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A VECTOR POTENTIAL EXPANSION AND THE CORRESPONDING EQUATIONS OF MOTION FOR A MARK V F.F.A.G. ACCELERATOR

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1. Vector Potential Expansions.

The Mark V F.F.A.G. magnetostatic field so far considered is governed by the following boundary conditions on the median plane:

$$
\begin{aligned}
&B_r(r,\theta,0) = 0\\
(1)\qquad &B_\theta(r,\theta,0) = 0\\
&B_z(r,\theta,0) = B_0\left(\frac{r}{r_0}\right)^k\left[1 + f\sin\left(N \ln\frac{r}{r_0} - N\theta\right)\right]
\end{aligned}
$$

The magnet poles for a field satisfying these conditions will have either a very large and inconvenient gap-widening in the radial direction, or will require a complicated and likewise inconvenient set of electrical poleface windings

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cemented to the iron surface if the gap width is to be kept constant or nearly constant. For this reason it may therefore be advantageous to consider also the field with the boundary conditions:

$$
\begin{aligned}
&B_r(r,\theta,0) = 0 \\
&B_\theta(r,\theta,0) = 0 \\
(2)\qquad &B_z(r,\theta,0) = B_0\left(\frac{r}{r_0}\right)^k e^{-\frac{k}{M}(N\theta - 2\pi m)}\left[1 + f\sin\left(M\ln\frac{r}{r_c} - N\theta\right)\right] \\
&m = \text{integer such that } 0 \leq N\theta - 2\pi m < 2\pi
\end{aligned}
$$

The spiralling feature of the fields governed by the conditions (1) and (2) is recognized by the set of logarithmic spirals

$$(3)\qquad r = r_0 e^{\frac{1}{M}(N\theta + C)}; \quad C = \text{arbitrary const.}$$

for which the argument of the sine-function reduces to the constant C. By combining the eqs. (2) and (3) one obtains the vertical component

$$(4)\qquad B_z\left(r_0 e^{\frac{1}{M}(N\theta + C)}, \theta, 0\right) = B_0 e^{\frac{k}{M}(C + 2\pi m)}\left[1 + f\sin C\right],$$

showing that on the median plane this field will be constant along any spiral inside a given magnet sector. Hence, for field (2) the unwanted feature of field (1) is practically removed.

Evidently, if one can obtain a vector potential for the complex field which satisfies the boundary conditions

$$
\begin{aligned}
&B_r(r,\theta,0) = 0 \\
(5)\qquad &B_\theta(r,\theta,0) = 0 \\
&B_z(r,\theta,0) = T r^\alpha e^{i(\beta\ln\frac{r}{r_c} - \gamma\theta - \delta)},
\end{aligned}
\qquad \left|\; \begin{aligned} &\alpha, \beta, \gamma, T \\ &= \text{constants}\end{aligned}\right.
$$

it will be an easy matter of superposition and proper choice of constants to construct the vector potentials for the fields governed by the conditions (1) or (2).

A vector potential for the field (5) may be determined on the form:

$$A_r(r,\theta,z) = T e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_u r^{\alpha+2+u} R_u(z)$$

(6) $$A_\theta(r,\theta,z) = T e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_u r^{\alpha+2+u} \Theta_u(z)$$

$$A_z(r,\theta,z) = 0 ,$$

where the functions R_u and Θ_u of the vertical coordinate z have to be determined such that both Maxwell's equations and the boundary conditions (5) are satisfied.

By the potential (6) the field $\bar{B} = \nabla \times \bar{A}$ is

$$B_r(r,\theta,z) = -T e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_u r^{\alpha+2+u} \Theta'_u(z)$$

(7) $$B_\theta(r,\theta,z) = T e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_u r^{\alpha+2+u} R'_u(z)$$

$$B_z(r,\theta,z) = T e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_u r^{\alpha+2+u} \Big[i\gamma R_{u+1}(z) + (\alpha+4+u+i\beta)\Theta_{u+1}(z) \Big]$$

Maxwell's equations require that $\nabla \times \bar{B} = 0$; which is true if simultaneously

$$R''_u(z) - \gamma^2 R_{u+2}(z) + i\gamma(\alpha+5+u+i\beta)\Theta_{u+2}(z) = 0$$

(8) $$\Theta''_u(z) + i\gamma(\alpha+3+u+i\beta) R_{u+2}(z) + (\alpha+3+u+i\beta)(\alpha+5+u+i\beta)\Theta_{u+2}(z) = 0$$

$$(\alpha+3+u+i\beta) R'_u(z) - i\gamma\, \Theta'_u(z) = 0$$

for all integers u .

The boundary conditions for these differential equations may be written down by inserting $z = 0$ in eqs. (7) and comparing the result with eqs. (5):

$$
\left.\begin{aligned} R_u'(0) &= 0 \\ \Theta_u'(0) &= 0 \end{aligned}\right\} \text{ for all integers } u
$$

(9)

$$
i\gamma R_u(0) + (\alpha + 3 + u + i\beta)\,\Theta_u(0) = \begin{cases} 0 \;; & u \neq -1 \\ 1 \;; & u = -1 \end{cases}
$$

For the system (8), (9) one readily finds the solution:

$$
R_u(z) = 0 \quad \text{for } u = \begin{cases} -1, 0, 1, 2, 3, \ldots\ldots \\ -2, -4, -6, -8, \ldots\ldots \end{cases}
$$

$$
R_{-3}(z) = -i\gamma \frac{z^2}{2}
$$

$$
R_{-2n-1}(z) = R_{-2n+1}(z)\left[\gamma^2 - (\alpha - 2n + 4 + i\beta)^2\right]\frac{z^2}{2n(2n-1)}
$$

$$
= -i\gamma\left[\gamma^2 - (\alpha + i\beta)^2\right]\left[\gamma^2 - (\alpha - 2 + i\beta)^2\right]\cdots
$$

$$
\cdots\left[\gamma^2 - (\gamma - 2n + 4 + i\beta)^2\right]\frac{z^{2n}}{(2n)!}\;;\;(n = 2, 3, 4, \ldots)
$$

(10)

$$
\Theta_u(z) = 0 \quad \text{for } u = \begin{cases} 0, 1, 2, 3, \ldots\ldots \\ -2, -4, -6, -8, \ldots\ldots \end{cases}
$$

$$
\Theta_{-1}(z) = \frac{1}{\alpha + 2 + i\beta}
$$

$$
\Theta_{-3}(z) = -(\alpha + i\beta)\frac{z^2}{2}
$$

$$\Theta_{-2n-1}(z) = \Theta_{-2n+1}(z)\,\frac{\alpha-2n+2+i\beta}{\alpha-2n+4+i\beta}\left[\gamma^2-(\alpha-2n+4+i\beta)^2\right]\frac{z^2}{2n(2n-1)}$$

$$= -(\alpha-2n+2+i\beta)\left[\gamma^2-(\alpha+i\beta)^2\right]\left[\gamma^2-(\alpha-2+i\beta)^2\right]\cdots$$

$$\cdots\left[\gamma^2-(\alpha-2n+4+i\beta)^2\right]\frac{z^{2n}}{(2n)!}\quad;(n=2,3,4,\cdots)$$

Inserting this solution into the series (6) we obtain the vector potential:

(11)
$$A_r(r,\theta,z) = -i\gamma T r^{\alpha+1} e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_{u=1}^{\infty} D_u \left(\frac{z}{r}\right)^{2u}$$

$$A_\theta(r,\theta,z) = T r^{\alpha+1} e^{i(\beta \ln \frac{r}{r_0} - \gamma\theta - \delta)} \sum_{u=0}^{\infty} E_u \left(\frac{z}{r}\right)^{2u}$$

$$A_z(r,\theta,z) = 0$$

where

(12)
$$D_1 = \frac{1}{2}$$

$$D_n = \frac{D_{n-1}}{2n(2n-1)}\left[\gamma^2-(\alpha-2n+4+i\beta)^2\right]$$

$$= \frac{1}{(2n)!}\left[\gamma^2-(\alpha+i\beta)^2\right]\left[\gamma^2-(\alpha-2+i\beta)^2\right]\cdots$$

$$\cdots\left[\gamma^2-(\alpha-2n+4+i\beta)^2\right]\quad,(n=2,3,4,\cdots)$$

$$E_0 = \frac{1}{\alpha+2+i\beta}$$

$$E_n = -(\alpha-2n+2+i\beta)D_n\quad;(n=1,2,3,\cdots)$$

The fields (1) and (2) are now regarded as superpositions of a flutter-field involving the flutter-factor f and a non-flutter field. The following table shows how the constants should be chosen in the eqs. (11) and (12) to give the different parts of the two types of fields considered.

Field	Part	α	β	γ	δ	T	
(1)	Non-flutter	k	0	0	0	$B_o r_o^{-k}$	
	Flutter	k	M	N	0	$B_o f r_o^{-k}$	Use imaginary part
(2)	Non-flutter	k	0	$-i\frac{kN}{M}$	$2\pi i\frac{km}{M}$	$B_o r_o^{-k}$	
	Flutter	k	M	$N(1-i\frac{k}{M})$	$2\pi i\frac{km}{M}$	$B_o f r_o^{-k}$	Use imaginary part

The vector potentials obtained in this manner may be expressed as follows:

For field (1):

$$A_r(r,\theta,z) = B_o r_o\left(\frac{r}{r_o}\right)^{k+1} \mathcal{R}e\left\{e^{i(M\ln\frac{r}{r_o}-N\theta)}\sum_{u=1}^{\infty} F_u^o\left(\frac{z}{r}\right)^{2u}\right\}$$

(13) $$A_\theta(r,\theta,z) = B_o r_o\left(\frac{r}{r_o}\right)^{k+1}\sum_{u=0}^{\infty}\left\{G_u^o + \mathcal{I}m\left[H_u^o e^{i(M\ln\frac{r}{r_o}-N\theta)}\right]\right\}\left(\frac{z}{r}\right)^{2u}$$

$$A_z(r,\theta,z) = 0$$

where

(14)
$$F_1^0 = -\frac{1}{2}Nf$$
$$F_n^0 = \frac{F_{n-1}^0}{2n(2n-1)}\left[N^2-(k-2n+4+iM)^2\right]$$
$$= -\frac{Nf}{(2n)!}\left[N^2-(k+iM)^2\right]\left[N^2-(k-2+iM)^2\right]\cdots$$
$$\cdots\left[N^2-(k-2n+4+iM)^2\right] \quad ; \quad (n=2,3,4,\cdots)$$

(15)
$$G_0^0 = \frac{1}{k+2}$$
$$G_n^0 = -G_{n-1}^0\,\frac{(k-2n+4)(k-2n+2)}{2n(2n-1)}$$
$$= \frac{(-1)^n}{(2n)!}\,k^2(k-2)^2(k-4)^2\cdots(k-2n+4)^2(k-2n+2);$$
$$(n=1,2,3,\cdots)$$

(16)
$$H_0^0 = \frac{f}{k+2+iM}$$
$$H_n^0 = \frac{k-2n+2+iM}{N}\,F_n^0 \quad ; \quad (n=1,2,3,\cdots)$$

For field (2):

$$A_r(r,\theta,z) = B_0 r_0 \left(\frac{r}{r_0}\right)^{k+1} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=1}^{\infty} \left\{ S_u^0 + \mathrm{Re}\left[T_u^0 e^{i\left(M \ln \frac{r}{r_0} - N\theta\right)} \right] \right\} \left(\frac{z}{r}\right)^{2u}$$

(17)
$$A_\theta(r,\theta,z) = B_0 r_0 \left(\frac{r}{r_0}\right)^{k+1} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=0}^{\infty} \left\{ U_u^0 + \mathrm{Im}\left[V_u^0 e^{i\left(M \ln \frac{r}{r_0} - N\theta\right)} \right] \right\} \left(\frac{z}{r}\right)^{2u}$$

$$A_z(r,\theta,z) = 0$$

where

$$S_1^0 = -\frac{1}{2}\frac{kN}{M}$$

$$S_n^0 = -\frac{S_{n-1}^0}{2n(2n-1)}\left[\left(\frac{kN}{M}\right)^2 + (k-2n+4)^2\right]$$

(18)
$$= \frac{(-1)^n}{(2n)!}\frac{kN}{M}\left[\left(\frac{kN}{M}\right)^2 + k^2\right]\left[\left(\frac{kN}{M}\right)^2 + (k-2)^2\right]\cdots$$

$$\cdots\left[\left(\frac{kN}{M}\right)^2 + (k-2n+4)^2\right];$$

$$(n = 2, 3, 4, \cdots)$$

(19)
$$
\begin{aligned}
T_1^\circ &= -\tfrac{1}{2} N f \left(1 - i\frac{k}{M}\right) \\
T_n^\circ &= \frac{T_{n-1}^\circ}{2n(2n-1)} \left[N^2\left(1-i\frac{k}{M}\right)^2 - (k-2n+4+iM)^2\right] \\
&= -\frac{Nf}{(2n)!}\left(1-i\frac{k}{M}\right)\left[N^2\left(1-i\frac{k}{M}\right)^2-(k+iM)^2\right]\left[N^2\left(1-i\frac{k}{M}\right)^2-(k-2+iM)^2\right]\cdots \\
&\quad \cdots\left[N^2\left(1-i\frac{k}{M}\right)^2-(k-2n+4+iM)^2\right] \quad ; \; (n=2,3,4,\dots)
\end{aligned}
$$

(20)
$$
\begin{aligned}
U_0^\circ &= \frac{1}{k+2} \\
U_n^\circ &= \frac{k+2n+2}{\frac{Nk}{M}} S_n^\circ \quad ; \; (n=1,2,3,\dots)
\end{aligned}
$$

(21)
$$
\begin{aligned}
V_0^\circ &= H_0^\circ \\
V_n^\circ &= \frac{k-2n+2+iM}{N\left(1-i\frac{k}{M}\right)} T_n^\circ \quad ; \; (n=1,2,3,\dots)
\end{aligned}
$$

The results (13) and (17) are exact. The power series in $\left(\frac{z}{r}\right)^2$ involved in these results will converge for $z < \infty$.

2. The Equations of Motion.

The equations of motion for a charged particle (mass m, charge q) moving in a magnetostatic field given by its vector potential $\bar{A}$ are most readily derived from Jacobi's principle:

(22) $$\delta \int_{(1)}^{(2)} \left(p\, ds + q \bar{A} \cdot d\bar{s} \right) = 0 ,$$

where $\bar{p} = m\,\bar{v}$ is the momentum of the particle and $d\,\bar{s}$ is the vector element of arc length along the orbit. In the case of a magnetostatic field the momentum has a constant modul $p = |\bar{p}|$; hence the variation (22) may be taken between definite limits.

Using cylindrical coordinates

(23) $$d\bar{s} = \bar{\imath}_r\, dr + \bar{\imath}_\theta\, r\, d\theta + \bar{\imath}_z\, dz$$
$$ds = \sqrt{r^2 + r'^2 + z'^2}\, d\theta , \quad r' = \frac{dr}{d\theta} , \quad z' = \frac{dz}{d\theta} ,$$

and introducing

(24) $$\bar{a} = \frac{q}{p} \bar{A}$$

one obtains from eq. (22):

(25) $$\delta \int_{(1)}^{(2)} \left[\sqrt{r^2 + r'^2 + z'^2} + r' a_r + r a_\theta + z' a_z \right] d\theta = 0 .$$

Here Jacobi's principle (22) in three dimensions (r, θ, z) is converted into a Hamilton's principle in two dimensions (r,z) with θ as independent variable. This variation principle is now converted to its canonical form

$$\delta \int_{(1)}^{(2)} \mathcal{H}(r, z, p_r, p_z, \theta)\, d\theta = 0$$

(26)
$$\mathcal{H} = p_r r' + p_z z' - \mathcal{L}$$

(27)
$$p_r = \frac{\partial \mathcal{L}}{\partial r'}, \quad p_z = \frac{\partial \mathcal{L}}{\partial z'},$$

$\mathcal{L}$ being the Lagrangian in eq. (25).

Solving the eqs. (27) for r' and z' gives

(28)
$$\frac{r'}{r_c} = (p_r - a_r) F \qquad F = \frac{\frac{r}{r_0}}{\sqrt{1 - (p_r - a_r)^2 - (p_z - a_z)^2}}$$
$$\frac{z'}{r_0} = (p_z - a_z) F$$

By introducing these velocities into the second eq. (26) one obtains the Hamiltonian

(29)
$$\mathcal{H} = -r\sqrt{1 - (p_r - a_r)^2 - (p_z - a_z)^2} - r a_\theta$$

which now give us the canonical equations of motion (28) and

(30)
$$p_r' = \frac{r}{r_0} F^{-1} + F\left[(p_r - a_r)\frac{\partial}{\partial r}(r_0 a_r) + (p_z - a_z)\frac{\partial}{\partial r}(r_0 a_z)\right] + \frac{\partial}{\partial r}(r a_\theta)$$
$$p_z' = F\left[(p_r - a_r)\frac{\partial}{\partial z}(r_c a_r) + (p_z - a_z)\frac{\partial}{\partial z}(r_0 a_z)\right] + \frac{\partial}{\partial z}(r a_\theta),$$

here written on dimensionless form.

If one, as in section 1., chooses a gage such that the a_z- component vanishes the eqs. (28), (30) may be written

$$\frac{r'}{r_0} = (p_r - a_r)F \qquad\qquad F = \frac{\frac{r}{r_0}}{\sqrt{1-(p_r-a_r)^2-p_z^2}}$$

$$\frac{z'}{r_0} = p_z F$$

(31) $$p_r' = \frac{1}{F}\frac{r}{r_0} + \frac{r'}{r_0}\frac{\partial}{\partial r}(r_0 a_r) + \frac{\partial}{\partial r}(r a_\theta)$$

$$p_z' = \frac{r'}{r_0}\frac{\partial}{\partial z}(r_0 a_r) + \frac{\partial}{\partial z}(r a_\theta)$$

Choosing the reference circle of radius r_0 in the customary way such that

(32) $$p = q B_0 r_0$$

we obtain from eq. (24)

(33) $$\bar{a} = \frac{\bar{A}}{B_0 r_0}$$

The five quantities a_r, $\frac{\partial}{\partial r}(r_0 a_r)$, $\frac{\partial}{\partial z}(r_0 a_r)$, $\frac{\partial}{\partial r}(r a_\theta)$ and $\frac{\partial}{\partial z}(r a_\theta)$ involved in the differential equations (31) may now be derived for the two types of fields here considered from the eqs. (13), (17) and (33). The result is listed in the following:

For field (1):

$$a_r = \left(\frac{r}{r_0}\right)^{k+1} \mathcal{R}e\left\{e^{i(M\ln\frac{r}{r_0}-N\theta)}\sum_{u=1}^{\infty} F_u^0 \left(\frac{z}{r}\right)^{2u}\right\}$$

$$\frac{\partial}{\partial r}(r_0 a_r) = \left(\frac{r}{r_0}\right)^{k} \mathcal{R}e\left\{e^{i(M\ln\frac{r}{r_0}-N\theta)}\sum_{u=1}^{\infty} F_u^1 \left(\frac{z}{r}\right)^{2u}\right\}$$

(34) $$\frac{\partial}{\partial z}(r_0 a_r) = \left(\frac{r}{r_0}\right)^{k} \mathcal{R}e\left\{e^{i(M\ln\frac{r}{r_0}-N\theta)}\sum_{u=1}^{\infty} F_u^2 \left(\frac{z}{r}\right)^{2u-1}\right\}$$

$$\frac{\partial}{\partial r}(r a_\theta) = \left(\frac{r}{r_0}\right)^{k+1} \sum_{u=0}^{\infty} \left\{G_u^1 + \mathcal{I}m\left[H_u^1 e^{i(M\ln\frac{r}{r_0}-N\theta)}\right]\right\}\left(\frac{z}{r}\right)^{2u}$$

$$\frac{\partial}{\partial z}(r a_\theta) = \left(\frac{r}{r_0}\right)^{k+1} \sum_{u=1}^{\infty} \left\{G_u^2 + \mathcal{I}m\left[H_u^2 e^{i(M\ln\frac{r}{r_0}-N\theta)}\right]\right\}\left(\frac{z}{r}\right)^{2u-1}$$

where F_n^0, G_n^0, H_n^0 are given by eqs. (14), (15), (16), and

$$\left.\begin{aligned} F_n^1 &= (k-2n+1+iM)\,F_n^0 \\ F_n^2 &= 2n\,F_n^0 \end{aligned}\right\} \quad (n = 1, 2, 3, \cdots)$$

(35) $$\left.\begin{aligned} G_n^1 &= (k-2n+2)\,G_n^0 \\ G_n^2 &= 2n\,G_n^0 \\ H_n^1 &= (k-2n+2+iM)\,H_n^0 \\ H_n^2 &= 2n\,H_n^0 \end{aligned}\right\} \quad (n = 0, 1, 2, \cdots)$$

For field (2):

$$a_r = \left(\frac{r}{r_o}\right)^{k+1} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=1}^{\infty} \left\{ S_u^0 + \mathcal{R}e\left[T_u^0 e^{i\left(M \ln \frac{r}{r_o} - N\theta\right)}\right]\right\} \left(\frac{z}{r}\right)^{2u}$$

$$\frac{\partial}{\partial r}(r_o a_r) = \left(\frac{r}{r_o}\right)^{k} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=1}^{\infty} \left\{ S_u^1 + \mathcal{R}e\left[T_u^1 e^{i\left(M \ln \frac{r}{r_o} - N\theta\right)}\right]\right\} \left(\frac{z}{r}\right)^{2u}$$

$$\frac{\partial}{\partial z}(r_o a_r) = \left(\frac{r}{r_o}\right)^{k} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=1}^{\infty} \left\{ S_u^2 + \mathcal{R}e\left[T_u^2 e^{i\left(M \ln \frac{r}{r_o} - N\theta\right)}\right]\right\} \left(\frac{z}{r}\right)^{2u-1} \tag{36}$$

$$\frac{\partial}{\partial r}(r a_\theta) = \left(\frac{r}{r_o}\right)^{k+1} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=0}^{\infty} \left\{ U_u^1 + \mathcal{I}m\left[V_u^1 e^{i\left(M \ln \frac{r}{r_o} - N\theta\right)}\right]\right\} \left(\frac{z}{r}\right)^{2u}$$

$$\frac{\partial}{\partial z}(r a_\theta) = \left(\frac{r}{r_o}\right)^{k+1} e^{-\frac{k}{M}(N\theta - 2\pi m)} \cdot \sum_{u=1}^{\infty} \left\{ U_u^2 + \mathcal{I}m\left[V_u^2 e^{i\left(M \ln \frac{r}{r_o} - N\theta\right)}\right]\right\} \left(\frac{z}{r}\right)^{2u-1}$$

where S_n^o, T_n^o, U_n^o, V_n^o are given by (18), (19), (20), (21), and

$$
(37)\qquad
\left.
\begin{aligned}
S_n^1 &= (k-2n+1)\,S_n^0 \\
S_n^2 &= 2n\,S_n^0 \\
T_n^1 &= (k-2n+1+iM)\,T_n^0 \\
T_n^2 &= 2n\,T_n^0
\end{aligned}
\right\}\ (n=1,2,3,\cdots)
$$

$$
\left.
\begin{aligned}
U_n^1 &= (k-2n+2)\,U_n^0 \\
U_n^2 &= 2n\,U_n^0 \\
V_n^1 &= (k-2n+2+iM)\,V_n^0 \\
V_n^2 &= 2n\,V_n^0
\end{aligned}
\right\}\ (n=0,1,2,\cdots)
$$

3. The Rate of Convergence. Truncation of the Series.

All power series in $\left(\frac{z}{r}\right)^2$ involved in the differential equations of motion (31) will converge for $z < r$; a condition which is necessarily fulfilled in any machine. However, on account of the large values of the parameters k, N and M, the convergence will be relatively slow, especially at the beginning of the series. In digital computor work it will therefore be necessary to carry a fairly large number of terms in the series before any truncation is permissible.

The truncation of the series should be performed by replacing the sums $\sum_{u=1}^{\infty}$ and $\sum_{u=0}^{\infty}$ of equations (34), (36) by $\sum_{u=1}^{q}$ and $\sum_{u=0}^{q}$ respectively; the series thereby being replaced by polynomials containing q or (q + 1) terms. By using this procedure the errors introduced by the truncation will not destroy the important Liouvillian character of the motion. One will thereby be calculating the Liouvillian motion of a charged particle in an approximately Maxwellian field. (One may if desired and with the same result use one number $q = q_1$ for the G, S and U series representing the influence of the non-flutter field, and a different number $q = q_2$ for the F, H, T and V series representing the flutter field.)

The rate of convergence of the F, H, T and V series may be studied in an approximate way by assuming that

(38) $M >> k$ and N,

which will be nearly true in any machine or model.

The truncated series involved in the calculation are then approximately the following:

$$(39) \qquad A \approx 1 + \frac{1}{2!}\left(M\frac{z}{r}\right)^2 + \frac{1}{4!}\left(M\frac{z}{r}\right)^4 + \cdots + \frac{1}{(2q)!}\left(M\frac{z}{r}\right)^{2q}$$

occuring in the expression for $\frac{\partial}{\partial r}(r a_\theta)$ for both field (1) and (2),

$$(40) \qquad B \approx 1 + \frac{1}{3!}\left(M\frac{z}{r}\right)^2 + \frac{1}{5!}\left(M\frac{z}{r}\right)^4 + \cdots + \frac{1}{(2q-1)!}\left(M\frac{z}{r}\right)^{2q-2}$$

occuring in the expressions for $\frac{\partial}{\partial z}(r_0 a_r)$ and $\frac{\partial}{\partial z}(r a_\theta)$ for both fields, and

$$(41) \qquad C \approx \frac{1}{2!} + \frac{1}{4!}\left(M\frac{z}{r}\right)^2 + \frac{1}{6!}\left(M\frac{z}{r}\right)^4 + \cdots + \frac{1}{(2q)!}\left(M\frac{z}{r}\right)^{2q-2}$$

occuring in the expressions for a_r and $\frac{\partial}{\partial r}(r_0 a_r)$ for both fields.

Each term in the series above will be approximately the same as in the exact series. The approximate series may therefore be used to determine the accuracy of the truncated exact series.

The series C need not be considered because this will converge faster than the series B and has the same number of terms as B.

Also, the G, S and U series need not be considered in this connection, as they will always converge more rapidly than the F, H, T and V series.

It follows from the above that the rate of convergence is solely determined by the magnitude of the parameter M. Also, it follows that the number q of terms necessary to obtain a certain degree of accuracy in the truncated series will depend on the largest value of $M \frac{z}{r}$ one wishes to handle with the equations.

The following table shows q as a function of this largest value of $M \frac{z}{r}$ and the least number of correct significant figures wanted in the truncated series:

TABLE OF q

Max. value of $(M \frac{z}{r})$	Min. no. of correct significant figures			
	4	6	8	10
1	3	4	5	6
2	5	6	7	8
4	7	9	10	12
8	10	12	14	16

The accuracy of the calculations performed with the eqs. (31) will of course depend on the choice of q. However, no analytical method has yet been found to predetermine q such that a certain degree of accuracy is obtained in the numerical results.