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Haakon A. Olsen
Institutt for fysikk
Norges teknisk-naturvitenskapelige universitet
N–7034 Trondheim, Norway

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Creation of relativistic positronium. Photoproduction cross sections including Coulomb corrections.

Haakon A. Olsen
Institute of Physics, University of Science and Technology, NTNU
N-7034 Trondheim, Norway

Abstract

Previous Born approximation calculations of photoproduction of singlet (para) positronium is extended to include Coulomb corrections. At the same time the cross section for triplet (ortho) positronium production — which does not exist in Born approximation — is obtained. The calculations and results are given in a form closely related to the well known high energy pair production cross section including Coulomb corrections. The relations to pair production are discussed.

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1. Introduction

The cross section for the production of singlet (para) positronium by photons in the field of an atom was obtained some time ago in the Born approximation [1, 2]. In the light of the recent interest in the experimental production of relativistic positronium beams, briefly described below, the present paper is a presentation of the complete production cross section to all orders in Z, the atomic number. We thereby obtain also the corresponding triplet (ortho) positronium production cross section, which can only be obtained by the exchange of at least two virtual photons with the atom, i.e. in the second Born approximation. It follows simply from total angular momentum conservation that the amplitude for production of singlet positronium is odd in Z, while the corresponding amplitude for triplet positronium production is even in Z.

The first and only experimental observation of relativistic positronium production was made by Alekseev et al. [3] almost 15 years ago, when one of the photons from a π^0 decay is converted to a Dalitz pair which in turn (which happens very rarely) is converted into a bound electron–positron state, positronium. Close to 200 positronium particles were recorded. This very important experimental work initiated interest in positronium production, first theoretical over many years [4], [5], then proposal of experiments, actually there is now a proposal to produce relativistic positronium beams at the planned REFER facility at Hiroshima University [5].

The paper is organized in the following way: In Sec. 2 the singlet and triplet positronium cross sections differential in the positronium polar and azimut angles are given, and in Sec. 3 the effects of photon linear and circular polarization are obtained. In Sec. 4 the total singlet and triplet cross sections are given and the results are presented in a way which demonstrates the relations to the pair production cross section. A relation connecting the Coulomb correction functions for positronium and pair production is derived in Sec. 5 which is of importance for obtaining simple and accurate formulae for singlet and triple positronium production cross sections. Comparison of production of pairs to production of positronium is discussed.

2. The singlet (para) and triplet (ortho) positronium production cross sections

The positronium production cross section including Coulomb corrections is obtained in the same way as for the case of the Born approximation in reference I: the pair cross section $d^5\sigma_{pair}$ for equal electron and positron momenta is multiplied with the appropriate ratio of phase space factors and by the inverse squared positronium normalization constant $N_{p_s} = \alpha^3 m_e^3/8\pi n^3$ for the n'the positronium energy state.

From the exact high energy pair production process [6] as summarized in Appendix A one finds the positronium production amplitude (A.5)

$$\begin{aligned} \vec{J}_\perp &= \frac{8\pi}{\omega}(1-F(q))\,\frac{\vec{u}\,\xi}{q^2}V(x)/V(1) \\ J_Z &= \frac{4\pi i a}{\omega}(1-F(q))\,\xi^2(2\xi-1)W(x)/V(1). \end{aligned} \tag{1}$$

By choosing appropriate spinor polarization combinations for singlet and triplet positronium one obtains from (A.1, A.2) the production cross sections for the n'th energy level

$$d^2\sigma_s^n = 16\pi\frac{Z^2\alpha^6}{m_e^2n^3}\left[1-F(q)\right]^2\frac{u^3du}{q^4}\xi^2\left(1-(\vec{e}\cdot\hat{u})^2\right)\left[V(x)/V(1)\right]^2\frac{d\varphi}{2\pi} \tag{2}$$

$$d^2\sigma_t^n = 4\pi\frac{Z^4\alpha^8}{m_e^2n^3}u\,du\,\xi^4\left(2\xi-1\right)^2\left(1+\vec{s}\cdot\hat{p}_{p_s}\;\hat{k}\cdot\vec{\xi}\right)\left[W(x)/V(1)\right]^2\frac{d\varphi}{2\pi} \tag{3}$$

where $\vec{p}_{p_s}=2\vec{p}_1$ is the positronium momentum, $\vec{q}=\vec{k}-\vec{p}_{p_s}$, $\vec{u}=\vec{p}_{p_s}^{\perp}/2$, $\xi=(1+u^2)^{-1}$, $x=1-q^2\xi^2$, $F(q)$ the screening function and $\vec{\xi}=i\vec{e}\times\vec{e^*}$ the circular photon polarization unit vector. In addition to the Coulomb corrected singlet, Born approximation cross section as given in reference I, we have here obtained the triplet cross section which cannot occur in Born approximation because of angular momentum conservation.

3. Effects of photon polarizations

From Eq. (2) follows that for a linearly polarized photon the singlet positronium is preferably emitted in a plane perpendicular to the polarization plane, with an azimuth angular distribution

$$d^2\sigma_s\,(\vec{e}\cdot\hat{u}=\cos\varphi)\,/\,\overline{d^2\sigma_s}\,(\vec{e}\cdot\hat{u})=\sin^2\varphi \tag{4}$$

with φ the angle between the two planes.

A circular polarization of the photon has an effect on the production of triplet positronium. A circular polarization $\vec{P}_c=\vec{\xi}P_c$ is directly transferred to the longitudinal positronium polarization P_{p_s}

$$P_{p_s}=\frac{d^2\sigma_t\,(\vec{s}\cdot\hat{p}_{p_s}=1)-d^2\sigma_t\,(\vec{s}\cdot\hat{p}_{p_s}=-1)}{d^2\sigma_t\,(\vec{s}\cdot\hat{p}_{p_s}=1)+d^2\sigma_t\,(\vec{s}\cdot\hat{p}_{p_s}=-1)}=\hat{k}\cdot\vec{\xi}P_c \tag{5}$$

It should be noted that the transverse positronium spin polarization is of the order m_e/E_1 and does not appear in this high energy calculation, as expected.

4. The total cross sections

The integrations of the cross sections over φ and u are performed as for pair production [6], and the integrations are greatly simplified by the fact that Coulomb corrections and screening effects are separable as explained in Appendix A. In fact while the singlet cross section obviously has contributions from $u \sim 1$, which implies Coulomb corrections and from $u \sim 1/\epsilon$, which implies screening corrections, the triplet cross section has only contributions from $u \sim 1$ and screening effects are absent. Therefore $F(q) = 0$ for the triplet cross section, Eq. (3).

We shall in the following keep close to the integrations for pair production reference [6]. However as we shall see, the integrals cannot in the case of positronium be performed in such an elegant form as in the case of pair production in the work of Davies, Bethe and Maximon [7].

For singlet positronium, we sort out the Born approximation term σ_{sing}^{Born} which was given in reference I, and write the total cross section in the form [9]

$$\sigma_s^n = \sigma_s^{n,Born} + \frac{\pi}{4}\frac{Z^2\alpha^6}{m_e^2 n^3}\left[\int\limits_0^{1-y_1}\frac{dx}{\sqrt{x}}\frac{1+x}{1-x}\left[V(x)/V(1)\right]^2 - 2(2\ln 2 - 1) + 2\ln y_1\right],$$

$$(y_1 \ll 1), \tag{6}$$

where σ^{Born} contains all screening effects, I, and the Coulomb corrections are contained in the last term of Eq. (5). This demonstrates completely the separation of screening effects and Coulomb corrections also for positronium.

For the triplet positronium cross section there is no Born approximation term, Eq. (3) integrated over φ and u is therefore

$$\sigma_t^n = \frac{\pi}{2}\frac{Z^4\alpha^8}{m_e^2 n^3}\int\limits_0^1 dx\,\sqrt{x}(1+x)(W(x)/V(1))^2, \tag{7}$$

with screening effects absent.

Since the sum over energy levels will be of more use to the experimentalist, we sum over all energy levels, giving

$$\sum 1/n^3 = \zeta(3) = 1.20205,$$

where $\zeta(p)$ is the Riemann ζ-function.

The total pair production cross section is given by [7], with

$$f(Z) = \sum_{n=1}^{\infty}\frac{a^2}{n\,(n^2+a^2)}, \quad a = \alpha Z,$$

$$\sigma = \frac{28}{9}\frac{Z^2\alpha^3}{m_e^2}\left[\ln\frac{2\omega}{m_e} - \frac{109}{42} - f(Z)\right] \quad \text{(no screening)} \tag{8}$$

$$\sigma = \frac{28}{9}\frac{Z^2\alpha^3}{m_e^2}\left[\ln\left(183\,Z^{-1/3}\right) - \frac{1}{42} - f(Z)\right] \quad \text{(complete screening)} \tag{9}$$

We write the positronium cross sections in the same style, from Eqs. (6) and (7)

$$\sigma_s = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\frac{\omega}{m_e} - 1 - f_s(Z)\right] \quad \text{(no screening)} \tag{10}$$

$$\sigma_s = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\left(242\,Z^{-1/3}\right) - 1 - f_s(Z)\right] \quad \text{(complete screening)} \tag{11}$$

$$\sigma_t = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)f_t(Z) \quad \text{(irrespective of screening)} \tag{12}$$

where the Born terms are taken from reference I and

$$f_s(Z) = \frac{1}{4}\Big[2(2\ln 2 - 1) - \ln y_1 - \int_0^{1-y_1}\frac{dx}{\sqrt{x}}\frac{1+x}{1-x}(V(x)/V(1))^2\Big], \quad (y_1 \ll 1) \tag{13}$$

$$f_t(Z) = \frac{1}{2}a^2\int_0^1 dx\,\sqrt{x}(1+x)[W(x)/V(1)]^2 \tag{14}$$

The numerical values of $f(Z)$, $f_s(Z)$ and $f_t(Z)$ for low values of Z are, $a = \alpha Z \ll 1$,

$$\begin{aligned} f(Z) &= \zeta(3)a^2 \\ f_s(Z) &= [-4(1-\ln 2) + \frac{7}{2}\zeta(3)]a^2 \\ f_t(Z) &= 8(1-\ln 2)a^2. \end{aligned} \tag{15}$$

For selected values of Z numerical values of $f(Z)$, $f_s(Z)$ and $f_t(Z)$ are given in Table I.

5. Arbitrary screening

We use the Thomas – Fermi – Molière model as in reference [6]

$$\frac{1-F(q)}{q^2} = \sum_{i=1}^{3} \frac{\alpha_i}{\beta_i^2 + q^2}$$

with

$$\alpha_1 = 0.10 \quad \alpha_2 = 0.55 \quad \alpha_3 = 0.35$$

$$\beta_i = \left(Z^{1/3}/121\right) b_i; \quad b_1 = 6.0, \quad b_2 = 1.20, \quad b_3 = 0.30.$$

The result is rather simple

$$\sigma_s = \frac{28}{9}\frac{Z^2\alpha^6}{m_e^2}\zeta(3)[\ln\frac{\omega}{m_e} - 1 - f_s(Z) + \mathcal{F}(2m_e/\omega)] \tag{16}$$

with

$$\begin{aligned}\mathcal{F}(2m_e/\omega) &= -\frac{1}{2}\sum_{i=1}^{3}\alpha_i^2 \ln(1+B_i) \\ &+ \sum_{i=1}^{3}\sum_{\substack{j=1 \\ i\neq j}}^{3} \alpha_i\alpha_j \left[\frac{1+B_j}{B_i - B_j}\ln(1+B_j) + \frac{1}{2}\right]\end{aligned} \tag{17}$$

where $B_i = \beta_i^2(2m_e/\omega)^2$. Values of $\mathcal{F}(2m_e/\omega)$ are given in Table II.

6. Relations to pair production

In Appendix B the relation

$$2f_s(Z) - f_t(Z) = 2f(Z) - g(Z) \tag{18}$$

is derived. This makes it possible to obtain numerical values avoiding the logarithmic singularity $\ln y_1$ in the integral in $f(Z)$, by calculating $f_t(Z)$ and $g(Z)$ which are both free of singularities, and deducing $f_s(Z)$ from Eq. (17).

For low values of Z, from Eq. (14) one obtaines the relation

$$2f_s(Z) - f_t(Z) = 2.915\, f(Z) \tag{19}$$

and for larger values of Z one obtaines the approximate relation, $g(Z) = -0.97\, f(Z)$, which gives

$$2f_s(Z) - f_t(Z) = 2.97\, f(Z) \tag{20}$$

which does not differ much from Eq. (16). This indicates that the relations between the three $f(Z)$ – functions are approximately constants. One finds

$$f_s(Z) = 2.25\, f(Z) \quad f_t(Z) = 1.75\, f(Z) \tag{21}$$

with an error in the cross section for singlet positronium which is of the order one percent or smaller. For triplet positronium the error is somewhat larger.

These results may be used to approximately relate the positronium production cross sections directly to the pair production cross section by writing for no screening Eqs. (10) and (11)

$$\begin{aligned} \sigma_s &= \pi \frac{Z^2\alpha^6}{m_e^2} \zeta(3)[\ln \frac{\omega}{m_e} - 1 - 2.25\, f(Z)] \\ \sigma_t &= \pi \frac{Z^2\alpha^6}{m_e^2} \zeta(3)\, 1.75\, f(Z), \end{aligned} \tag{22}$$

with $\ln{(\omega/m_e)}$ replaced by $\ln{(242\, Z^{-1/3})}$ for complete screening. The total cross section for positronium production

$$\sigma_s + \sigma_t = \pi \frac{Z^2\alpha^6}{m_e^2} \zeta(3)[\ln \frac{\omega}{m_e} - 1 - 0.50\, f(Z)] \tag{23}$$

shows that the Coulomb correction effect is smaller when a bound state, the neutral particle positronium, is produced than when the separate electron and positron are produced, Eqs. (8) and (9). This is what would be expected, producing an almost neutral particle would show almost no final state charge effects. The Coulomb correction effect in Eq. (23) gives a picture of the creation process of positronium, which is a "large" particle, twice as large as the Hydrogen atom and therefore represents a charge distribution of the size of the interaction volume. It is interesting to note that the observed magnitude of the Coulomb correction to positronium creation gives an insight into the creation process.

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$$\vec{q} = \vec{k} - \vec{p_1} - \vec{p_2}. \tag{A.7}$$

The variable $x = 1 - \xi\eta q^2$ has the property that for small values of q, $q \sim q_{min} = \omega/2\epsilon_1\epsilon_2 \ll 1$, $x = 1$ and the cross section is given by the Born approximation as can be seen from Eq. (2): the dependence on a drops out for $x = 1$. In this region where screening may be important, Coulomb corrections are absent. Conversely for larger values of q, $q \sim 1$, where Coulomb corrections are important, screening effects are negligible. This shows that Coulomb corrections and screening effects are separable. Accordingly, we have included the screening effect in Eq. (2) by multiplying with the screening factor $1 - F(q)$.

Appendix B

The differential equation for the hypergeometric function $V(x) = F(ia, -ia; 1; x)$ [10],

$$(1-x)\frac{d}{dx}\left(x\frac{dV}{dx}\right) = a^2 V$$

gives with $dV/dx = a^2 W$

$$\frac{1}{1-x}V^2(x) + a^2 x W^2(x) = \frac{d}{dx}\left(xV(x)W(x)\right)$$

Multiplication with $(1+x)/\sqrt{x}$ and integration gives directly the functions f_s and f_t

$$\begin{aligned} -4f_s(Z) + 2f_t(Z) &= -V^2(1)\int_0^{1-y_1}\frac{dx}{x}\frac{1+x}{1-x} + (1+x)\sqrt{x}V(x)W(x)\Big|_0^{1-y_1} \\ &+ \int_0^1 dx\frac{1-x}{\sqrt{x}}V(x)W(x) \end{aligned}$$

Using $W(1-y_1) = V(1)(\ln y_1 + 2f(Z))$, $y_1 \ll 1$ [7] one finds the desired relation

$$2f_s(Z) - f_t(Z) = 2f(Z) - g(Z) \tag{B.1}$$

with

Table I

	Z	$f(Z)$	$f_s(Z)$	$f_t(Z)$
C	6	0.0023	0.0057	0.0047
Al	13	0.0107	0.0265	0.0218
Fe	26	0.0420	0.100	0.0828
Kr	36	0.0784	0.185	0.1504
Sn	50	0.144	0.325	0.2632
Pt	78	0.303	0.655	0.4952
Pb	82	0.332	0.705	0.538
U	92	0.395	0.815	0.595

The Coulomb correction functions $f(Z)$ [9], $f_s(Z)$ and $f_t(Z)$.

Table II

$\sqrt{B_1}$	0.5	2.0	8.0	20.0	30.0	60.0	80.0	100.0
$-\mathcal{F}(2m_e/\omega)$	0.014	0.140	0.676	1.37	1.73	2.39	2.68	2.90

Thomas – Fermi – Molière screening functions, with $\sqrt{B_1} = 6\frac{Z^{1/3}}{121}\frac{\omega}{2m_e}$.