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REMARKS ON THE APPROXIMATION OF DYNAMICAL SYSTEMS. ACTIONS OF AMENABEL GROUPS. T-PROPERTY [1)]

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1.INTRODUCTION

As in constructive function theory one studies the relationship between the properties of functions and the speed of their approximations by functions of some particular fixed class, in the same spirit one can study how to approximate dynamical systems, and thus from the speed of approximation to derive some properties of the system.

These problems have been extensively studied for actions of **Z**. We study the same problem for actions of any amenabel group.

For some subclass of amenabel groups we derive the formula for the speed of approximation.

The style of this paper is informal. We present several results, also we raise several questions which are, in our opinion, interesting.

1) **Here T stays for tile; this is not Kazhdan's T-property.**

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Suppose G is a topological group acting on a probability space (X, Σ, μ). We'll denote this by $G : (X, \Sigma, \mu)\circlearrowleft$ or $T^G : (X, \Sigma, \mu)\circlearrowleft$

We assume that G is locally compact and unimodular, i.e. the left and right invariant Haar measures coincide. Obviously the second condition is the strong restriction; one can find even Lie Groups which is not unimodular.

Whenever G is not discrete, for a space (X, Σ, μ) we assume a Lebesgue structure. This is a special structure which gives you "a feeling of coninuity in a measurable space" [7].

Definition 1. A dynamical system $T^G : (X, \Sigma, \mu)\circlearrowleft$ is called periodic if there exists a finite measurable partition $\xi = (C_1, ..., C_n)$ of (X, Σ, μ) such that $\forall g \in G\ \exists \sigma_g \in S_n$ such that $\forall i \in \{1, ..., n\}$, $gC_i = C_{\sigma_g(i)}$, and also if $\sigma_g(i) = i$ then $gx = x, \forall x \in C_i$.

When G is discrete one of the possible definitions for approximation could be the following

Definition 2. Let $f : N \mapsto R_+$ be the function, $f(n)$ decreases to 0.

We say that a dynamical system $T^G : (X, \Sigma, \mu)\circlearrowleft$ can be approximated by a periodic dynamical system $T^G_{per} : (X, \Sigma, \mu)\circlearrowleft$ with speed $f(n)$ if for any compact $K \subset G$ there exists a sequence of finite measurable partitions $\xi_m = \{C_m^{(1)}, ..., C_m^{(i_m)}\}$ such that ξ_m decreases to 0 [this means that $SUP_{j\in\{1,...,i_m\}}\mu(C_m^j) \to 0$] and

$$\sum_{g\in K}\sum_{i=1}^{i_m} \mu(T^g(C_i) \triangle T^g_{per}(C_i)) < f(n)$$

2. AMENABEL GROUPS

A locally compact group G is called amenabel if there is a left invariant mean on the space of bounded continuous functions on G. For unimodular groups, however, and in particular for discrete groups, starting from left invariant mean one can construct a biinvariant mean.

It has been proved by **Banach** that Z is amenabel. The systematic study of amenability, however, starts from **von Neumann**.

In 1950's there has been found a simple equivalent condition to amenability, namely the existence of Folner sequence **[1]**. From this result it comes out trivially (or easyly) that Z, R, Z^n, R^n and any commutative group is amenabel.

To find non-amenabel group is not diffucult: F_2 (free group on two generators) is not amenabel. However, for a long time it was an open question whether there exists a non-amenabel group which doesn't contain F_2 as a subgroup. The first example has been constructed by **A.Olshanski** **[8]** in 1970's.

The following definition we will need in the sequel.

Definition: A minimal class of groups containing all cyclic and commutative groups and closed under the following operations is called elementary.

(i) passing to subgroup, (ii) taking factorgroup, (iii)taking increasing union, (iv) taking compact extensions.

THEOREM (von Neumann). Elementary groups are amenabel.

Note that all solvable and even almost solvable groups are in the class of elementary groups.

Not all amenabel groups are elementary, however. This too, was a long resisting question until R.Grigorchuk **[9]** constructed the first example. The last examples of this type are even finitely presented.

This question is very related to J.Milnor's problem about the exsistence of intermediate growth in finitely generated groups. For example, any group with 2 generaters has either polynomial growth or exponential growth. There is nothing in between. This interesting fact is not true, however, in general. Intermediate growth does not appear also among the finitely generated groups with one relation.

QUESTION. Is there any reflection group (i.e. group generated by reflections in an n-dimensional, not-necesssarily Euclidean, space) with intermediate growth.

The answer is no, because reflection groups are linear and they do not have intermediate growth because of famous Tits alternative.

However one can ask this question for groups that are a sort of generalization of Reflection Groups, for example for the so called Generalized Coxeter Groups. This question seems interesting also because it remains still unclear if there exists any **finitely presented** group with intermediate growth.

3.ROHLIN LEMMA AND TILE COVERINGS OF AMENABEL UNIMODULAR GROU

In its classical form Rohlin Lemma looks like the following

Rohlin Lemma [3]. Let $T : (X, \Sigma, \mu)$ be an aperiodic dynamical system, i.e. $\mu\{x : T^n x = x \text{ for some } n\} = 0$ then $\forall \varepsilon > 0$, and $\forall n \in N \ \ \exists A \in \Sigma$ such that $T^i A \cap T^j A = \emptyset, i \neq j, \quad i, j \in \{1, ..., n\}$ and $\mu(\cup_{k=1}^{n} T^k A) > 1 - \varepsilon$

There exist many interesting versions of this lemma among which we remind the classical Strong Version of Rohlin Lemma [3] and ε-free Rohlin Lemma [4].

The set $\{1, ..., n\}$ in the acting group **Z** turns out to be very fundamental. Instead of this set we could take, for example, $\{1, 2, 5, 6\}$ or, in general, $\{1, ..., n, 2n+1, ..., 3n\}$ but not $\{1, 2, 4, 5\}$ The reason is that the last set is not a tile (see definition below) for the group **Z**.

Definition 1. Suppose G is any group and $K \subset G$. We say K is a tile for G iff $\{c_i\}_{i \in I} \subset G$ such that $Kc_i \cap Kc_j = \emptyset \ \ i \neq j$, and $\cup_{i \in I} Kc_i = G$

Here I is a countable set.

Definition 2. Suppose the group G acts on (X, Σ, μ) and $K \subset G$. Then a set $A \in \Sigma$ is called a base of a K-tower iff $k_1 A \cap k_2 A = \emptyset \ \ \forall k_1, k_2 \in K, \quad k_1 \neq k_2$

These definitions now give rise to the following generalization of Rohlin Lemma up to the case of discrete amenabel groups.

Theorem 1 (D.Ornstein, B.Weiss [2]) If K is a tile for a discrete amenabel group G acting on (X, Σ, μ), then $\forall \varepsilon > 0$ there exists $A \in \Sigma$ such that A is a base of a K-tower and $\mu(K \cdot A) > 1 - \varepsilon$.

Whether or not this theorem remains valid for any discrete group is an open question. It has not been understood even for the simplest tiles of F_2.

The theorem above shows how important is a notion of tile. Because of this the following questions are interesting

QUESTION 1. Does every discrete group have a tile with as big cardinality as we wish?

QUESTION 2. Does every amenabel group possess a tile as invariant as we wish (w.r.t the fixed compact set).

QUESTION 3. Does every topologicàl group (amenabel, unimodular) group have a tile (any tile!)?

QUESTION 4. Suppose G is an amenabel, unimodular topological group with a fixed Haar measure. If G has a cocompact lattice, does it possess a tile with Haar measure as small as we wish?

The questions 2 and 3 have been raised in **[2]**. With the questions 1 and 4 one oftenly meets in certain problems of amenabel group actions.

As it has been remarked in **[2]**, the answer to QUESTION 2 is positive whenever the amenabel group G is elementary. This is because, cyclic groups do have tiling Folner sets, and starting from this one can build up tiling Folner sets for all abelian, solvable groups, their increasing unions, finite extensions, etc. that is for all elementary amenabel groups.

Definition 3. Let G be a discrete group. We say it has a T-property if for any finite $F \subset G$ there exists a tile $T \supseteq F$.

It still remains open whether any group or at least any amenabel group has this property. In **[10]** it has been proved that the class of groups with T-property is closed under the operations (i), (ii), (iii) and (iv) of **von Neumann Theorem** and that all elementary groups have T-property. **[10]** also contains some other results about the class of groups with T-property. We wish to add the following Lemma which can be easily obtained by slightly developing ideas of that paper.

Lemma. Let G be a discrete group such that for each finite $F \subset G$ there exists a collection of normal subgroups $K_1, ..., K_n$ such that G/K_i has a T-property $\forall i \in \{1, ..., n\}$ and $\forall x, y \in F \;\; \exists i = i(x, y) \in \{1, ..., n\}$ such that $xK_i \neq yK_i$.

Then G has T-property.

In **[2]** they prove also the exsistence of quasi-tiles (see definition below) for any amenabel group. This is extremely useful result and in many situations it becomes sufficient to proceed as much as you would have in your hand the result of existence of tiling Folner sets.

Definition 4. We say that the collection of sets $T_1, ..., T_N$ of a discrete amenabel group G is an ε-quasitile iff $e \in T_1 \subset T_2 \subset ... \subset T_N$ and $\forall$ finite $D \subset G \;\; \exists C_i, 1 \leq i \leq N$ such that

(i) $\{T_i c : c \in C_i\}$ are ε-disjoint.

(ii) If $i \neq j$ then $T_i C_i \cap T_j C_j = \emptyset$

(iii)The sets $T_i C_i, i \in \{1, ..., N\}$, $(1 - \varepsilon)$-cover D.

Theorem 2. (D.Ornstein, B.Weiss [2] For any $\varepsilon > 0$ $\exists N = N(\varepsilon)$ such that for any compact K and any $\delta > 0$ there exists an ε-quasitile $\{T_1, ..., T_N\}$

It is important to notice that N depends only on ε. It is universal for all K and δ.

The Rohlin Lemma for this quasitile looks like the following

ROHLIN LEMMA. Let G be any discrete amenabel group acting almost freely on (X, Σ, μ). If $\{T_1, ..., T_N\}$ is an ε-quasitile and each $T_i, i = 1, ..., N$ is sufficiently invariant, then $\exists V_i^l, 1 \leq i \leq N, 1 \leq l \leq L_i$ such that

(i) $T_i V_i^l$ is a T_i-tower

(ii) $T_i V_i^l \quad 1 \leq l \leq L_i$ are ε-disjoint

(iii)for $i \neq j \quad T_i V_i^l \cap T_j V_j^{l'} = \emptyset$

(iv)$\mu(\cup_{i=1}^{k} \cup_{l=1}^{L_i} T_i V_i^l) \geq 1 - \varepsilon$

4.APPROXIMATION RESULTS FOR AMENABEL GROUP ACTIONS

We present here two theorems.

THEOREM 1. Any action of a discrete amenabel group can be approximated by periodic actions of that group.

We do not present here the proof. In the proof we use quasi-tile covering results and the corresponding Rohlin Lemma. The essential idea is from **[5]**, but we need to solve technical diffculties which arise even in the case when the group has a strict sufficiently invariant tile.

THEOREM 2. Let G be a discrete, elementary amenabel group with subexponential growth. Then by known result **[6]** the growth will be polynomial.

Suppose d is the degree of the polynomial which bounds the growth. Then any action of G can be approximated by the periodic action with speed $f(n) = \frac{a_n}{(log n)^{1/d}}$ where a_n is any increasing unbounded sequence of positive real numbers.

For d=1 this result has been proved in **[5]**.

Intuitively, as smaller is the entropy of the dynamical system as better it can be approximated by periodic systems. In **[5]** this has been justified by the result that in THEOREM 2 instead of a_n one can take $h(T) + \delta$ where $h(T)$ is the entropy of the system and δ is any positive number. One can prove the same result for actions of amenabel groups because we do have a **Shannon-Macmillan type theorem** also for this case.

If the dynamical system (in the case of action of Z) can be approximated by cyclic periodic dynamical systems then this imposes a strong restriction on the dynamics of the system. There exist many interesting results in this direction. Because of this it would be interesting to have an answer for the following

QUESTION. How to define the cyclic periodic dynamical systems for the actions of amenabel groups.

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