5$ defined by: $X = D \cup \bigcup_{n\geq 2} \Gamma_n^{-1}(B_n)$.

We claim that X is not cohomologically 3-complete.

Let f be the holomorphic function on X defined by $f(z, w) = z$, *and let* $X_0 = \{(z, w) \in X : f(z, w) = 0\}$. *Then* $X_0 \simeq \{w \in \mathbb{C}^4 - \{w_1 = w_2 = 0\} : H(0) - log|w| < 0\}$.

We first prove that $H^3(X_0, O) \mp 0$. Here O is the sheaf of germs of holomorphic functions on $\mathbb{C}^4$.

We write $X_0 = Y_0 \cap Z_0$ *with* $Y_0 = \mathbb{C}^4 - \{w_1 = w_2 = 0\}$, *and* $Z_0 = \mathbb{C}^4 - \{w \in \mathbb{C}^4 : |w| \leq e^{H(0)}\}$

From theorem 15, [1] page 254, We deduce that the restriction map:

$$H^r(\mathbb{C}^4, O) \longrightarrow H^r(Z_0, O)$$

is an isomorphism onto if $r < dih(O) - 1$. But since $dih(O) = 4$, *and* Z_0 is not a domain of holomorphy, it follows that $H^r(Z_0, O) = 0$ *if* $r = 1, 2$ *and* $H^3(Z_0, O) \mp 0$.

From the Mayer-Vietoris exact sequence of sheaves, we obtain the induced exact sequence of cohomology groups

$$\longrightarrow H^3(Y_0 \cup Z_0, O) \longrightarrow H^3(Y_0, O) \oplus H^3(Z_0, 0) \longrightarrow H^3(X_0, O) \longrightarrow 0$$

where $Y_0 \cup Z_0 = \mathbb{C}^4 - \{|w| \leq exp(H(0)), w_1 = w_2 = 0\} = Y_0 \cup Z_0'$ with $Z_0' = \mathbb{C}^4 - \{|w_3|^2 + |w_4|^4 \leq e^{2H(0)}\}$. Z_0' is clearly 2-complete, because the function $\phi(z) = |z|^2 + \frac{1}{|z_3|^2+|z_4|^2-e^{2H(0)}}$ is a strongly 2-convex exhaustion on Z_0'.

From the exact sequence of cohomology groups:

$$\to H^2(Y_0, O) \oplus H^2(Z_0', O) \to H^2(Y_0 \cap Z_0') \to H^3(Y_0 \cup Z_0') \to H^3(Y_0, O_0) \oplus H^3(Z_0', O_{Z_0'}) \to$$

We deduce that: $H^2(Y_0 \cap Z_0', O) \simeq H^3(Y_0 \cup Z_0, O)$.

But since $\psi(z) = Max(|z|^2 + \frac{1}{|z_1|^2+|z_2|^2}, |z|^2 + \frac{1}{|z_3|^2+|z_4|^2-e^{2H(0)}})$ is a strongly 2-convex exhaustion function on:

$Y_0 \cap Z_0' = \mathbb{C}^4 - (\{|w_3|^2 + |w_4|^2 \leq e^{2H(0)}\} \cup \{w_1 = w_2 = 0\})$,

then $H^3(Y_0 \cup Z_0, O) = 0$. It follows, from the first exact sequence of cohomology, that $H^3(X_0, O_{X_0}) \mp 0$.

Consider now the exact sequence of sheaves

$$0 \longrightarrow O_X \xrightarrow{\phi} O_X \longrightarrow O_X/fO_X \longrightarrow 0.$$

where $(z, w) = z$, *and* $\phi(g) = f_x g$ *for any* $g \in O_{X,x}, x \in X$.

We get the induced exact sequence of cohomology groups

$$H^2(X_o, O_{X_o}) \longrightarrow H^3(X, O_X) \longrightarrow H^3(X, O_X) \longrightarrow H^3(X_o, O_{X_o}) \longrightarrow H^4(X, O_X)$$

If $H^p(X, O_X) = 0$ for $p \geq 3$, it follows from the exact sequence of cohomology groups that $H^3(X_o, O_{X_o}) = 0$ which is a contradiction. We conclude that X is not cohomologically 3-complete.

Theorem 1 -*There exists a sequence of 3-complete open subsets* $(X_j)_{j\geq 2}$ *of* $X \subset \mathbb{C}^5$ *such that*

$$X_2 \subset X_3 \subset \cdots \subset \bigcup_{j\geq 2} X_j = X$$

Proof

Define, for each $k \geq 2$,

$X_k = X - \bigcup_{n \geq k} \{(\frac{1}{n}, 0, 0)\} \times \mathbb{C} \times \mathbb{C}^*$

The subspaces X_k are strongly 3-complete. In fact, the function defined by:

$|z|^2 + |w|^2 + exp(\frac{-1}{H(z)-log|w|}) + \frac{1}{1-|z|} + \sum_{p<k} |z - \frac{1}{p}|(|w_1|^2 + |w_2|^2 + |w_4|^2)^{-k_p}$

$+ \prod_{p<k} \frac{|z - \frac{1}{p}|^{k_p}}{|w_1|^2 + |w_2|^2}$ is a strongly 3-convex exhaustion function on X_k.

But X is not 3-complete.

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