THE INFLUENCE OF RADIATION ON OSCILLATING ELECTRON ORBITS IN AN IDEAL BETATRON

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In the following an attempt is made to solve the differential equations governing the electron motion in an ideal betatron, when also the radiation reaction is taken into account. The first order effect of the radiation is well known, being a slow contraction of the equilibrium orbit. However, it is not yet known whether the radiation will result in a decrease or an increase in the first order damping of the betatron oscillations due to the rising magnetic field. As there exists evidence in the theory of certain radiation oscillators for submillimeter waves that the latter might be true, this investigation seems justified.

The inclusion of radiation terms in the equations of motion complicates these to such an intent, that a general approximate solution showing the effect on the betatron oscillations seems impossible, but it will be shown that at least in a special case the radiating electron will receive an additional damping. Only radial oscillations will be considered.

It is found that the calculation is most easily carried through in a flat four-dimensional manifold; the usual three space coordinates being extended with the time dependent $\alpha = ict$. Hereby all coefficients in our differential equations become constants. Also the equations will apply equally well both for low and high energies.

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In addition to this a special system of units is introduced, whereby all coefficients reduce to unity. This system has the electronic rest mass m_0 and charge e as mass and charge units. The length and time units are

$$1L = \frac{\mu e^2}{6\pi m_0} \approx 1.9\cdot 10^{-15}\,\text{meter}, \quad 1T = \frac{\mu e^2}{6\pi m_0 c} \approx 6.3\cdot 10^{-24}\,\text{sec.},$$

μ being the permeability of free space and c the velocity of light.

The equations of motion for an electron in an electromagnetic field are now in four-dimensional tensor notation:

$$(1) \qquad F^i{}_\alpha u^\alpha = u^i{}_{;\alpha} u^\alpha - (u^i{}_{;\alpha} u^\alpha)_{;\beta} u^\beta + u^i u^\beta u^\delta u_{\alpha;\beta} u^\alpha{}_{;\delta} \;,$$

the two last terms representating the radiation reaction. Here

$$(2) \qquad F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}$$

is the electromagnetic field tensor given as the curl of the four-potential A_k and

$$(3) \qquad u^k = \dot{x}^k = \frac{dx^k}{d\tau}$$

is the four-velocity; the derivatives of the four space coordinates with respect to the eigentime τ. The four-velocity is subject to the to the restriction

$$(4) \qquad u^\alpha u_\alpha = -1$$

The electronic motion in a betatron is most conveniently studied in cylindrical coordinates (r, θ, z, α), and the field described by an axial symmetric four-potential:

$$(5) \qquad \begin{aligned} A_r = A_z = A_\alpha = 0 \;, \\ A_\theta = \frac{1}{2\pi}\,\Phi(r,z,\alpha) \,, \end{aligned}$$

i.e. the magnetic flux at time α through the circle r = const., z = const. divided by 2π.

The five equations (1), (4) are dependent in such a way that any one of the four eqs. (1) may be derived from the remaining three and eq. (4). We may therefore disregard one of the eqs. (1) and choose the fourth where $u^i = \dot{\alpha}$.

We shall assume that the potential A_θ has a plane of symmetry at z = 0, thereby enabling us to restrict ourselves to considering only orbits in this plane, and to disregard also the third of the eqs. (1) with $u^i = \dot{z}$.

The remaining theee equations are now evaluated in terms of the coordinates and $A = A_\theta(r, 0, \alpha)$, giving

$$
\begin{aligned}
&\frac{\partial A}{\partial r}\Omega = \ddot{r} - r\Omega^2 \underline{- \dddot{r} + 3r\Omega\dot{\Omega} + 3\dot{r}\Omega^2 + \dot{r}Q}\,, \\
(6) \qquad &\dot{A} = -r^2\dot{\Omega} - 2r\dot{r}\Omega \underline{+ r^2\ddot{\Omega} + 3r\ddot{r}\Omega + 3r\dot{r}\dot{\Omega} - r^2\Omega^3 - r^2\Omega Q}\,, \\
&\dot{r}^2 + r^2\Omega^2 - W^2 + 1 = 0\,,
\end{aligned}
$$

where

$$
\begin{aligned}
(7) \qquad Q = u^\beta u^\delta u_{\alpha;\beta}\, u^\alpha{}_{;\delta} &= \ddot{r}^2 + r^2\dot{\Omega}^2 - \dot{W}^2 - 2r\Omega^2\ddot{r} \\
&\quad + 4r\dot{r}\Omega\dot{\Omega} + r^2\Omega^4 + 4\dot{r}^2\Omega^2\,,
\end{aligned}
$$

and

$$
(8) \qquad \Omega = \dot{\theta}\,, \quad \dot{\alpha} = i\dot{t} = iW\,,
$$

W being the energy of the electron in our special system of units. The underlined terms in the eqs. (6) are due to radiation.

A solution of eqs. (6) is impossible without disregarding a large number of terms. It is therefore necessary to study the order of magnitude of the different terms. This is done here by introducing the well known first order solutions obtained by disregarding all radiation terms. Using the relation

(9) $$B_z = B_z(r,\alpha) = \frac{1}{r}\frac{\partial A}{\partial r}$$

and noticing that the second eq. (6) then may be integrated once, we are left with

(10) $$\begin{aligned} r B_z \Omega &= \ddot{r} - r\Omega^2 , \\ A &= - r^2\Omega + C , \quad C = constant, \\ \dot{r}^2 &+ r^2\Omega^2 - W^2 + 1 = 0 . \end{aligned}$$

Assuming that B_z and A are proportional in their time dependence and choosing $C = 0$, these equations have the circular solution

(11) $$\begin{aligned} r &= r_0 = \sqrt{\frac{A_0}{B_0}} , & A_0 &= A(r_0, 0, \alpha) , \\ \Omega &= \Omega_0 = -B_0 , & B_0 &= B_z(r_0, \alpha) . \\ W &= W_0 = \sqrt{1 + r_0^2 B_0^2} , \end{aligned}$$

By eq. (5) the first of these equations gives the usual result that at the circular orbit the field B_0 is half the average field inside the orbit.

Following the customary procedure we now study the solutions in the immediate vicinity of the circular solution (11), writing

(12) $$r = r_0(1+x) , \quad \Omega = \Omega_0(1+\varepsilon)$$

and assuming x and $\varepsilon \ll 1$.

Also, we shall assume a magnetic field varying linearly with radius:

(13) $$B_z = B_0(1 - mx) ; \quad 0 < m < 1 .$$

By eqs. (5), (11) we then obtain

(14) $$A = A_0 + \int_{r_0}^{r} B_z r\,dr = r_0^2 B_0 \left(1 + x + \frac{1-m}{2} x^2 - \frac{m}{3} x^3\right).$$

Introducing these relations (12), (13), (14) in the eqs. (10) and neglecting terms of second or higher order in the variables x, ε , we obtain the well-known first order solutions

$$x = \frac{C_1}{B_0} + \frac{C_2}{\sqrt{B_0}} \cos\left[\int_0^\tau \Omega_r d\tau + C_3\right]$$

(15) $\Omega_r = |B_0| \sqrt{1-m}$

$$\varepsilon = -m\frac{C_1}{B_0} - \frac{C_2}{\sqrt{B_0}} \cos\left[\int_0^\tau \Omega_r d\tau + C_3\right]$$

subject to the conditions

(16) $$\dot{B}_0 << B_0^2 \ , \quad \ddot{B}_0 << B_0^3$$

which will always be fulfilled in a practical betatron. C_1, C_2, C_3 are the constants of integration, C_1 being related to the constant in eqs. (10) by

(17) $$C_1 = -\frac{C}{(1-m)\, r_0^2}$$

The eigentime integrals in eqs. (15) may be converted into ordinary time integrals by

(18) $$\int_0^\tau \Omega_r d\tau = \int_0^t \omega_r dt \ , \quad \omega_r = \sqrt{1-m}\left|\frac{d\theta}{dt}\right|_{r=r_0} .$$

The first order radiation correction to the above solution is readily found by inserting this solution in the radiation terms of eqs. (6) and then studying the orders of magnitude of these terms. It is then found that by assuming

(19) $$\dddot{B}_0 \lesssim \ddot{B}_0 \ , \quad B_0^2 << C_1 << B_0$$

all radiation terms except the two terms

(20) $$- r_0^2 \Omega_0^3 - r_0^4 \Omega_0^5 = - r_0^2 B_0^3 W_0^2$$

may be neglected, as these will be the only ones comparable with the first order non-radiation terms of eqs. (10). The second of these equations must therefore be interchanged with

$$A = -r^2\Omega - \int_0^{\tau} r_0^2 B_0^3 W_0^2 d\tau + C. \tag{21}$$

Again inserting the relations (12), (13), (14) and solving for x, ε we now find

$$\begin{aligned} x &= \frac{C_1}{B_0} - \frac{1}{(1-m)B_0}\int_0^{\tau} B_0^3 W_0^2 d\tau + \frac{C_2}{\sqrt{B_0}}\cos\left[\int_0^{\tau}\Omega_r d\tau + C_3\right] \\ \varepsilon &= -m\frac{C_1}{B_0} + \frac{m}{(1-m)B_0}\int_0^{\tau} B_0^3 W_0^2 d\tau - \frac{C_2}{\sqrt{B_0}}\cos\left[\int_0^{\tau}\Omega_r d\tau + C_3\right] \end{aligned} \tag{22}$$

where one may insert eq. (18) and

$$\int_0^{\tau} B_0^3 W_0^2 d\tau = \int_0^{t} B_0^3 W_0 dt \tag{23}$$

to obtain the solutions in terms of the ordinary time.

To a first approximation the radiation reaction will therefore not introduce any additional damping factor to the radial betatron oscillations. To obtain such a damping factor one must sustain at least the second order terms in x, ε and their eigentime derivatives in our equations. However, as our equations then become hopelessly involved, there is little hope to obtain a solution unless we by some means introduce a very drastic simplification.

Let us therefore investigate the damping of the radial oscillations in a betatron with a constant guide field B_z and a slowly increasing acceleration field, such that a circular solution $r = r_0$ is possible. We shall calculate the oscillations in x about this circular orbit.

By the eqs. (6) the circular orbit at $r = r_0$ exists if

$$\dot{A}_0 = r_0^2 B_0^3 (1 + r_0^2 B_0^2). \tag{24}$$

Then, as before

$$\Omega_o = -B_o \qquad (25)$$
$$W_o = \sqrt{1 - r_o^2 B_o^2}\ ,$$

these quantities now being constants. As before we introduce x, ε by the relations (12), (13), while now (14) must be replaced by

$$A = A_o + r_o^2 B_o\left(x + \frac{1-m}{2}x^2 - \frac{m}{3}x^3\right). \qquad (26)$$

Then, assuming the conditions

$$B_o \ll a^2 \ll r_o^2 B_o^3 \ll a \ll 1 \qquad (27)$$

or, using MKS - units,

$$10^{-12} B_o \ll a^2 \ll 10^{-7} r_o^2 B_o^3 \ll a \ll 1\ , \qquad (27a)$$

a being the amplitude of the first order solution,

$$x = -\varepsilon = a\cos(\Omega_r \tau)\ , \qquad (28)$$

the two first eqs. (6) reduce to

$$\ddot{x} - B_o^2(1+x)(1+\varepsilon)(mx+\varepsilon) + r_o^2 B_o^4 \dot{x} = \mathcal{O}[\ll a^3 B_o^2]\ , \qquad (29)$$

$$\dot{x} + \dot{\varepsilon} + 2(x\dot{\varepsilon} + \dot{x}\varepsilon) + (1+m)x\dot{x} + mx^2\dot{x} + x^2\dot{\varepsilon} + 2x\dot{x}\varepsilon$$
$$-2r_o^2 B_o^2 \ddot{x} + r_o^2 B_o^4(4x + 5\varepsilon) - \mathcal{O}[\ll a^3 B_o]. \qquad (30)$$

Eq. (29) is now solved for ε which inserted into eq. (30) leaves us with

$$\dddot{x} + B_o^2(1-m)\dot{x} + B_o^2(1-3m)x\dot{x} + (1+m)(\dot{x}\ddot{x} + x\dddot{x}) - 2\frac{\ddot{x}\dddot{x}}{B_o^2}$$
$$+ 4a^2 B_o^4 \ddot{x} + (4-5m) r_o^2 B_o^6 x + \mathcal{O}[a^3 B_o^3] = 0 \qquad (31)$$

This equation may be solved in a relatively simple manner if we chose

$$m = 4/5 \qquad (32)$$

and disregard the terms of order $a^3 B_o^3$. Then one integration may be performed at once giving:

$$\ddot{x} + \frac{1}{5} B_o^2 x - \frac{7}{10} B_o^2 x^2 + \frac{9}{5} x\ddot{x} - \frac{\ddot{x}^2}{B_o^2} + 4 r_o^2 B_o^4 \dot{x} = C. \tag{33}$$

As we are only interested in oscillations about the stable circular orbit, $r = r_o$, we may choose the constant $C = 0$. Solving eq. (33) for $\ddot{x}$ we obtain

$$\ddot{x} + B_o^2 \left(\frac{1}{5} x - \frac{11}{10} x^2 + 4 r_o^2 B_o^2 \dot{x} \right) = \mathcal{O}\left[\ll r_o^2 a B_o^5 \right]. \tag{34}$$

The only radiation term left in this equation is the smallest term $4 r_o^2 B_o^4 \dot{x}$, which will give us the radiation damping of the oscillation. This term is an order of magnitude less than the term in x^2. Therefore the solution of eq. (34) will not deviate much from the solution of the equation

$$\ddot{x} + B_o^2 \left(\frac{1}{5} x - \frac{11}{10} x^2 \right) = \mathcal{O}\left[r_o^2 a B_o^5 \right] \tag{35}$$

which we shall denote $x = F(\tau)$.

Then let us try a solution to eq. (34) of the form

$$x = g(\tau) F(\tau), \tag{36}$$

$g(\tau)$ being the wanted damping factor.

Inserting and using eq. (35) for $F(\tau)$, we arrive at the equation:

$$2\dot{g}\dot{F} + 4 r_o^2 B_o^4 g \dot{F} = \frac{11}{10} B_o^2 F^2 g(g-1) - \left(4 r_o^2 B_o^4 \dot{g} - \ddot{g} \right) F. \tag{37}$$

As $x = gF$ is close to the solution of eq. (35), the function $g(\tau)$ must be close to unity and its derivatives very small. We may therefore in the first approximation neglect the terms to the right in eq. (37), whereby

(38) $$g(\tau) = \text{const.}\, e^{-2r_0^2 B_0^4 \tau},$$

and by suitable choice of the time $\tau=0$ the constant may be set equal to unity. Checking back on the neglected terms in eq. (37) we find that the solution (38) is correct provided

(39) $$\tau << \frac{1}{aB_0} \quad \text{or} \quad t << \frac{W_0}{aB_0}.$$

The oscillations will therefore receive a positive damping, which, inside a certain time interval may be taken to be exponential.

It remains to solve the eq. (35) for $x=F(\tau)$. Substituting

(40) $$\dot{x} = \frac{1}{z}, \quad \ddot{x} = -\frac{1}{z^3}\frac{dz}{dx}$$

we obtain $$\frac{dz}{z^3} = B_0^2\left(\frac{1}{5}x - \frac{11}{10}x^2\right)dx,$$

and upon integration

(41) $$\frac{B_0}{\sqrt{5}}z = \frac{B_0}{\sqrt{5}}\frac{d\tau}{dx} = \left[C_2^2 - x^2 + \frac{11}{3}x^3\right]^{-1/2}$$
$$= \left[C_2^2 - x^2\right]^{-1/2} - \frac{11}{6}x^3\left[C_2^2 - x^2\right]^{-3/2} + \ldots$$

Inserting $\Omega_r = -\frac{B_0}{\sqrt{5}}$ and integrating once more gives

(42) $$\Omega_r\tau + C_3 = \arccos\frac{x}{C_2} + \frac{11}{6}\;\frac{2C_2^2 - x^2}{\sqrt{C_2^2 - x^2}} + \ldots$$

Here C_2 and C_3 are the integration constants.

Observing that the second term on the right will be small compared to the first term, this transcendental equation may be solved for x by inserting the first order solution

$$x = C_2 \cos\left(\Omega_r\tau + C_3\right)$$

in the small second term; hence

$$x = C_2 \cos\left[\Omega_r\tau + C_3 - \frac{11C_2}{6}\,\frac{1+\sin^2(\Omega_r\tau + C_3)}{\sin(\Omega_r\tau + C_3)} + \cdots\right],$$

or approximately

$$x = F(\tau) = C_2 \cos(\Omega_r\tau + C_3) + \frac{11C_2^2}{6}\left[1 + \sin^2(\Omega_r\tau + C_3)\right]. \tag{43}$$

Actually we have made an error in our computation as the series (41) and (42) will diverge in the vicinity of $x=C_2$. We must therefore check our solution (43) by insertion in the original eq. (35). Doing this we find that the solution (43) is indeed correct under the conditions (27).

Combining eqs. (36), (38) and (43), we may write the solution of eq. (34):

$$x = C_2 e^{-2r_0^2 B_0^4 \tau}\left\{\cos(\Omega_r\tau + C_3) + \frac{11C_2}{6}\left[1 + \sin^2(\Omega_r\tau + C_3)\right]\right\}. \tag{44}$$

On the basis of the above calculation it does not seem unreasonable to assume that also in a betatron with increasing field B_z and $m \neq \frac{4}{5}$, the radiation reaction will cause an additional damping in accordance with the function (38). Of course this radiation damping will not be comparable with the damping caused by the increasing magnetic field. Changing to conventional units and ordinary time the damping factor (38) becomes

$$g(t) = e^{-2.4 \cdot 10^{-12} r_0 B_0^3 t} \tag{45}$$

(B_0 in gauss, r_0 in cm, t in sec.)

For a 250 MeV Betatron with r_0=100 cm and B_0= 20,000 the relaxation time for the radiation damping then will be about ½ millisec. at the highest energies.