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S.Manoff*, B.Dimitrov

$(\bar{L}_n, g)$-SPACES. SPECIAL TENSOR FIELDS

*Permanent address: Institute for Nuclear Research and Nuclear Energy, Department of Theoretical Physics, Blvd. Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

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1. Introduction

The spaces with different (not only by sign) contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-spaces] have, on the one hand, all properties necessary for their use as models of the space-time [2] - [4] and, on the other hand, they have interesting properties which distinguish them from the usual spaces with one affine connection and metrics [(L_n, g)-spaces]. At the same time, they provide a differential-geometric background for considerations analogous in some sense to that of the bi-connection theory of gravitation [1].

Contravariant and covariant tensor fields considered over $(\overline{L}_n, g)$-spaces have as in (L_n, g)-spaces tensor bases constructed by means of tensor products of bases of one and the same vector space [tensors of $\otimes^l{}_x(M)$ are constructed by the use of a $\otimes^l$ tensor product of basic vectors of $T_x(M)$, tensors of $\otimes_l{}_x(M)$ are constructed by the use of $\otimes_l$ tensor products of basic vectors of $T^*_x(M)$]. If the tensor basis of a tensor at a given point $x \in M$ are constructed by means of basic vectors of two different vector spaces at $x \in M$, i. e. if the tensor basis can be represented in the form

$$[\otimes^k(e_\alpha)] \otimes [\otimes_l(e^\beta)] \ , \quad k,\ l \in N \ ,$$

where

$$\otimes^k(e_\alpha) = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_k} \ , \quad e_{\alpha_i}{}_{/x} \in N_x(M) \ ,$$
$$\otimes_l(e^\beta) = e^{\beta_1} \otimes e^{\beta_2} \otimes \ldots \otimes e^{\beta_l} \ , \quad e^{\beta_j}{}_{/x} \in L_x(M) \ ,$$
$$\dim N_x = k \ , \quad \dim L_x = l \ , \quad i = 1, \ldots, k \ , \quad j = 1, \ldots, l \ ,$$

and a tensor K in the given basis at a point $x \in M$ can be represented as

$$K_x = K^A{}_B(x).e_{A/x} \otimes e^B{}_{/x} \ ,$$
$$e_{A/x} = (e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_k})_x \ , \quad e^B{}_{/x} = (e^{\beta_1} \otimes e^{\beta_2} \otimes \ldots \otimes e^{\beta_l})_x \ ,$$

then K_x is called *mixed tensor field* of rank $\otimes^k{}_l$ at a point $x \in M$. If $N_x(M) \equiv T_x(M)$ and $L_x(M) \equiv T^*_x(M)$, the K_x is called *mixed tensor field with contravariant rank k and covariant rank l* at a point $x \in M$.

Definition 1. *A mixed tensor field K with contravariant rank k and covariant rank l. A tensor field $K : x \to K_x(M)$, $K_x(M) = K(x) \in \otimes^k{}_l{}_{/x}(M)$ [or $K_x(M) \in (k, l)_x(M)$], $\otimes^k{}_l(M) = [\otimes^k T(M)] \otimes [\otimes_l T^*(M)]$.*

In a co-ordinate basis a mixed tensor field K can be written in the form

$$K = K^A{}_B . \partial_A \otimes dx^B \ , \quad \partial_A = \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \ ,$$
$$dx^B = dx^{j_1} \otimes \ldots \otimes dx^{j_l} \ , \quad A = i_1 \ldots i_k \ , \quad B = j_1 \ldots j_l \ . \tag{1}$$

In a non-co-ordinate basis K will have the form

$$K = K^A{}_B . e_A \otimes e^B \ , \quad e_A = e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \ ,$$
$$e^B = e^{\beta_1} \otimes \ldots \otimes e^{\beta_l} \ , \quad A = \alpha_1 \ldots \alpha_k \ , \quad B = \beta_1 \ldots \beta_l \ . \tag{2}$$

Mixed tensor fields of one and the same type (with equal contravariant rank and equal covariant rank) fulfil relations determined by the properties of the tensor product $\otimes$ and by the properties of the vector spaces to which the tensor fields appear as sections:

1. $(K_1 + K_2) \otimes V = K_1 \otimes V + K_2 \otimes V$, $K_1, K_2 \in \otimes^k{}_l(M)$, $V \in \otimes^i{}_m(M)$ [distribution law with respect to $\otimes^k{}_l(M)$].

2. $K \otimes (V_1 + V_2) = K \otimes V_1 + K \otimes V_2$, $V_1, V_2 \in \otimes^i{}_m(M)$ [distribution law with respect to $\otimes^i{}_m(M)$].

3. $\alpha . K \otimes V = \alpha . (K \otimes V) = K \otimes \alpha . V$, $\alpha \in R$ [or C, or $C^r(M)$], $K \in \otimes^k{}_l(M)$, $V \in \otimes^i{}_m(M)$.

4. $K \otimes \beta . V = \beta . (K \otimes V) = \beta . K \otimes V$, $\beta \in R$ [or C, or $C^r(M)$], $K \in \otimes^k{}_l(M)$, $V \in \otimes^i{}_m(M)$.

5. $K_1 + K_2 = K_3$, $K_i \in \otimes^k{}_l(M)$, $\quad i = 1, 2, 3$.

6. $(\alpha + \beta) . K = \alpha . K + \beta . K$, $\quad K \in \otimes^k{}_l(M)$, $\quad \alpha, \beta \in R$ [or C, or $C^r(M)$].

There are some special tensor fields related to the action of the contraction operator on mixed tensor fields.

2. SPECIAL MIXED TENSOR FIELDS

2.1. Kronecker tensor field.

Definition 2. *Kronecker tensor field Kr. A mixed tensor field of second rank which components in a given basis are equal to the Kronecker symbol*

$$Kr = g^i_j . \partial_i \otimes dx^j = g^\alpha_\beta . e_\alpha \otimes e^\beta \ , \quad Kr \in \otimes^1{}_1(M) \ ,$$

$$g^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad g^\alpha_\beta = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta . \end{cases}$$

The properties of the Kronecker tensor Kr are determined by its specific constant components in a co-ordinate or in a non-co-ordinate basis

$$Kr = g^\alpha_\beta . e_\alpha \otimes e^\beta = g^{\alpha'}_{\beta'} . e_{\alpha'} \otimes e^{\beta'} = e_\alpha \otimes e^\alpha =$$
$$= g^i_j . \partial_i \otimes dx^j = g^{i'}_{j'} . \partial_{i'} \otimes dx^{j'} = \partial_i \otimes dx^i \ . \tag{3}$$

(a)

$$S(Kr) = g^\alpha_\beta . S(e_\alpha \otimes e^\beta) = g^\alpha_\beta . S(e_\alpha , e^\beta) = g^\alpha_\beta . f^\beta{}_\alpha = f^\alpha{}_\alpha = \overline{f} .$$

(b)

$$Kr(u) = S^\beta{}_\gamma (Kr \otimes u) = S^\beta{}_\gamma (g^\alpha_\beta . e_\alpha \otimes e^\beta , u^\gamma . e_\gamma) =$$
$$= g^\alpha_\beta . u^\gamma . S(e^\beta , e_\gamma) . e_\alpha = g^\alpha_\beta . u^\gamma . f^\beta{}_\gamma . e_\alpha = u^\gamma . f^\alpha{}_\gamma . e_\alpha = f^\alpha{}_\gamma . u^\gamma . e_\alpha =$$
$$= u^{\overline{\alpha}} . e_\alpha = \overline{u} \ , \quad u^{\overline{\alpha}} = f^\alpha{}_\gamma . u^\gamma \ , \quad u, \ \overline{u} \in T(M) .$$

(c)

$$\begin{aligned} p(Kr) = S^{\gamma}{}_{\alpha}(p \otimes Kr) = S^{\gamma}{}_{\alpha}(p_{\gamma}.e^{\gamma} \otimes g^{\alpha}_{\beta}.e_{\alpha} \otimes e^{\beta}) = \\ = p_{\gamma}.g^{\alpha}_{\beta}.S^{\gamma}{}_{\alpha}(e^{\gamma}, e_{\alpha}).e^{\beta} = p_{\gamma}.g^{\alpha}_{\beta}.f^{\gamma}{}_{\alpha}.e^{\beta} = p_{\gamma}.f^{\gamma}{}_{\beta}.e^{\beta} = \\ = p_{\overline{\beta}}.e^{\beta} = \overline{p}, \quad p_{\overline{\beta}} = f^{\gamma}{}_{\beta}.p_{\gamma}, \quad p, \ \overline{p} \in T^{*}(M). \end{aligned}$$

If $S = C$, then the properties of Kr determine the well known properties of the Kronecker symbol:

$$\begin{aligned} &(\mathrm{a}_1) \quad C(Kr) = g^{\alpha}_{\beta}.g^{\beta}_{\alpha} = n, \quad \dim M = n. \\ &(\mathrm{b}_1) \quad (Kr)_C(u) = u, \quad u \in T(M). \\ &(\mathrm{c}_1) \quad p_C(Kr) = p, \quad p \in T^{*}(M), \end{aligned} \tag{4}$$

where

$$(Kr)_C(u) = C^{\beta}{}_{\gamma}(Kr \otimes u), \quad p_C(Kr) = C^{\gamma}{}_{\alpha}(p \otimes Kr).$$

$C^{\beta}{}_{\gamma}$ is the contraction operator C acting on the basic vector fields e^{β} and e_{γ}.

Action of the covariant differential operator on the Kronecker tensor field. The action of the covariant differential operator on the Kronecker tensor field is determined by the general conditions for its acting on mixed tensor fields. On the grounds of the specific constant components of the Kronecker tensor, the covariant derivative along a basic vector field determines the relations between the components of the contravariant and the covariant affine connections.

The covariant derivative of the components of the Kronecker tensor field along a basic vector field determine the relation between the components of the covariant affine connection P and the components of the contravariant affine connection Γ by the condition

$$\begin{aligned} P^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\beta\gamma} = g^{\alpha}_{\beta/\gamma} \quad &\text{(in a non-co-ordinate basis)}, \\ P^{i}_{jk} + \Gamma^{i}_{jk} = g^{i}_{j;k} \quad &\text{(in a co-ordinate basis)}. \end{aligned} \tag{5}$$

The proof of the last statement follows from the relations

$$\begin{aligned} \nabla_{\xi} Kr = \nabla_{\xi}(g^{\alpha}_{\beta}.e_{\alpha} \otimes e^{\beta}) = g^{\alpha}_{\beta/\gamma}.\xi^{\gamma}.e_{\alpha} \otimes e^{\beta} = \\ = \nabla_{\xi}(g^{i}_{j}.\partial_i \otimes dx^{j}) = g^{i}_{j;k}.\xi^{k}.\partial_i \otimes dx^{j}, \quad \xi \in T(M), \end{aligned} \tag{6}$$

$$\begin{aligned} g^{\alpha}_{\beta/\gamma} = e_{\gamma} g^{\alpha}_{\beta} + \Gamma^{\alpha}_{\delta\gamma}.g^{\delta}_{\beta} + P^{\delta}_{\beta\gamma}.g^{\alpha}_{\delta} = P^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\beta\gamma}, \\ e_{\gamma} g^{\alpha}_{\beta} = 0 \quad \text{(in a non-co-ordinate basis)}, \end{aligned} \tag{7}$$

$$\begin{aligned} g^{i}_{j;k} = g^{i}_{j,k} + \Gamma^{i}_{lk}.g^{l}_{j} + P^{l}_{jk}.g^{i}_{l} = \Gamma^{i}_{jk} + P^{i}_{jk}, \\ g^{i}_{j,k} = 0 \quad \text{(in a co-ordinate basis)}. \end{aligned} \tag{8}$$

From (7) and (8), it follows that *the necessary and sufficient conditions for the existence of a covariant affine connection P different only by sing from a contravariant affine connection Γ* (i. e. $P^{\alpha}_{\beta\gamma} = -\Gamma^{\alpha}_{\beta\gamma}$, $P^{i}_{jk} = -\Gamma^{i}_{jk}$) *are the conditions*

$$\begin{aligned} g^{\alpha}_{\beta/\gamma} = 0 \quad &\text{(in a non-co-ordinate basis)}, \\ g^{i}_{j;k} = 0 \quad &\text{(in a co-ordinate basis)}. \end{aligned} \tag{9}$$

From the properties of the Kronecker tensor field and its covariant derivative along an arbitrary given contravariant vector field ξ we can find relations between the components of the Kronecker tensor and its covariant derivative.

In a co-ordinate basis

$$g^i_l . g^l_{j;k} = g^i_{j;k} \ , \quad g^j_k . (g^i_j . g^k_l)_{;m} = 2 . g^i_{l;m} \ , \tag{10}$$

$$(g^i_j . g^k_l)_{;m} = g^i_{j;m} . g^k_l + g^i_j . g^k_{l;m} \ , \quad g^j_k . (g^i_j . g^k_l)_{;m} = g^i_{j;m} . g^j_l + g^i_k . g^k_{l;m} \ , \tag{11}$$

$$\nabla_\xi [C^j{}_k (Kr \otimes Kr)] = \nabla_\xi Kr \ , \quad S = C \ , \\ C^j{}_k (Kr \otimes Kr) = C^j{}_k (g^i_j . \partial_i \otimes dx^j \otimes g^k_l . \partial_k \otimes dx^l) \ , \tag{12}$$

$$C^j{}_k (Kr \otimes \nabla_\xi Kr) = \nabla_\xi Kr \ , \quad S = C \ , \\ C^j{}_k (Kr \otimes \nabla_\xi Kr) = C^j{}_k (g^i_j . \partial_i \otimes dx^j \otimes g^k_{l;m} . \xi^m . \partial_k \otimes dx^l) \ , \tag{13}$$

$$C_i{}^l (\nabla_\xi K \otimes Kr) = K^{Ak}{}_{B;j} . \xi^j . \partial_A \otimes dx^B \otimes \partial_k \ , \quad S = C \ , \\ K^{Ai}{}_{B;j} . g^k_i = K^{Ak}{}_{B;j} \ , \\ Kr = g^k_l . \partial_k \otimes dx^l \ , \quad K = K^{Ai}{}_B . \partial_A \otimes \partial_i \otimes dx^B = K^C{}_B . \partial_C \otimes dx^B \ . \tag{14}$$

$g^i_{j;k} = P^i_{jk} + \Gamma^i_{jk}$ is called covariant derivative of the components g^α_β of the Kronecker tensor Kr in a co-ordinate basis.

Special case: $f^i{}_j = e^\varphi . g^i_j : S = e^\varphi . C$:

$$g^i_{j;k} = P^i_{jk} + \Gamma^i_{jk} = \varphi_{,k} . g^i_j \ , \ \nabla_\xi Kr = \phi(\xi) . Kr \ , \quad \phi(\xi) = \varphi_{,k} . \xi^k \ . \tag{15}$$

The last relation follows from the condition for $f^i{}_j : f^i{}_{j,k} = P^i_{lk} . f^l{}_j + \Gamma^l_{jk} . f^i{}_l$ and its explicit form.

Action of the Lie differential operator on the Kronecker tensor field. The action of the Lie differential operator on the Kronecker tensor field Kr is determined by its action on the tensor basis of Kr

$$\pounds_\xi Kr = \pounds_\xi (g^\alpha_\beta . e_\alpha \otimes e^\beta) = (\xi g^\alpha_\beta) . e_\alpha \otimes e^\beta + \\ + g^\alpha_\beta . (\pounds_\xi e_\alpha \otimes e^\beta + e_\alpha \otimes \pounds_\xi e^\beta) = g^\alpha_\beta . (\pounds_\xi e_\alpha \otimes e^\beta + e_\alpha \otimes \pounds_\xi e^\beta) = \\ = \pounds_\xi e_\alpha \otimes e^\alpha + e_\alpha \otimes \pounds_\xi e^\alpha \ , \quad \xi g^\alpha_\beta = 0 \ . \tag{16}$$

On the other side,

$$g^\alpha_\beta . \pounds_\xi e_\alpha = (- g^\alpha_\beta . e_\alpha \xi^\gamma + g^\alpha_\beta . \xi^\delta . C_{\delta\alpha}{}^\gamma) . e_\gamma = \\ = -(e_\beta \xi^\gamma - \xi^\delta . C_{\delta\beta}{}^\gamma) . e_\gamma = \pounds_\xi e_\beta \ , \\ g^\alpha_\beta . \pounds_\xi e^\beta = g^\alpha_\beta . k^\beta{}_\gamma(\xi) . e^\gamma = k^\alpha{}_\beta(\xi) . e^\beta = \\ = [e_{\underline{\beta}} \xi^{\overline{\alpha}} + (P^\alpha_{\beta\gamma} + \Gamma^{\overline{\alpha}}_{\underline{\beta}\gamma} + C_{\underline{\beta}\gamma}{}^{\overline{\alpha}}) . \xi^\gamma] . e^\beta = \pounds_\xi e^\alpha \ , \tag{17}$$

$$\pounds_\xi Kr = \pounds_\xi e_\alpha \otimes e^\alpha + e_\alpha \otimes \pounds_\xi e^\alpha = -(e_\alpha \xi^\beta - \xi^\gamma . C_{\gamma\alpha}{}^\beta) . e_\beta \otimes e^\alpha + \\ + e_\alpha \otimes [e_{\underline{\beta}} \xi^{\overline{\alpha}} + (P^\alpha_{\beta\gamma} + \Gamma^{\overline{\alpha}}_{\underline{\beta}\gamma} + C_{\underline{\beta}\gamma}{}^{\overline{\alpha}}) . \xi^\gamma] . e^\beta = (\pounds_\xi g^\alpha_\beta) . e_\alpha \otimes e^\beta \ , \\ \pounds_\xi g^\alpha_\beta = -(e_\beta \xi^\alpha - \xi^\gamma . C_{\gamma\beta}{}^\alpha) + e_{\underline{\beta}} \xi^{\overline{\alpha}} + (P^\alpha_{\beta\gamma} + \Gamma^{\overline{\alpha}}_{\underline{\beta}\gamma} + C_{\underline{\beta}\gamma}{}^{\overline{\alpha}}) . \xi^\gamma \ , \tag{18}$$

$$\pounds_\xi g^\alpha_\beta = g^\alpha_{\beta/\gamma} . \xi^\gamma - g^\gamma_\beta . (\xi^\alpha{}_{/\gamma} - T_{\gamma\delta}{}^\alpha . \xi^\delta) + g^\alpha_\gamma . (\xi^{\overline{\gamma}}{}_{/\underline{\beta}} - T_{\underline{\beta}\delta}{}^{\overline{\gamma}} . \xi^\delta) = \\ = g^\alpha_{\beta/\gamma} . \xi^\gamma - (\xi^\alpha{}_{/\beta} - T_{\beta\gamma}{}^\alpha . \xi^\gamma) + (\xi^{\overline{\alpha}}{}_{/\underline{\beta}} - T_{\underline{\beta}\gamma}{}^{\overline{\alpha}} . \xi^\gamma) \ . \tag{19}$$

In a co-ordinate basis

$$\pounds_\xi Kr = (\pounds_\xi g^i_j) . \partial_i \otimes dx^j \ , \\ \pounds_\xi g^i_j = g^i_{j;k} . \xi^k - g^k_j . (\xi^i{}_{;k} - T_{kl}{}^i . \xi^l) + g^i_k . (\xi^{\overline{k}}{}_{;\underline{j}} - T_{\underline{j}l}{}^{\overline{k}} . \xi^l) = \\ = g^i_{j;k} . \xi^k - (\xi^i{}_{;j} - T_{jl}{}^i . \xi^l) + (\xi^{\overline{i}}{}_{;\underline{j}} - T_{\underline{j}l}{}^{\overline{i}} . \xi^l) \ . \tag{20}$$

Special case: $f^i{}_j = e^{\varphi}.g^i_j : S = e^{\varphi}.C$:

$$f_j{}^k = e^{-\varphi}.g^k_j \ , \quad \pounds_\xi g^i_j = g^i_{j;k}.\xi^k \ , \quad \pounds_\xi Kr = \nabla_\xi Kr \ . \tag{21}$$

Action of the curvature operator on the Kronecker tensor field . The action of the curvature operator on the Kronecker tensor field determines the relation between the components of the contravariant and the covariant curvature tensors.

In a co-ordinate basis

$$\begin{aligned}[R(\partial_k, \partial_l)]Kr = [R(\partial_k, \partial_l)](g^i_j.\partial_i \otimes dx^j) = (R^i{}_{jkl} + P^i{}_{jkl}).\partial_i \otimes dx^j = \\ = (g^i_{j;l;k} - g^i_{j;k;l} - g^i_{j;m}.T_{lk}{}^m).\partial_i \otimes dx^j \ ,\end{aligned} \tag{22}$$

$$\begin{aligned}R^i{}_{jkl} + P^i{}_{jkl} = g^i_{j;l;k} - g^i_{j;k;l} - g^i_{j;m}.T_{lk}{}^m \ , \\ P^i{}_{jkl} = -R^i{}_{jkl} + g^i_{j;l;k} - g^i_{j;k;l} - g^i_{j;m}.T_{lk}{}^m \ .\end{aligned} \tag{23}$$

Action of the deviation operator on the Kronecker tensor field . The action of the deviation operator on the Kronecker tensor field determines the relation between the Lie derivatives of the contravariant and the covariant affine connections.

In a non-co-ordinate basis

$$\begin{aligned}[\pounds\Gamma(\xi, e_\gamma)]Kr = [\pounds\Gamma(\xi, e_\gamma)](g^\alpha_\beta.e_\alpha \otimes e^\beta) = \\ = g^\alpha_\beta.[\pounds\Gamma(\xi, e_\gamma)](e_\alpha \otimes e^\beta) = g^\alpha_\beta.\{[\pounds\Gamma(\xi, e_\gamma)]e_\alpha\} \otimes e^\beta + \\ + g^\alpha_\beta.e_\alpha \otimes \{[\pounds\Gamma(\xi, e_\gamma)]e^\beta\} = g^\alpha_\beta.\pounds_\xi\Gamma^\delta_{\alpha\gamma}.e_\delta \otimes e^\beta + g^\alpha_\beta.e_\alpha \otimes \pounds_\xi P^\beta_{\delta\gamma}.e^\delta = \\ = (\pounds_\xi\Gamma^\alpha_{\beta\gamma} + \pounds_\xi P^\alpha_{\beta\gamma}).e_\alpha \otimes e^\beta \ ,\end{aligned} \tag{24}$$

where

$$[\pounds\Gamma(\xi, e_\gamma)]e_\alpha = \pounds_\xi\Gamma^\delta_{\alpha\gamma}.e_\delta \ , \quad [\pounds\Gamma(\xi, e_\gamma)]e^\beta = \pounds_\xi P^\beta_{\delta\gamma}.e^\delta \ . \tag{25}$$

In a co-ordinate basis

$$[\pounds\Gamma(\xi, \partial_k)]Kr = (\pounds_\xi\Gamma^i_{jk} + \pounds_\xi P^i_{jk}).\partial_i \otimes dx^j \ . \tag{26}$$

The action of the covariant differential operator on a product of an invariant (invariant function, form-invariant function) and the Kronecker tensor field is determined by its action on a product of a function and a mixed tensor field

$$\begin{aligned}\nabla_\xi(L.Kr) = (\nabla_\xi L).Kr + L.\nabla_\xi Kr = (\xi L).Kr + L.(\nabla_\xi Kr) \ , \\ L'(x^{k'}) = L(x^k) \in C^r(M) \ ,\end{aligned} \tag{27}$$

where $L(x^k)$ is invariant function with respect to diffeomorphisms, which are local transformations of the co-ordinates $\{x^k\}$.

In a co-ordinate basis

$$\nabla_\xi(L.Kr) = (L.g^i_j)_{;k}.\xi^k.\partial_i \otimes dx^j = (g^i_j.L_{,k} + L.g^i_{j;k}).\xi^k.\partial_i \otimes dx^j \ , \tag{28}$$

$$(L.g^i_j)_{;k} = g^i_j.L_{,k} + L.g^i_{j;k} \ , \quad L_{,k} = L_{;k} \ , \tag{29}$$

$$(L.g^j_i)_{;j} = L.g^j_{i;j} + L_{,i} \ , \quad L_{,i} = (L.g^j_i)_{;j} - L.g^j_{i;j} \ . \tag{30}$$

The action of the Lie differential operator on a product of an invariant function and the Kronecker tensor field is determined by its action on a product of a function and a mixed tensor field

$$\pounds_\xi(L.Kr) = (\pounds_\xi L).Kr + L.\pounds_\xi Kr = (\xi L).Kr + L.(\pounds_\xi Kr) \ , \qquad (31)$$
$$L'(x^{k'}) = L(x^k) \in C^r(M) \ .$$

In a co-ordinate basis

$$\pounds_\xi(L.g^i_j) = L_{,k}.\xi^k.g^i_j + \qquad (32)$$
$$+ L.[g^i_{j;k}.\xi^k - (\xi^i{}_{;j} - T_{jk}{}^i.\xi^k) + (\xi^{\bar{i}}{}_{;\underline{j}} - T_{\underline{j}k}{}^{\bar{i}}.\xi^k)] \ .$$

Special case: $f^i{}_j = e^\varphi.g^i_j : S = e^\varphi.C$:

$$\pounds_\xi(L.Kr) = \nabla_\xi(L.Kr) \ , \qquad (33)$$
$$\pounds_\xi(L.g^i_j) = (L_{,k}.g^i_j + L.g^i_{j,k}).\xi^k = (L.g^i_j)_{,k}.\xi^k \ .$$

2.2. Contraction tensor field. The functions $f^\alpha{}_\beta(x^k)$ and $f^i{}_j(x^k)$ obtained as a result of the action of the contraction operator on basic vector fields of non-co-ordinate or co-ordinate basis have transformation properties of components of a mixed tensor field over the manifold M. On this basis the notion of contraction tensor field can be introduced.

Definition 3. *The mixed tensor field* $Sr = f^\alpha{}_\beta.e_\alpha \otimes e^\beta = f^i{}_j.\partial_i \otimes dx^j$ *is called contraction tensor field* Sr.

From the action of the Kronecker tensor field Kr on itself, i. e. from

$$Kr(Kr) = (g^\alpha_\kappa.e_\alpha \otimes e^\kappa)(g^\sigma_\beta.e_\sigma \otimes e^\beta) = S^\kappa{}_\sigma(g^\alpha_\kappa.e_\alpha \otimes e^\kappa, g^\sigma_\beta.e_\sigma \otimes e^\beta) =$$
$$= g^\alpha_\kappa.g^\sigma_\beta.S(e^\kappa, e_\sigma).e_\alpha \otimes e^\beta = g^\alpha_\kappa.g^\sigma_\beta.f^\kappa{}_\sigma.e_\alpha \otimes e^\beta = f^\alpha{}_\beta.e_\alpha \otimes e^\beta = Sr \ ,$$

the relation between Kr and Sr follows

$$Sr = Kr(Kr) \ . \qquad (34)$$

The components of the contraction tensor field Sr obey the conditions for the commutation of the contraction operator S with the covariant differential operator

$$e_\gamma f^\alpha{}_\beta = P^\alpha_{\delta\gamma}.f^\delta{}_\beta + \Gamma^\delta_{\beta\gamma}.f^\alpha{}_\delta \ , \qquad f^i{}_{j,k} = P^i_{lk}.f^l{}_j + \Gamma^l_{jk}.f^i{}_l \ . \qquad (35)$$

The action of the covariant differential operator on the contraction tensor field Sr is determined by its action on mixed tensor fields as well as by the relations which have to be obeyed by the components of Sr in a given basis.

The action of the Lie differential operator on the contraction tensor field Sr is determined by its action on mixed tensor field of rank (1, 1) [or $\otimes^1{}_1(M)$].

In a co-ordinate basis

$$Sr = f^i{}_j.\partial_i \otimes dx^j \ , \quad \pounds_\xi Sr = (\pounds_\xi f^i{}_j).\partial_i \otimes dx^j \ , \qquad (36)$$
$$\pounds_\xi f^i{}_j = f^i{}_{j;k}.\xi^k - f^k{}_j.(\xi^i{}_{;k} - T_{kl}{}^i.\xi^l) + f^i{}_k.(\xi^{\bar{k}}{}_{;\underline{j}} - T_{\underline{j}l}{}^{\bar{k}}.\xi^l) \ .$$

The action of the curvature operator on the contraction tensor field Sr is determined by its standard action on mixed tensor field of second rank $(1,1)$. At the same time, on the basis of this action, the integrability conditions for the equations of the components of the contraction tensor field can be found, if the components of the contravariant and covariant affine connections are considered as known (given) functions.

In a co-ordinate basis

$$\begin{aligned}[R(\partial_k, \partial_l)]Sr = [R(\partial_k, \partial_l)](f^i{}_j . \partial_i \otimes dx^j) = \\ = (f^m{}_j . R^i{}_{mkl} + f^i{}_m . P^m{}_{jkl}) . \partial_i \otimes dx^j = \\ = (f^i{}_{j;l;k} - f^i{}_{j;k;l} - f^i{}_{j;m} . T_{lk}{}^m) . \partial_i \otimes dx^j \ ,\end{aligned} \tag{37}$$

$$\begin{aligned}R^i{}_{\bar{j}kl} + P^{\bar{i}}{}_{jkl} = f^i{}_{j;l;k} - f^i{}_{j;k;l} + f^i{}_{j;m} . T_{kl}{}^m \ , \\ R^i{}_{\bar{j}kl} = f^m{}_j . R^i{}_{mkl} \ , \qquad P^{\bar{i}}{}_{jkl} = f^i{}_m . P^m{}_{jkl} \ .\end{aligned} \tag{38}$$

After some computations we obtain the conditions

$$\begin{aligned}(R^\kappa{}_{\beta\gamma\delta} . g^\alpha_\sigma + P^\alpha{}_{\sigma\gamma\delta} . g^\kappa_\beta) . f^\sigma{}_\kappa = R^\kappa{}_{\beta\gamma\delta} . f^\alpha{}_\kappa + P^\alpha{}_{\sigma\gamma\delta} . f^\sigma{}_\beta = \\ = R^{\overline{\alpha}}{}_{\beta\gamma\delta} + P^\alpha{}_{\overline{\beta}\gamma\delta} = 0 \ .\end{aligned} \tag{39}$$

In a co-ordinate basis, the last relation has the form

$$\begin{aligned}(R^m{}_{jkl} . g^i_n + P^i{}_{nkl} . g^m_j) . f^n{}_m = R^m{}_{jkl} . f^i{}_m + P^i{}_{nkl} . f^n{}_j = \\ = R^{\bar{i}}{}_{jkl} + P^i{}_{\bar{j}kl} = 0 \ .\end{aligned} \tag{40}$$

From the integrability conditions for the equations (35)

$$f^i{}_{j,k,l} - f^i{}_{j,l,k} = 0 \ , \tag{41}$$

where

$$\begin{aligned}f^i{}_{j,k,l} = (P^i_{nk,l} + P^i_{mk} . P^m_{nl}) . f^n{}_j + (\Gamma^n_{jk,l} + \Gamma^m_{jk} . \Gamma^n_{ml}) . f^i{}_n + \\ + (P^i_{nl} . \Gamma^m_{jk} + P^i_{nk} . \Gamma^m_{jl}) . f^n{}_m \ .\end{aligned} \tag{42}$$

after some simple calculations we can find the relations

$$\begin{aligned}f^i{}_{j,k,l} - f^i{}_{j,l,k} = R^m{}_{jlk} . f^i{}_m + P^i{}_{nlk} . f^n{}_j = \\ = -(R^m{}_{jkl} . f^i{}_m + P^i{}_{nkl} . f^n{}_j) = 0 \ .\end{aligned} \tag{43}$$

Therefore, the conditions (40) appear as integrability conditions for the equations (35), determining the action of the contraction operator if it commutes with the covariant differential operator. These conditions can be found directly from the conditions (41), if the structure of the components of the contravariant and the covariant curvature tensors is well known, or they can be obtained if the conditions, following from the commutation relations between the contraction operator and the covariant differential operator, are impose on the components of the contraction tensor.

The action of the deviation operator on the contraction tensor field is determined by its action on mixed tensor fields of second rank $(1,1)$.

$$[\pounds\Gamma(\xi, \partial_k)]Sr = (f^l{}_j . \pounds_\xi \Gamma^i_{lk} + f^i{}_l . \pounds_\xi P^l_{jk}) . \partial_i \otimes dx^j \ . \tag{44}$$

2.3. Multi-Kronecker tensor field.

Definition 4. *The mixed tensor field* $MKr = g^A_B.e_A \otimes e^B = g^C_D.\partial_C \otimes dx^D$ *is called multi-Kronecker tensor field of rank* l *(*$A = i_1...i_l$*,* $B = j_1...j_l$*,* $C = k_1...k_l$*,* $D = m_1...m_l$*,* $l = 1, 2, ..., N$*,* $l = \text{rank } g^B_A$*).*

The properties of the multi-Kronecker tensor field are determined by the properties of the multi-Kronecker symbol. If we introduce the abbreviations

$$\begin{array}{c} MKr = g^A_B.e_A \otimes e^B \ , \quad K = K^C{}_D.e_C \otimes e^D \ , \\ \text{rank } MKr = l \ , \quad \text{rank } K = \otimes^l{}_l(M) = (l, l) \ , \end{array} \tag{45}$$

the properties of MKr can be proved:

(a) $C^B{}_C(MKr \otimes K) = K = C^B{}_C(MKr, K)$. $C^B{}_C$ is the contraction operator $S = C$ acting on the indices B and C and $C^B{}_C(e^B, e_C) = g^B_C$.

(b) $C^D{}_A(K \otimes MKr) = K$.

(c) $C_A{}^D(MKr \otimes K) = \overline{K} = K^C{}_D.e^D \otimes e_C$.

(d) $C_A{}^B(MKr) = n^l$, $n = \dim M$.

(e) $S_A{}^B(MKr) = g^A_B.S_A{}^B(e_A, e^B)$, $S_A{}^B(e_A, e^B) = f^B{}_A$, $f^B{}_A = f^{j_1}{}_{i_1}...f^{j_l}{}_{i_l}$, $\overline{f} = g^{i_l}_{j_l}.f^{j_l}{}_{i_l}$, $S^A{}_B(MKr) = \overline{f}^l$.

(f) $C^B{}_C(MKr \otimes MKr) = MKr$.

If we introduce the abbreviations

$$\begin{array}{c} MKr = g^A_B.e_A \otimes e^B \ , \quad \text{rank } MKr = l \ , \\ K = K^C{}_D.e_C \otimes e^D \ , \quad \text{rank } K = (l, l) \ , \end{array}$$

then the properties of the multi-Kronecker tensor field can be proved:

(a) $\nabla_\xi[C^B{}_C(MKr \otimes K)] = C^B{}_C(MKr \otimes \nabla_\xi K) = \nabla_\xi K$, $(g^A_C.K^C{}_B)_{/\gamma} = g^A_C.K^C{}_{B/\gamma} = K^A{}_{B/\gamma}$, $(g^A_C.K^C{}_B)_{;k} = g^A_C.K^C{}_{B;k} = K^A{}_{B;k}$.

(b) $C^D{}_A(\nabla_\xi K \otimes MKr) = \nabla_\xi[C^D{}_A(K \otimes MKr)] = \nabla_\xi K$,

$C^D{}_A(K \otimes MKr) = K$, $(K^C{}_D.g^D_B)_{/\gamma} = K^C{}_{D/\gamma}.g^D_B = K^C{}_{B/\gamma}$,

$(K^A{}_C.g^C_B)_{;k} = K^A{}_{C;k}.g^C_B = K^A{}_{B;k}$.

2.4. Multi-contraction tensor field.

Definition 5. *The mixed tensor field*

$$\begin{array}{c} MVr = S_{B\alpha}{}^{A\beta}.e_A \otimes e_\beta \otimes e^B \otimes e^\alpha = \\ = S_{Bi}{}^{Aj}.\partial_A \otimes \partial_j \otimes dx^B \otimes dx^i \ , \\ A = i_1...i_l, \ B = j_1...j_l \ , \\ \partial_A = \partial_{i_1} \otimes ... \otimes \partial_{i_l} \ , \quad dx^B = dx^{j_1} \otimes ... \otimes dx^{j_l} \ . \end{array}$$

Definition 6. *is called multi-contraction tensor field of rank* $l + 1$.

The properties of the multi-contraction tensor field MVr are determined by the properties of the multi-contraction symbol.

Let

$$\begin{array}{c} MVr = S_{B\alpha}{}^{A\beta}.e_A \otimes e_\beta \otimes e^B \otimes e^\alpha \ , \quad Q = Q^C{}_D{}^\gamma.e_C \otimes e^D \otimes e_\gamma \ , \\ A = \alpha_1...\alpha_l \ , \quad B = \beta_1...\beta_l \ , \quad C = \gamma_1...\gamma_l \ , \quad D = \delta_1...\delta_l \ , \end{array}$$

be given. The the following relations are fulfilied for the tensor product $MVr \otimes Q$

$$C_A{}^D(MVr \otimes Q) = S_{B\alpha}{}^{D\beta}.Q^C{}_D{}^\gamma.e_\beta \otimes e^B \otimes e^\alpha \otimes e_C \otimes e_\gamma \; . \tag{46}$$

$$\begin{aligned} C_A{}^{DB}{}_C(MVr \otimes Q) &= S_{B\alpha}{}^{D\beta}.Q^B{}_D{}^\gamma.e_\beta \otimes e^\alpha \otimes e_\gamma = \\ &= \overline{Q}_\alpha{}^{\gamma\beta}.e_\beta \otimes e^\alpha \otimes e_\gamma \; , \quad \overline{Q}_\alpha{}^{\gamma\beta} = S_{B\alpha}{}^{D\beta}.Q^B{}_D{}^\gamma \; . \end{aligned} \tag{47}$$

The properties of the components of the multi-contraction tensor field MVr could be used in the construction of a Lagrangian formalism for tensor fields over differentiable manifolds with contravariant and covariant affine connections.

By the use of the tensor fields

$$MVr = S_{B\alpha}{}^{A\beta}.e_A \otimes e_\beta \otimes e^B \otimes e^\alpha \; , \quad Q = Q^C{}_D{}^\gamma.e_C \otimes e^D \otimes e_\gamma \; ,$$

the following relations can be found:

$$\nabla_\xi[C_A{}^{DB}{}_C(MVr \otimes Q)] = C_A{}^{DB}{}_C(MVr \otimes \nabla_\xi Q) \; , \tag{48}$$

where $\overline{Q}_\alpha{}^{\gamma\beta} = S_{B\alpha}{}^{D\beta}.Q^B{}_D{}^\gamma$.

In an analogous way,

$$\overline{P}_\alpha{}^{\gamma\beta}{}_{/\rho} = (S_{C\alpha}{}^{A\beta}.P^C{}_A{}^\gamma)_{/\rho} = S_{C\alpha}{}^{A\beta}.P^C{}_A{}^\gamma{}_{/\rho} \; , \tag{49}$$

where $MVr \in \otimes^{k+1}{}_{k+1}(M)$, $P \in \otimes^{k+1}{}_k(M)$, $A = \alpha_1 ... \alpha_k$, $C = \beta_1 ... \beta_k$.

$$\pounds_\xi[C_A{}^{DB}{}_C(MVr \otimes Q)] = C_A{}^{DB}{}_C(MVr \otimes \pounds_\xi Q) \; , \tag{50}$$

$$\pounds_\xi \overline{Q}_\alpha{}^{\gamma\beta} = \pounds_\xi(S_{B\alpha}{}^{D\beta}.Q^B{}_D{}^\gamma) = S_{B\alpha}{}^{D\beta}.\pounds_\xi Q^B{}_D{}^\gamma \; . \tag{51}$$

In a co-ordinate basis

$$\overline{Q}_i{}^{kj}{}_{;l} = (S_{Bi}{}^{Dj}.Q^B{}_D{}^k)_{;l} = S_{Bi}{}^{Dj}.Q^B{}_D{}^k{}_{;l} \; , \tag{52}$$

$$\begin{aligned} g^l_k.\overline{Q}_i{}^{kj}{}_{;l} &= \overline{Q}_i{}^{kj}{}_{;l}.g^l_k = \overline{Q}_i{}^{kj}{}_{;k} = \overline{Q}_i{}^{lj}{}_{;l} = \\ &= (S_{Bi}{}^{Dj}.Q^B{}_D{}^k)_{;k} = S_{Bi}{}^{Dj}.Q^B{}_D{}^k{}_{;k} \; , \end{aligned}$$

$$\overline{P}_i{}^{kj}{}_{;l} = (S_{Ci}{}^{Aj}.P^C{}_A{}^k)_{;l} = S_{Ci}{}^{Aj}.P^C{}_A{}^k{}_{;l} \; ,$$

$$g^l_k.\overline{P}_i{}^{kj}{}_{;l} = \overline{P}_i{}^{kj}{}_{;l}.g^l_k = \overline{P}_i{}^{kj}{}_{;k} = (S_{Ci}{}^{Aj}.P^C{}_A{}^k)_{;k} = S_{Ci}{}^{Aj}.P^C{}_A{}^k{}_{;k} \; . \tag{53}$$

$$\pounds_\xi \overline{Q}_i{}^{kj} = S_{Bi}{}^{Dj}.\pounds_\xi Q^B{}_D{}^k \; , \quad \pounds_\xi \overline{P}_i{}^{kj} = S_{Ci}{}^{Aj}.\pounds_\xi P^C{}_A{}^k \; , \tag{54}$$

where

$$\overline{Q}_i{}^{kj} = S_{Bi}{}^{Dj}.Q^B{}_D{}^k \; , \quad \overline{P}_i{}^{kj} = S_{Ci}{}^{Aj}.P^C{}_A{}^k \; . \tag{55}$$

By the use of the special mixed tensor fields different structure can be found, connected with a Lagrangian formalism and its applications in theories constructed on the grounds of tensor fields over differentiable manifolds with different contravariant and covariant affine connections [2] - [6].

3. Symmetric tensor fields

Let B be a covariant tensor field of rank k : $B \in \otimes_k(M)$:

$$B = B_A.dx^A = B_C.e^C = B_{i_1...i_k}.dx^{i_1} \otimes ... \otimes dx^{i_k} = B_{\alpha_1...\alpha_k}.e^{\alpha_1} \otimes ... \otimes e^{\alpha_k}$$

and let $\overline{B}$ be a contravariant tensor field of rank k : $\overline{B} \in \otimes^k(M)$:

$$\overline{B} = \overline{B}^A.\partial_A = \overline{B}^C.e_C = \overline{B}^{i_1...i_k}.\partial_{i_1} \otimes ... \otimes \partial_{i_k} = \overline{B}^{\alpha_1...\alpha_k}.e_{\alpha_1} \otimes ... \otimes e_{\alpha_k} .$$

For every covariant (or contravariant) tensor field a full symmetric covariant (or contravariant) tensor field ${}_sB$ (or ${}_s\overline{B}$)

$$\begin{aligned} {}_sB &= B_{(i_1...i_k)}.dx^{i_1}....dx^{i_k} = B_{(\alpha_1...\alpha_k)}.e^{\alpha_1}....e^{\alpha_k} , \\ {}_s\overline{B} &= \overline{B}^{(i_1...i_k)}.\partial_{i_1}...\partial_{i_k} = \overline{B}^{(\alpha_1...\alpha_k)}.e_{\alpha_1}....e_{\alpha_k} , \end{aligned} \tag{56}$$

can be found. The components of ${}_sB$ (or ${}_s\overline{B}$) are not changing under a change of the order of the indices. The same is valid for the basic tensor fields. They do not change if the place of the basic vector fields $[(dx^{i_l} , e^{\alpha_l})$ or $(\partial_{i_l}, e_{\alpha_l})]$ in the tensor basis $[(dx^{i_1}....dx^{i_k}, e^{\alpha_1}....e^{\alpha_k})$ or $(\partial_{i_1}...\partial_{i_k}, e_{\alpha_1}....e_{\alpha_k})]$ are changing their places.

$B_{(i_1...i_k)}$ are the components of the full symmetric covariant tensor field ${}_sB$ in a co-ordinate basis. They can be constructed by the use of the s. c. symmetric Bach brackets [7] (p. 41). The following decomposition formulas are valid for the symmetric Bach brackets:

$$\begin{aligned} B_{(i_1...i_k)} = \frac{1}{k}.[B_{(i_1 i_2...i_{k-1})i_k} + B_{(i_1 i_2...i_{k-2} i_k)i_{k-1}} + B_{(i_1 i_2...i_{k-3} i_{k-1} i_k)i_{k-2}} + \\ ... + B_{(i_2 i_3...i_k)i_1}] = (B_{(i_1...i_{k-1})i_k})_{(i_1...i_k)} . \end{aligned} \tag{57}$$

The transposition (interchange of pairs) of indices does not change the value of the corresponding components of B:

$$B_{(i_1 i_2...i_k)} = B_{(i_2 i_1...i_k)} = B_{(i_1 i_2...i_{k-1} i_k)} = B_{(i_1 i_2...i_k i_{k-1})} . \tag{58}$$

Since a permutation of symbols may be obtained from a sequence of permutations [8] (p.93) the components of ${}_sB$ does not change under a change of the row of the indices (a change of the order of the basic vector fields building the corresponding tensor basis). A full symmetric co-ordinate (or non-co-ordinate) tensor basis is denoted as $dx^{(A)}$

$$\begin{aligned} dx^{(A)} = dx^{i_1}.dx^{i_2}.....dx^{i_k} , \qquad & e^{(A)} = e^{\alpha_1}.e^{\alpha_2}.....e^{\alpha_k} , \\ \partial_{(A)} = \partial_{i_1}...\partial_{i_k} , \qquad & e_{(A)} = e_{\alpha_1}....e_{\alpha_k} . \end{aligned} \tag{59}$$

Remark 1. *If some of the indices are not included in the procedure of symmetrisation they are put between straight lines:* $(i_1 i_2... \mid i_{l-1} i_l i_{l+1} \mid i_{l+2}...i_k)$.

If we consider the procedure of a symmetrisation as an action of an operator *Sym* (called symmetrisation operator), then we can define the operator *Sym* and its action in the following way [8] (pp. 88-91):

Definition 7. *Symmetrisation operator. The operator*

$$Sym : B \to SymB = {}_sB \; , \qquad B, \; {}_sB \in \otimes_k(M) \; ,$$

Definition 8. *is called symmetrisation operator.*

Here,

$$B = B_{i_1 \ldots i_k}.dx^{i_1} \otimes \ldots \otimes dx^{i_k} = B_{\alpha_1 \ldots \alpha_k}.e^{\alpha_1} \otimes \ldots \otimes e^{\alpha_k} \; ,$$
$${}_sB = B_{(i_1 \ldots i_k)}.dx^{i_1} \ldots . dx^{i_k} = B_{(\alpha_1 \ldots \alpha_k)}.e^{\alpha_1} \ldots . e^{\alpha_k} \; ,$$

$$B_{(i_1 \ldots i_k)} = \frac{1}{k!} \cdot \sum_{(i_1 \ldots i_k)} B_{i_1 \ldots i_k} \; , \qquad dx^{i_1} \ldots . dx^{i_k} = \frac{1}{k!} \cdot \sum_{(i_1 \ldots i_k)} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \; , \qquad (60)$$

$$B_{(\alpha_1 \ldots \alpha_k)} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \ldots \alpha_k)} B_{\alpha_1 \ldots \alpha_k} \; , \quad e^{\alpha_1} \ldots . e^{\alpha_k} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \ldots \alpha_k)} e^{\alpha_1} \otimes \ldots \otimes e^{\alpha_k} \; . \qquad (61)$$

$(i_1 \ldots i_k)$ or $(\alpha_1 \ldots \alpha_k)$ denote all permutations of the indices i_l or α_l $(l = 1, \ldots, k)$. Thus, ${}_sB$ is invariant with respect to the permutation of the indices in a given co-ordinate or non-co-ordinate basis.

Analogous considerations and formulas can be found for contravariant tensor fields:

$$Sym : \overline{B} \to Sym\overline{B} = {}_s\overline{B} \; , \qquad \overline{B}, \; {}_s\overline{B} \in \otimes^k(M) \; ,$$

where

$$\overline{B} = \overline{B}^{i_1 \ldots i_k}.\partial_{i_1} \otimes \ldots \otimes \partial_{i_k} = \overline{B}^{\alpha_1 \ldots \alpha_k}.e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \; ,$$
$${}_s\overline{B} = \overline{B}^{(i_1 \ldots i_k)}.\partial_{i_1} \ldots . \partial_{i_k} = \overline{B}^{(\alpha_1 \ldots \alpha_k)}.e_{\alpha_1} \ldots . e_{\alpha_k} \; , \qquad (62)$$

$$\overline{B}^{(i_1 \ldots i_k)} = \frac{1}{k!} \cdot \sum_{(i_1 \ldots i_k)} \overline{B}^{i_1 \ldots i_k} \; , \qquad \partial_{i_1} \ldots . \partial_{i_k} = \frac{1}{k!} \cdot \sum_{(i_1 \ldots i_k)} \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \; , \qquad (63)$$

$$\overline{B}^{(\alpha_1 \ldots \alpha_k)} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \ldots \alpha_k)} \overline{B}^{\alpha_1 \ldots \alpha_k} \; , \quad e_{\alpha_1} \ldots . e_{\alpha_k} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \ldots \alpha_k)} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \; . \qquad (64)$$

Remark 2. *The action of the symmetrisation operator Sym can be extended to every geometrical object and on the whole to quantities having indices of one and the same type.*

The symmetry property of a covariant (or contravariant) tensor field is independent of the change of the tensor basis or the co-ordinates of the manifold. This is obvious from the construction of a full symmetric tensor field.

Example 9. *Full symmetric covariant tensor field of rank 2:* $B = B_{i_1 i_2}.dx^{i_1} \otimes dx^{i_2}$, $B \in \otimes_2(M)$.

$${}_sB = B_{(i_1 i_2)}.dx^{i_1}.dx^{i_2} \; , \quad B_{(i_1 i_2)} = \frac{1}{2}.(B_{i_1 i_2} + B_{i_2 i_1}) \; ,$$
$$dx^{i_1}.dx^{i_2} = \frac{1}{2}.(dx^{i_1} \otimes dx^{i_2} + dx^{i_2} \otimes dx^{i_1} \; . \qquad (65)$$

Example 10. *Full symmetric covariant tensor field of rank 3:* $B = B_{i_1 i_2 i_3}.dx^{i_1} \otimes dx^{i_2} \otimes dx^{i_3}$, $B \in \otimes_3(M)$.

$$\begin{gathered} {}_sB = B_{(i_1 i_2 i_3)}.dx^{i_1}.dx^{i_2}.dx^{i_3} \ , \\ B_{(i_1 i_2 i_3)} = \frac{1}{3.2}.(B_{i_1 i_2 i_3} + B_{i_2 i_1 i_3} + B_{i_3 i_2 i_1} + B_{i_1 i_3 i_2} + B_{i_3 i_1 i_2} + B_{i_2 i_3 i_1}) \ , \\ dx^{i_1}.dx^{i_2}.dx^{i_3} = \frac{1}{3.2}.(dx^{i_1} \otimes dx^{i_2} \otimes dx^{i_3} + dx^{i_2} \otimes dx^{i_1} \otimes dx^{i_3} + \\ + dx^{i_3} \otimes dx^{i_2} \otimes dx^{i_1} + dx^{i_1} \otimes dx^{i_3} \otimes dx^{i_2} + dx^{i_3} \otimes dx^{i_1} \otimes dx^{i_2} + \\ + dx^{i_2} \otimes dx^{i_3} \otimes dx^{i_1}) \ . \end{gathered} \tag{66}$$

The full symmetric tensor fields obey the rule of a symmetric algebra [8] (pp. 87-91). The symmetric multiplication "." is

(a) Commutative: ${}_sA.{}_sB = {}_sB.{}_sA$.

(b) Associative: $({}_sA.{}_sB).{}_sC = {}_sA.({}_sB.{}_sC)$.

(c) Distributive: $({}_sA + {}_sB).{}_sC = {}_sA.{}_sC + {}_sB.{}_sC$.

${}_sA$, ${}_sB$ and ${}_sC$ are full symmetric covariant (or contravariant) tensor fields. For designation of the set of all full symmetric covariant (or contravariant) tensor fields of rank k we will use the abbreviation ${}^s \otimes_k (M)$ [or ${}^s \otimes^k (M)$].

The symmetrisation operator acts as a linear operator on tensor fields

$$\begin{gathered} Sym(\alpha.B) = \alpha.SymB = \alpha.{}_sB \ , \qquad \alpha \in R \text{ (or } C), \\ Sym(\alpha.B_1 + \beta.B_2) = \alpha.SymB_1 + \beta.SymB_2 = \alpha.{}_sB_1 + \beta.{}_sB_2 \ , \\ B_i \in \otimes_k(M) \ , \quad i = 1, 2 \ , \qquad \alpha, \beta \in R \text{ (or } C). \end{gathered} \tag{67}$$

Sym commutes with the covariant differential operator and with the Lie differential operator

$$Sym \circ \nabla_{\partial_i} = \nabla_{\partial_i} \circ Sym \ , \qquad Sym \circ \pounds_{\partial_i} = \pounds_{\partial_i} \circ Sym \ . \tag{68}$$

Proof:

$$\begin{gathered} Sym[\nabla_{\partial_i} B] = Sym[\nabla_{\partial_i}(B_{i_1 \dots i_k}.dx^{i_1} \otimes \dots \otimes dx^{i_k})] = \\ = Sym[B_{i_1 \dots i_k;i}.dx^{i_1} \otimes \dots \otimes dx^{i_k}] = B_{i_1 \dots i_k;i}.dx^{i_1}.\dots.dx^{i_k} = \\ = B_{(i_1 \dots i_k);i}.dx^{i_1}.\dots.dx^{i_k} = B_{(A);i}.dx^{(A)} \ . \end{gathered} \tag{69}$$

$$\begin{gathered} \nabla_{\partial_i}[SymB] = \nabla_{\partial_i}[\tfrac{1}{k!} \cdot \textstyle\sum_{(i_1 \dots i_k)} B_{i_1 \dots i_k} \cdot \tfrac{1}{k!} \cdot \sum_{(i_1 \dots i_k)} dx^{i_1} \otimes \dots \otimes dx^{i_k}] = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(i_1 \dots i_k)} B_{i_1 \dots i_k,i} \cdot \tfrac{1}{k!} \cdot \sum_{(i_1 \dots i_k)} dx^{i_1} \otimes \dots \otimes dx^{i_k} + \\ + \tfrac{1}{k!} \cdot \textstyle\sum_{(i_1 \dots i_k)} B_{i_1 \dots i_k} \cdot \tfrac{1}{k!} \cdot \sum_{(i_1 \dots i_k)} \nabla_{\partial_i}(dx^{i_1} \otimes \dots \otimes dx^{i_k}) = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(A)} B_{A,i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A + \tfrac{1}{k!} \cdot \sum_{(A)} B_A \cdot \tfrac{1}{k!} \cdot \sum_{(A)} \nabla_{\partial_i}(dx^A) = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(A)} B_{A,i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A + \tfrac{1}{k!} \cdot \sum_{(A)} B_A \cdot \tfrac{1}{k!} \cdot \sum_{(A)} \Gamma^A_{Bi}.dx^B = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(A)} B_{A,i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A + \tfrac{1}{k!} \cdot \sum_{(B)} B_B \cdot \tfrac{1}{k!} \cdot \sum_{(B)} \Gamma^B_{Ai}.dx^A = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(A)} B_{A,i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A + \tfrac{1}{k!} \cdot \sum_{(B)} B_B \cdot \Gamma^{(B)}_{Ai} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A = \\ = [B_{(A),i} + \Gamma^{(B)}_{(A)i}].dx^{(A)} = B_{(A);i}.dx^{(A)} = B_{(i_1 \dots i_k);i}.dx^{i_1}.\dots.dx^{i_k} \ , \\ \text{because of } \Gamma^{(B)}_{Ai} = \Gamma^{(B)}_{(A)i} . \end{gathered} \tag{70}$$

The proof also follows from the linearity property of the operators *Sym* and ∇_{∂_i}

$$\begin{gathered} \nabla_{\partial_i}[SymB] = \nabla_{\partial_i}[\tfrac{1}{k!} \cdot \tfrac{1}{k!} \cdot \textstyle\sum_{(A)(1)} \cdot \sum_{(A)(2)} B_{A(1)} \cdot dx^{A(2)}] = \\ = \tfrac{1}{k!} \cdot \tfrac{1}{k!} \cdot \textstyle\sum_{(A)(1)} \cdot \sum_{(A)(2)} B_{A(1);i} \cdot dx^{A(2)} = \\ = \tfrac{1}{k!} \cdot \textstyle\sum_{(A)(1)} B_{A(1);i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)(2)} dx^{A(2)} = \tfrac{1}{k!} \cdot \sum_{(A)} B_{A;i} \cdot \tfrac{1}{k!} \cdot \sum_{(A)} dx^A = Sym(\nabla_{\partial_i} B) \ , \end{gathered} \tag{71}$$

where $(A)(1)$ means the permutations (A) of the first term $B_{A(1)} = B_A$ and $(A)(2)$ means the permutations of the second term $dx^{A(2)} = dx^A$ in the sum.

The proof for $Sym \circ \pounds_{\partial_i} = \pounds_{\partial_i} \circ Sym$ is analogous to that for $Sym \circ \nabla_{\partial_i} = \nabla_{\partial_i} \circ Sym$.

4. Anti-symmetric tensor fields. Differential forms

One can construct a covariant or contravariant full anti-symmetric tensor field in analogous way as this was done for a symmetric tensor field.

Let B be a covariant tensor field of rank k:

$$B = B_A.dx^A = B_C.e^C = B_{i_1...i_k}.dx^{i_1} \otimes ... \otimes dx^{i_k} = B_{\alpha_1...\alpha_k}.e^{\alpha_1} \otimes ... \otimes e^{\alpha_k}$$

and let $\overline{B}$ be a contravariant tensor field of rank k:

$$\overline{B} = \overline{B}^A.\partial_A = \overline{B}^C.e_C = \overline{B}^{i_1...i_k}.\partial_{i_1} \otimes ... \otimes \partial_{i_k} = \overline{B}^{\alpha_1...\alpha_k}.e_{\alpha_1} \otimes ... \otimes e_{\alpha_k} .$$

For every covariant (or contravariant) tensor field a full anti-symmetric covariant (or contravariant) tensor field ${}_aB$ (or ${}_a\overline{B}$)

$$\begin{aligned} {}_aB &= B_{[i_1...i_k]}.dx^{i_1} \wedge ... \wedge dx^{i_k} = B_{[\alpha_1...\alpha_k]}.e^{\alpha_1} \wedge ... \wedge e^{\alpha_k} \ , \\ {}_a\overline{B} &= \overline{B}^{[i_1...i_k]}.\partial_{i_1} \wedge ... \wedge \partial_{i_k} = \overline{B}^{[\alpha_1...\alpha_k]}.e_{\alpha_1} \wedge ... \wedge e_{\alpha_k} \ , \end{aligned} \tag{72}$$

can be found. The components of ${}_aB$ (or ${}_a\overline{B}$) are changing or not changing their sign under a change of the order of the indices after an odd or even permutation respectively. The same is valid for the basic tensor fields. They change their sign (respectively their orientation) if the places of the basic vector fields $[(dx^{i_l} \ , e^{\alpha_l})$ or $(\partial_{i_l}, e_{\alpha_l})]$ in the tensor basis $[(dx^{i_1}....dx^{i_k} \ , e^{\alpha_1}....e^{\alpha_k})$ or $(\partial_{i_1}...\partial_{i_k}, e_{\alpha_1}....e_{\alpha_k})]$ are changing after an odd permutation of their indices. They do not change if the places of the basic vector fields $[(dx^{i_l} \ , e^{\alpha_l})$ or $(\partial_{i_l}, e_{\alpha_l})]$ in the tensor basis $[(dx^{i_1}....dx^{i_k} \ , e^{\alpha_1}....e^{\alpha_k})$ or $(\partial_{i_1}...\partial_{i_k}, e_{\alpha_1}....e_{\alpha_k})]$ are changing after an even permutation of their indices.

$B_{[i_1...i_k]}$ are the components of the full anti-symmetric covariant tensor field ${}_aB$ in a co-ordinate basis. They can be constructed by the use of the s. c. anti-symmetric Bach brackets [7] (p. 41) . The following decomposition formulas are valid for the anti-symmetric Bach brackets:

$$\begin{aligned} B_{[i_1...i_k]} &= \tfrac{1}{k}.[B_{(i_1 i_2...i_{k-1})i_k} - B_{[i_1 i_2...i_{k-2}i_k]i_{k-1}} + B_{[i_1 i_2...i_{k-3}i_{k-1}i_k]i_{k-2}} + ...+ \\ &+ (-1)^{k-1}.B_{[i_2 i_3...i_k]i_1}] = (B_{[i_1...i_{k-1}]i_k})_{[i_1...i_k]} \ , \text{ or} \\ (B_{i_1[i_2...i_k]})_{[i_1...i_k]} &= B_{[i_1...i_k]} = \tfrac{1}{k}.[B_{i_1[i_2...i_k]} - B_{i_2[i_1 i_3...i_k]} + B_{i_3[i_1 i_2 i_4...i_k]} + \\ &+ ... + (-1)^{k-1}.B_{i_k[i_1...i_{k-1}]} \ . \end{aligned} \tag{73}$$

We can consider the operation of the anti-symmetrisation as the action of an operator called anti-symmetrisation operator (alternating operator) [8] (pp. 91-97) that maps a covariant (or contravariant tensor field) into its full anti-symmetric (skew-symmetric) part.

Definition 11. *Anti-symmetrisation operator. The operator*

$$\begin{aligned} Asym : B \to {}_aB = AsymB \ , \quad & B \in \otimes_k(M) \ , \quad {}_aB \in {}^a\otimes_k(M) \equiv \Lambda^k(M) \ , \\ {}_aB &= B_{[i_1...i_k]}.dx^{i_1} \wedge ... \wedge dx^{i_k} = B_{[A]}.d\widehat{x}^A = \\ &= B_{[\alpha_1...\alpha_k]}.e^{\alpha_1} \wedge ... \wedge e^{\alpha_k} = B_{[C]}.\widehat{e}^C \ , \end{aligned} \tag{74}$$

Definition 12. *where*

$$B_{[i_1 \dots i_k]} = \frac{1}{k!} \cdot \sum_{(i_1 \dots i_k)} sgn(i_1 \dots i_k) . B_{i_1 \dots i_k} = B_{[A]} \ , \tag{75}$$

$$B_{[\alpha_1 \dots \alpha_k]} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \dots \alpha_k)} sgn(\alpha_1 \dots \alpha_k) . B_{\alpha_1 \dots \alpha_k} = B_{[C]} \ , \tag{76}$$

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{k!} \cdot \sum_{(i_1 \dots i_k)} sgn(i_1 \dots i_k) . dx^{i_1} \otimes \dots \otimes dx^{i_k} \ , \tag{77}$$

$$e^{\alpha_1} \wedge \dots \wedge e^{\alpha_k} = \frac{1}{k!} \cdot \sum_{(\alpha_1 \dots \alpha_k)} sgn(\alpha_1 \dots \alpha_k) . e^{\alpha_1} \otimes \dots \otimes e^{\alpha_k} \ . \tag{78}$$

is called anti-symmetrisation operator.

The sum is over $k!$ permutations $(i_1 \dots i_k)$ [or $(\alpha_1 \dots \alpha_k)$] of the numbers from 1 to k. In an analogous way one can define the action of the anti-symmetrisation operator *Asym* on contravariant tensor fields. The set of all covariant full anti-symmetric tensor fields of rank k over the manifold M is designated with the symbol $\Lambda^k(M)$ or with ${}^a\otimes_k(M)$. We will use further the generally accepted symbol $\Lambda^k(M)$. For the set of all contravariant full anti-symmetric tensor fields of rank k over M the indications ${}^a\otimes^k(M)$ or $\Lambda_k(M)$ can be introduced. We will use the indication $\Lambda_k(M)$.

Remark 3. *In the symbols* $\otimes^k$ *or* $\otimes_k$ *the upper or lower indices k are used with respect to the components of the tensor fields of rank k [i.e.* $\otimes^k$ *means a set of contravariant tensor fields of rank k* $(B \in \otimes^k : B = B^{i_1 \dots i_k} . \partial_{i_1} \otimes \dots \otimes \partial_{i_k})$ *in contrast to* $\otimes_k$ *that means a set of covariant tensor fields of rank k* $(C \in \otimes_k : C = C_{j_1 \dots j_k} . dx^{j_1} \otimes \dots \otimes dx^{j_k})$*]. In the symbols* Λ^k *or* Λ_k *the upper or lower indices are introduced with respect to the basis of the tensor fields of rank k [i.e.* Λ_k *means a set of full anti-symmetric contravariant tensor fields of rank k* $(B \in \otimes^k : B = B^{i_1 \dots i_k} . \partial_{i_1} \otimes \dots \otimes \partial_{i_k})$ *in contrast to* Λ^k *that means a set of full anti-symmetric covariant tensor fields of rank k* $(C \in \otimes_k : C = C_{j_1 \dots j_k} . dx^{j_1} \otimes \dots \otimes dx^{j_k})$*]. Many authors use instead of* $\otimes^k$ *and* $\otimes_k$ *the abbreviations* $(k, 0)$ *or* $(0, k)$ *and instead of* Λ^k *and* Λ_k *they use* $\Lambda(0, k)$ *and* $\Lambda(k, 0)$.

The property of anti-symmetry of a full anti-symmetric tensor field B (or $\overline{B}$) is independent of the change of the tensor basis as well as the change of the co-ordinates of the manifold. It appears as an intrinsic property of the anti-symmetric tensor fields.

The *external (exterior) product* of two full anti-symmetric tensor fields is defined [8] (p. 94) as

$${}_aA \wedge {}_aB = {}_a({}_aA \otimes {}_aB) \ . \tag{79}$$

The full anti-symmetric tensor fields obey the rules of an anti-symmetric algebra (exterior algebra, Grassmann algebra). The anti-symmetric multiplication (exterior or alternating, Grassmann, wedge product) $[\wedge]$ is

(a) Anti-commutative: ${}_aA \wedge {}_aB = (-1)^{k.l} . {}_aB \wedge {}_aA$, ${}_aA \in \Lambda^k(M)$, ${}_aB \in \Lambda^l(M)$ [or ${}_aA \in \Lambda_k(M)$, ${}_aB \in \Lambda_l(M)$].

(b) Associative: $({}_aA \wedge {}_aB) \wedge {}_aC = {}_aA \wedge ({}_aB \wedge {}_aC)$.

(c) Distributive: $({}_aA + {}_aB) \wedge {}_aC = {}_aA \wedge {}_aC + {}_aB \wedge {}_aC$.

${}_aA$, ${}_aB$ and ${}_aC$ are full anti-symmetric covariant (or contravariant) tensor fields of the type respectively

$$
\begin{array}{c}
{}_aA = A_{[i_1 \dots i_k]}.dx^{i_1} \wedge \dots \wedge dx^{i_k} = A_{[\alpha_1 \dots \alpha_k]}.e^{\alpha_1} \wedge \dots \wedge e^{\alpha_k} \ , \quad k = 1, \dots, N \ , \\
{}_aA = A^{[i_1 \dots i_k]}.\partial_{i_1} \wedge \dots \wedge \partial_{i_k} = A^{[\alpha_1 \dots \alpha_k]}.e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k} \ .
\end{array} \tag{80}
$$

Remark 4. *For designation of the set of all full anti-symmetric covariant (or contravariant) tensor fields of rank k we will use the abbreviation $\Lambda^k(M)$ [or $\Lambda_k(M)$].*

Definition 13. *Differential form. A full anti-symmetric covariant tensor field ${}_aA \in \Lambda^k(M)$ is called differential form.*

Definition 14. *k-form. A differential form ${}_aA$ of rank k : ${}_aA \in \Lambda^k(M)$ is called k-form.*

The operator $Asym$ obeys the relations:

$$
\begin{array}{c}
Asym(A \otimes B) = {}_a(A \otimes B) = {}_aA \wedge {}_aB = AsymA \wedge AsymB \ , \\
A \in \otimes_k(M) \ , \qquad B \in \otimes_l(M) \ ,
\end{array} \tag{81}
$$

$$
\begin{array}{c}
Asym(\alpha.B_1 + \beta.B_2) = \alpha.AsymB_1 + \beta.AsymB_2 = \alpha.{}_aB_1 + \beta.{}_aB_2 \ , \\
B_i \in \otimes_k(M) \ , \quad i = 1, 2 \ , \qquad \alpha, \ \beta \in R \ (\text{or } C),
\end{array} \tag{82}
$$

$$
Asym(f) = id(f) = f \ , \quad Asym(dx^i) = id(dx^i) = dx^i \ , \quad Asym(e^\alpha) = e^\alpha \ . \tag{83}
$$

Example 15. $A = A_{i_1 i_2}.dx^{i_1} \otimes dx^{i_2}$, $A \in \otimes_2(M)$.

$$
\begin{array}{c}
AsymA = {}_aA = A_{[i_1 i_2]}.dx^{i_1} \wedge dx^{i_2} \ , \\
A_{[i_1 i_2]} = \frac{1}{2}.(A_{i_1 i_2} - A_{i_2 i_1}) \ , \qquad dx^{i_1} \wedge dx^{i_2} = \frac{1}{2}.(dx^{i_1} \otimes dx^{i_2} - dx^{i_2} \otimes dx^{i_1}) \ .
\end{array} \tag{84}
$$

The properties and applications of the differential forms are considered in the theory of the differential forms [9].

The anti-symmetry of the tensor fields leads to specific forms of the action of the operators acting on tensor fields.

4.1. Action of the contraction operator on a contravariant vector field and a full anti-symmetric tensor field.

Action of the contraction operator S on a contravariant basic vector field ∂_j and a full anti-symmetric covariant basic tensor field $d\widehat{x}^A$. Let we consider now the action of the contraction operator S on ∂_j and $d\widehat{x}^A$.

$$
d\widehat{x}^A = dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{k!}.\sum_{(i_1 \dots i_k)} sgn(i_1 \dots i_k).dx^{i_1} \otimes \dots \otimes dx^{i_k} \ . \tag{85}
$$

$$
\begin{aligned}
S(\partial_j, d\widehat{x}^A) &= S(\partial_j, dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \\
&= S(\partial_j, \frac{1}{k!}.\sum_{(i_1 \dots i_k)} sgn(i_1 \dots i_k).dx^{i_1} \otimes \dots \otimes dx^{i_k}) =
\end{aligned}
$$

$$= \frac{1}{k!} \cdot \sum_{(i_1 \ldots i_k)} sgn(i_1 \ldots i_k).S(\partial_j, dx^{i_1}).dx^{i_2} \otimes \ldots \otimes dx^{i_k} =$$

$$= \frac{1}{k} \cdot \sum_{(i_1 \ldots i_k)} sgn(i_1 \ldots i_k).S(\partial_j, dx^{i_1}).\frac{1}{(k-1)!}.dx^{i_2} \otimes \ldots \otimes dx^{i_k} =$$

$$= \frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).S(\partial_j, dx^{i_1}).$$

$$\cdot \frac{1}{(k-1)!} \cdot \sum_{(i_2 \ldots i_k)} sgn(i_2 \ldots i_k).dx^{i_2} \otimes \ldots \otimes dx^{i_k} \ ,$$

$$S(\partial_j, d\widehat{x}^A) = \frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \ldots \wedge dx^{i_k} \ , \qquad (86)$$

where

$$dx^{i_2} \wedge \ldots \wedge dx^{i_k} = \frac{1}{(k-1)!} \cdot \sum_{(i_2 \ldots i_k)} sgn(i_2 \ldots i_k).dx^{i_2} \otimes \ldots \otimes dx^{i_k} \ , \qquad (87)$$

$$S(\partial_j, dx^{i_1}) = f^{i_1}{}_j \ .$$

Therefore,

$$S(\partial_j, d\widehat{x}^A) = \frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).f^{i_1}{}_j.dx^{i_2} \wedge \ldots \wedge dx^{i_k} = \qquad (88)$$

$$= (f^{i_1}{}_j.dx^{i_2} \wedge \ldots \wedge dx^{i_k})_{[i_1 i_2 \ldots i_k]} \ .$$

Action of the contraction operator S on a contravariant vector field ξ and a full anti-symmetric covariant basic tensor field $d\widehat{x}^A$. Let $\xi = \xi^i.\partial_i = \xi^\alpha.e_\alpha$ be a contravariant vector field [$\xi \in T(M)$] and $d\widehat{x}^A = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ be a tensor basis of rank k. Then

$$S(\xi, d\widehat{x}^A) = S(\xi^j.\partial_j, d\widehat{x}^A) = \xi^j.S(\partial_j, d\widehat{x}^A) =$$

$$= \xi^j.\frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).f^{i_1}{}_j.dx^{i_2} \wedge \ldots \wedge dx^{i_k} =$$

$$= \frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).f^{i_1}{}_j.\xi^j.dx^{i_2} \wedge \ldots \wedge dx^{i_k} =$$

$$= \frac{1}{k} \cdot \sum_{(i_1(i_2 \ldots i_k))} sgn(i_1(i_2 \ldots i_k)).\xi^{\overline{i}_1}.dx^{i_2} \wedge \ldots \wedge dx^{i_k} \ ,$$

$$= (\xi^{\overline{i}_1}.dx^{i_2} \wedge \ldots \wedge dx^{i_k})_{[i_1 i_2 \ldots i_k]} \ , \qquad \xi^{\overline{i}_1} = f^{i_1}{}_j.\xi^j \ . \qquad (89)$$

Action of the contraction operator S on a contravariant vector field ξ and a full anti-symmetric covariant tensor field ${}_aA$. Let $\xi = \xi^i.\partial_i = \xi^\alpha.e_\alpha$ be a contravariant vector field $[\xi \in T(M)]$ and ${}_aA = A_{[A]}.d\widehat{x}^A = A_{[i_1...i_k]}.dx^{i_1} \wedge ... \wedge dx^{i_k}$ be a full anti-symmetric tensor field of rank k. Then,

$$\begin{aligned} S(\xi, {}_aA) &= S(\xi^j.\partial_j, A_{[A]}.d\widehat{x}^A) = \xi^j.A_{[A]}.S(\partial_j, d\widehat{x}^A) = A_{[A]}.S(\xi, d\widehat{x}^A) = \\ &= A_{[i_1...i_k]}.f^{i_1}{}_j.\xi^j.dx^{i_2} \wedge ... \wedge dx^{i_k} = A_{[i_1...i_k]}.\xi^{\bar{i}_1}.dx^{i_2} \wedge ... \wedge dx^{i_k} = \\ &= A_{[\bar{j}...i_k]}.\xi^j.dx^{i_2} \wedge ... \wedge dx^{i_k} = A_{[i_1...i_k]}.S(\xi, dx^{i_1}).dx^{i_2} \wedge ... \wedge dx^{i_k} . \end{aligned} \tag{90}$$

Action of the contraction operator S on a contravariant basic vector field ∂_j and an external product $d\widehat{x}^A \wedge d\widehat{x}^B$ of two full anti-symmetric covariant tensor bases $d\widehat{x}^A$ and $d\widehat{x}^B$. Let $d\widehat{x}^A = dx^{i_1} \wedge ... \wedge dx^{i_k}$ and $d\widehat{x}^B = dx^{j_1} \wedge ... \wedge dx^{j_l}$ be two full anti-symmetric covariant tensor basis of rank k and l. Then

$$d\widehat{x}^A \wedge d\widehat{x}^B = dx^{i_1} \wedge ... \wedge dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_l} ,$$

and

$$\begin{aligned} S(\partial_j, d\widehat{x}^A \wedge d\widehat{x}^B) &= S(\partial_j, \frac{1}{k!}.\sum_{(i_1...i_k)} sgn(i_1...i_k).dx^{i_1} \otimes ... \otimes dx^{i_k} \wedge \\ &\wedge \frac{1}{l!}.\sum_{(j_1...j_l)} sgn(j_1...j_l).dx^{j_1} \otimes ... \otimes dx^{j_l}) = \\ &= \frac{1}{k!}.\sum_{(i_1...i_k)} sgn(i_1...i_k).S(\partial_j, dx^{i_1}).dx^{i_2} \otimes ... \otimes dx^{i_k} \wedge \\ &\wedge \frac{1}{l!}.\sum_{(j_1...j_l)} sgn(j_1...j_l).dx^{j_1} \otimes ... \otimes dx^{j_l} + \\ &+ \frac{1}{k!}.\sum_{(i_1...i_k)} sgn(i_1...i_k).dx^{i_1} \otimes ... \otimes dx^{i_k} \wedge \\ &\wedge (-1)^k.\frac{1}{l!}.\sum_{(j_1...j_l)} sgn(j_1...j_l).S(\partial_j, dx^{j_1}).dx^{j_2} \otimes ... \otimes dx^{j_l} = \\ &= S(\partial_j, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\partial_j, d\widehat{x}^B) . \end{aligned} \tag{91}$$

Therefore,

$$S(\partial_j, d\widehat{x}^A \wedge d\widehat{x}^B) = S(\partial_j, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\partial_j, d\widehat{x}^B) .$$

The contraction operator S acts on a basic contravariant vector field and an external product as a differential operator not obeying the Leibniz rule but obeying the *rule for anti-differentiation.*

Action of the contraction operator S on a contravariant vector field ξ and an external product $d\widehat{x}^A \wedge d\widehat{x}^B$ of two full anti-symmetric covariant tensor bases $d\widehat{x}^A$ and $d\widehat{x}^B$. By the use of the above relations we can now find the explicit form of the expression $S(\xi, d\widehat{x}^A \wedge d\widehat{x}^B)$

$$\begin{aligned} S(\xi, d\widehat{x}^A \wedge d\widehat{x}^B) &= S(\xi^j.\partial_j, d\widehat{x}^A \wedge d\widehat{x}^B) = \xi^j.S(\partial_j, d\widehat{x}^A \wedge d\widehat{x}^B) = \\ &= \xi^j.[S(\partial_j, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\partial_j, d\widehat{x}^B)] = \\ &= S(\xi^j.\partial_j, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\xi^j.\partial_j, d\widehat{x}^B) = \\ &= S(\xi, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\xi, d\widehat{x}^B) \ . \end{aligned} \tag{92}$$

Therefore,

$$S(\xi, d\widehat{x}^A \wedge d\widehat{x}^B) = S(\xi, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\xi, d\widehat{x}^B) \ . \tag{93}$$

Action of the contraction operator S on a contravariant vector field ξ and an external product ${}_aA \wedge {}_aB$ of two full anti-symmetric covariant tensor fields ${}_aA$ and ${}_aB$. Let ${}_aA = A_{[A]}.d\widehat{x}^A = A_{[i_1...i_k]}.dx^{i_1} \wedge ... \wedge dx^k$ and ${}_aB = B_{[B]}.d\widehat{x}^B = B_{[j_1...j_l]}.dx^{j_1} \wedge ... \wedge dx^{j_l}$ be two full anti-symmetric covariant tensor fields of rank k and l. Then, the external product ${}_aA \wedge {}_aB$ can be written in the forms

$$\begin{aligned} {}_aA \wedge {}_aB &= A_{[i_1...i_k]}.dx^{i_1} \wedge ... \wedge dx^{i_k} \wedge B_{[j_1...j_l]}.dx^{j_1} \wedge ... \wedge dx^{j_l} = \\ &= A_{[i_1...i_k]}.B_{[j_1...j_l]}.dx^{i_1} \wedge ... \wedge dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_l} = \\ &= A_{[A]}.B_{[B]}.d\widehat{x}^A \wedge d\widehat{x}^B \ . \end{aligned} \tag{94}$$

The result of the action of the contraction operator S on $\xi \in T(M)$ and ${}_aA \wedge {}_aB$ can be found by the use of the relations

$$\begin{aligned} S(\xi, {}_aA \wedge {}_aB) &= S(\xi^j.\partial_j, A_{[A]}.B_{[B]}.d\widehat{x}^A \wedge d\widehat{x}^B) = \\ &= \xi^j.A_{[A]}.B_{[B]}.S(\partial_j, d\widehat{x}^A \wedge d\widehat{x}^B) = A_{[A]}.B_{[B]}.S(\xi, d\widehat{x}^A \wedge d\widehat{x}^B) = \\ &= A_{[A]}.B_{[B]}.[S(\xi, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge S(\xi, d\widehat{x}^B)] = \\ &= A_{[A]}.B_{[B]}.S(\xi, d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.A_{[A]}.B_{[B]}.d\widehat{x}^A \wedge S(\xi, d\widehat{x}^B) = \\ &= S(\xi, A_{[A]}.d\widehat{x}^A) \wedge B_{[B]}.d\widehat{x}^B + (-1)^k.A_{[A]}.d\widehat{x}^A \wedge S(\xi, B_{[B]}.d\widehat{x}^B) = \\ &= S(\xi, {}_aA) \wedge {}_aB + (-1)^k.{}_aA \wedge S(\xi, {}_aB) \ , \end{aligned} \tag{95}$$

in the form

$$S(\xi, {}_aA \wedge {}_aB) = S(\xi, {}_aA) \wedge {}_aB + (-1)^k.{}_aA \wedge S(\xi, {}_aB) \ . \tag{96}$$

The contraction operator S fulfils also the relation

$$S(\xi, S(\xi, {}_aA)) = 0 \ . \tag{97}$$

Proof:

$$\begin{aligned} S(\xi, S(\xi, {}_aA)) &= A_{[i_1 i_2 i_3 ... i_k]}.\xi^{\bar{i}_1}.\xi^{\bar{i}_2}.dx^{i_3} \wedge ... \wedge dx^{i_k} = \\ &= A_{[i_2 i_1 i_3 ... i_k]}.\xi^{\bar{i}_2}.\xi^{\bar{i}_1}.dx^{i_3} \wedge ... \wedge dx^{i_k} = -A_{[i_1 i_2 i_3 ... i_k]}.\xi^{\bar{i}_1}.\xi^{\bar{i}_2}.dx^{i_3} \wedge ... \wedge dx^{i_k} = 0 \ . \end{aligned}$$

4.2. Inner (internal) product of a contravariant vector field and a full anti-symmetric tensor field. In the theory of the differential forms one of the important operators is the operator of the internal product used instead of a contraction operator acting on a contravariant vector field and a full anti-symmetric covariant tensor field.

Definition 16. *Inner (internal, interior) product [8] (pp. 170-173). The operator i_ξ for every contravariant vector field ξ mapping a full anti-symmetric covariant tensor field ${}_aA$ of rank k (k-form) in a full anti-symmetric covariant tensor field of rank $k-1$ ($k-1$-form) in the form*

$$i_\xi : {}_aA \to i_\xi({}_aA) = k.A_{[i_1 \dots i_k]}.S(\xi, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^k \ ,$$
$${}_aA \in \Lambda^k(M), \quad i_\xi({}_aA) \in \Lambda^{k-1}(M) \ , \qquad \xi \in T(M) \ ,$$
$$k.A_{[i_1 \dots i_k]} = \frac{1}{(k-1)!}.\sum_{(i_1 \dots i_k)} sgn(i_1 \dots i_k).A_{i_1 \dots i_k} \ , \qquad A \in \otimes_k(M) \ .$$

is called internal (inner, interior) product by ξ.

The operator i_ξ acts on a full anti-symmetric covariant tensor field like the contraction operator S [compare with

$$S(\xi, {}_aA) = A_{[i_1 \dots i_k]}.S(\xi, dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \frac{1}{k}.i_\xi({}_aA) \ ,$$

where in contrast to $i_\xi({}_aA)$ the contraction operator S in $S(\xi, {}_aA)$ acts on ξ and $d\widehat{x}^A$, and not only on dx^{i_1} as it is the case in $S(\xi, dx^{i_1})$].

The internal product has the properties:

(a) Action on a basic covariant anti-symmetric tensor field $d\widehat{x}^A = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$i_\xi(d\widehat{x}^A) = k.S(\xi, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} \ ,$$
$$i_{\partial_j}(d\widehat{x}^A) = k.S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} = k.f^{i_1}{}_j.dx^{i_2} \wedge \dots \wedge dx^{i_k} =$$
$$\frac{1}{k-1}.S(\partial_j, dx^{i_1}).\sum_{(i_2 \dots i_k)} sgn(i_2 \dots i_k).dx^{i_2} \otimes \dots \otimes dx^{i_k} \ .$$

(b) Action on an external product $d\widehat{x}^A \wedge d\widehat{x}^B$, $d\widehat{x}^B = dx^{j_1} \wedge \dots \wedge dx^{j_l}$

$$i_{\partial_j}(d\widehat{x}^A \wedge d\widehat{x}^B) = (k+l).S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} =$$
$$= k.S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} +$$
$$+ l.S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} =$$
$$= i_{\partial_j}(d\widehat{x}^A) \wedge d\widehat{x}^B + l.S(\partial_j, dx^{j_1}).dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_l} =$$
$$= i_{\partial_j}(d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge i_{\partial_j}(d\widehat{x}^B) \ ,$$

where

$$S(\partial_j, dx^{i_1}).dx^{i_2} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} =$$
$$= -S(\partial_j, dx^{i_2}).dx^{i_1} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l} =$$
$$= (-1)^k.S(\partial_j, dx^{j_1}).dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_l} \ ,$$

$$i_\xi(d\widehat{x}^A \wedge d\widehat{x}^B) = i_\xi(d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge i_\xi(d\widehat{x}^B) \ , \qquad \xi \in T(M) \ .$$

(c) Action on an external product ${}_aA \wedge {}_aB$ of two anti-symmetric covariant tensor fields ${}_aA$ and ${}_aB$

$$i_\xi({}_aA \wedge {}_aB) = i_\xi({}_aA) \wedge {}_aB + (-1)^k.{}_aA \wedge i_\xi({}_aB) \ .$$

Proof:

$$\begin{aligned} i_\xi({}_aA \wedge {}_aB) &= i_\xi(A_{[A]}.d\widehat{x}^A \wedge B_{[B]}.d\widehat{x}^B) = A_{[A]}.B_{[B]}.i_\xi(d\widehat{x}^A \wedge d\widehat{x}^B) = \\ &= A_{[A]}.B_{[B]}.[i_\xi(d\widehat{x}^A) \wedge d\widehat{x}^B + (-1)^k.d\widehat{x}^A \wedge i_\xi(d\widehat{x}^B)] = \\ &= i_\xi(A_{[A]}.d\widehat{x}^A) \wedge B_{[B]}.d\widehat{x}^B + (-1)^k.A_{[A]}.d\widehat{x}^A \wedge i_\xi(B_{[B]}.d\widehat{x}^B = \\ &= i_\xi({}_aA) \wedge {}_aB + (-1)^k.{}_aA \wedge i_\xi({}_aB) \ . \end{aligned}$$

The internal product i_ξ acts on a basic contravariant vector field and an external product as a differential operator not obeying the Leibniz rule but obeying the *rule for anti-differentiation*.

The operator i_ξ has also the property

$$i_\xi(i_\xi({}_aA)) = 0 \ .$$

Proof:

$$\begin{aligned} i_\xi(i_\xi({}_aA)) &= i_\xi(k.A_{[i_1 \ldots i_k]}.\xi^{\overline{i}_1}.dx^{i_2} \wedge \ldots \wedge dx^{i_k}) = \\ &= k.(k-1).A_{[i_1 \ldots i_k]}.\xi^{\overline{i}_1}.\xi^{\overline{i}_2}.dx^{i_3} \wedge \ldots \wedge dx^{i_k} = 0 \ , \end{aligned}$$

because of $\xi^{\overline{i}_1}.\xi^{\overline{i}_2} = \xi^{\overline{i}_2}.\xi^{\overline{i}_1}$ and $A_{[i_1 i_2 \ldots i_k]} = -A_{[i_2 i_1 \ldots i_k]}$.

The properties of symmetric and especially of antisymmetric tensor fields over $(\overline{L}_n, g)$-spaces are important for the working out of the symplectic geometry of differential forms over these spaces.

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References

[1] Chernikov N. A., *Different questions in the theory of relativity.* Elementary particles and atomic nucleus **18** (1987) 5, 1000-1034 (in russ.); *A background connection as a necessary object in general relativity theory.* Preprint JINR Dubna P2-88-778 (1988) (in russ.); *Einstein's theory of gravitation from a point of view of the tensor analysis.* Comm. JINR Dubna **P2-90-399** (1990) 1-17 (in russ.)

[2] Manoff S., *Kinematics of vector fields.* In *Complex Structures and Vector Fields.* eds. Dimiev St., Sekigawa K. (World Sci. Publ., Singapore, 1995), pp. 61-113

[3] Manoff S., *Geodesic and autoparallel equations over differentiable manifolds.* Intern. J. Mod. Phys. **A 11** (1996) 21, 3849-3874

[4] Manoff S., *Fermi-Walker transports over spaces with affine connections and metrics.* JINR Rapid Communications **1 [81]** (1997) 5-12

[5] Manoff S., *Fermi derivative and Fermi-Walker transports over (L_n, g)-spaces.* Class. Quantum Grav. **15** (1998) 2, 465-477

[6] Manoff S., *Lagrangian formalism for tensor fields*. In *Topics in Complex Analysis, Differential Geometry and Mathematical Physics*. eds. Dimiev St., Sekigawa K. (World Sci. Publ., Singapore, 1997), pp. 177-218

[7] Schmutzer E., *Relativistische Physik (Klassische Theorie)* (B.G. Teubner Verlagsgesellschaft, Leipzig, 1968)

[8] Bishop R. L., Goldberg S. I., *Tensor Analysis on Manifolds* (The Macmillan Company, New York, 1968)

[9] von Westenholz C., *Differential Forms in Mathematical Physics* (North-Holland Publ. Co., Amsterdam, 1978)

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Манов С., Димитров Б. E5-98-182

($\overline{L}_n$, g)-пространства. Специальные тензорные поля

Рассмотрены тензорное поле Кронекера, контракционное тензорное поле, мульти-кронекеровое и мульти-контракционное тензорные поля на пространствах с контравариантной и ковариантной аффинными связностями. Определены действие и соответствующие производные ковариантного дифференциального оператора, дифференциального оператора Ли, оператора кривизны и девиационного оператора на этих полях. Рассмотрены коммутационные соотношения операторов *Sym* и *Asym* с ковариантным дифференциальным оператором и с дифференциальным оператором Ли на пространствах с контравариантной и ковариантной связностями.

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Manoff S., Dimitrov B. E5-98-182

($\overline{L}_n$, g)-Spaces. Special Tensor Fields

The Kronecker tensor field, the contraction tensor field, as well as the multi-Kronecker and multi-contraction tensor fields are determined and the action of the covariant differential operator, the Lie differential operator, the curvature operator, and the deviation operator on these tensor fields is established. The commutation relations between the operators *Sym* and *Asym* and the covariant and Lie differential operators are considered acting on symmetric and antisymmetric tensor fields over ($\overline{L}_n$, g)-spaces.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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