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CROSSING RESTRICTIONS ON $\pi\pi$ PARTIAL WAVES *)

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A B S T R A C T

We find a parametrization for the most general $\pi\pi$ partial wave amplitudes consistent with crossing symmetry and isospin invariance. This enables us to write all the constraints on the partial wave amplitudes which follow from, and which ensure, crossing symmetry. We then discuss how these relations might be useful, and apply them in particular to Brown and Goble's model for $\pi\pi$ scattering.

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1. INTRODUCTION

In a previous paper [1], we solved the Balachandran-Nuyts [2] crossing equations for $\pi^0\pi^0$ scattering, which enabled us to write the most general partial wave expansion, convergent in the Mandelstam triangle $(s \geq 0,\ t \geq 0,\ u \geq 0)$, consistent with crossing symmetry. In the course of the analysis we found all the constraints on the partial wave amplitudes in the region $0 \leq s \leq 4m_\pi^2$ which follow from and which ensure crossing symmetry.

In this paper we generalize that analysis to $\pi\pi$ scattering with isospin. We again find the most general partial waves in the region $0 \leq s \leq 4m_\pi^2$ consistent with crossing and isospin invariance. The general form gives rise to constraints on the partial wave amplitudes, some of which have been reported elsewhere [3]. We find two constraints involving only s waves, three more involving s and p waves, and the number increases rapidly with the number of waves. As an example, the s wave constraints are (in units where $4m_\pi^2 = 1$)

$$\int_0^1 (1-s)\left(2 f_0^{(0)}(s) - 5 f_0^{(2)}(s)\right) ds = 0$$

$$\int_0^1 (1-s)(3s-1)\left(f_0^{(0)}(s) + 2 f_0^{(2)}(s)\right) ds = 0 \tag{1.1}$$

where $f_0^{(I)}(s)$ denotes the s wave with isospin I. These constraints are not only necessary but sufficient. That means, given $f_0^{(0)}$ and $f_0^{(2)}$ satisfying (1.1), there always exist amplitudes with the proper crossing properties, of which $f_0^{(0)}$, $f_0^{(2)}$ are the $I = 0,\ 2$ s waves respectively. Similarly, the five constraints on s and p waves are sufficient to ensure the existence of amplitudes with the proper crossing properties, of which the given waves are the s and p waves.

These relations always involve the amplitudes in an unphysical region. But using dispersion relations, it is possible to transform them into constraints on the imaginary parts of partial waves in the physical region. However, in this form the constraints are not very useful because they involve an infinite number of partial waves.

The chief use of the constraints will be to models for the low partial waves of $\pi\pi$ scattering which do not have crossing built in. By adjusting some parameters in the model, one can try to ensure the validity of our relations, thus guaranteeing that crossing is satisfied for the partial waves under consideration. Or else, given models for the partial wave amplitudes, one could insert them in our relations to get a feeling for how badly crossing symmetry is violated in the models.

The paper is organized as follows. In Section 2, we review some kinematic aspects of $\pi\pi$ scattering. In Section 3, we outline the Nuyts-Balachandran approach to this problem, and derive the form of the crossing equations, which are solved in Section 4. In Section 5, we apply this solution to derive constraints on the amplitudes. In Section 6, we discuss how these results could be applied to various models of $\pi\pi$ scattering, and in Section 7 we apply them to the Brown and Goble model, in an attempt to use our crossing relations to determine the σ mass from the ρ mass. The attempt is unsuccessful and reasons for the failure are discussed. Finally, the conclusions are presented in Section 8.

2. KINEMATICS OF $\pi\pi$ SCATTERING

The amplitude for $\pi\pi$ scattering in the s channel can be written as [4)]

$$T_{\alpha\beta,\gamma\delta}(s,t,u) = \delta_{\alpha\beta}\delta_{\gamma\delta}A(s,t,u) + \delta_{\alpha\gamma}\delta_{\beta\delta}B(s,t,u) + \delta_{\alpha\delta}\delta_{\beta\gamma}C(s,t,u) \quad (2.1)$$

where $\alpha\beta$ $(\gamma\delta)$ denote the isospin indices of the outgoing (incoming) pions (see the Figure). By crossing symmetry,

$$A(s,t,u) = A(s,u,t) \quad (2.2)$$

$$B(s,t,u) = A(t,s,u) \quad (2.3)$$

$$C(s,t,u) = A(u\,t\,s)\,. \quad (2.4)$$

In the s channel, the isospin amplitudes are given by

$$T^{(0)}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,s,t) \tag{2.5}$$

$$T^{(1)}(s,t,u) = A(t,s,u) - A(u,s,t) \tag{2.6}$$

$$T^{(2)}(s,t,u) = A(t,s,u) + A(u,s,t) \quad . \tag{2.7}$$

The amplitudes $T^{(0)}$, $T^{(1)}$ and $T^{(2)}$ will be compatible with crossing symmetry and isospin invariance if and only if we can find a function $A(s,t,u)$ subject to (2.2), such that (2.5)-(2.7) are valid.

As a function of three variables, A can be written as a linear combination of functions which transform irreducibly under the permutation group operating on the variables s, t and u. It is shown in Appendix A that, in view of (2.2), the most general A can be written as

$$A(s,t,u) = f(s,t,u) + (2s-t-u)g(s,t,u) + (2s^2-t^2-u^2)h(s,t,u) \tag{2.8}$$

where f, g and h are totally symmetric in s,t,u. In terms of these new functions, we can write

$$T^{(0)}(s,t,u) = 5f(s,t,u) + 2(2s-t-u)g(s,t,u) + 2(2s^2-t^2-u^2)h(s,t,u) \tag{2.9}$$

$$T^{(1)}(s,t,u) = 3(t-u)g(s,t,u) + 3(t^2-u^2)h(s,t,u) \tag{2.10}$$

$$T^{(2)}(s,t,u) = 2f(s,t,u) + (t+u-2s)g(s,t,u) + (t^2+u^2-2s^2)h(s,t,u). \tag{2.11}$$

The amplitudes $T^{(0)}$, $T^{(1)}$, $T^{(2)}$ will be consistent with crossing and isospin invariance if and only if there exist three totally symmetric functions f, g, h such that (2.9)-(2.11) hold. In the next Sections, we shall exploit the ideas of Balachandran and Nuyts to examine the implications of (2.9)-(2.11) for the partial wave amplitudes.

3. THE BALACHANDRAN-NUYTS APPROACH

The investigations of Balachandran and Nuyts [2] have indicated that to simplify crossing symmetry, it is useful to expand the amplitudes in the Mandelstam triangle $(s \geqslant 0,\ t \geqslant 0,\ u \geqslant 0)$, in terms of the functions

$$S^{\ell}_{\sigma-\ell}(s,t) \equiv (1-s)^{\ell}\, P^{(2\ell+1,0)}_{\sigma-\ell}(2s-1)\, P_{\ell}(z_s) \quad , \qquad (3.1)$$

where P_ℓ is the Legendre polynomial, $P^{(2\ell+1,0)}_{\sigma-\ell}$ is the Jacobi polynomial [5], z_s is the scattering angle in the s channel

$$z_s = 1 + \frac{2t}{s-1} \quad , \qquad (3.2)$$

σ is an integer running from ℓ to ∞, and we have taken units such that

$$4m_\pi^2 = 1 \quad . \qquad (3.3)$$

We shall denote the Mandelstam triangle by Δ .

We write

$$T^{(i)}(s,t,u) = \sum_{\ell=0}^{\infty} \sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\left(a^{(i)}_{\sigma}\right)_{\ell}\, S^{\ell}_{\sigma-\ell}(s,t) \qquad (3.4)$$

where $(a^{(i)}_\sigma)_\ell$ are constants to be determined. Since $S^{\ell}_{\sigma-\ell}(s,t)$ form a complete orthogonal set in Δ , any $\mathcal{L}^2$ function can be expanded in terms of them. [We assume that $T^{(i)}(s,t,u)$ has no poles in Δ , so that the expansion (3.4) is possible.] The advantages of this particular complete set are twofold. First, the partial wave expansion is easily obtained, namely

$$f^{(i)}_{\ell}(s) = \sum_{\sigma=\ell}^{\infty} 2(\sigma+1)\left(a^{(i)}_{\sigma}\right)_{\ell} (1-s)^{\ell}\, P^{(2\ell+1,0)}_{\sigma-\ell}(2s-1) \quad , \qquad (3.5)$$

where $f_\ell^{(i)}$ is the ℓ^{th} partial wave of isospin i. Secondly there exists a self-adjoint operator [2] O, symmetric under permutations of s,t,u of which all the functions $S_{\sigma-\ell}^{\ell}(s,t)$ are eigenfunctions with eigenvalues $\sigma(\sigma+2)$, dependent only on σ and not on ℓ. This simplifies the crossing relations enormously, as we now show.

The usual crossing relations can be formulated as

$$T^{(i)}(t,s,u) = \sum_{j=0}^{2} A_{ij}\, T^{(j)}(s,t,u) \qquad , \tag{3.6}$$

where A_{ij} is the crossing matrix. Substituting (3.4) gives

$$\begin{aligned}\sum_{\ell'=0}^{\infty}\sum_{\sigma'=\ell'}^{\infty} 2(\sigma'+1)(2\ell'+1)\left(a_{\sigma'}^{(i)}\right)_{\ell'} S_{\sigma'-\ell'}^{\ell'}(t,s) \\ = \sum_{\ell=0}^{\infty}\sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\sum_{j=0}^{2} A_{ij}\left(a_{\sigma}^{(j)}\right)_{\ell} S_{\sigma-\ell}^{\ell}(s,t).\end{aligned} \tag{3.7}$$

Using the orthogonality relation [2]

$$\begin{aligned}\left(S_{\sigma-\ell}^{\ell}, S_{\tau-m}^{m}\right) &\equiv \iint_{\Delta} ds\,dt\, S_{\sigma-\ell}^{\ell}(s,t)\, S_{\tau-m}^{m}(s,t) \\ &= \frac{1}{2(\sigma+1)(2\ell+1)}\,\delta_{\ell m}\,\delta_{\sigma\tau}\,,\end{aligned} \tag{3.8}$$

we find

$$\left(a_{\sigma'}^{(i)}\right)_{\ell'} = \sum_{\ell=0}^{\infty}\sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\sum_{j=0}^{2} A_{ij}\left(a_{\sigma}^{(j)}\right)_{\ell}\left(T_{\sigma'-\ell'}^{\ell'}, S_{\sigma-\ell}^{\ell}\right), \tag{3.9}$$

where

$$T_{\sigma'-\ell'}^{\ell'}(s,t) = S_{\sigma'-\ell'}^{\ell'}(t,s) \quad . \tag{3.10}$$

Since both $T_{\sigma'-\ell'}^{\ell'}$, $S_{\sigma-\ell}^{\ell}$ are eigenfunctions of O with eigenvalues $\sigma'(\sigma'+2)$ and $\sigma(\sigma+2)$ respectively, the scalar product on the right-hand side of (3.9) vanishes unless $\sigma = \sigma'$. Thus, the crossing relations, expressed as constraints on the $(a_\sigma^{(i)})_\ell$ take the form of finite dimensional matrix equations involving only those a's with the same value of σ. They are :

$$\left(a_\sigma^{(i)}\right)_{\ell'} = \sum_{j=0}^{2} \sum_{\ell=0}^{\sigma} G^{\sigma}_{i\ell',j\ell} \left(a_\sigma^{(j)}\right)_\ell \quad , \tag{3.11}$$

where

$$G^{\sigma}_{i\ell',j\ell} = 2(\sigma+1)(2\ell+1)\, A_{ij}\left(T^{\ell'}_{\sigma-\ell'}, S^{\ell}_{\sigma-\ell}\right). \tag{3.12}$$

The explicit form of G can be worked out but is not necessary for what follows.

The problem of finding the most general amplitude consistent with crossing symmetry is the problem of finding all the eigenvectors of G of eigenvalue 1. We shall now do this by exploiting the results of Section 2.

4. SOLUTION OF THE CROSSING PROBLEM

In this Section, we follow the techniques of I very closely. Briefly, the idea is the following : the amplitudes $T^{(i)}$ given by (2.9)-(2.11) will satisfy crossing symmetry for any choice of symmetric functions f,g,h and will therefore have an expansion of the form (3.4) with $(a_\sigma^{(i)})_\ell$ satisfying the crossing equation (3.11). By choosing a complete set of functions for f,g,h we will generate a complete basis for the solutions of (3.11). In particular, we shall take f,g,h to be symmetric polynomials in s,t,u, which are also polynomials in the variables

$$x = st + su + tu \tag{4.1}$$

$$y = stu \tag{4.2}$$

$$w = s+t+u \equiv 1 \quad . \tag{4.3}$$

As an example, suppose

$$\begin{aligned} f(s,t,u) &= x^m y^q \\ g(s,t,u) &= h(s,t,u) = 0 \quad . \end{aligned} \tag{4.4}$$

Then

$$T^{(0)}(s,t,u) = 5x^m y^q = \sum_{\ell=0}^{\infty} \sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\left(a^{(0)}_{mq,\sigma}\right)_\ell S^{\ell}_{\sigma-\ell}(s,t) \tag{4.5}$$

$$T^{(1)}(s,t,u) = 0 = \sum_{\ell=0}^{\infty} \sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\left(a^{(1)}_{mq,\sigma}\right)_\ell S^{\ell}_{\sigma-\ell}(s,t) \tag{4.6}$$

$$T^{(2)}(s,t,u) = 2x^m y^q = \sum_{\ell=0}^{\infty} \sum_{\sigma=\ell}^{\infty} 2(\sigma+1)(2\ell+1)\left(a^{(2)}_{mq,\sigma}\right)_\ell S^{\ell}_{\sigma-\ell}(s,t) . \tag{4.7}$$

Since both sides of (4.5)-(4.7) are polynomials in s and t, the sums on the right are finite sums; there is no question of convergence, and the equality holds for all s and t.

Now consider the limit $s \to \infty$, z_s fixed. In this limit

$$t \approx \frac{s}{2}(z-1) \tag{4.8}$$

$$u \approx -\frac{s}{2}(z+1), \tag{4.9}$$

so that

$$x \approx -s^2\left(\frac{z^2+3}{4}\right) \tag{4.10}$$

$$y \approx -s^3\left(\frac{z^2-1}{4}\right), \tag{4.11}$$

so that the left-hand side of (4.5) goes as

$$5\, s^{2m+3q} (-1)^{m+q} \left(\frac{z^2+3}{4}\right)^m \left(\frac{z^2-1}{4}\right)^q .$$

Each term on the right-hand side of (4.5) is a polynomial of degree σ in s, so that asymptotically, the terms with the highest value of σ will dominate, and the largest σ must be of the form

$$\sigma = 2m + 3q \quad . \tag{4.12}$$

For this value of σ , using the asymptotic result [6]

$$(1-s)^{\ell} P_{\sigma-\ell}^{(2\ell+1,0)}(2s-1) \approx (-1)^{\ell} \frac{(2\sigma+1)!}{(\sigma-\ell)!(\sigma+\ell+1)!} s^{\sigma} , \tag{4.13}$$

and equating the leading terms in s, we find

$$5(-1)^{m+q} \left(\frac{z^2+3}{4}\right)^m \left(\frac{z^2-1}{4}\right)^q = \sum_{\ell=0}^{\sigma} 2(\sigma+1)(2\ell+1)\left(a^{(0)}_{mq,\sigma}\right)_{\ell} (-1)^{\ell} \frac{(2\sigma+1)! P_{\ell}(z)}{(\sigma-\ell)!(\sigma+\ell+1)!} . \tag{4.14}$$

This relation is easily inverted to give

$$\left(a^{(0)}_{mq,\sigma}\right)_{\ell} = (-1)^{\ell+m+q} \frac{(\sigma-\ell)!(\sigma+\ell+1)!}{(2\sigma+2)!} \frac{5}{2} \int \left(\frac{z^2+3}{4}\right)^m \left(\frac{z^2-1}{4}\right)^q P_{\ell}(z) dz , \tag{4.15}$$

with σ given by (4.12). Similarly we find, for this value of σ

$$\left(a^{(2)}_{mq,\sigma}\right)_{\ell} = \frac{2}{5} \left(a^{(0)}_{mq,\sigma}\right)_{\ell} \tag{4.16}$$

$$\left(a^{(1)}_{mq,\sigma}\right)_{\ell} = 0 \quad . \tag{4.17}$$

Equations (4.15)-(4.17) give one solution of the crossing equation (3.11) when σ is given by (4.12). For fixed σ , there are as many solutions of this kind as there are different solutions of (4.12) for non-negative integer m,q. As was shown in I, these solutions can be put in one to one correspondence with the integers in the closed interval $[\sigma/3, \sigma/2]$.

One can repeat the same kind of calculation by taking $f=h=0$ and $g=x^m y^q$ or $f=g=0$ and $h=x^m y^q$. In that way, we generate different solutions of (3.11). The arguments of I show that these solutions are linearly independent, and form a complete basis for the solutions of (3.11). After doing the simple calculations, one finds that the most general solution of (3.11) can be written as

$$\left(a_\sigma^{(0)}\right)_\ell = \sum_{p=\{\sigma/3\}}^{[\sigma/2]} 5\, c_p^\sigma \left(\alpha_p^\sigma\right)_\ell + \sum_{p=\{\frac{\sigma-1}{3}\}}^{[\frac{\sigma-1}{2}]} 2\, d_p^\sigma \left(\beta_p^\sigma\right)_\ell + \sum_{p=\{\frac{\sigma-2}{3}\}}^{[\frac{\sigma-2}{2}]} 2\, e_p^\sigma \left(\gamma_p^\sigma\right)_\ell \qquad (4.18)$$

$$\left(a_\sigma^{(1)}\right)_\ell = \sum_{p=\{\frac{\sigma-1}{3}\}}^{[\frac{\sigma-1}{2}]} d_p^\sigma \left(\delta_p^\sigma\right)_\ell + \sum_{p=\{\frac{\sigma-2}{3}\}}^{[\frac{\sigma-2}{2}]} 3\, e_p^\sigma \left(\epsilon_p^\sigma\right)_\ell \qquad (4.19)$$

$$\left(a_\sigma^{(2)}\right)_\ell = \sum_{p=\{\sigma/3\}}^{[\sigma/2]} 2\, c_p^\sigma \left(\alpha_p^\sigma\right)_\ell - \sum_{p=\{\frac{\sigma-1}{3}\}}^{[\frac{\sigma-1}{2}]} d_p^\sigma \left(\beta_p^\sigma\right)_\ell - \sum_{p=\{\frac{\sigma-2}{3}\}}^{[\frac{\sigma-2}{2}]} e_p^\sigma \left(\gamma_p^\sigma\right)_\ell \qquad (4.20)$$

with c_p^σ, d_p^σ, e_p^σ arbitrary constants and

$$\left(\alpha_p^\sigma\right)_\ell = (\sigma-\ell)!\,(\sigma+\ell+1)! \int_{-1}^{1} P_\ell(z)\,(z^2+3)^{3p-\sigma}\,(1-z^2)^{\sigma-2p}\,dz \qquad (4.21)$$

$$\left(\beta_p^\sigma\right)_\ell = (\sigma-\ell)!\,(\sigma+\ell+1)! \int_{-1}^{1} P_\ell(z)\,(z^2+3)^{3p-\sigma+1}\,(1-z^2)^{\sigma-2p-1}\,dz \qquad (4.22)$$

$$\left(\gamma_p^\sigma\right)_\ell = (\sigma-\ell)!\,(\sigma+\ell+1)! \int_{-1}^{1} P_\ell(z)\,(z^2+3)^{3p-\sigma+2}\,(1-z^2)^{\sigma-2p-2}\left(\frac{3-z^2}{2}\right)dz \qquad (4.23)$$

$$(\delta_p^\sigma)_\ell = -(\sigma-\ell)!(\sigma+\ell+1)! \int_{-1}^{1} P_\ell(z)(z^2+3)^{3p-\sigma+1}(1-z^2)^{\sigma-2p-1} z\, dz \tag{4.24}$$

$$(\epsilon_p^\sigma)_\ell = (\sigma+\ell)!(\sigma+\ell+1)! \int_{-1}^{1} P_\ell(z)(z^2+3)^{3p-\sigma+2}(1-z^2)^{\sigma-2p-2} z\, dz . \tag{4.25}$$

We have used the notation :

$\{\sigma/3\} \equiv$ (smallest non-negative integer $\geq \sigma/3$)

$[\sigma/2] \equiv$ (largest integer $\leq \sigma/2$).

5. RESULTS

We insert the solutions (4.18)-(4.20) into the expansion (3.5). This will give us the most general expression for the partial wave amplitudes in Δ which have the proper crossing properties. The point of these solutions is that the arbitrary constants c_p^σ , d_p^σ , e_p^σ do not depend on ℓ , which gives rise to correlations among the partial wave amplitudes.

Suppose, for example, that the s waves for $I = 0, 2$ are known. By the completeness and orthogonality of $P_\sigma^{(1,0)}(2s-1)$ on the interval $0 \leq s \leq 1$, this fixes $(a_\sigma^{(0)})_0$ and $(a_\sigma^{(2)})_0$ for each σ , i.e., it fixes

$$(a_\sigma^{(0)})_0 = \sum_{p=\{\sigma/3\}}^{[\sigma/2]} 5 c_p^\sigma (\alpha_p^\sigma)_0 + \sum_{p=\{\frac{\sigma-1}{3}\}}^{[\frac{\sigma-1}{2}]} 2 d_p^\sigma (\beta_p^\sigma)_0 + \sum_{p=\{\frac{\sigma-2}{3}\}}^{[\frac{\sigma-2}{2}]} 2 e_p^\sigma (\gamma_p^\sigma)_0 \tag{5.1}$$

and

$$\left(a_\sigma^{(2)}\right)_0 = \sum_{p=\{\sigma/3\}}^{[\sigma/2]} 2c_p^\sigma (\alpha_p^\sigma)_0 - \sum_{p=\{\frac{\sigma-1}{3}\}}^{[\frac{\sigma-1}{2}]} d_p^\sigma (\beta_p^\sigma)_0 - \sum_{p=\{\frac{\sigma-2}{3}\}}^{[\frac{\sigma-2}{2}]} e_p^\sigma (\gamma_p^\sigma)_0 \quad . \tag{5.2}$$

Suppose $\sigma = 0$. Then

$$\left(a_0^{(0)}\right)_0 = 5c_0^0 (\alpha_0^0)_0 \tag{5.3}$$

$$\left(a_0^{(2)}\right)_0 = 2c_0^0 (\alpha_0^0)_0 \quad , \tag{5.4}$$

i.e.,

$$\left(a_0^{(0)}\right)_0 = \frac{5}{2} \left(a_0^{(2)}\right)_0 \quad . \tag{5.5}$$

But from (3.5) and the orthogonality of $P_\sigma^{(1,0)}(2s-1)$, we can write

$$\left(a_0^{(I)}\right)_0 = \int_0^1 (1-s)\, f_0^{(I)}(s)\, ds \quad , \tag{5.6}$$

so that (5.5) becomes

$$\int_0^1 (1-s)\, f_0^{(0)}(s)\, ds = \frac{5}{2} \int_0^1 (1-s)\, f_0^{(2)}(s)\, ds \quad . \tag{5.7}$$

This is one constraint on the s waves of $\pi\pi$ scattering. Now suppose $\sigma = 1$. Then

$$\left(a_1^{(0)}\right)_0 = 2d_0^1 (\beta_0^1)_0 \tag{5.8}$$

$$\left(a_1^{(2)}\right)_0 = -d_0^1 (\beta_0^1)_0 \quad , \tag{5.9}$$

so that

$$\left(a_1^{(0)}\right)_0 + 2\left(a_1^{(2)}\right)_0 = 0 \quad . \tag{5.10}$$

Writing this as an integral relation, we have

$$\int_0^1 (1-s)(3s-1)\left(f_0^{(0)}(s) + 2f_0^{(2)}(s)\right) ds = 0 \quad . \tag{5.11}$$

Since the $\pi^o\pi^o$ amplitude is $\frac{1}{3}(T^{(0)}+2T^{(2)})$, this result can also be written as

$$\int_0^1 (1-s)(3s-1)\, f_0^{oo}(s) ds = 0 \tag{5.12}$$

where $f_0^{oo}(s)$ is the $\pi^o\pi^o$ s wave. This equation was already derived in I.

One can see that there are no more restrictions on the s waves from crossing alone, because for $\sigma \geqslant 2$, at least two arbitrary constants enter into $(a_\sigma^{(0)})_0$ and $(a_\sigma^{(2)})_0$, so that no relation exists between them. If the p wave is also considered, we find one more relation for $\sigma = 1$, another for $\sigma = 2$ and another for $\sigma = 3$. These are

$$\left(a_1^{(0)}\right)_0 + 2\left(a_1^{(1)}\right)_1 = 0 \tag{5.13}$$

$$2\left(a_2^{(0)}\right)_0 - 5\left(a_2^{(2)}\right)_0 = 6\left(a_2^{(1)}\right)_1 \tag{5.14}$$

$$2\left(a_3^{(0)}\right)_0 - 5\left(a_3^{(2)}\right)_0 = -15\left(a_3^{(1)}\right)_1 \quad . \tag{5.15}$$

The integral form of these relations can be written as

$$\int_0^1 (1-s)(3s-1)\, f_0^{(0)}(s) ds = -2\int_0^1 (1-s)^2 f_1^{(1)}(s)\, ds \tag{5.16}$$

$$\int_0^1 (1-s)(10s^2-8s+1)\left(2f_0^{(0)}(s)-5f_0^{(2)}(s)\right)ds = 6\int_0^1 (1-s)^2(5s-1)\,f_1^{(1)}(s)\,ds \tag{5.17}$$

$$\int_0^1 (1-s)(35s^3-45s^2+15s-1)\left(2f_0^{(0)}(s)-5f_0^{(2)}(s)\right)ds = -15\int_0^1 (1-s)^2(21s^2-12s+1)\,f_1^{(1)}(s)\,ds \; . \tag{5.18}$$

For arbitrary σ, the unknowns are c^{σ}_{ρ}, d^{σ}_{ρ} and e^{σ}_{ρ} for appropriate values of ρ. The number of unknowns is $[\sigma/2]+1$. The knowledge of all partial waves $\ell \leq L$, for all isospins, determines $[(3L)/2]+2$ conditions. Consequently, given all partial waves for $\ell \leq L$ for all isospins, the number of constraints for a given σ is (assuming $\sigma \geq L$)

$$\left[\frac{3L}{2}\right]-\left[\frac{\sigma}{2}\right]+1 \tag{5.19}$$

if this number is positive. Thus for $L=1$, we find two constraints for $\sigma=1$, one for $\sigma=2$, and one for $\sigma=3$. Going to the d waves yields ten more constraints, and higher waves lead to even more.

It should be stressed that the constraints we have derived are not only necessary but sufficient. That is, given functions $f_0^{(0)}$ and $f_0^{(2)}$ satisfying (5.7) and (5.11), there exist crossing symmetric amplitudes $T^{(i)}(s,t,u)$ of which $f_0^{(0)}$ and $f_0^{(2)}$ are the $I=0$ and $I=2$ s waves, respectively. In this sense, (5.7) and (5.11) express the full content of crossing symmetry when restricted to s waves. Similarly, the addition of (5.16)-(5.18) to these two relations expresses the full content of crossing symmetry applied to s and p waves.

Using the Froissart-Gribov representation for the partial waves, the integral equalities can be rewritten as constraints on the absorptive parts. For $\ell \geq 2$, we have [7)]

$$f_\ell^{(i)}(s) = (-1)^\ell \frac{2}{\pi} \frac{2}{1-s} \int_1^\infty A_t^{[i]}(s,t)\, Q_\ell\left(\frac{2t}{1-s} - 1\right) dt\,, \tag{5.20}$$

where $A_t^{[i]}(s,t)$ is the absorptive part in the t channel with isospin i in the s channel. Because of the possibility of subtractions, the representation may not be valid for $\ell = 0$ or $\ell = 1$. But, following Martin [8], we show in Appendix B that the s and p waves can be determined in terms of the absorptive parts up to the arbitrariness

$$f_0^{(0)}(s) \longrightarrow f_0^{(0)}(s) + 5a + 2b(3s-1) \tag{5.21}$$

$$f_1^{(1)}(s) \longrightarrow f_1^{(1)}(s) \qquad - b(1-s) \tag{5.22}$$

$$f_0^{(2)}(s) \longrightarrow f_0^{(2)}(s) + 2a - b(3s-1) \tag{5.23}$$

where a and b are arbitrary constants.

Having expressed the partial waves in terms of the absorptive parts, we can rewrite all the constraints like (5.7), (5.11) in terms of integrals over the absorptive parts $A_t^{[i]}(s,t)$ or $(d/ds)A_t^{[i]}(s,t)$. where the integration domain is $0 < s < 1$, $1 < t < \infty$. Recalling that

$$A_t^{[i]}(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{Im}\, f_\ell^{[i]}(t)\, P_\ell\left(1 + \frac{2s}{t-1}\right)\,, \tag{5.24}$$

these constraints become integral constraints of the form

$$\int_1^\infty dt \sum_{i=0}^{2} \sum_{\ell=0}^{\infty} \mathrm{Im}\, f_\ell^{[i]}(t)\, B_{\ell,n}^{(i)}(t) = 0 \tag{5.25}$$

where $B_{\ell,n}^{(i)}(t)$ are known functions. These equations (for different n) are the necessary and sufficient conditions on the absorptive parts in the physical region to ensure crossing symmetry [provided $f_\ell(t)$ is such that the amplitude has the requisite analyticity so that the usual dispersion relations are satisfied]. Moreover, from the derivation, it follows that the relations are independent for different n.

For example, the relation (5.11) can be rewritten as

$$\int_1^\infty dt \int_0^1 ds \left[A_t^{[0]}(s,t) + 2A_t^{[2]}(s,t)\right] \left\{ (1-s)(3s-1)\left[\frac{1}{t+2s-1} + \frac{1}{t-s} - \frac{2}{1-s} \ln \frac{t}{t+s-1}\right] + 2s(1-s)^2 \left[\frac{1}{(t+2s-1)^2} - \frac{1}{(t-s)^2}\right] \right\} = 0 . \tag{5.26}$$

In principle, expanding $A_t^{[0]} + 2A_t^{[2]}$ in terms of partial waves in the t channel, as in (5.24), one could perform the integral over s to obtain a relation like (5.25), but it is difficult to get a closed form expression for $B_\ell^{(i)}(t)$.

6. APPLICABILITY

The chief application of the results of Section 5 will be to models for the low partial waves of $\pi\pi$ scattering, which do not have crossing symmetry built in. For example, in a model for s and p waves, one could try to choose some of the parameters in the model so that (5.7), (5-11) and (5.16)-(5.18) were satisfied. One would then have incorporated the full content of crossing symmetry into these models. (We are implicitly assuming that the models can be continued into the unphysical region $0 \leq s \leq 1$.)

For example, Wanders and his collaborators 9) construct models for the s waves of $\pi\pi$ scattering with free parameters which they fit so that the amplitudes are consistent with all the constraints which have been derived on the basis of analyticity, unitarity and crossing. Although Atkinson 10) has shown that there is a very large class of functions, labelled by a symmetric function of two variables, consistent with these general principles, the work of Martin and his co-workers 11) has shown that for low partial waves, at least, the amplitudes are numerically constrained below threshold. The authors of Ref. 9) choose a particular parametrization of the s wave amplitudes which automatically satisfies elastic unitarity, and which has free parameters. They find that the constraints of Ref. 11) severely restrict their parameters, and lead to phase shifts in the physical region which are in rough agreement with experiment.

Our results represent further constraints. Since they are equalities, they serve to eliminate some of the parameters, rather than to restrict their range, as do the inequalities of Ref. 11). Furthermore, Wanders [12] has pointed out that since the aim of his work is to obtain results on the general shape of the phase shifts, our constraints, involving integrals over the amplitudes, are perhaps better suited to this purpose than those of Martin, which often take the form of inequalities on the amplitudes at certain isolated values of s. It should be pointed out, though, that Martin has really derived inequalities over a range of s, but that they are sharpest at the isolated values which he tabulates.

Another application could be to the theory of Padé approximants applied to the partial waves of $\pi\pi$ scattering, derived in some Lagrangian framework [13]. By taking the Padé approximant of the partial wave amplitudes, one can satisfy unitarity, but one violates crossing. By comparing various terms in our equations, one has a measure of how badly crossing symmetry is violated in this way. This check has been carried out by Basdevant et al. [14] for the Padé treatment of the $\lambda\, \phi^4$ theory, and indicates that crossing is very well satisfied.

Still another application could be to models which attempt to unitarize crossing symmetric amplitudes, e.g., Lovelace's work [15] on $\pi\pi$ scattering based on the Veneziano model. Lovelace interprets the Veneziano amplitude as the K matrix. In this way, he automatically satisfies unitarity and violates crossing. But since the left-hand cut in his model is largely arbitrary, one could try to determine it in the low partial waves to be consistent with our crossing relations. A similar problem will be investigated in more detail in the next Section.

7. THE MODEL OF BROWN AND GOBLE

Brown and Goble [16] have given a particular prescription for unitarizing the current algebra results on $\pi\pi$ scattering. They write, for the s and p waves

$$f_\ell^{(i)}(s) = \frac{f_\ell^{(i)C.A.}(s)}{1 + h_\ell^{(i)}(s)\, f_\ell^{(i)C.A.}(s)} , \tag{7.1}$$

where $f_\ell^{(i)CA}(s)$ is the current algebra result [17]. From elastic unitarity, one has

$$\mathrm{Im}\; h_\ell^{(i)}(s) = -\frac{k}{\sqrt{s}} \qquad 4m_\pi^2 < s < 16 m_\pi^2 \tag{7.2}$$

where

$$s = 4(k^2 + m_\pi^2) . \tag{7.3}$$

In order not to build in a singularity at $s = 0$, they chose

$$h_\ell^{(i)}(s) = -\frac{ik}{\sqrt{s}} + \frac{2k}{\pi\sqrt{s}} \ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right) + k_\ell^{(i)}(s) , \tag{7.4}$$

where $k_\ell^{(i)}(s)$ must be a real analytic function without cuts in the range $0 < s < 16m_\pi^2$. Their initial choice was to take $k_\ell^{(i)} = 0$ for the s waves, and a constant for the p wave. The constant was chosen so that $f_1^{(1)}(s)$ has a resonance at the mass of the ρ. They then found that the ρ width was well predicted (~130 MeV). This success can perhaps be taken as evidence that their form for the p wave is reasonable from threshold up to the ρ mass.

However, recent data [18] have shown evidence for a resonance in the $I = 0$ s wave as well. This was not obtained in their initial formula, which took $k_0^{(i)}(s) = 0$. In analogy with the p wave, they then [19] took $k_0^{(i)}(s)$ to be a constant, whose value was chosen to reproduce the σ mass. They have shown that for m_σ in the range

$$13 m_\pi^2 < m_\sigma^2 < 20 m_\pi^2 , \tag{7.5}$$

their formula is consistent with the phase shifts of Ref. 18).

However, by unitarizing the amplitudes, they have destroyed the crossing symmetry of the current algebra result. We have attempted to restore the crossing symmetry by using the relations of Section 5. If one is interested only in the $I = 0$ s wave and $I = 1$ p wave, there is only one relation

$$\int_0^1 (1-s)(3s-1) f_0^{(0)}(s) = -2 \int_0^1 (1-s)^2 f_1^{(1)}(s) ds \tag{7.6}$$

in units where $4m_\pi^2 = 1$. (Since we shall discuss only the $I = 0$ s wave and $I = 1$ p waves, we can drop the isospin superscript.) If one takes Brown and Goble's form for f_1, and parametrizes f_0 as in (7.1), (7.4), with k_0 an arbitrary constant, one can try to solve for k_0 by imposing (7.6). One would then have a prediction on the σ mass based on the p wave and crossing.

The integrals cannot be done analytically, but we have compared both sides of (7.6) numerically for different values of m_σ. The result is that it is not possible to satisfy that relation for any value of the σ mass above threshold. This means that the parametrization of Brown and Goble is not satisfactory in the region $0 \leq s \leq 1$.

One can explain the failure of the parametrization as follows. The unitarization has given the amplitudes the right analytic structure above $s = 1$, but their parametrization has no left-hand cut at all. But we know from crossing and unitarity [20] that $f_\ell^{(i)}(s)$ must have a cut beginning at $s = 0$, and with a discontinuity behaving like

$$\mathrm{Im}\, f_0^{(0)}(s) \sim \left(\frac{2}{9} |f_0^{(0)}(1)|^2 + \frac{10}{9} |f_0^{(2)}(1)|^2 \right) (-s)^{3/2} \tag{7.7}$$

$$\mathrm{Im}\, f_1^{(1)}(s) \sim \left(-\frac{2}{9} |f_0^{(0)}(1)|^2 + \frac{5}{9} |f_0^{(2)}(1)|^2 \right) (-s)^{3/2} \tag{7.8}$$

for $s < 0$, $|s|$ small. Because of the factors $(1-s)$, $(1-s)^2$, (7.6) is more sensitive to the amplitudes near $s = 0$ than to their values near $s = 1$, and since the amplitudes have the wrong analytic form near $s = 0$, it is not surprising that (7.6) is violated.

One can try to choose $k_\ell(s)$ to have a branch point at $s = 0$ with the proper discontinuity. The results will depend on how one parametrizes the function. We chose two different parametrizations of $k_\ell(s)$, which were motivated by the following considerations. $k_\ell(s)$

must have a discontinuity behaving like $s^{\frac{3}{2}}$ near $s=0$. Moreover, we want $k_\ell(s)$ to tend to a constant at large s, so that the singularity at $s=0$ does not affect the form of the amplitude near the σ or ρ meson. Finally, we want the current algebra result to be approximately valid in the region $0 \leq s \leq 1$, which is the smoothness assumption of PCAC. From (7.1) this will be true if

$$h_\ell(s)\, f_\ell^{C.A.}(s) \ll 1 \qquad 0 \leq s \leq 1. \tag{7.9}$$

Numerically, $f_\ell^{CA}(s)$ is small, so that (7.9) will be satisfied if $h_\ell(s)$ does not have a pole in the region $0 \leq s \leq 1$ or if the pole of $h_\ell(s)$ coincides with the zero of f_ℓ^{CA}.

The first choice of $k_\ell(s)$ was

$$k_\ell(s) = c_\ell + \frac{\alpha_\ell}{s^{3/2} - s_\ell^{3/2}} \tag{7.10}$$

where c_ℓ, α_ℓ are constants to be determined, and s_ℓ is the location of the zero of the current algebra amplitude f_ℓ^{CA}. The parameters α_ℓ were fitted to satisfy (7.7), (7.8) where $f_0^{(i)}(1)$ was taken to be the current algebra value. This choice of $f_0^{(i)}(1)$ was reasonable in view of (7.9). c_1 was then fitted to reproduce the ρ resonance and c_0 was determined from (7.6). The result was again that no value of m_σ above threshold was consistent with (7.6).

The second parametrization was

$$k_\ell(s) = \frac{s^{3/2}}{a_\ell + b_\ell s^{3/2}} \tag{7.11}$$

where a_ℓ, b_ℓ are constants to be determined. a_ℓ was again determined by (7.7) and (7.8), and b_1 chosen to produce the ρ pole. a_1 and b_1 then have the same sign so that $k_1(s)$ is not singular in the region $0 \leq s \leq 1$. However, if b_0 is to give rise to a σ resonance, we find that a_0 and b_0 must have opposite signs and that $k_0(s)$ must have a pole in the region $0 \leq s \leq 1$. The location of this pole does not coincide with the current algebra zero, so that condition (7.9) is violated. This solution for b_0 would give rise to two zeros of $f_0^{(0)}(s)$ in the region $0 < s < 1$, whereas current algebra predicts only one. This would violate the spirit of PCAC.

In summary, we have not found a parametrization of the $I = 0$ s wave which is consistent with unitarity, the validity of the current algebra and PCAC, the crossing relations (7.6)-(7.8), and the existence of a ρ and σ resonance.

One can interpret this in two possible ways. First, perhaps none of the parametrizations we gave was really adequate. Since (7.6) is sensitive to the amplitudes near $s = 0$, one should perhaps have taken greater care in determining $\mathrm{Im}\, f_\ell(s)$ on the left-hand cut. In fact, given an assumed form for the s and p waves near threshold, the arguments leading to (7.7) and (7.8) can also fix the coefficients of $(-s)^{5/2}$, $(-s)^{7/2}$ and $(-s)^{9/2}$ in $\mathrm{Im}\, f_\ell^{(i)}(s)$. The inclusion of these higher terms will define the left-hand cut much more precisely, and it is possible that a parametrization which satisfies these constraints would be consistent with all the principles outlined above.

It may also be that no simple parametrization of the amplitudes will have the features of the ρ and σ resonances, the current algebra, unitarity and crossing symmetry. This view has some support from the investigations of Ref. 9) into $\pi\pi$ amplitudes consistent with Martin's results [11] and satisfying elastic unitarity. In their parametrization they cannot reproduce Weinberg's scattering lengths predictions. There is no doubt that there exist functions consistent with all the properties listed above. For example, Iliopoulos' [21] solution a) is consistent with crossing, current algebra, the constraints of Martin, and unitarity to a given order in $(1-s)$. Although this model cannot incorporate resonances, one could extrapolate it to higher s in many ways, so as to yield a σ and a ρ. But the hope of Brown and Goble was more ambitious. They wanted a simple parametrization (however badly defined that may be) valid from threshold up to the ρ mass at least, satisfying elastic unitarity exactly, consistent with current algebra and predicting the resonances. The constraints of crossing may mean that their functions have to be quite complicated.

One point should be emphasized in these considerations. Since the amplitudes should be well approximated by the current algebra results, one might argue that changes in $k_{\ell}(s)$ could hardly affect the validity of (7.6). However, since the current algebra results are built to be consistent with crossing, they automatically satisfy (7.6), so that one should really interpret that equation as a relation on the deviations from the current algebra predictions. Then changes in $k_{\ell}(s)$ play a significant role.

8. CONCLUSIONS

We have found the necessary and sufficient conditions on the $\pi\pi$ partial wave amplitudes in the region $0 \leqslant s \leqslant 4m_{\pi}^2$ which follow from, and ensure, crossing symmetry. Moreover, we have found the most general parametrization of the $\pi\pi$ amplitudes in the Mandelstam triangle which is consistent with crossing symmetry.

It follows from this parametrization that if, for example, we are given s and p waves consistent with the relations (5.9), (5.11), (5.16)-(5.18), then it is possible to construct many different amplitudes with the proper crossing properties of which the given waves are the s and p waves. It will usually be the case, however, that none of these amplitudes will have other desirable properties like unitarity or analyticity. That is to say, these other principles restrict the s and p waves further than our relations do. Examples of such restrictions are given in Ref. 11).

In this respect our work is complementary to that of Yndurain [22], who has found the necessary and sufficient conditions on, for example, the $\pi^0\pi^0$ partial wave amplitudes below threshold which follow from and ensure the proper analyticity and the positivity of the absorptive part. Combining his constraints with our general parametrization will give the necessary and sufficient conditions to ensure crossing, analyticity, and the positivity of the absorptive part. So far, however, it has been very difficult to extract any significant consequences from the combination of the two approaches.

One of the limitations of our results is that the constraints are formulated in the region $0 \leq s \leq 4m_\pi^2$. Using dispersion relations, we have shown how these constraints can be rewritten as constraints on the imaginary parts of the partial wave amplitudes in the physical region. But the equations do not seem very useful in this form, since each involves an infinite number of partial waves.

We have suggested that these constraints would be most useful in models for the partial wave amplitudes which do not have crossing built in, for it might be possible to adjust certain parameters to guarantee that crossing is not violated in the low partial waves. In these models, we could also test how badly crossing symmetry is violated, by comparing various terms in our equations.

We have applied our ideas to the model for $\pi\pi$ scattering due to Brown and Goble. We have tried to use the freedom of their model to satisfy the one relation between the $I = 0$ s wave and the $I = 1$ p wave in an attempt to determine the σ mass from the ρ mass. But we have not found a simple parametrization for which this rather naïve "bootstrap" is successful, and consistent with current algebra and PCAC. We have suggested that the failure can be traced to an improper treatment of the left-hand cut, or else that our attempt to find simple parametrizations for the s and p waves, consistent with unitarity, current algebra, the existence of the ρ and σ , and crossing, is overly ambitious. We hope to pursue this matter further in a later work.

The next problem that arises is the generalization of these techniques to processes with unequal mass, or with spin. The general framework for the unequal mass case, or the case of πN scattering, has been developed by Balachandran et al. [23)]. However, since the application of these techniques will probably be to models for the low partial wave amplitudes, it may not be necessary to find the general solution to the crossing equations. It is important to find all the constraints on the low partial waves, and to write them out explicitly, so that they can be readily applied. The technique for this has been given in Ref. 14), for any elastic scattering process.

It is also possible to derive results on arbitrary processes relating the low partial waves of $AB \to CD$ to those of $A\bar{C} \to \bar{B}D$. This depends on the existence of a region R in the s,t plane with the following property : for fixed s, z_s varies from -1 to 1; for fixed t, z_t varies from -1 to 1, and similarly for fixed u. Such a region always exists [24] if the scattering particles are stable, and not massless. We continue the full amplitudes of $AB \to CD$ or $A\bar{C} \to \bar{B}D$ into this region, where the crossing condition implies that they are equal. We then multiply the two amplitudes by the same polynomial in s and t, and integrate over R. The results must still be equal. Since for fixed s, t is linear in z_s, and for fixed t, s is linear in z_t, we then get integral relations between a finite number of partial waves in the two channels. Crossing symmetry in R is equivalent to satisfying these constraints, for arbitrary polynomials in s and t, because a function orthogonal to all polynomials in R necessarily vanishes.

APPENDIX A

We wish to show that if $F(s,t,u)$ is an analytic function in a domain D where D is invariant under all permutations on s,t,u, then on D we can uniquely write $F(s,t,u)$ as

$$\begin{aligned} F(s,t,u) = {} & f(s,t,u) + (2s-t-u)g_1(s,t,u) + (t-u)g_2(s,t,u) \\ & + (2s^2-t^2-u^2)h_1(s,t,u) + (t^2-u^2)h_2(s,t,u) + (s-t)(t-u)(u-s)j(s,t,u), \end{aligned} \tag{A.1}$$

where f, g_1, g_2, h_1, h_2, j are symmetric functions which are analytic in D.

This decomposition of F is the decomposition into irreducible representations under the group S_3 of permutations of s,t,u where f and j correspond to the symmetric and antisymmetric terms, respectively, while the terms with g_1 and g_2 or h_1 and h_2 form bases for the mixed representation in which the (t,u) permutation is taken to be of the form

$$(tu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.2}$$

To prove the decomposition, we first project out those pieces of F transforming according to a given row of a given representation. So we define

$$S(s,t,u) = \frac{1}{6}\left[F(s,t,u) + F(t,u,s) + F(u,s,t) + F(s,u,t) + F(t,s,u) + F(u,t,s)\right] \tag{A.3}$$

$$A(s,t,u) = \frac{1}{6}\left[F(s,t,u) + F(t,u,s) + F(u,s,t) - F(s,u,t) - F(t,s,u) - F(u,t,s)\right] \tag{A.4}$$

$$M_1(s,t,u) = \frac{1}{6}\left[2F(s,t,u) - F(t,u,s) - F(u,s,t) + 2F(s,u,t) - F(t,s,u) - F(u,t,s)\right] \tag{A.5}$$

$$M_2(s,t,u) = \frac{1}{6}\left[2F(s,t,u) - F(t,u,s) - F(u,s,t) - 2F(s,u,t) + F(t,s,u) + F(u,t,s)\right], \tag{A.6}$$

so that

$$F = S + A + M_1 + M_2 \; . \tag{A.7}$$

We now identify

$$f(s,t,u) = S(s,t,u) \tag{A.8}$$

$$(s-t)(t-u)(u-s)\, j(s,t,u) = A(s,t,u) \tag{A.9}$$

$$(2s-t-u)\, g_1(s,t,u) + (2s^2-t^2-u^2)\, h_1(s,t,u) = M_1(s,t,u) \tag{A.10}$$

$$(t-u)\, g_2(s,t,u) + (t^2-u^2)\, h_2(s,t,u) = M_2(s,t,u) \; . \tag{A.11}$$

It is clear that $f(s,t,u)$ is totally symmetric and analytic in D. Similarly, $j(s,t,u)$, being the quotient of two totally antisymmetric functions is symmetric. Moreover, it is analytic because A is analytic and antisymmetric, so that $A/(s-t)(t-u)(u-s)$ is also analytic. It remains to show that M_1, M_2 have the representations (A.10) and (A.11). We write

$$g_1(s,t,u) = \frac{M_1(s,t,u)\left[s^2+u^2-2t^2\right] - M_1(t,s,u)\left[t^2+u^2-2s^2\right]}{3(t-s)(s-u)(u-t)} \tag{A.12}$$

$$h_1(s,t,u) = \frac{M_1(s,t,u)\left[2t-s-u\right] - M_1(t,s,u)\left[2s-t-u\right]}{3(t-s)(s-u)(u-t)} \tag{A.13}$$

$$g_2(s,t,u) = \frac{M_2(s,t,u)\left[s^2-u^2\right] - M_2(t,s,u)\left[t^2-u^2\right]}{(t-s)(s-u)(u-t)} \tag{A.14}$$

$$h_2(s,t,u) = \frac{M_2(s,t,u)(u-s) - M_2(t,s,u)(u-t)}{(t-s)(s-u)(u-t)} \quad . \tag{A.15}$$

It is easy to show that this ansatz satisfies (A.10) and (A.11), and using (A.5) and (A.6), it also follows that g_1, g_2, h_1, h_2 are symmetric and analytic in D.

To prove the uniqueness of the decomposition, it is sufficient to prove the linear independence of the terms on the right-hand side of (A.1), i.e., that if $F(s,t,u)$ vanishes identically, so do f, g_1, g_2, h_1, h_2, j. Using the result that terms corresponding to different representations, or different rows of the same representation, are independent, it remains to show that if

$$(2s-t-u)g_1(s,t,u) + (2s^2-t^2-u^2)\,h_1(s,t,u) = 0 \tag{A.16}$$

and

$$(t-u)g_2(s,t,u) + (t^2-u^2)\,h_2(s,t,u) = 0 \quad , \tag{A.17}$$

then g_1, h_1, g_2, h_2 all vanish. Suppose for example that $g_1 \not\equiv 0$. Then

$$\frac{2s-t-u}{2s^2-t^2-u^2} = -\,\frac{h_1(s,t,u)}{g_1(s,t,u)} \quad . \tag{A.18}$$

The right-hand side of this equation is totally symmetric; the left-hand side is not, which is a contradiction. Thus g_1 must vanish, and similarly, g_2, h_1, h_2 also vanish.

It is also clear that if $F(s,t,u)$ is symmetric in t and u, then g_2, h_2, j all vanish.

APPENDIX B

Given the Froissart-Gribov representation for partial waves for $\ell \geq 2$,

$$f_\ell^{(i)}(s) = (-1)^\ell \frac{2}{\pi} \frac{2}{1-s} \int_1^\infty A_t^{[i]}(s,t)\, Q_\ell\left(\frac{2t}{1-s} - 1\right) dt \,, \tag{B.1}$$

we wish to solve for the partial waves $f_0^{(0)}$, $f_0^{(2)}$, $f_1^{(1)}$ in terms of the absorptive parts $A_t^{[i]}(s,t)$.

For $i = 0,2$, we write

$$T^{(i)}(s,t) = \sum_{\ell \text{ even}}^{\infty} (2\ell+1)\, f_\ell^{(i)}(s)\, P_\ell(z_s) \tag{B.2}$$

$$= f_0^{(i)}(s) + \sum_{\ell \text{ even} \geq 2}^{\infty} (2\ell+1)\, f_\ell^{(i)}(s)\, P_\ell(z_s) \,. \tag{B.3}$$

Inserting (B.1) into (B.3), and using the Darboux-Christoffel formula

$$\sum_{\ell \text{ even} \geq 2}^{\infty} (2\ell+1) P_\ell(z) Q_\ell(W) = \frac{1}{2}\left[\frac{1}{W-Z} + \frac{1}{W+Z} - \ln\left(\frac{W+1}{W-1}\right)\right], \tag{B.4}$$

we find

$$T^{(i)}(s,t) = f_0^{(i)}(s) + \frac{1}{\pi}\int_1^\infty dt'\ A_t^{[i]}(s,t')\left[\frac{1}{t'-t} + \frac{1}{t'+t+s-1} - \frac{2}{1-s} \ln \frac{t'}{t'+s-1}\right]. \tag{B.5}$$

Similarly, we can write for $i = 1$

$$T^{(1)}(s,t) = 3 f_1^{(1)}(s)\left(1+\frac{2t}{s-1}\right)$$
$$+ \frac{1}{\pi}\int_1^\infty dt' A_t^{[1]}(s,t')\left[\frac{1}{t'-t} - \frac{1}{t'+t+s-1} + \frac{6(1-s-2t)}{(1-s)^2}\left\{\frac{2t'+s-1}{1-s}\ln\frac{t'}{t'+s-1} - 2\right\}\right].$$

(B.6)

In the t channel, we have $T^{(0)}(t,s)$ and $T^{(2)}(t,s)$ symmetric under interchange of s and u, while $T^{(1)}(t,s)$ is antisymmetric. This implies

$$T^{(1)}(t,s)\big|_{s=u} = 0 \tag{B.7}$$

$$\frac{d}{ds}T^{(0)}(t,s)\Big|_{s=u} = \frac{d}{ds}T^{(2)}(t,s)\Big|_{s=u} = 0 \quad . \tag{B.8}$$

But since $s+t+u=1$, treating s and t as independent we find that

$$t = 1-2s \tag{B.9}$$

assures that

$$s = u \quad . \tag{B.10}$$

Using the equations (B.7) and (B.8), the crossing relations

$$T^{(i)}(t,s) = \sum_{j=0}^{2} A_{ij}\, T^{(j)}(s,t) \quad , \tag{B.11}$$

where

$$A_{ij} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} \quad , \tag{B.12}$$

and the representations (B.5) and (B.6), we find equations for

(a) $\frac{1}{3} f_0^{(0)}(s) - \frac{5}{6} f_0^{(2)}(s) + \frac{3}{2} f_1^{(1)}(s)\left(1 + \frac{2(1-2s)}{s-1}\right)$

(b) $\frac{1}{3} \frac{df_0^{(0)}(s)}{ds} + \frac{5}{3} \frac{df_0^{(2)}(s)}{ds} + 3\left\{ \frac{df_1^{(1)}(s)}{ds}\left(1+\frac{2(1-2s)}{s-1}\right) - 2f_1^{(1)}(s)\frac{(1-2s)}{(1-s)^2}\right\}$

(c) $\frac{1}{3} \frac{df_0^{(0)}(s)}{ds} + \frac{1}{6} \frac{df_0^{(2)}(s)}{ds} - \frac{3}{2}\left\{ \frac{df_1^{(1)}(s)}{ds}\left(1+\frac{2(1-2s)}{s-1}\right) - 2f_1^{(1)}(s)\frac{(1-2s)}{(1-s)^2}\right\}$

in terms of the absorptive parts. These determine $f_0^{(0)}$, $f_0^{(2)}$ and $f_1^{(1)}$ up to solutions of the homogeneous equations

$$(a) = (b) = (c) = 0 \quad . \tag{B.13}$$

The solution of (B.13) can be obtained by adding (b)+2(c) to obtain

$$f_0^{(0)}(s) + 2f_0^{(2)}(s) = \text{Const.} \quad , \tag{B.14}$$

solving for $f_1^{(1)}$ from (a), inserting this solution into (b) and using (B.14) to eliminate $f_0^{(2)}$ to get an elementary equation for $f_0^{(0)}$. The result is that the amplitudes $f_0^{(0)}$, $f_0^{(2)}$, $f_1^{(1)}$ are determined by the absorptive parts up to the ambiguity

$$f_0^{(0)}(s) \longrightarrow f_0^{(0)}(s) + 5a + 2b(3s-1) \tag{B.15}$$

$$f_1^{(1)}(s) \longrightarrow f_1^{(1)}(s) \qquad - b(1-s) \tag{B.16}$$

$$f_0^{(2)}(s) \longrightarrow f_0^{(2)}(s) + 2a - b(3s-1) \tag{B.17}$$

where a, b are arbitrary constants.

It is easy to see from our expressions (4.18)-(4.20) that the ambiguity in $f_0^{(0)}$, $f_0^{(2)}$, $f_1^{(1)}$ is given by (B.15)-(B.17). Since the absorptive part fixes all partial waves for $\ell \geq 2$, all the coefficients c_p^σ, d_p^σ, e_p^σ are fixed for $\sigma \geq 2$. The coefficients for $\sigma = 0$ and $\sigma = 1$ are c_0^0 and d_0^1 respectively. They enter only into the s and p waves, and are therefore not determined by the absorptive part. Leaving them arbitrary, and using (3.5), (4.18)-(4.22) and (4.24) gives the result (B.15)-(B.17).

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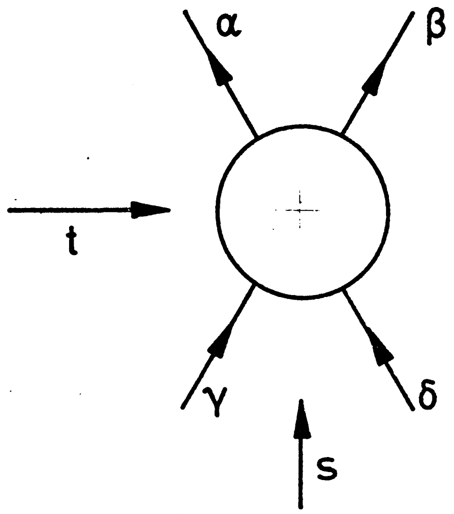

Schematic diagram of $\pi\pi$ scattering,
with isospin indices $\alpha, \beta, \gamma, \delta$.