NON-LINEAR SPACE CHARGE EFFECTS IN BEAM DYNAMICS

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Introduction

It has been customary in the design of transport and acceleration systems for positive ion or electron beams either to neglect the space charge effects or to assume them to be linear (Vladimirskij and Kapchinskij, 1959). In a beam of circular cross-section this assumption of linearity imposes a constant charge density over the beam radius. Experimental results, however, indicate that this is not the case (Taylor, 1963) a typical distribution being bell-shaped with a maximum on the axis. For this, non-uniform distribution, non-linear space charge forces are present and these can have an important influence on the dynamical behaviour of the beam.

Taylor (1969) has pointed out that there are two main consequences of the field variation across a non-uniform beam. The first, due to the variation of the horizontal field E_x with x will cause an initially elliptic emittance diagram in the phase plane (x, x′) to be become distorted (Fig. 1 a), while the second, produced by the variation of E_x with y, will introduce a coupling between the transverse motions. This coupling has the effect of producing a continuous spreading in the phase plane of the projected beam distribution. Equi-density contours in the projection will no longer enclose a constant area (as would be the case for a uniform beam) and this manifests itself as an apparent increase in the emittance in the projection. This is illustrated in Fig. 1 b which shows that the fraction of the beam current lying inside a given area of the phase plane is diminishing with time. Alternatively we can say that the phase plane area enclosing a given current is increasing.

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The present paper describes the work that has been carried out in the PSLinac Group at CERN on non-linear space charge effects and summarises the preliminary results that have been obtained.

Theoretical Considerations

In the study of the dynamics of charged particle beams the fundamental transport equation is the Boltzmann equation:

$$\frac{\partial F}{\partial t} + \sum_{\substack{\text{spatial} \\ \text{co-ordinates}}} v_i \frac{\partial F}{\partial x_i} + \sum_{\substack{\text{velocity} \\ \text{co-ordinates}}} \frac{f_i}{m} \frac{\partial F}{\partial v_i} = \left(\frac{\partial F}{\partial t}\right)_{\text{collisions}} \tag{1}$$

where $F(x_1, x_2, x_3, v_1, v_2, v_3, t)\, dx_1\, dx_2\, dx_3\, dv_1\, dv_2\, dv_3$ is the number of particles in the phase space volume $dx_1\, dx_2\, dx_3\, dv_1\, dv_2\, dv_3$. The f_i are the components of the so-called "field forces" acting on a particle of mass m and charge e.

In the Vlasov approach to the study of beam dynamics the effect of the space charge field is included in the f_i, that is it is considered as a macroscopic field derivable from a potential. The term on the right hand side of (1) represents the change in F due to binary collisions, including ionisation and similar effects. At the densities normally encountered in a proton beam this collision term is negligible (Lapostolle et al, 1968) so that the right hand side of thus (1) is zero. It is then interesting to note that the left hand side may be written:

$$\frac{DF}{Dt} = 0$$

where D denotes differentiation in a frame of reference following the element of phase space under consideration. Thus (1) has the same form as Liouville's equation and the phase density F remains constant if we follow the motion of an element in phase space.

A common method of studying beam dynamics is the so-called Eulerian approach in which it is assumed that the beam is monoenergetic, that is at any point in space the beam can be described by a single density ρ and a single velocity field $\bar{u}$. The basic equations to be satisfied are the continuity equation

$$\operatorname{div}(\bar{u}\,\rho) = -\frac{\partial \rho}{\partial t} \tag{2}$$

and the momentum equation

$$m \frac{D\bar{u}}{Dt} = e\,\bar{E} \tag{3}$$

where $\bar{E}$ is the sum of the external field and the space charge field of the beam. Again D/Dt denotes a differentiation in a frame of reference. following the motion of the element of the beam under consideration.

All magnetic effects are assumed to be negligible and the motion is taken to be non-relativistic.

Let us now consider a rotationally symmetric beam whose divergence is small so that longitudinal effects may be neglected. We assume that the axial beam velocity and the axial components of the external field are constant over the cross-section of the beam. Then, in the absence of rotational motion the equations describing the radial motion of the beam are from (2) and (3)

$$\frac{\partial \rho}{\partial t}+u_r\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_r}{\partial r}=0 \tag{4}$$

and

$$m\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}\right)=eE_r. \tag{5}$$

One may see the relation between these last two expressions and the Boltzmann equation by taking the first and second moments of the latter (Davies, 1969) when one obtains:

$$\frac{\partial}{\partial r}\left(\rho\,\bar{u}_r\right)+\frac{\partial \rho}{\partial t}=0 \tag{6}$$

and

$$m\left(\frac{\partial \bar{u}_r}{\partial t}+\bar{u}_r\frac{\partial \bar{u}_r}{\partial r}\right)=eE_r-\frac{1}{\rho}\frac{\partial}{\partial r}(k\rho T) \tag{7}$$

where

$$\frac{1}{2}kT=\frac{1}{2}\overline{m\,(u_r-\bar{u}_r)^2}\,.$$

One sees that these last two relations are identical in form with (5) and (6), the difference being the use of the average velocity $\bar{u}_r$ rather than the local particle velocity and the appearance of the "temperature" term on the right hand side of (6). When $T=0$, $\bar{u}_r=u_r$ and the solution of the Boltzmann equation will yield the same values for p and $\bar{u}_r$ as the solution hydrodynamic eguations (4) and (5).

We may thus conclude that for beams of small emittance (i. e. a small velocity spread and thus a small "temperature" so that the last term on the right hand side of (7) is negligible) the density function and particle velocities computed from (4) and (5) will be very good approximations to the mean densities and velocities.

In general solutions of the hydrodynamic equations have only been considered for the case of laminar flow when there is no "overtaking" of particle trajectories, that is a particle having an initial radial co-ordinate less than that of another particle will continue to have a smaller radial co-ordinate.

Davies (1969) has however described a method by which the crossing of particle trajectories may be taken into consideration. For a rotationally symmetric beam of zero emittance the state of the beam is completely specified by a "zero emittance line" in the phase plane rr' (Fig. 2). Whereas the charge density distribution can discontinuous if it is taken to be a function of the radial co-ordinate, the distribution along the zero emittance line is always continuous and no overtaking can occur along this line. The basic idea of the method is to trace the motion of the zero emittance line and the distribution of charge along it. This distribution is only projected on to the r axis in order to compute the space charge force on any element of the beam.

Davies has also shown that a very good approximation to the motion of the beam envelope of a finite emittance beam can be obtained if the space charge force on a typical particle on the envelope is computed from the mean motion of the beam, that is from the charge density distribution along the zero emittance line.

Although this method can give a great deal of information about the beam dynamics and only uses a relatively small amount of computer storage and time it does not trace the full four-dimensional phase distribution of the beam which is essential if, for example, one wishes to project the distribution on to any one of the six phase planes.

Tanguy (1969) has developed an alternative approach to the problem in which he traces the four-dimensional phase space distribution by a Lagrangian method.

Again the beam is taken to be rotationally symmetric and longitudinal effects are assumed to be negligible. The beam is represented as approximately 6000 groups of charge and the trajectories of these groups are traced, at each stage the space charge at a given radius being computed by determining the number of these groups inside that radius.

Tangy's procedure can be summarized as follow:

a) From the equations of motion:

$$
\begin{aligned}
x'' &= \frac{q}{\varepsilon_0 m v^2}\,(1+x'^2+y'^2)\frac{x}{r^2}\int_0^r r\rho(r)dr \\
y'' &= \frac{q}{\varepsilon_0 m v^2}\,(1+x'^2+y'^2)\frac{x}{r^2}\int_0^r r\rho(r)dr
\end{aligned}
\tag{8}
$$

the unknown integrals on the right hand side are expressed in terms of the current I(r) flowing inside a circle of radius r by:

$$\int_0^r r\rho(r)dr \simeq \frac{I(r)}{2\pi v} \tag{9}$$

where v denotes the velocity of the particles.

b) In terms of the total current I_t carried by the beam the equations of motion (8) become

$$x''=\frac{qI_t}{2\pi\varepsilon_0 mv^3}(1+x'^2+y'^2)\frac{x}{r^2}\frac{I(r)}{I_t}$$
$$y''=\frac{qI_t}{2\pi\varepsilon_0 mv^3}(1+x'^2+y'^2)\frac{x}{r^2}\frac{I(r)}{I_t} \quad (10)$$

c) Equations (10) are integrated for a great number of particles N, whose initial co-ordinates and slopes are distributed according to the desired distribution in the initial four-dimensional phase space. For each particle j ($j=1, \cdots N$) and at each step of integration ratio $I(r_j)/I_t$ is given by:

$$\frac{I(r_j)}{I_t}=\frac{N_j}{N} \quad (11)$$

where N_j denotes the number of particles whose distance to the axis of the beam is equal to or less than r_j.

d) The computation has been made possible when the number N of particles is very large by using the following manoeuvre: let us suppose that at the abscissa z the N particles are arranged so that their distance to the axis of the beam are:

$$r_j \leqslant r_j+1, \quad j=1, \cdots, N \quad (12)$$

Thus the ratio is given by:

$$\frac{I(r_j)}{I_t}=\frac{N_j}{N}=\frac{j}{N}, \quad (13)$$

As we are not dealing with a laminar flow the arrangement as defined in (12) no longer holds, but as the integration step is small the degree of this disarrangement is therefore small and a subroutine is able to restore the arrangement of (12) in a very short time. Finaly let us note that the method used has allowed us to reduce the computation time by a factor of around 150.

In order to follow the evolution of the current r emittance function the output of this programme can be analysed by the method described by Warner (Taylor, Warner et al, 1966) which gives the equi-density contours in the emittance projection and inregtares the areas and currents to produce the density curve (Fig. 1 b.)

In both the above methods the effects of linear* and acceleration in the logitudinal direction can be taken into consideration.

Results

In Fig 3 a — e are shown the results of the calculations for the case of a beam having an initial Gaussian distribution in principal axes

* Non linear external forces and any kind of rotationally symmetric density distributions in the four-dimensional hypervolume can be treated by Tanguy,s procedure.

(Fig. 3 a) drifting tn a drift space with no external fields. Fig. 3b gives the results of the hydrodynamic rr' calculation for drifts of 1 m and 1.5 m, and for comparison the ringed points show the proejction on to the xx' plane of the y=0, y'=0 section as obtained from the Lagrangian calculation. From this figure we see that particle overtaking commences at about 1.3 metres. Figs 3 c ond 3 d show the density distributions in real space obtained by both methods. Here we see the formation of first a hump towards the outer radius of the beam which finally tends to a singularity when overtaking of particle trajectories has occured. Fig. 3 e shows the corresponding curves for the traction of the current I_r inside radius r as a function of r.

We see from the emittance diagram corresponding to 1.5 metres drift in Fig. 3 b that the curve may be divided into two regions. The first, between the origin and about 1.6 cm, approximates to a long thin ellipse with nearly constant charge density. If the rest of the beam were absent we would thus expect to be able to focus the beam with linear lenses without introducing any further aberrations since the space charge forces will be nearly linear (in this case an ellipse will always transform into an ellipse). The outer region beyond 16 cm display a larger amount of aberration and contains 50% of the charge in the beam. Thus if one introduced a diaphragm to remove this outer region one would also lose 50% of the beam current.

To illustrate the large aberrations that may be introduced by the nonlinear space charge the computation was performed for the case of a converging beam with same emittance as used previously Fig. 4 a—c. Again the emittance curves obtained by the two methods are practically identical.

Also shown dotted in Fig. 4c is the beam envelope one would obtain on the assumption that the current is uniformly distributed over any cross-section. We see that the neglect of the non-linear effects may give a completely unrealistic picture of the beam dynamics. In particular, whereas in the case of a uniform distribution the charge would all be contained inside a radius of about 11·2 cm, in the non-linear case only 39% of the charge lies inside this radius.

Finally concerning the increase in the area of a given amount of charge mentioned in the Introdnction, the analysis of the Tanguy programme output shows that the inner equi-density contours increase by the order of 30—40% in a drift of 1.5 m, whereas the total area increases by almost a factor of two. This effect represents a real reduction in the emittance projection, while the aberrations can lead to an effective reduction.

Conclusinos

The above examples demonstrated that one is unjustified in neglecting or linearising the space charge forces in a beam having a non-uniform distribution in the transverse direction.

The hydrodynamic or zero emittance method for studying the beam dynamics can give a great deal of information about the beam dynamics and because it is economical as regards computer storage and space is very convenient for making a preliminary investigation and design of a transport or accelerating system.

The Lagrangian method, although requiring considerable computing facilities, is able to trace the four-dimensional phase space distribution and can give a more detailed final analysis.

Both these methods are to be used in the study of the proposed new pre-injector and tank 1 of the CERN Linac.

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ДИСКУССИЯ

Капчинский: Нелинейные эффекты в принципе могут приводить к расширению фазового объема только в тех случаях, когда пучок не согласован с фокусирующим каналом. Фазовые распределения, вызывающие нелинейные кулоновские силы, не приводят к увеличению эмиттанса, если распределения стационарны. В этой связи следует подчеркнуть важность согласования пучка с каналом, которое пока на всех действующик протонных ускорителях осуществляется недостаточно хорошо.

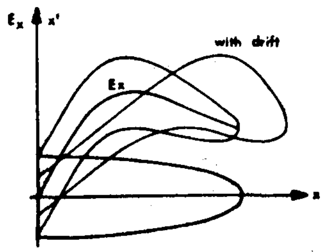


Fig. 1a Effect of variaton of E_x with x.

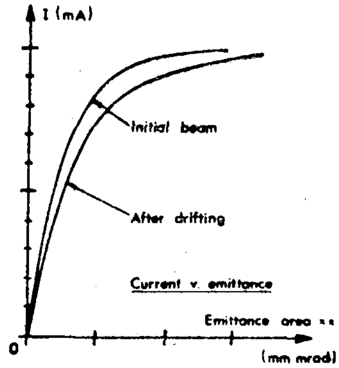
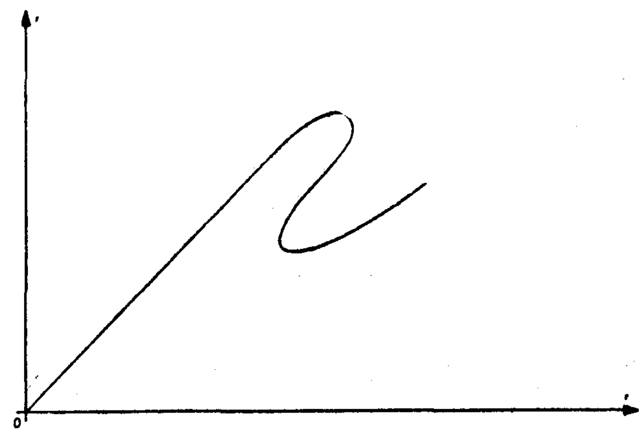


Fig. 1b. Effect of coupling



Eig 2. Zero emittance line,

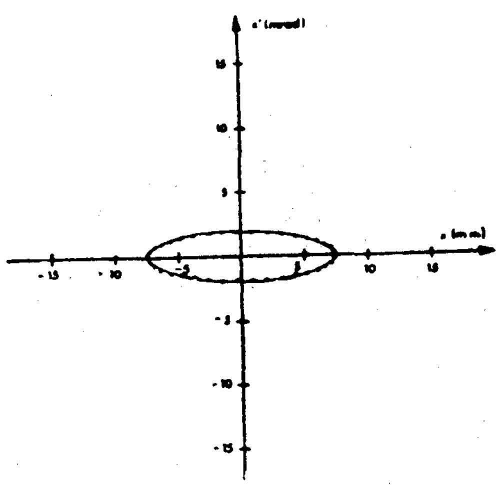


Fig. 3a. Initial beam.

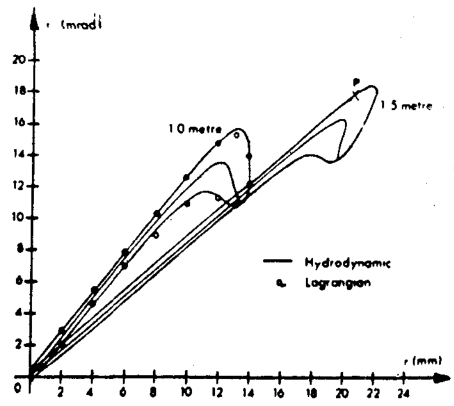


Fig. 3b. rr' proection after drifting.

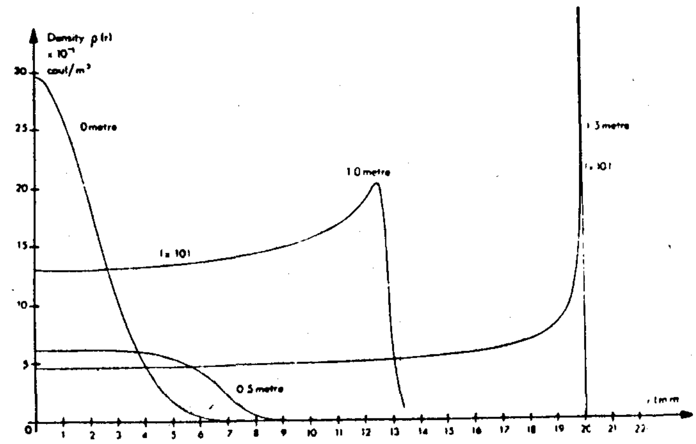


Fig. 3c. Real space density distributions by zero emitance calculation.

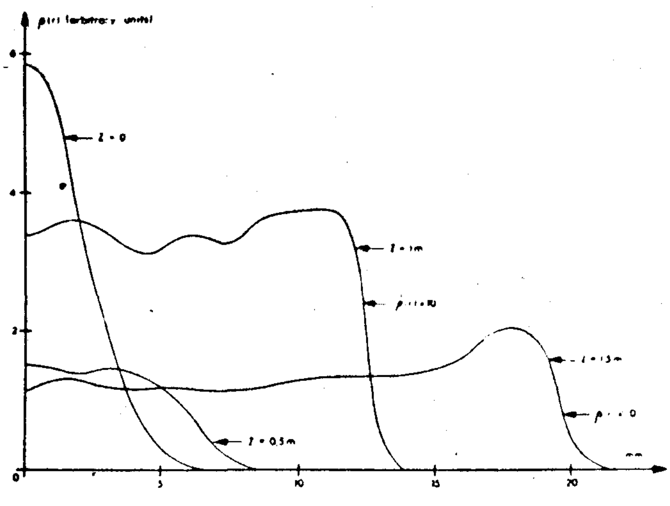


Fig 3d. Real space density by Lograngian calculation

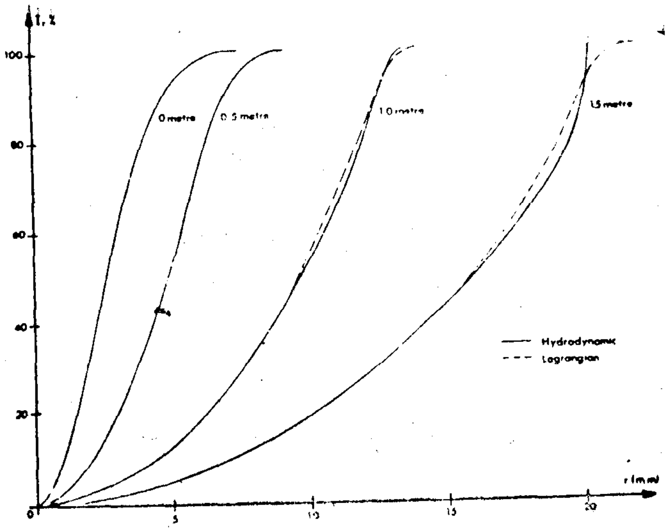


Fig. 3e. Fraction of current within a given radius.

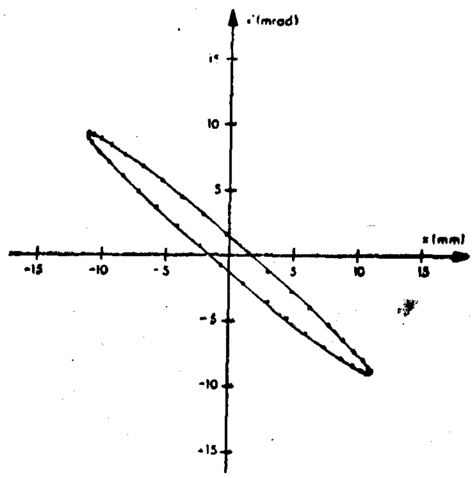


Fig. 4a, Initial beam.

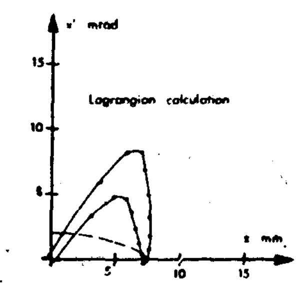


Fig. 4b. xx' projection (y=0, y'=0) after drifting 0.75 m.

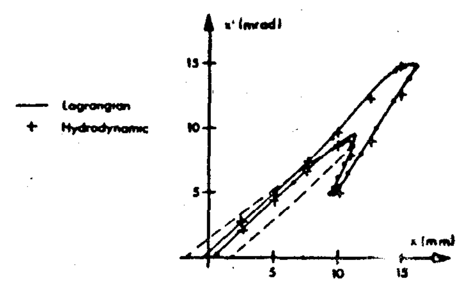


Fig. 4c. xx′ projection (y=0, y′=0) after 1. 5m.