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Higher dimensional supersymmetric quantum mechanics and Dirac equation

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We exhibit the supersymmetric quantum mechanical structure of the full Dirac equation for both cases of vanishing and nonvanishing mass. We also discuss the 'physical' significance of superpartner states.

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Supersymmetric (SUSY) quantum mechanics, particularly in 1-dimension, has been a subject of extensive study in the past[1]. Recently, Das, Okubo and Pernice[2] have generalised SUSY quantum mechanics to higher dimensions and have noticed an interesting property that such generalisation forces a spin structure into the theory. Adopting a different approach, Vahle and Ram[3] have established a relationship between the 1+1-dimensional Dirac equation and 1-dimensional SUSY quantum mechanics. We, in this letter, exhibit a 3-dimensional SUSY quantum mechanical structure of the full 3+1 dimensional Dirac equation for both zero and non-zero mass cases. Our result provides a concrete realisation of the conclusions of Das et al[2] and generalises the work of Vahle and Ram[3].

We start with mass m = 0 case.

m = 0 case :

The Dirac equation in this case has the familiar form

$$\vec{\alpha}\,.\vec{p}\;\Psi = i\,\frac{\partial}{\partial t}\,\Psi \tag{1}$$

with $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ where $\vec{\sigma}$'s are the standard Pauli matrices. $\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}$ with Ψ_a and Ψ_b each being two component spinors. Taking the time dependence of Ψ to be given by e^{-iEt} and with usual substitution $\vec{p} = -i\vec{\nabla}$, Eqn.(1) is written as a pair of first-order coupled differential equations

$$-i\,\partial_-\,\Psi_2 - i\,\partial_3\,\Psi_1 = E\,\Psi_1 \tag{2a}$$

and

$$-i\,\partial_+\,\Psi_1 + i\,\partial_3\,\Psi_2 = E\,\Psi_2 \tag{2b}$$

We have used the obvious notation $\partial_{\pm} = \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2}$ and $\partial_3 = \frac{\partial}{\partial x_3}$

Uncoupling eqn. (2) yields,

$$- \nabla^2 \Psi_{1,2} = \bar{E} \Psi_{1,2}, \text{ with } \bar{E} = E^2 \tag{3}$$

These are the Schrödinger equations corresponding to vanishing supersymmetric partner potentials. One can, however, go ahead and define the SUSY Hamiltonian H as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p^2 \tag{4}$$

with $H_1 = H_2 = p^2$. Defining the SUSY charges as

$$Q = \vec{\sigma}.\vec{p} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } Q^+ = \vec{\sigma}.\vec{p} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5}$$

one checks that the SUSY algebra

$$[Q,H] = [Q^+, H] = 0$$

$$\{ Q, Q \} = \{ Q^+, Q^+ \} = 0 \tag{6}$$

and $\{ Q, Q^+\} = H$

is satisfied. One further notices that Q converts an upper component spinor $\begin{pmatrix} \Psi_1 \\ 0 \end{pmatrix}$ to a lower one and Q^+ does the opposite. Identification $\vec{\sigma}.\vec{p}\ \Psi_{1,2} = E\ \Psi_{2,1}$ is consistent with the Schrödinger equation. Furthermore, one also verifies that if ϕ hapens to be an eigenstate of $H_1(H_2)$, $Q\phi$ $(Q^+ \phi)$ is that of the $H_2(H_1)$ with equal 'energy'. Infact, the work of Das et al[2] was suggestive of the Q and Q^+ structure that we have employed. The difficulty in factorising the Laplacian noted by them is circumvented here by the presence of a two component wave function or in other words spin structure of the theory.

Thus, we have obtained the three-dimensional free particle supersymmetric quantum mechanics starting from zero mass Dirac equation. Recalling that the solutions of the zero mass Dirac equation are the two independent helicity states of the massless particle, our discussion above leads to the interesting conclusion that these two helicity states can be viewed as the superpartners of each other in the SUSY quantum mechanical formalism.

We now go over to the nonzero mass case.

(ii) **m ≠ 0 case** :

Here the Dirac equation is

$$(\gamma_\mu \partial_\mu + m)\,\Psi = 0 \tag{7}$$

To make the discussion clear we proceed in two steps; we first take the coordinate dependence of the four-component Dirac wave function as

$$\Psi\,(X,\ t) = \Psi(X_3)\ e^{i(p_1X_1+p_2X_2-Et)} \tag{8}$$

and finally consider the 3-dimensional generalisation of it.

Consideration of eqn. (8) reduces the Dirac eqn. (7) to the following form

$$[i\ \gamma_1 p_1 + i\ \gamma_2 p_2 + \gamma_3\ \partial_3) - \gamma_4 E + m]\,\Psi = 0 \tag{9}$$

Multiplying γ_3 from left to the above equation gives us

$$[i\ \gamma_3\gamma_1 p_1 + i\ \gamma_3\gamma_2 p_2 + \partial_3 + \gamma_3(m - \gamma_4 E)]\,\Psi = 0 \tag{10}$$

We now take $m = m(X_3)$ to be a Lorentz scalar potential[4] . Differentiating eqn. (10) with respect to X_3 once more and substituting for ∂_3 from eqn. (10), one arrives at the equation,

$$[\ \partial_3^2 + \gamma_3 m' + E^2 - m^2]\ \Psi = 0 \tag{11}$$

with $m' = \partial_3 m(X_3)$.

Using $\gamma_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$ and writting $\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$ as before eqn. (11) splits to two coupled equations

$$(-\partial_3^2 + m^2)\,\psi_a + i\,\sigma_3\, m'\psi_b = \bar{E}\,\psi_a$$

and $$(-\partial_3^2 + m^2)\,\psi_b - i\,\sigma_3 m'\,\psi_a = \bar{E}\,\psi_b \qquad (12)$$

One immediately notices that these equations decouple in the space of $\chi_1 = \psi_a + i\,\psi_b$ and $\chi_2 = \psi_a - i\,\psi_b$ to equations

$$(-\partial_3^2 + m^2 + \sigma_3 m')\,\chi_1 = \bar{E}\,\chi_1$$

and $$(-\partial_3^2 + m^2 - \sigma_3 m')\chi_2 = \bar{E}\,\chi_2 \qquad (13)$$

So χ_1 and χ_2 can now be looked upon as the superpartners with superpotentials $\pm\,\sigma_3\,\partial_3 m$. The Hamiltonian now can be written as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p^2+m^2) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sigma_3\,\partial_3 m = \begin{pmatrix} Q^+Q & 0 \\ 0 & QQ^+ \end{pmatrix} \qquad (14)$$

with $$Q = \begin{pmatrix} 0 & 0 \\ \sigma_3 p_3 + im & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & \sigma_3 p_3 - im \\ 0 & 0 \end{pmatrix} \qquad (15)$$

The Q and Q^+ defined above satisfy the SUSY algebra (eqn. (6)).

To see the structure at the full 4-component level of ψ let us take $\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\psi_b = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$. Eqn. (13) immediately shows that there are now two sets of superpartners $\psi_1 + i\,\psi_3(\psi_2 - i\psi_4)$ and $\psi_2 + i\,\psi_4(\psi_1 - i\,\psi_3)$ satisfying 1-dimensional Schrödinger equation with superpartner potential m^2+m' and m^2-m' respectively as obtained by Vahle and Ram[3]. The construction of SUSY Hamiltonian, charges and resulting discussion all follow as in their case. We, additionally, note here the interesting feature that the superpartner states $(\Psi_1 \pm i\Psi_3)$ and $(\Psi_2 \pm i\Psi_4)$ are infact superposition of positive (Ψ_1 and Ψ_3) and negative (Ψ_2 and Ψ_4) helicity states of Dirac equation. This degeneracy is a reflection of the degeneracy present at the $\bar{E}(=E^2)$ level for the solutions involved.

Going over to the 3-dimensional generalisation is rather straightforward. We now take coordinate dependence as

$$\Psi(X, t) = \Psi(X)\, e^{-iEt} \tag{16}$$

The Dirac equation (7) now reduces to,

$$[\gamma_i \partial_i - \gamma_4 E + m(X)]\, \Psi' = 0 \tag{17}$$

Multiplying this equation by γ_4 from left and operating the resulting equation by $\gamma_j \partial_j$ from left one gets

$$[\gamma_j \gamma_i \partial_j \partial_i + (E^2 - m^2 + \gamma_j \partial_j m)]\, \Psi = 0 \tag{18}$$

Interchanging $i \leftrightarrow j$ in this equation and adding the resulting equation to it, we obtain

$$[\nabla^2 + E^2 - m^2 + \vec{\gamma}.\vec{m}\,']\, \Psi = 0 \tag{19}$$

In arriving at eqn. (19) we have used the Dirac gamma matrix properties $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$. Eqn. (19) is indeed the 3-dimensional generalisation of eqn. (11).

Now using $\vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\sigma & 0 \end{pmatrix}$ and $\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}$, one immediately obtains the generalised version of eqn. (13) as

$$(-\nabla^2 + m^2 + \vec{\sigma}.\vec{\nabla}\, m)\chi_1 = \bar{E}\,\chi_1$$

and $$(-\nabla^2 + m^2 - \sigma.\nabla\, m)\chi_2 = \bar{E}\,\chi_2 \tag{20}$$

χ_1 and χ_2 are now the superpartners with super potentials $\pm\ \vec{\sigma}.\vec{\nabla}\,m$. Defining Q and Q^+ as generalisation of eqn. (15) like

$$Q = \begin{pmatrix} 0 & 0 \\ \vec{\sigma}.\vec{\nabla}+m & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & \vec{\sigma}.\vec{\nabla}-m \\ 0 & 0 \end{pmatrix} \tag{21}$$

one easily verifies that Hamiltonian can be, once again, written as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p^2+m^2)+\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\vec{\sigma}.\vec{\nabla}\,m = \begin{pmatrix} Q^+Q & 0 \\ 0 & QQ^+ \end{pmatrix} \tag{22}$$

and the SUSY algebra (eqn. (6)) is satisfied.

At this stage, just writting $\Psi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ and $\Psi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}$ does not lead to a set of decoupled equations at the 4-component level as it happened earlier. If one, however, orients the derivative of m(x) along the direction of $\vec{\sigma}$ instead of leaving it arbitrary i.e. takes

$$\vec{m}' = \vec{\nabla}\, m(x) = \vec{\sigma}\,/\sqrt{3}\ m'(x) \tag{23}$$

the desired decoupling results in the form of

$$(-\nabla^2 + m^2 + \sqrt{3}\ m')\chi_1 = \bar{E}\,\chi_1$$

and $$(-\nabla^2 + m^2 - \sqrt{3}m')\chi_2 = \bar{E}\,\chi_2 \tag{24}$$

Once again the degeneracy appears at the 4-components level with $\Psi_1 + i\,\Psi_3(\Psi_2 + i\,\Psi_4)$ and $\Psi_2 - i\,\Psi_4(\Psi_1 - i\,\Psi_3)$ forming the two sets of superpartners each satisfying a 3-dimensional Schrodinger equation with superpotentials $\pm\sqrt{3}\ m'$.

We have thus deciphered a 3-dimensional SUSY quantum mechanical structure starting from the full Dirac equation involving a Lorentz scalar potential (m(x)). It is to be emphasized that our work involves a Lorentz scalar potential as opposed to a vector potential used by Das et al.[2] and thus can be viewed as complementary to their work. Since the Dirac equation has an intrinsic spin structure associated with it, our result can be regarded as concrete realisation of the observation of Das et al[2] that higher dimensional SUSY quantum mechanics introduces a spin structure into the theory. We have further observed that the linear super-position of helicity states of similar kind basically form superpartners of each other in the C-number formalism of SUSY quantum mechanics. This was infact ensured by our representation of m(x) as taken in eqn. (23) in the general case[5]. In the process, we have also generalised the work of Vahle and Ram[3], as stated earlier.

Further, if Schrödinger equations (24) can be solved for a class of supersymmetric partner potentials $V_\pm(x) = m^2(x) \pm \sqrt{3}\, m'(x)$, then so can be the Dirac equation for the corresponding Lorentz scalar potential m(x). This brings out a greater usefulness of SUSY quantum mechanics.

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