SISSA ISAS

THE CLASSIFICATION OF ORBITS OF BRAID GROUP ACTION ON THE SET OF GENERATORS OF FINITE COXETER GROUPS.

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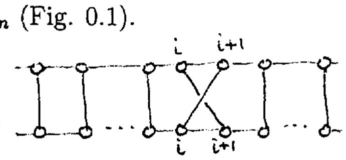
We classify completely the orbits of braid group action on the set of generators of finite Coxeter groups. This leads eventually to the classification of Frobenius manifolds with polynomial potentials. On this we prepare another paper together with Prof. Boris A. Dubrovin.

* * *

Suppose $\{e_1, ..., e_n\}$ is a basis of R^n which is endowed by a bilinear form $(.,.)$, not necessarily positive definite. Then the braid group Br_n acts on the set of basises in the following way:

$\sigma_i(e_1, ..., e_n) = (e'_1, ..., e'_n)$ where $i \in \{1, ..., n-1\}, e'_j := e_j$ if $j \neq i, i+1, e'_i := e_{i+1}, e'_{i+1} := e_i - \frac{2(e_i,e_j)}{(e_j,e_j)} e_j$.

Here $\sigma_1, ..., \sigma_{n-1}$ are the generators of Br_n (Fig. 0.1).

Fig. 0.1.  σ_i

One easily verifies that this is in fact the action of braid group Br_n. We remind the relations of this group: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, ..., n-1;$ $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1, i, j \in \{1, ..., n-1\}$

Except the action of braid group we have another operations on a given basis, namely we allow to take any pair of vectors from the basis (i-th and j-th vectors) and then to reflect one of the vectors with respect to another obtaining a new basis or a new generator system of a given Coxeter Group. Also the permutation (before

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reflection) of the vectors in the pair is allowed. This we call in the sequel the action of elementary braid σ_{ij} (or σ_{ji} if we have permuted the vectors) although it corresponds to the real braid action only if $j = i+1$. All these can be covered by the naturally extended action $Br_n \times S_n$ but the proof we present here works for much weaker actions.

We classify completely the orbits (i.e. we find out how many orbits there are) of of this action on the set of generators of finite Coxeter groups, i.e. $A_n(n \geq 2), B_n(n \geq 3), C_n(n \geq 3), D_n(n \geq 4), E_8, E_7, E_6, F_4, G_2, H_3, H_4$ and finally $I_2(k)(k \geq 2)$. All these reflection groups are in a Euclidean space. In non-Euclidean geometry, as it is known, we don't have finite reflection groups.

The result is shown in the table below

$A_n(n \geq 3)$	$B_n(n \geq 3)$	$C_n(n \geq 3)$	$D_n(n \geq 4)$				
1	1	1	1				
E_8	E_7	E_6	F_4	G_2	H_3	H_4	$I_2(k)$.
1	1	1	1	4	3	10	$\frac{1}{4}(3+(-1)^k)\mu(k)$

Here $\mu(k)$ is number of natural numbers less or equal than k which are mutually prime with k.

It is possible, probably, to solve these problems also using singularity theory (for some of the groups). Instead we have chosen a combinatorical way. This approach has it's own advantage. For example, from the proof it comes out that exactly which set of basises generate the given Coxeter group. This fact, which, of course, is not the main result of our paper, to our surprise was not known even to specialists whom we contacted.

The order we have chosen to prove the claim of the table is the following
$A_n \rightarrow D_n \rightarrow B_n, C_n \rightarrow E_8, E_7, E_6 \rightarrow F_4 \rightarrow G_2, H_3, H_4, I_2(k)$.

For the action of braid group itself one obtains a compact classification picture only for the action on the space of Gram matrises of a given generator system. We will present this result in our next publication.

PROPOSITION A. Any two generating root system of A_n are equivalent.

Proof. The root system of A_n is $\Phi = \{\epsilon_i - \epsilon_j : i, j \in \{1, ..., n+1\}, i \neq j\}$ where $\{\epsilon_1, ..., \epsilon_{n+1}\}$ is a standard basis of R^{n+1}. $\Delta_{st} = \{\epsilon_1 - \epsilon_2, ..., \epsilon_n - \epsilon_{n+1}\}$ is standard generating system.

Let Δ be any generating system. We associate to it a graph $\Gamma(\Delta) = \{V(\Delta), E(\Delta)\}$ where $V(\Delta) = \{1, ..., n+1\}, E(\Delta) = \{((i,j),k) : \epsilon_i - \epsilon_j$ or $\epsilon_j - \epsilon_i$ is the k-th vector of $\Delta\}$.

So, each vertice and each edge of our graph is labelled.

Observation: Δ is a generator iff $\Gamma(\Delta)$ is a connected tree.

The elementary braid σ_{ij} acts in the way shown in Figure . These operations in $\Gamma(\Delta)$ we'll call elementary operations.

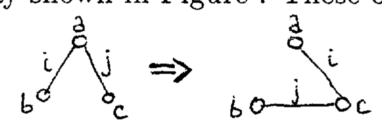


Fig. 1.

Lemma 1. Under elementary operations the graph $\Gamma(\Delta)$ is equivalent to

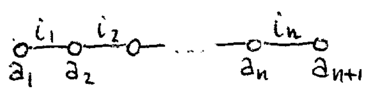


Fig. 2.

Here $\{i_1, ..., i_n\} \equiv \{1, ..., n\}, \{a_1, ..., a_{n+1}\} \equiv \{1, ..., n+1\}$

Proof. For $\Gamma(\Delta) = \{E(\Delta), V(\Delta)\}$ let $W_\Gamma := (\rho_l - 1)$ where ρ_l is the number of edges incident to the vertice l.

We prove the lemma by induction in W_Γ. .

If there is a vertex l with $\rho_l = 2$ then the graph can be divided into two parts by taking off this vertex. So, we can assume $\rho_l \neq 2$ for any $l \in \{1, ..., n+1\}$. We claim that there exist such vertices a, b, c that $\rho_a = \rho_c = 1$, where $(a,b), (c,b) \in E(\Delta)$.

Indeed, otherwise every endpoint of the tree $\Gamma(\Delta)$ is connected with the inner vertex which is not connected to any other endpoint. By the assumption, all the inner vertices l have the weight, i.e. the number of vertices incident to the given vertex, $\rho_l \geq 3$. Denote

$$n = E(\Delta),$$

k :=Card{endpoints of $\Gamma(\Delta)\}, k + m$:=Card{inner vertices of $\Gamma(\Delta)\}$. We have Card$V(\Delta) = n + 1$. So, $2k + m = n + 1 (*)$.

From the obvious formula

$$2\text{Card } E(\Delta) = \rho_j$$

we derive an inequality $2n \geq k + 3(k+m)$. This contradicts $(*)$.

Now, applying the elementary operation to the triple (a, b, c) we lower W_Γ by 1. This completes the proof by induction.

Lemma 2 Any graph $\{\{1, 2, ..., n, n+1\}, \{((i_1, i_2), j_1), ..., ((i_n, i_{n+1}), j_n)\}$ can be brought by elementary operations to the form

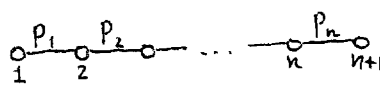


Fig. 3

Here $\{i_1, ..., i_{n+1}\} = \{1, ..., n+1\}, \{j_1, ..., j_n\} = \{p_1, ..., p_n\} = \{1, ..., n\}$

Proof. We will use induction in $n = \mathrm{Card}E(\Delta)$.

Let the vertex 1 stand in the k-th place when looking from one of the edges of the tree. If $k = 0$ or $k = n+1$ then we can apply induction to the rest of the graph. Otherwise we can bring the vertex 1 to the end of the standard tree using the sequence of elementary operations shown in Fig

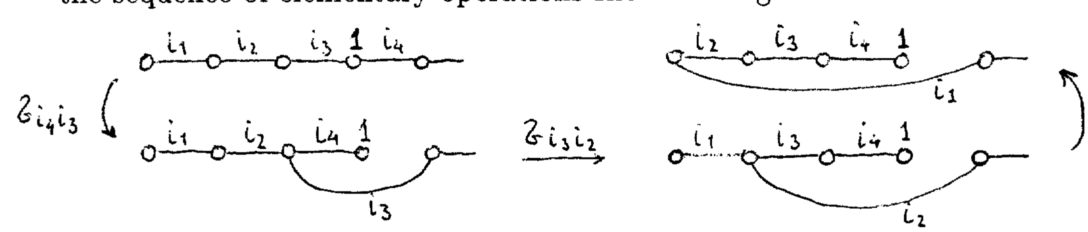


Fig. 4

Lemma 3. The claim of the proposition is true for A_2 (up to permutation).

Proof. Take the standard system of generators in $W(A_2)$ and apply the braids σ_1^k for any k. By straghtforward computation we check that the orbit exhausts all possible unordered roots of generators.

Now we introduce another object for Δ, oriented graph: $\Gamma_{or}(\Delta) = \{V(\Delta), E_{or}(\Delta)\}$, where $E_{or}(\Delta) := \{((i, j), k) : (\epsilon_i - \epsilon_j)$ is the k-th vector of $\Delta\}$.

Elementary operations for elementary braids can be introduced in a similar way.

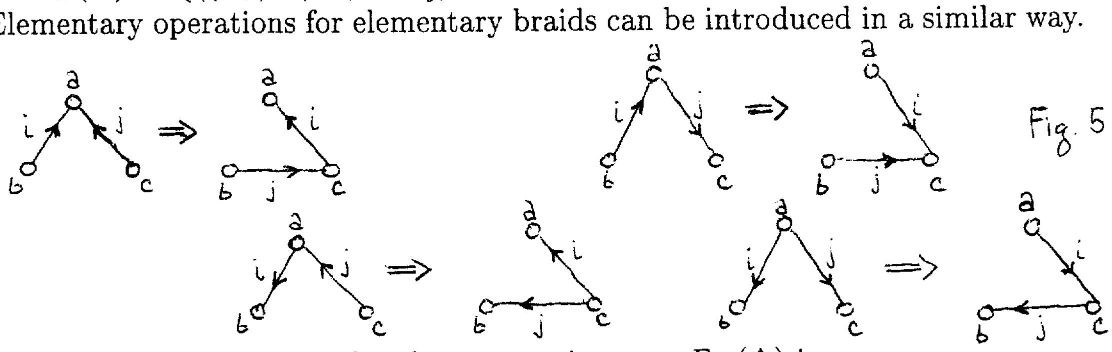


Fig. 5

We need to prove that under these operations any $\Gamma_{or}(\Delta)$ is

equivalent to

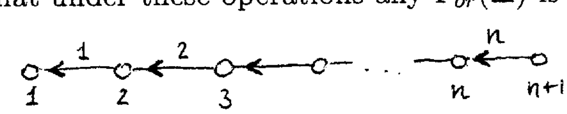


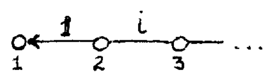
Fig. 6

However, from Lemma1, Lemma2 we see that it's enough to prove only the following

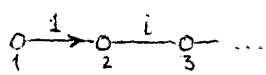
Lemma 4. The or-graph having the topology of the standard tree

with the standard numerations of the vertices, but with arbitrary orientations and labels of the edges, is equivalent to the oriented graph of the standard system of generators of $W(A_n)$

Proof. We proceed by induction, which starts from $n = 2$ in Lemma A3. If the graph has the form



then we can apply the induction in n. If, however, the graph has the form



then, applying Lemma 3 to the subsystem of A_2-type generated by e_1 and e_i, we invert the orientation of e_1. If the edge 1 stands on the k-th place of the graph, $k > 1$, looking from the left, and the edge with the vectors $(i-1, i)$ corresponding to the vector e_i, then we apply again Lemma 3 to the root system (e_i, e_1) to move e_1 to the $(k-1)$-th position.

This construction completes the inductive proof of the lemma and so, proposition is proved.

PROPOSITION D. Any two generating root system of D_n are equivalent.

Proof. The root system of D_n is $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i, j \in \{1, ..., n\}, i \neq j\}$.

$\Delta_{st} := \{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n\}$ is a standard generating system.

Again we associate to Δ a combinatorical object $\Gamma^{sign}(\Delta) = \{V(\Delta), E^{sign}(\Delta)\}$ where $V(\Delta) = \{1, ..., n\}, E^{sign}(\Delta) = \{((i,j), k; +) : \epsilon_i + \epsilon_j$ or $-(\epsilon_i + \epsilon_j)$ is the k-th vector of $\Delta\} \cup \{((i,j), k; -) : (\epsilon_i - \epsilon_j) or (\epsilon_j - \epsilon_i)$ is the k-th vector of $\Delta\}$.

So, vertices and edges of our graph are labelled and each edge has a sign.

Lemma D1. Δ is a generator of D_n iff $\Gamma^{sign}(\Delta)$ is connected, it has only one cycle and the number of + edges on this cycle is odd.

Proof. If the graph is disconnected then the spans of two components of it are orthogonal subspaces. This contradicts irreducibility of $W(D_n)$.

Our graph has n vertices and n edges. So it has exactly one cycle. Let $i_1, i_2, ..., i_k, i_{k+1} := i_1$ be the vertices of the cycle of the graph. So, our basis contains the vectors $e_s = \pm(\epsilon_{i_s} + \delta_s \epsilon_{s+1}), s = 1, ..., k$.

Let us form a $k \times k$ matrix A consisting of the coordinates of the vectors $e_1, ..., e_s$ w.r.t. the basis $\pm\epsilon_{i_1}, ..., \pm\epsilon_{in}$.

$$A = \begin{bmatrix} 1 & \delta_1 & & & \\ 0 & 1 & \delta_2 & & \\ & & 1 & & \\ & & & \ddots & \\ \delta_k & & & & 1 \end{bmatrix}$$

The determinant

$$detA = 1 + (-1)^{k+1}\delta_1 ... \delta_n$$

doesn't vanish iff $\delta_1 ... \delta_k = (-1)^{k+1}$. This means that the number of $+1$ among $\delta_1, ..., \delta_k$ is odd.

Lemma is proved.

The elementary braid acts in the way showen at the picture

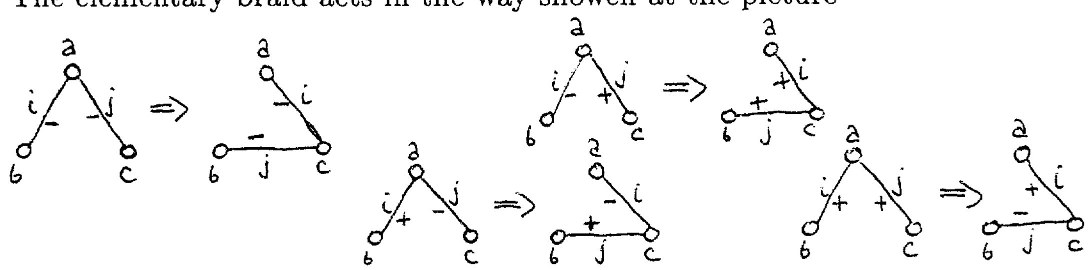
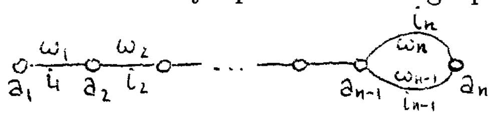


Fig. 7

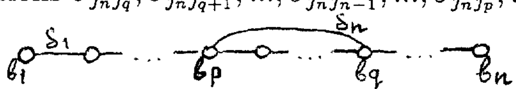
These operations in $\Gamma^{sign}(\Delta)$ we'll call elementary operations.

Lemma D2. Under elementary operations the graph $\Gamma^{sign}(\Delta)$ is equivalent to $(*)$.

(∗) 

Here $\{i_1, ..., i_n\} = \{a_1, ..., a_n\} = \{1, ..., n\}, \omega_k$ is + or - and only one of the ω_{n-1} and ω_n. is a + sign.

Proof.If we exclude one edge of $\Gamma^{sign}(\Delta)$ which is in the cycle then we get a tree. Because of Lemma A.1 this shows that $\Gamma^{sign}(\Delta)$ is equivalent to the graph of the form (xx), where $\{b_1, ..., b_n\} = \{j_1, ..., j_n\} = \{1, ..., n\}, \delta_k$ is + or - , the number of + signs among $\delta_p, ..., \delta_{q-1}, \delta_n$ is odd. In its turn (xx) can be broght to the form $(*)$ by the following elementary operations $\sigma_{j_n j_q}, \sigma_{j_n j_{q+1}}, ..., \sigma_{j_n j_{n-1}}, ..., \sigma_{j_n j_p}, ..., \sigma_{j_n j_{n-2}}$.

 (xx)

Lemma D3. The signed graph $(*)$ is equivalent to $\diamond$.

Proof is similar to that of Lemma A.

Lemma D4. The claim of proposition is true for D_3

Proof. The root system of D_3 type is isomorphic to the one of the A_3 type. Thus lemma follows from Proposition A.

We introduce now more subtle combinatorical object for Δ; oriented signed graph: $\Gamma^{sign}_{or}(\Delta)$. To define this on each edge of $\Gamma^{sign}(\Delta)$ we put an orientaion in the foolowing way; if the edge stays for the vector

(a)$\epsilon_i - \epsilon_j$ then the edge is oriented from i to j.

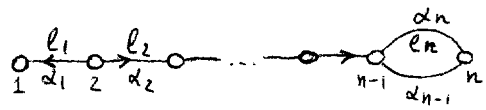
(b)$\epsilon_j - \epsilon_j$ then the edge is oriented from j to i.

(c)$\epsilon_i + \epsilon_j$ then the edge is oriented from i to j if $i < j$, and from j to i if $i > j$.

Elementary operations on this graph we define in a similar way.

The following lemma is the last step in the proof of proposition.

Lemma D5. Any graph of the type $\diamond$ is equivalent under elementary operations to the standard graph. Here orientations are arbitrary, $\{l_1, ..., l_n\} \equiv \{1, ..., n\}, \alpha_i$ is

(◇)  Fig 8

+ or - , $i = 1, ..., n$.

Proof is by induction in n. The base of induction we have already established in Lemma 4.

Suppose now the vector 1 stays at k-th place (Fig 8). Then by elementary operations as we used in the proof of Lemma 2 we can get the graph of the type $\diamond_1$.

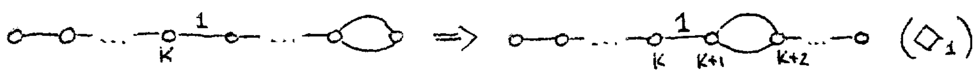


Fig. 9

Then applying Lemma 4 to the triple $\{k, k+1, k+2\}$ we get $\diamond_2$ and then again applying Lemma 4 to the triple $\{k-1, k, k+1\}$ we get $\diamond_3$ such that the vector 1 is now in $(k-1)$-th place with standard orientation.

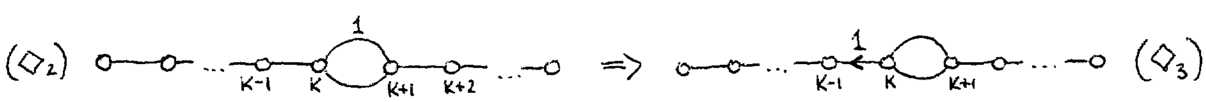


Fig. 10

Continuing this process we get a graph $\diamond_0$ in which the vector 1 is in the first place with standard orientatin. Then the claim of the lemma follows by induction.

Proposition is proved.

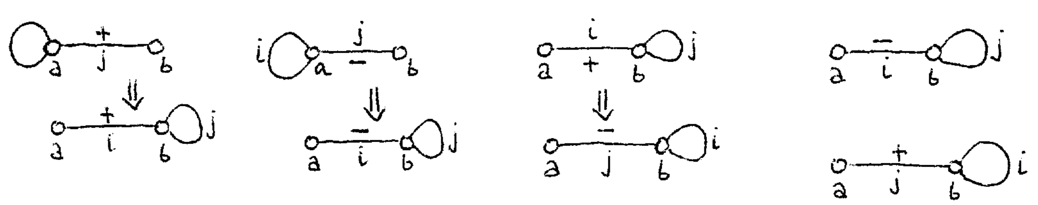
PROPOSITION B. Any two generating root system for B_n are equivalent.

Proof. The root system of B_n is $\Phi = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, i \neq j, i, j \in \{1, ..., n\}\}$.
$\Delta_{st.} = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$ is a standard generating system.
We associate to Δ a graph $\Gamma^{sign}(\Delta) = \{V(\Delta), E^{sign}(\Delta)\} = \{\{1, ..., n\}, \{\{((i, j), k; +); \epsilon_i$ ϵ_j or $-(\epsilon_i + \epsilon_j)$ is the k-th vector of $\Delta\} \cup \{((i, j), k; -); \epsilon_i - \epsilon_j$ or $\epsilon_j - \epsilon_i$ is the k-th vector of $\Delta\} \cup \{((i, i), k); \epsilon_i$ or $-\epsilon_i$ is the k-th vector of $\Delta\}$.
So, all vertices and edges are labelled, and each edge has a sign except if it is a loop.

Lemma B1. Δ is generator system of B_n iff $\Gamma^{sign}(\Delta)$ is a connected cycle-free graph with exactly one loop.

Proof. The graph must be connected because of irreducibilty of $W(B_n)$. It must contain at least one loop, because otherwise it is of D_n type because of Proposition D. Let's assume that it contains at least two loops. Then the remaining part of the graph will contain n vertices and at most $n - 2$ edges. Such a graph, however, can never be connected. Lemma is proved.

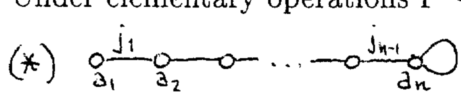
The elementary braid σ_{ij} acts in the way shown in Fig.7 and in Fig.11



These operations we'll call elementary operations.

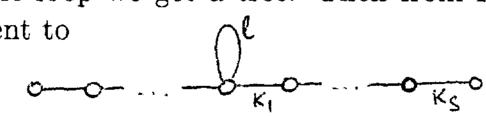
Fig. 11

Lemma B2 Under elementary operations $\Gamma^{sign}(\Delta)$ is equivalent to $(*)$.



Here $\{a_1, ..., a_n\} \equiv \{1, 2, .., n\}, \{j_1, ..., j_{n-1}\} \equiv \{1, ..., n-1\}$, ω_k is $+$ or $-$.

Proof. If we delete the single loop we get a tree. Then from Proposition A it follows that $\Gamma^{sign}(\Delta)$ is equivalent to



Applying elementary braids $\sigma_{lk_1}, \sigma_{k_1k_2}, ..., \sigma_{k_{s-1}k_s}$ successively from the last graph we obtain a graph of type $(*)$.

Lemma B3 The graph (*) is equivalent to 

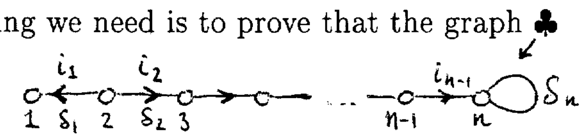
Proof is similar to the proof of analogical Lemma A (induction in n).

We need again to introduce oriented signed graphs for $\Delta : \Gamma_{or}^{sign}(\Delta) = \{V(\Delta), E_{or}^{sign}(\Delta)\}$. If the edge is not a loop then the orientation of this edge is defined as for the oriented graph of generator system of D_n.We do not assign orientation to loops. The loop has $+$ $(-)$ sign if it represents $+\epsilon_i(-\epsilon_i)$ for some $i \in \{1, 2, ..., n\}$.Elementary braid operations on $\Gamma_{or}^{sign}(\Delta)$ are also defined similarly.

Lemma B4. The claim of Proposition is true for B_3

Proof is by direct check.

Now, the only thing we need is to prove that the graph



is equivalent to $\Gamma_{or}^{sign}(\Delta_{st})$

This can be done by induction in n. Indeed, applying each time Lemma B3 after a finite step we'll succeed to bring the edge 1 with standard sign and orientation to the 1st position, i.e. the edge 1 will connect the vertices 1 and 2 . Along the process we never change the order of vertices. Then we proceed by induction.

Proposition is proved.

PROPOSITION E. Any two generating root system of E_8 are equivalent.

PROOF. Let $\Phi_1 = \{\pm\epsilon_i \pm \epsilon_j, i,j \in \{1,...,8\}, i \neq j\}, \Phi_2 = \{1/2(\pm\epsilon_1 \pm ... \pm \epsilon_8)$: even number of $+\}.\Phi_1 \cup \Phi_2$ is a root system of E_8.

$\Delta_{st} = \{1/2(\epsilon_1 - \epsilon_2 - ... - \epsilon_7 + \epsilon_8), \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, ..., \epsilon_7 - \epsilon_6\}$ is a standard generating system. We need to prove that Δ is equivalent to Δ_{st}.

The algorithm of bringing Δ (by successive actions of elementary braids) to Δ_{st} will be as following; First, we find $\Delta' \sim \Delta$ such that $\text{Card}\{\Delta' \cap \Phi_2\} = 1$. So we have a single vector in Δ' from Φ_2 (obviously, $\{\Delta \cap \Phi_2\} \geq 1$ for any Δ). Then we "adjust" the signs of this vector, i.e. to make them as of $\Delta_{st} \cap \Phi_2$, maybe except two.

[Along the process we'll have always one vector from Φ_2 in the generator system].

After this we proceed using some lemmas that we are going to prove.

Let, now, $A := \{\epsilon_1, ..., \epsilon_8\}$. We define a map $T : 2^{\Phi_1} \to 2^A$ in the following way: for $X \in 2^{\Phi_1}$ put $T(X) = \{\epsilon_i : \exists j \neq i, \{\pm\epsilon_i \pm \epsilon_j\}\} \cap X \neq \emptyset$.

Here are two simple observations

(a) Let $e_i = u \in \Delta \cap \Phi_1, e_i = \pm\epsilon_p \pm \epsilon_q$, and $e_j = v \in \Delta \cap \Phi_2$ be two nonorthogonal vectors. Applying the braid σ_{ij} we obtain a new basic vectors $e'_i = v', e'_j = u$ such that exactly the p-th and the q-th coordinates of v' and v have the opposite signs.

(b)If $u, v \in \Phi_2$ are not orthogonal then the reflection of one of them w.r.t. another one will be a vector of Φ_1.

Lemma 1. If $\text{Card}\{\Delta \cap \Phi_2\} = 1$ then $\text{Card}T(\Delta \cap \Phi_1) = 7$ or 8.

Proof. Indeed, otherwise the system ,i.e. Δ, is linearly dependent.

Lemma 2. There exists $\Delta' \sim \Delta$ s.t. $\text{Card}\{\Delta' \cap \Phi_2\} = 1$.

Proof. Let $n(\Delta) := Card\{\Delta \cap \Phi_2\}$. Then $1 \leq n(\Delta) \leq 8$.

If $n(\Delta) = 1$ then there is nothing to prove, so suppose $n(\Delta) \geq 2$. We'll prove that $\exists \Delta''$ such that

$$n(\Delta'') = n(\Delta) - 1$$

(obviously this is enough for the lemma). This follows from (b) if $\exists u, v \in \Phi_2 \cap \Delta$ such that $(u, v) \neq 0$ or, otherwise, $\exists w \in \Phi_1 \cap \Delta, \exists u, v \in \Phi_2 \cap \Delta$ such that $(u, w) \neq 0, (v, w) \neq 0$.

Indeed, let us consider the case when all the vectors $u_1, ..., u_k \in \Delta \cap \Phi_2$ are orthogonal pairwise, and every vector $w \in \Delta \cap \Phi_1$ is orthogonal to all of them but one. Then $\Delta \cap \Phi_1$ splits into k subsets

$$\Delta \cap \Phi_1 = \Delta_1 \cap ... \Delta_k, \qquad \Delta_i \cap \Delta_j = \emptyset$$

for $i \neq j, w \in \Delta_j$ iff $(w, u_j) \neq 0$

If Δ_i and Δ_j are orthogonal for any $i \neq j$ then the system splits into k pairwise orthogonal parts $\Delta_i \cup \{u_i\}, i = 1, ..., k$. This, however, is not allowed.

Otherwise we can find two vectors

$$w_i \in \Delta_i, w_j \in \Delta_j, \qquad (w_i, w_j) \neq 0.$$

Applying an appropriate braid we transform this pair to

$$w_i' = w_i, w_j' = w_i - (w_i, w_j)w_j$$

Then the new vector w_j' is not orthogonal to both u_i and u_j.

The lemma is proved.

Lemma 3. If Card$\{\Delta \cap \Phi_2\} = 1$ then $\exists \Delta' \sim \Delta$ such that $\Delta' \cap \Phi_2 = \Delta_{st} \cap \Phi_2$ or $\Delta' \cap \Phi_2 = (-\Delta_{st}) \cap \Phi_2$.

Proof. From Lemma 1 it follows that Card$A \setminus T(\Delta \cap \Phi_1) = 1$ or 0.

For the first case, suppose $A \setminus T(\Delta \cap \Phi_1) = \{\omega \epsilon_i\}$ where $\omega \in \{+1, -1\}$. The $i-$th sign of the single vector $e \in \{\Delta \cap \Phi_2\}$, i.e. the i-th coordinate's sign, may or may not coincide with the $i-$th sign of $\Delta_{st} \cap \Phi_2 = 1/2(\epsilon_1 - \epsilon_2 - ... - \epsilon_7 + \epsilon_8)$. But since $\Delta \cap \Phi_1$ is a generator system for D_7 we can adjust 6 of the other 7 signs using the procedure (a).

The case Card$A \setminus T(\Delta \cap \Phi_1) = 0$ we handle in the following

Lemma 4. If Card$\{\Delta \cap \Phi_2\} = 1$, and Card$T(\Delta \cap \Phi_1) = 8$ then $\exists \Delta' \sim \Delta$ such that

Card$\{\Delta' \cap \Phi_2\} = 1$ and $T(\Delta' \cap \Phi_1) = 7$

Proof. Let $\Delta = \{u, v_1, ..., v_7\}$ where $u \in \Phi_2, v_1, ..., v_7 \in \Phi_1$. We can assume that $(u, v_k) \neq 0$ for any $k \in \{1, ..., 7\}$.Then $\exists i \in \{1, ..., 8\}$ such that Card$\{T^{-1}\{\epsilon_i\} \cap \Delta\} = 1$.

Let Card$T^{-1}\{\epsilon_i\} \cap \Delta = v_7 := \epsilon_i \pm \epsilon_j$ Then the elementary braid corresponding to u and v_7 maps Δ to $\Delta_1 = \{u, u', v_1, ...v_6\}$, where $u' \in \Phi_2$ and u' differs from u

only in the signs of ϵ_i and ϵ_j. Since $(v_k, u) \neq 0$ for $k \in 1, ..., 6$ applying elementary braids to the pairs $(u, v_{k_1}), ..., (u, v_{k_l})$ successively

[here $\{v_{k_1}, ..., v_{k_l}\} \subset \{v_1, ..., v_6\}$; l may range from 2 to 6]

we get $\Delta_1 \sim \Delta_2 = \{w, u', v_1, ..., v_6\}$ where $w \in \Phi_2$ differs from u not only in the signs of ϵ_i, ϵ_j and also in 6 more signs. Then applying elementary braid to the pair w, u' we obtain finally $\Delta_2 \sim \Delta_3 = \{u', v', v_1, ..., v_6\}$ where $v' \in \Phi_1$ and ϵ_i is not in $T(\Delta_3 \cap \Phi_1)$.

The lemma is proved.

Lemma 5. Let Δ be an unordered basis of E_8-type such that

1)$\Delta \cap \Phi_2 = 1/2(\epsilon_1 - \epsilon_2 - ... - \epsilon_7 + \epsilon_8) := e_1$

2)Card$T(\Delta \cap \Phi_1) = 7$

Then Δ is equivalent to Δ_{st}

Proof. The vectors $e_2, ..., e_8 \in \Delta \cap \Phi_1$ is a basis of the D_7-type. Let $\epsilon_i \notin T(\Delta \cap \Phi_1)$. If i=8 then, reducing the D_7-type system $e_2, ..., e_8$ to the standard one, we obtain a reduction of the orthogonal system to the standard E_8-type. If, however, $i \in 2, ..., 7$ then changing if necessary the basis $e_2, ..., e_8$ to an equivalent basis of the D_7-type we may assume that

$$e_8 = \epsilon_i - \epsilon_j \in \Delta \cap \Phi_1$$

where $j \in \{2, ..., 7\}, j \neq i$, and that

$$e_2, ..., e_7 \in span\{\{\epsilon_1, ..., \epsilon_7\} \setminus \epsilon_i\}$$

is a basis of the D_6-type. We can change the D_6-basis $e_2, ..., e_7$ to an equivalent one containing the vectors $e'_2 = \epsilon_p + \epsilon_q, e'_3 = \epsilon_k + \epsilon_l$ where p, q, k, l are all distinct, $\{\epsilon_p, \epsilon_q, \epsilon_k, \epsilon_l\} \subset \{\epsilon_2, ..., \epsilon_7\} \setminus \{\epsilon_i, \epsilon_j\}$. Applying the reflections

$$e_8 \mapsto R_{e'_2} R_{e'_3}(e'_8) =: e''_8$$

we obtain a vector e''_8 such that

$$e''_8 = 1/2(-\epsilon_1 - \epsilon - 2 - ... + \epsilon_i - ... - \epsilon_7 + \epsilon_8)$$

and

$$R_{e_1}(e''_8) = \epsilon_8 - \epsilon_i =: e'''_8$$

For the basis $\Delta''' = \{e_1, e_2, ..., e_7, e'''_8\}$,

$$T(\Delta''' \cap \Phi_1) = \epsilon_1, ..., \epsilon_7.$$

Reducing the D_7 basis $\Delta''' \cap \Phi_1$ to the canonical D_7-form, we obtain a reduction of Δ to the canonical E_8 form.

If i=1 then, first, we obtain an equivalent system $\Delta' \cap \Phi_2$, with $e_8 = \epsilon_8 - \epsilon_2$, and the vectors

$$e_2, ..., e_7 \in span(\epsilon_2, ..., \epsilon_7)$$

being a system of D_6-type containing $e_2 = \epsilon_4 + \epsilon_5, e_3 = \epsilon_6 + \epsilon_7$. The system $\Delta'' = \{e_1, e_2, ..., e_7, R_{e_3} R_{e_2} R_{e_1}(e_8)\}$ has $\Delta'' \cap \Phi_1, T(\Delta \cap \Phi_1) = \{\epsilon_1, ..., \epsilon_7\}$.

Lemma is proved.

Lemma 5$'$. If $\Delta \cap \Phi_2 = -1/2(\epsilon_1 + ... + \epsilon_8)$, and Card$T(\Delta \cap \Phi_1) = 7$, then Δ is equivalent to Δ'

Proof. Let us begin with the basis

$$e_i = 1/2(\epsilon_1 - \epsilon_2 - ... - \epsilon_7 + \epsilon_8), e_2 = \epsilon_1 + \epsilon_8, e_3, ..., e_8$$

where $e_3, ..., e_8 \in span\{\epsilon_3, ..., \epsilon_6\}$. Applying the transformation

$$e_1 \mapsto R_{e_2}(e_1) = -1/2(\epsilon_1 + ... + \epsilon_8)$$

we obtain a basis Δ with $\Delta \cap \Phi_2 = -1/2(\epsilon_1 + ... + \epsilon_8)$,

and

$$\Delta \cap \Phi_1 = \{\epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_1, ..., \epsilon_6 - \epsilon_5, \epsilon_8 - \epsilon_6\}$$

is a system of the D_7 type. Let us redenote $\epsilon_1 \mapsto -\epsilon_1, \epsilon_8 \mapsto -\epsilon_8$, and then apply to the basis the construction of Lemma 5. This gives a new E_8-basis Δ' such that $\Delta' \cap \Phi_2 = -1/2(\epsilon_1 + ... + \epsilon_8)$, with an arbitrary subset $T(\Delta' \cap \Phi_1)$ consisting of 7 elements.

Lemma is proved.

We have done everything but only up to permutation.

Lemma 6. Any permutation of Δ_{st} is equivalent to Δ_{st}.

Proof is similar to the anological lemma for the standard generator of F_4 which will be the next proposition. We apologize from the reader for this inconvinence.

REMARK: From the proof of the Proposition E it comes out that, any two generating systems of E_6 or E_7 are also equivalent.

PROPOSITION F. Any two generating system of F_4 are equivalent.

Proof. Let $\Phi_0 = \{\pm\epsilon_i : i \in \{1,2,3,4\}\}, \Phi_1 = \{\pm\epsilon_i \pm \epsilon_j : i \neq j, i, j \in \{1,2,3,4\}\}, \Phi_2 = \{1/2(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$.

$\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2$ is a root system, $\Delta_{st} = \{\epsilon_4, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, 1/2(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ is a standard generating system.

We again define a map $T : 2^{\Phi_0 \cup \Phi_1} \mapsto 2^{\epsilon_1, \ldots, \epsilon_4}$ as following: for any $X \in 2^{\Phi_0 \cup \Phi_1}$ we put $T(X) = \{\epsilon_i : \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, j \neq i\} \cap X \neq \emptyset\}$

Lemma 1. For a generator system Δ, Card$\{\Delta \cap \Phi_1\}$ is invariant under elementary operations.

Proof. Let's take any pair $\{u, v\}$ and elementary braid operation on this ordered pair. u and also v may belong to Φ_0, Φ_1, or Φ_2.

So, we have 9 cases. But in each case, one verifies that, Card$\{\{u,v\} \cap \Phi_1\}$ doesn't change. Lemma is proved.

Let now Δ be a generating system. We are going to proceed by similar strategy to that of the case E_8

Lemma 2. $\exists \Delta' \sim \Delta$ such that Card$\{\Delta' \cap \Phi_2\} = 1$

Proof. Denote $n(\Delta) :=$ Card$\{\Delta \cap \Phi_2\}$. We'll prove that if $n(\Delta) > 1$ then $\Delta' \sim \Delta$ with $n(\Delta') = n(\Delta) - 1$. [Obviuosly this is enough for the lemma]

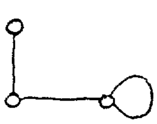
Indeed, let $u, v \in \Delta \cap \Phi_2$. If $(u, v) \neq 0$ or if, otherwise, $\exists w \in \Delta \cap \Phi_2$ such that $(w, u) \neq 0, (w, v) \neq 0$ then $\exists \Delta' \sim \Delta$ with $n(\Delta') = n(\Delta) - 1$.

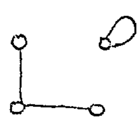
Suppose the above is not the case. Then exactly as in the case of Proposition E we can still lower $n(\Delta)$ by 1.

Lemma 3. Let Δ be a generating system with Card$\{\Delta \cap \Phi_2\} = 1$. Then

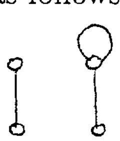
1) The system $\Delta \cap (\Phi_0 \cup \Phi_1)$ generates one of the following groups: $W(B_3)$ or $W(A_2) \times W(A_1)$

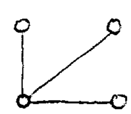
2) The system $\Delta \cap (\Phi_0 \cup \Phi_1)$ is equivalent under the action of the braid group B_3 to one of the following

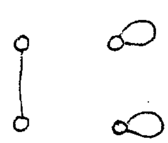
B_3:


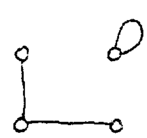
$A_2 \times A_1$:


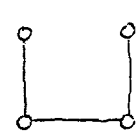
Proof. If Card$T(\Delta \cap \Phi_0 \cup \Phi_1) \leq 2$ then the three vectors in $\Delta \cap \Phi_2$ are linearly dependent. If $T(\Delta \cap \Phi_0 \cup \Phi_1) = 4$ then the possibilities for the graph of $(\Delta \cap \Phi_0 \cup \Phi_1)$ are as follows:


(a)


(b)


(c)


(d)


(e)

Without loss of generality we may assume that

$$\Delta \cap \Phi_2 = 1/2(\epsilon_1 + ... + \epsilon_4) =: e_1$$

For the graph Fig 13a), the rest part of $\Delta \cap \Phi_0 \cup \Phi_1$ has the form, up to renumrating,

$$e_2 = \pm(\epsilon_1 + \epsilon_2), e_3 = \pm\epsilon_3 \pm \epsilon_4, e_4 = \epsilon_4$$

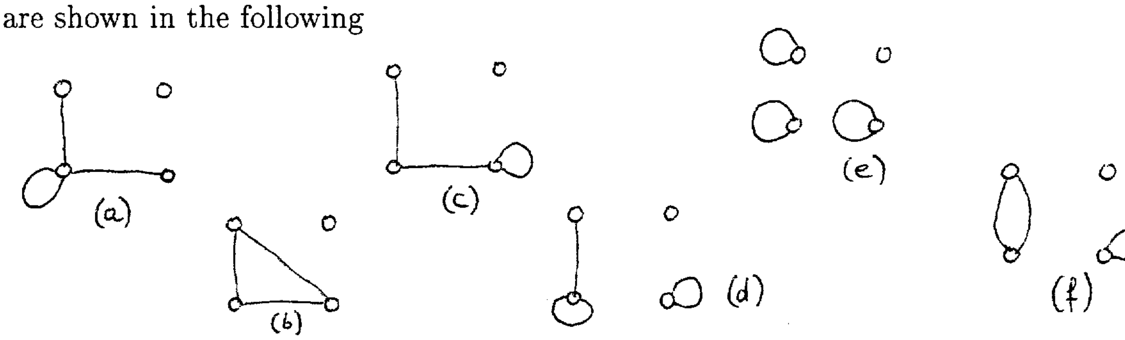
The full system $\{e_1, ..., e_4\}$ will be linearly dependent.

In a similar way we prove that the graph in Fig 13c) cannot be ralized.

The graphs Fig 13b) and Fig 13e) both correspond to a real system of the type A_3. We will show below that, in this case , Δ doesn't generate $W(F_4)$.

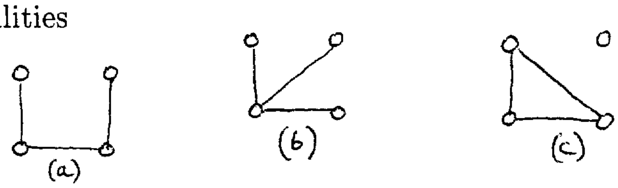
The graph Fig 13d) correspond to the direct product $A_2 \times A_1$.

If Card$T(\Delta \cap \{\Phi_0 \cup \Phi_1\}) = 3$ then the possibilities for the graph of $\Delta \cap \Phi_0 \cup \Phi_1$ are shown in the following



The graphs c), d), and e) correspond to $B_3, B_2 \times A_1, A_1 \times A_1 \times A_1$ root systems respectively. The graph a) corresponds to a system of generators of B_3 type. Using Proposition B we can transform it to the graph c). The graph b) corresponds to a system of the A_3 type.

Let us show now that, if $\Delta \cap \{\Phi_0 \cup \Phi_1\}$ generates the group $W(A_3)$ then Δ generates $W(B_4) \subset W(F_4)$. For the graph of $\Delta \cap \{\Phi_0 \cup \Phi_1\}$ in this case we have the following three possibilities



For the graphs shown in Fig a),b) the system $\Delta \cap \Phi_0 \cup \Phi_1$ is of the type A_3. All such systems of generators are equivalent w.r.t. the action of Br_3. For the graph a) without loss of generality we may assume that $\Delta = \{e_1, ..., e_4\}$ has the form

$$e_1 = 1/2(\epsilon_1 + ... + \epsilon_4), e_2 = \epsilon_1 + \epsilon_2, e_3 = \epsilon_2 + \epsilon_3, e_4 = \epsilon_3 - \epsilon_4$$

or

$$e_1 = 1/2(\epsilon_1 + ... + \epsilon_4), e_2 = \epsilon_1 - \epsilon_2, e_3 = \epsilon_2 - \epsilon_3, e_4 = \epsilon_3 - \epsilon_4$$

Let us introduce a new orthogonal basis in R^4 putting

$f_1 = 1/2(\epsilon_1+...+\epsilon_4), f_2 = 1/2(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4), f_3 = 1/2(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4), f_4 = 1/2(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)$

In this basis our root vectors have respectively the form

$$e_1 = f_1, e_2 = f_1 + f_2, e_3 = f_1 - f_4, e_4 = f_3 - f_4$$

or

$$e_1 = f_1, e_2 = f_3 + f_4, e_3 = f_2 - f_3, e_4 = f_3 - f_4$$

In both cases we obtain a root system of the B_4 type.

For the graph shown in Fig c) we have the following possibilities for $\Delta = \{e_1, ..., e_4\}$

$$e_1 = 1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4), e_2 = \epsilon_1 + \epsilon_2, e_3 = \epsilon_2 + \epsilon_4, e_4 = \epsilon_3 + \epsilon_1$$

or

$$e_1 = 1/2(\epsilon_1 + ... + \epsilon_4), e_2 = \epsilon_1 + \epsilon_2, e_3 = \epsilon_2 - \epsilon_3, e_4 = \epsilon_1 - \epsilon_3$$

With respect to the above new basis $f_1, ..., f_4$ these are equal to

$$e_1 = f_1, e_2 = f_1 + f_2, e_3 = f_1 - f_2, e_4 = f_1 + f_3$$

or

$$e_1 = f_1, e_2 = f_1 + f_3, e_3 = f_2 - f_3, e_4 = f_2 + f_4.$$

We again obtain a root system of the B_4-type.

Let us now prove that Δ generates $W(C_4)$ if $\Delta \cap \Phi_0 \cup \Phi_1$ generates the group $W(B_2) \times W(A_1)$. The graph of $\Delta \cap \Phi_0 \cup \Phi_1$ must have the form shown in Fig.

The basis Δ will read

$$e_1 = 1/2(\epsilon_1 + ... + \epsilon_4), e_2 = \epsilon_1 + \epsilon_2, e_3 = \pm\epsilon_1, e_4 = \pm\epsilon_4$$

Introducing the new orthonormal basis

$$f_1 = 1/2(\epsilon_1 + \epsilon_2), f_2 = 1/2(\epsilon_3 + \epsilon_4), f_3 = 1/2(\epsilon_1 - \epsilon_2), f_4 = 1/2(\epsilon_3 - \epsilon_4)$$

we obtain

$$e_1 = f_1 + f_2, e_2 = 2f_1 \text{ or } e_2 = 2f_3, e_3 = \pm(f_1 + f_3), e_4 = \pm(f_2 \pm f_4).$$

This is equivalent to the standard root system of the C_4-type.

The last possibility to be excluded is that $\Delta \cap \Phi_0 \cup \Phi_1$ is a root system of $A_1 \times A_1 \times A_1$ type. Indeed, the graph of $\Delta \cap \Phi_0 \cup \Phi_1$ has the form

So the basis is

$$e_1 = 1/2(\epsilon_1 + ... + \epsilon_4), e_2 = \pm\epsilon_1, e_3 = \pm\epsilon_2, e_4 = \pm\epsilon_3$$

This root system generates $W(D_4) \subset W(F_4)$.
Lemma is proved.

Lemma 4. Any generator system Δ of type (1) is equivalent to the generator system of type (2).

Proof. Indeed, let $\Delta = u_1, v_1, v_2, v_3$ where $u \in \Phi_2, v_3 \in \Phi_0$. If both v_1, v_2 are orthogonal to v_3 then applying elementary braid to the pair $\{v_3, u\}$ we get $\{u', v_1, v_2, u\}$, where $u' \in \Phi_2$ and one of the vectors v_1, v_2 is not orthogonal to u'.
Let it be v_1. Applying elementary braid operation to the pair $\{u', v_1\}$ we get

$$v_1, u", v_2, u, u" \in \Phi_2$$

and then the elemementary braid action for the pair {u",u} will give us a system of type (2). Lemma is proved.

Lemma 5. Any two generators Δ_1 and Δ_2 of type (2) are equivalent up to permutation.

Proof. We know that Card$T(\Delta_i \cap \Phi_0 \cup \Phi_1) = 3$ for $i = 1, 2$. If

$$\{\epsilon_1, ..., \epsilon_4\} \setminus T(\Delta_1 \cap \{\Phi_0 \cup \Phi_1\}) = \{\epsilon_1, ..., \epsilon_4\} \setminus T(\Delta_2 \cap \{\Phi_0 \cup \Phi_1\})$$

then the claim follows from the fact that $\Delta_i \cap (\Phi_0 \cup \Phi_1)$ is a generator system of $B_3 (i = 1, 2)$.
In opposite case, let

$$\epsilon_1 \notin T(\Delta_1 \cap \{\Phi_0 \cup \Phi_1\}), \epsilon_2 \notin T(\Delta_2 \cap \{\Phi_0 \cup \Phi_1\})$$

Then from Δ_1 we get Δ_1' (using the action of Br_3 on $\Delta_1 \cap \{\Phi_0 \cup \Phi_1\}$) such that
$\exists u \in \Delta_1', \quad \epsilon_2 \notin T((\Delta_1 \cap \{\Phi_0 \cup \Phi_1\}) \setminus \{u\})$ and $(u, v) \neq 0$

where

$$\{v\} \equiv \{\Delta'_1 \cap \Phi_2\} = \{\Delta_1 \cap \Phi_2\}$$

Then the elementary braid for the pair $\{u, v\}$ gives us Δ''_1 for which $\epsilon_1 \in T(\Delta''_1 \cap \{\Phi_0 \cup \Phi_1\})$ and $\epsilon_2 \in T(\Delta''_1 \cap \{\Phi_0 \cup \Phi_1\}$.

In it's turn $\Delta'_1 \sim \Delta_2$.

Lemma is proved.

We have proved the equivalence in the claim of Proposition only up to permutation. So, the last Lemma completes the proof.

Lemma 6. Any permutation of Δ_{st} is equivalent to it.

Proof. Let $\Delta = \beta(\Delta_{st})$ for $\beta \in S_4$. Denote $u := \{\Delta \cap \Phi_2\}$. Let u stand at the 4-th position. Since the 3 vectors of $\Delta \cap \{\Phi_0 \cup \Phi_1\}$ generate B_3 we can reorder them arbitrarily by Br_3 action.

But if u is not at 4-th position then again by the action of Br_3 on $\Delta \cap \{\Phi_0 \cup \Phi_1\}$ we bring $\epsilon_3 - \epsilon_4$ to the 4-th position and then applying elementary braid to the orthogonal pair $\{\epsilon_3 - \epsilon_4, 1/2(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ we bring u to the 4-th position. This proves the Lemma.

Proposition is proved.

The cases H_3 and H_4 also could be handled in a combinatorial way. (Singularity theory in this case is probably helpless). However, we used computers and found 3 orbits for H_3 and 10 orbits for H_4. The last we still need to check.

The cases G_2 and $I_2(k)$ are obviously easy to handle and we have showen the answer in the table above.

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