HIGHER ORDER CORRECTIONS TO THE COHERENT PRODUCTION OF VECTOR BOSONS IN THE COULOMB FIELDS OF A NUCLEUS

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A B S T R A C T

The wave function for a vector boson in a Coulomb field is obtained in a high energy approximation. The Furry wave function for a lepton in a Coulomb field is reconsidered and extended by taking into account a term hitherto neglected. Both wave functions are then applied to the coherent production of vector-bosons and muons by neutrinos in the Coulomb field of a nucleus. For low-energy muons the matrix element is improved by taking into account all second order effects for the muon. The effect of nuclear structure is briefly discussed.

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1. INTRODUCTION

Recently, Lee, Markstein and Yang [1] calculated the coherent production cross section of intermediate vector bosons by neutrinos in lowest order in the e.m. coupling constant. Their calculations indicate a rather strong dependence of the cross section on the mass and the magnetic moment of the vector boson and it will thus be probable that mass and magnetic moment are determined from experimental cross sections. Clearly any uncertainty in the theoretical cross section will come out then as an uncertainty in these parameters and it seems desirable to investigate higher order corrections. The process in question is very similar to electron-positron pair creation by a photon, which has been studied extensively. For the case of large Z where higher order effects in Ze^2 are important, the pair production has been calculated in a high energy approximation by Bethe and Maximon [2], and it is the purpose of the present paper to do the same type of calculation for production of vector bosons (hereafter called W). This problem can be split into two parts, i) the calculation of the wave function of a vector boson in a Coulomb field, and ii) the application of this wave function to a special problem, for instance neutrino induced production or muon induced production. Clearly, in attacking the problem in this way, we have the advantage of finding a wave function which is of use independently of the way other particles (muon or electron) enter the problem. For instance, such a wave function can be used also in high energy neutrino induced processes, where the muon (or electron) comes out with low velocity, a situation which we shall see to be of practical interest.

In Section 2, kinematical considerations will reveal that coherent production necessarily involves a relativistic vector boson and a momentum transfer whose magnitude is small with respect to the energy of the W. In Section 3, we solve the equation of motion of a vector boson in a Coulomb field for high energies and small momentum transfer. Section 4 treats the Dirac equation in an analogous way. In Section 5, the wave functions of Sections 3 and 4 are applied to the problem of neutrino induced W production, both outgoing particles being relativistic. To improve the matrix element in the case of a slow outgoing muon certain second order terms are added as discussed in Section 6. In Section 7 the important nuclear structure effects are discussed. Section 8, finally, summarizes the results of Sections 5 to 7.

2. KINEMATICS

For definiteness we consider a specific process :

$$\nu + Z \rightarrow \mu + W + Z$$

where ν = neutrino, Z = nucleus of charge Ze, μ = muon, W = vector boson. Here and in what follows we will work in the laboratory system of reference and neglect the recoil energy of the nucleus. Putting $\hbar = c = 1$ and denoting the momentum and energy of the neutrino, μ-meson and vector boson by $(\vec{q}, E)$, (p, ε) and (k, ω) respectively, we have the relations :

$$\vec{Q} = \vec{q} - \vec{k} - \vec{p} \qquad q = |\vec{q}| = E \qquad p = |\vec{p}| = \sqrt{\varepsilon^2 - m^2}$$

$$k = |\vec{k}| = \sqrt{\omega^2 - M^2} \qquad E = \varepsilon + \omega \qquad Q = |\vec{Q}|$$

where $\vec{Q}$ is the momentum absorbed by the nucleus and m and M are the masses of muon and vector boson. For given initial energy E the minimum amount of momentum to be absorbed is given by

$$Q_m = E - \sqrt{E^2 - (M + m)^2}$$

This is when $\vec{q}$, $\vec{k}$ and $\vec{p}$ are all in the same direction, while ε and ω are given by

$$\varepsilon = m \frac{E}{M+m} \qquad \omega = M \frac{E}{M+m}$$

The restriction to the coherent process means $Q \lesssim [\text{nucl. dim.}]^{-1}$, which gives for instance for lead $Q \lesssim 40$ MeV/c. However, larger momentum transfers will also occur, although at a much reduced rate depending on the details of the nuclear form factor. Taking for definiteness $Q \lesssim 100$ MeV/c we get with $M = 600$ MeV and $m = 100$ MeV that $E > 2500$ MeV. This gives to ε and ω relativistic values which can be considered large with respect to Q.

Suppose now that we want to observe muons coming out in directions orthogonal to the incident ν. The region of special interest is then the low muon energy region where $\varepsilon \sim m$ and $Q \sim Q_m + m$. Of course, this Q will be above the upper limit $[\text{nucl. dim.}]^{-1}$, but, depending on the special form of the nuclear form factor, the excess will not be so strong as to suppress all processes of this form. The vector boson can never be slow, however, as its mass is much higher so that $Q_m + M$ is always far above threshold.

3. VECTOR BOSON IN COULOMB FIELD

The equation of motion of a W particle in the Coulomb field of a point charge Ze is (in addition to $\hbar = c = 1$ we set $Ze^2 = a$) :

$$\frac{\partial^2}{\partial x_\mu^2} \varphi_\nu - \frac{\partial^2}{\partial x_\nu \partial x_\mu} \varphi_\mu - M^2 \varphi_\nu = \frac{2a}{r} \frac{\partial \varphi_\nu}{\partial x_4} - \frac{a}{r} \frac{\partial \varphi_4}{\partial x_\nu} - \frac{a}{r} \delta_{\nu 4} \frac{\partial \varphi_\mu}{\partial x_\mu} \tag{3.1}$$

$$-\lambda \delta_{\nu 4} \left(\frac{\partial \frac{a}{r}}{\partial x_i}\right) \varphi_i - (1-\lambda) \left(\frac{\partial \frac{a}{r}}{\partial x_\nu}\right) \varphi_4 - \frac{a^2}{r^2} \varphi_\nu + \frac{a^2}{r^2} \delta_{\nu 4} \varphi_4$$

Latin indices will always run from 1 to 3, while Greek ones take the values 1 to 4. λ is the parameter for the magnetic moment μ_W, defined by

$$\mu_W = \frac{e\hbar}{2Mc} \lambda$$

There is a subsidiary condition (given in (3.3) below) that can be found by applying the operator $\frac{\partial}{\partial x_\nu}$ to Eq. (3.1) and summing over ν . This condition can be used to eliminate the second term in the left-hand side of (3.1). After some algebraic manipulations one finds :

$$(\Box - M^2) \varphi_\nu = \frac{2a}{r} \frac{\partial \varphi_\nu}{\partial x_\nu} - \frac{(2-\lambda)}{M^2} \left[\frac{\partial^2}{\partial x_\nu \partial x_i}, \frac{a}{r}\right] \frac{\partial \varphi_i}{\partial x_4} - \lambda \delta_{\nu 4} \left[\frac{\partial}{\partial x_i}, \frac{a}{r}\right] \varphi_i$$

$$+ \frac{(1-\lambda)}{M^2} \left[\frac{\partial^3}{\partial x_\nu \partial x_i^2}, \frac{a}{r}\right] \varphi_4 - \frac{(1-\lambda)}{M^2} \left[\frac{\partial}{\partial x_\nu}, \frac{a}{r}\right] \left(\frac{\partial^2 \varphi_4}{\partial x_4^2} + M^2 \varphi_4\right)$$

$$+ \frac{\lambda}{M^2} \left[\frac{\partial^2}{\partial x_\nu \partial x_i}, \frac{a}{r}\right] \frac{\partial \varphi_4}{\partial x_i} - \frac{a^2}{r^2} \varphi_\nu + \frac{(1-\frac{1}{2}\lambda)}{M^2} \left[\frac{\partial^2}{\partial x_\nu \partial x_i}, \frac{a^2}{r^2}\right] \varphi_i$$

$$+ \frac{1}{M^2} \left[\frac{\partial}{\partial x_\nu}, \frac{a}{r}\right] \left\{ \frac{a^2}{r^2} \varphi_4 - \lambda \frac{a}{r} \frac{\partial \varphi_4}{\partial x_4} - \lambda \left[\frac{\partial}{\partial x_i}, \frac{a}{r}\right] \varphi_i + (1-\lambda) \left(\frac{\partial}{\partial x_4} + \frac{a}{r}\right) [A] \right\}$$

$$- \frac{(1-\frac{1}{2}\lambda)}{M^2} \left[\frac{\partial}{\partial x_\nu}, \frac{a^2}{r^2}\right] \frac{\partial \varphi_i}{\partial x_i} - \frac{a}{r} \delta_{\nu 4} [A] \tag{3.2}$$

with

$$[A] = -\frac{(2-\lambda)}{M^2}\frac{\partial\frac{a}{r}}{\partial x_i}\frac{\partial\varphi_i}{\partial x_4} + \frac{(1-\lambda)}{M^2}\left[\frac{\partial^2}{\partial x_i^2}, \frac{a}{r}\right]\varphi_4 + \frac{(1-\frac{1}{2}\lambda)}{M^2}\frac{\partial\frac{a^2}{r^2}}{\partial x_i}\varphi_i$$

$$+ \frac{\lambda}{M^2}\frac{\partial\frac{a}{r}}{\partial x_i}\frac{\partial\varphi_4}{\partial x_i}$$

$i = 1, 2, 3$; $\nu = 1, 2, 3, 4$.

As far as is possible the right hand side has been written in terms of commutators of ar^{-1} with differential operators in order to facilitate the discussion below. The subsidiary condition is

$$\frac{\partial\varphi_\mu}{\partial x_\mu} = \frac{a}{r}\varphi_4 + [A] \qquad (3.3)$$

One notes that at $r = \pm\infty$ this goes over into the subsidiary condition for a free particle.

We must now find a wave function that obeys this wave equation in the low momentum transfer limit. This corresponds to the limit of large r in (3.2) and, neglecting all terms of order r^{-2}, one is left with the equation :

$$(\Box - M^2)\varphi_\nu - \frac{2a}{r}\frac{\partial\varphi_\nu}{\partial x_4} = 0 \qquad (3.4)$$

The monochromatic solutions of this equation are well known and can be written in terms of a confluent hypergeometric function [3]. They differ from each other in the asymptotic behaviour for $r \to \infty$. We restrict ourselves to the particular solution needed when the W occurs in the final state of a collision. This means that we have to take the solution with ingoing spherical waves [4] :

$$\varphi_\nu^{(o)}(x) = e^{i\omega t}\, \varphi_\nu^{(o)}(\vec{x}) = \tag{3.5}$$

$$= M(\omega) e_\nu(k) e^{-i\vec{k}\vec{r}+i\omega t}\, {}_1F_1(-ia_1; 1; i\vec{k}\vec{r}+ikr)$$

$$M(\omega) = e^{-\frac{\pi a_1}{2}} \Gamma(1-ia_1) \qquad \omega^2 = k^2+M^2 \qquad a_1 = a\frac{\omega}{k}$$

$e_\nu(k)$ is a polarisation vector depending on $\vec{k}$ and ω (see (3.7)). Clearly ${}_1F_1(-ia_1; 1; i\vec{k}\vec{r} + ikr)$ fulfils :

$$(\nabla^2 - 2ik\vec{\nabla} - \frac{2a_1 k}{r})\, {}_1F_1(-ia_1; 1; i\vec{k}\vec{r} + ikr) = 0 \tag{3.6}$$

One way of seeing that this is the solution with the correct asymptotic behaviour, is by taking the first order term in a of its Fourier transform. This Fourier transform can only be defined if we include a damping term $\exp(-\lambda' r)$ with very small λ' :

$$\int d_3 r\, e^{i(\vec{q}-\vec{k})\vec{r} - \lambda' r}\, {}_1F_1(-ia_1; 1; i\vec{k}\vec{r} + ikr) =$$

$$= \left[\frac{8\pi\lambda'(1+ia_1)}{((\vec{q}-\vec{k})^2+\lambda'^2)} - \frac{1}{(q^2-k^2-2ik\lambda'+\lambda'^2)}\, \frac{8\pi a_1(k+i\lambda')}{(\vec{q}-\vec{k})^2+\lambda'^2}\right].$$

$$\left(\frac{(\vec{q}-\vec{k})^2+2(\vec{k},\vec{q}-\vec{k})+\lambda'^2-2ik\lambda'}{(\vec{q}-\vec{k})^2+\lambda'^2}\right)^{ia_1}$$

For $\vec{q} \neq \vec{k}$ and $\lambda' \rightarrow 0$ only the second term in square brackets is non-zero, and one sees that in first order in a it coincides with the first Born approximation for the potential $2a\omega r^{-1}$ and an ingoing spherical wave. The subsidiary condition gives a restriction on the polarisation vector $e_\nu(k)$:

$$k_\nu \, e_\nu(k) = 0 \tag{3.7}$$

Clearly one can find 3 independent unit vectors that fulfil (3.7). This corresponds to the 3 polarisation possibilities.

The wave function (3.5) is often not accurate enough for direct application. Therefore, we try to improve it by calculating the next higher order contribution. We do this by treating the r^{-2} terms in Born approximation, where (3.4) is considered as the unperturbed equation. Dropping all terms of order a^2r^{-3} in (3.2), inserting $\varphi_\nu^{(o)}$ in all r^{-2} terms and substituting $\varphi_\nu(x) = (\exp i\omega t)\,\varphi_\nu(x)$ we are left with :

$$\begin{aligned}(\nabla^2 + k^2 - \frac{2a_1k}{r})\varphi_j(\vec{x}) = &- \frac{(2-\lambda)}{2M^2}\left[\frac{\partial^2}{\partial x_j \partial x_i}, \frac{2a_1k}{r}\right]\varphi_i^{(o)}(\vec{x}) + \\ &+ \frac{(1-\lambda)}{2\omega M^2}\left[\frac{\partial^3}{\partial x_j \partial x_i^2}, \frac{2a_1k}{r}\right]\varphi_4^{(o)}(\vec{x}) - \frac{(1-\lambda)}{2\omega M^2}\left[\frac{\partial}{\partial x_j}, \frac{2a_1k}{r}\right](\omega^2+M^2)\varphi_4^{(o)}(\vec{x}) \\ &- \frac{a^2}{r^2}\varphi_j^{(o)}(\vec{x}) - \frac{i\lambda k_i}{2\omega M^2}\left[\frac{\partial^2}{\partial x_j \partial x_i}, \frac{2a_1k}{r}\right]\varphi_4^{(o)}(\vec{x})\end{aligned} \tag{3.8}$$

$$\begin{aligned}(\nabla^2 + k^2 - \frac{2a_1k}{r})\varphi_4(\vec{x}) = &- \frac{(2-\lambda)\omega}{2M^2}\left[\frac{\partial}{\partial x_i}, \frac{2a_1k}{r}\right]\varphi_i^{(o)}(\vec{x}) + \\ &- \frac{\lambda}{2\omega}\left[\frac{\partial}{\partial x_i}, \frac{2a_1k}{r}\right]\varphi_i^{(o)}(\vec{x}) + \frac{(1-\lambda)}{2M^2}\left[\frac{\partial^2}{\partial x_i^2}, \frac{2a_1k}{r}\right]\varphi_4^{(o)}(\vec{x}) + \\ &- \frac{a^2}{r^2}\varphi_4^{(o)}(\vec{x}) - \frac{i\lambda k_i}{2M^2}\left[\frac{\partial}{\partial x_i}, \frac{2a_1k}{r}\right]\varphi_4^{(o)}(\vec{x})\end{aligned} \tag{3.9}$$

where we have used $\frac{\partial}{\partial x_i} \varphi_4^{(o)} = ik_i \varphi_4^{(o)} +$ terms of order $\frac{a}{r} \varphi_4^{(o)}$. One notes that the right hand side contains a r^{-2} term and terms which have a commutator of $\vec{\nabla}$ and r^{-1} as a factor. We will treat them separately mainly because the book-keeping becomes somewhat simpler. Consider the commutator terms first. They require the solution of an equation of the form

$$(\nabla^2 + k^2 - \frac{2a_1 k}{r})\varphi_j(\vec{x}) = \left[f_{ji}(\vec{\nabla}), \frac{2a_1 k}{r}\right] \varphi_i^{(o)}(\vec{x})$$

$$= M(\omega) e^{-i\vec{k}\vec{r}} \left[f_{ji}(\vec{\nabla} - i\vec{k}), \frac{2a_1 k}{r}\right] e_i(k)\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) \tag{3.10}$$

where f is some function that can be given in terms of a power series. A solution of this equation is :

$$\varphi_j(\vec{x}) = M(\omega) e^{-i\vec{k}\vec{r}} \left\{f_{ji}(\vec{\nabla} - i\vec{k}) - f_{ji}(-i\vec{k})\right\} \cdot$$

$$e_i(\vec{k})\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

as can be seen immediately by insertion in (3.10). Since the derivative of ${}_1F_1$ behaves asymptotically as r^{-1} this $\varphi_j(\vec{x})$ has the asymptotic behaviour required for our problem. By this method we have now the solution of (3.8) and (3.9) as far as the commutator terms are concerned. Consider next

$$(\nabla^2 + k^2 - \frac{2a_1 k}{r})\varphi_j(\vec{x}) = \frac{\alpha}{r^2} M(\omega) e^{i\vec{k}\vec{r}}\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) \tag{3.11}$$

α being some constant, We make the ansatz [5)]

$$\varphi_j(\vec{x}) = \frac{\alpha M(\omega)}{4\pi} e_j(k) \int d_3k' \frac{e^{-i\vec{k}'\vec{r}}}{(k^2-k'^2+io)} \cdot \tag{3.12}$$

$$\cdot \frac{1}{|\vec{k}-\vec{k}'|} {}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr)$$

We insert this ansatz into the left-hand side of (3.11). In order to permit differentiation under the integral sign we make the $\vec{k}'$ integration uniformly convergent by adding a term $-\lambda' |\vec{k}-\vec{k}'|$ in the exponent. By using the equation for a confluent hypergeometric function

$$\left(x\frac{d^2}{dx^2} + (b-x)\frac{d}{dx} - a\right) {}_1F_1(a;b;x) = 0$$

we get without difficulty :

$$(\nabla^2 + k^2 - \frac{2a_1k}{r})\varphi_j(x) =$$

$$= \frac{\alpha M(\omega)}{4\pi} e_j(k) \int d_3k' \frac{e^{-i\vec{k}'\vec{r}-\lambda'|\vec{k}'-\vec{k}|}}{|\vec{k}-\vec{k}'|} \left\{ {}_1F_1''' - 2\,{}_1F_1' + {}_1F_1 \right\}$$

To calculate this we use the well-known representation :

$${}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr) =$$

$$= \frac{1}{2\pi i} \int^{o^+,1^+} dt \;\; t^{-ia_1-1}(t-1)^{ia_1} \, e^{t(i\vec{k}'\vec{r}+ikr)} \tag{3.13}$$

The integrand has a cut from $t = o$ to $t = 1$ and we are in the sheet where the arguments of t and $t-1$ are zero on the real axis to the right of 1. The integration contour encircles the points o and 1 anti-clockwise. Taking the contour sufficiently close around the cut we can, for a given $\lambda' \neq 0$, exchange the t and k' integration. Performing the k' integration we get :

$$(\nabla^2 + k^2 - \frac{2a_1 k}{r})\varphi_j(\vec{x}) = \frac{\alpha M(\omega)}{8\pi i} e_j(k) e^{-i\vec{k}\vec{r}} \cdot$$

$$\int_{o^+}^{1^+} dt\; t^{-ia_1-1}(t-1)^{ia_1+2}\; e^{i\vec{k}\vec{r}t+ikrt} \frac{4\pi}{r^2(t-1)^2+\lambda'^2}$$

The t integrand has now acquired two poles which approach the point $t = 1$ as $\lambda' \to 0$ so that the contour is pinched from two sides. Therefore, we expand first the contour so that the poles come inside by adding the residues of the poles. These residues are seen to go to zero if $\lambda' \to 0$, and performing this limit we get the desired result (3.11). The solution (3.12) goes over into the normal plane wave Born approximation if we let $a_1 \to 0$. One establishes without difficulty that (3.12) has the proper asymptotic behaviour.

We can now write down the complete solution of (3.8) and (3.9) :

$$\varphi_\nu(x) = \varphi_c + \varphi_d + \varphi_a \tag{3.14}$$

with

$$\varphi_c = \frac{M(\omega)}{\sqrt{2\omega V}} e^{-i\vec{k}\vec{r}+i\omega t} e_\nu(k)\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

$$\varphi_d = \frac{M(\omega)}{\sqrt{2\omega V}} e^{-i\vec{k}\vec{r}+i\omega t}\, \Omega_{\nu\mu}\, e_\mu(k)\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

$$\varphi_a = \frac{a^2}{4\pi} \frac{M(\omega)}{\sqrt{2\omega V}} e_\nu(k) \int d_3k' \frac{e^{-i\vec{k}'\vec{r}}}{(k'^2-k^2-io)} \frac{1}{|\vec{k}-\vec{k}'|} \cdot$$

$${}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr)$$

where we have added the usual normalization factor $(2\omega V)^{-\frac{1}{2}}$ 6). Ω is given by :

$$\Omega_{ji} = -\frac{(2-\lambda)}{2M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\frac{\partial}{\partial x_j} - ik_i\right) + k_j k_i\right\}$$

$$\Omega_{j4} = \frac{(1-\lambda)}{2\omega M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\nabla^2 - 2ik_i\frac{\partial}{\partial x_i} - k^2\right) ik_j k^2 + \right.$$

$$\left. - (\omega^2 + M^2)\frac{\partial}{\partial x_j}\right\} - \frac{\lambda}{2\omega M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\frac{\partial}{\partial x_i} - ik_i\right)ik_i + ik_j k^2\right\}$$

$$\Omega_{4i} = -\frac{(2-\lambda)\omega}{2M^2}\frac{\partial}{\partial x_i} - \frac{\lambda}{2\omega}\frac{\partial}{\partial x_i}$$

$$\Omega_{44} = \frac{(1-\lambda)}{2M^2}\left\{\nabla^2 - 2ik_i\frac{\partial}{\partial x_i}\right\} - \frac{\lambda}{2M^2} ik_i\frac{\partial}{\partial x_i}$$

With the help of Eq. (3.6) for ${}_1F_1$ and the condition (3.7) we finally reduce Ω to :

$$\Omega_{ji} = -\frac{(2-\lambda)}{2M^2}\left\{\frac{\partial^2}{\partial x_j \partial x_i} - ik_j\frac{\partial}{\partial x_i}\right\}$$

$$\Omega_{j4} = \frac{(1-\lambda)}{M^2}\left(\frac{\partial}{\partial x_j} - ik_j\right)\frac{a_1 k}{r\omega} + \frac{\lambda}{2\omega M^2}\left\{-ik_i\frac{\partial^2}{\partial x_i \partial x_j} - k_j k_i\frac{\partial}{\partial x_i} + \right.$$

$$\left. + M^2\frac{\partial}{\partial x_j}\right\}$$

(3.15)

$$\Omega_{4i} = -\frac{2\omega^2 - \lambda k^2}{2M^2\omega}\frac{\partial}{\partial x_i}$$

$$\Omega_{44} = \frac{(1-\lambda)}{M^2}\frac{a_1 k}{r} - \frac{i\lambda}{2M^2}k_i\frac{\partial}{\partial x_i}$$

4. DIRAC PARTICLE IN COULOMB FIELD

The Dirac equation for a particle in a Coulomb field is :

$$(\gamma^{\mu}\frac{\partial}{\partial x_{\mu}} - \gamma^{4}\frac{a}{r} - m)\psi(x) = 0 \qquad (4.1)$$

Multiplying with $(-\gamma^{\mu}\partial/\partial x_{\mu} + \gamma^{4} a r^{-1} - m)$ we have :

$$(\Box - m^{2})\psi(x) = (\frac{2a}{r}\frac{\partial}{\partial x_{4}} + a\gamma^{4}\gamma^{i}\frac{x_{i}}{r^{3}} - \frac{a^{2}}{r^{2}})\psi(x) \qquad (4.2)$$

There seems to be some confusion in the literature [7] concerning the use of (4.2) instead of (4.1). In Appendix A we show that in second order the diagrams of perturbation theory generated by (4.2) are the same as those generated by (4.1). For the complete proof of the equivalence of (4.2) and (4.1) we refer to the work of L. Brown and M. Tonin [8].

In a way completely analogous to the calculation in Section 3, we find the solution $\psi(x)$ which takes into account exactly the first term in the right-hand side of (4.2) and in first approximation the terms of order r^{-2}. Again specializing to the case of an outgoing particle we have :

$$\bar{\psi}(x) = \bar{\psi}_{c} + \bar{\psi}_{d} + \bar{\psi}_{a} \qquad (4.3)$$

$$\bar{\psi}_{c} = \frac{N(\varepsilon)}{\sqrt{V}} e^{-i\vec{p}\vec{r}+i\varepsilon t}\,\bar{u}(p)\,{}_{1}F_{1}(ia_{2};1;i\vec{p}\vec{r}+ipr)$$

$$\bar{\psi}_{d} = \frac{N(\varepsilon)}{\sqrt{V}} e^{-i\vec{p}\vec{r}+i\varepsilon t}\,\bar{u}(p)\frac{1}{2\varepsilon}\gamma^{4}\gamma^{i}\frac{\partial}{\partial x_{i}}\,{}_{1}F_{1}(ia_{2};1;i\vec{p}\vec{r}+ipr)$$

$$\bar{\psi}_{a} = \frac{a^{2}N(\varepsilon)}{4\pi\sqrt{V}}\bar{u}(p)\int d_{3}p'\frac{e^{-i\vec{p}'\vec{r}}}{(p'^{2}-p^{2})|\vec{p}-\vec{p}'|}\,{}_{1}F_{1}(ia_{2};1;i\vec{p}'\vec{r}+ipr)$$

$$a_{2} = \frac{a\varepsilon}{p} \qquad N(\varepsilon) = e^{\frac{\pi}{2}a_{2}}\Gamma(1+ia_{2}) \qquad \varepsilon^{2} = p^{2} + m^{2}$$

$\bar{u}(p)$ is the Dirac spinor for a free muon of four impules p, cf. Ref. [6].

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5. THE MATRIX ELEMENT FOR NEUTRINO INDUCED W PRODUCTION

In the foregoing we discussed the wave functions of vector boson and lepton in a Coulomb field of arbitrary strength in the low momentum transfer limit. We shall use now these functions for the particular case of W production by a neutrino in the Coulomb field of a nucleus where by the nucleus is assumed to remain in the same state (coherent process). The weak interaction will be treated in lowest order. The matrix element for this process corresponding to the diagram of Fig. 1, will depend on the four-momenta $\underline{q}$, $\underline{p}$ and $\underline{k}$ of the neutrino, muon and vector boson, and on the polarisation of these particles.

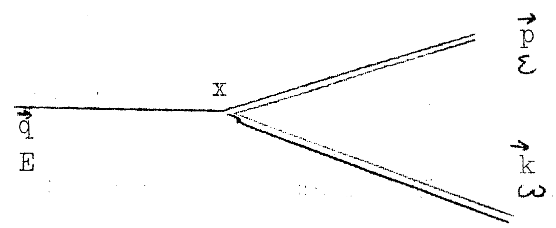

Fig. 1

Taking the neutrino to be in a definite state of helicity we need two indices t and s to indicate the polarisation of muon and vector boson respectively. Denoting the matrix element by $M(\underline{q}, \underline{p}, \underline{k}, t, s)$ we have :

$$M(\underline{q},\underline{p},\underline{k},t,s) = ig\int d_4x \langle p,t|\bar{\Psi}^m(x)|0\rangle \ \gamma^\mu(1+\gamma^5)\langle 0|\Psi^n(x)|q\rangle . \langle k,s|\varphi_\mu(x)|0\rangle \quad (5.1)$$

In here $\bar{\Psi}^m$, Ψ^n, φ_μ stand for muon, neutrino and vector boson field. We must now insert the wave functions calculated above together with a plane wave for the neutrino. The wave functions from Sections 3 and 4 have the form :

$$\langle k,s|\varphi_\mu(x)|0\rangle = \varphi_c + \varphi_d + \varphi_a$$

$$\langle p,t|\bar{\Psi}^m(x)|0\rangle = \bar{\Psi}_c + \bar{\Psi}_d + \bar{\Psi}_a$$

where all φ and $\bar{\psi}$ are as indicated in (3.14) and (4.3), i.e., φ_c and $\bar{\psi}_c$ stand for the solution of the equations of motion with the Coulomb field r^{-1} alone (c.f. (3.4)), φ_d and $\bar{\psi}_d$ for the corrections arising from that part of the equation which could be written in terms of commutators of r^{-1} with ∇, and φ_a and $\bar{\psi}_a$ finally for the correction arising from the ar^{-2} term (c.f. (3.11)). Clearly, φ_d, φ_a are small compared with φ_c, and $\bar{\psi}_d$, $\bar{\psi}_a$ are small in comparison to $\bar{\psi}_c$. Inserting these expressions into (5.1) we have :

$$M(\underline{q},\underline{p},\underline{k},t,s) = \frac{ig}{\sqrt{V}}\int d_4x\,\{\bar{\psi}_c + \bar{\psi}_d + \bar{\psi}_a\}\gamma^{\mu}(1+\gamma^5)\{u^n(\underline{q})e^{i\underline{qx}}\,.$$

$$\{\varphi_c + \varphi_d + \varphi_a\}$$

$u^n(q)$ is the Dirac spinor for a free neutrino of four impuls q.

Evaluating this expression we will neglect the term involving $(\bar{\psi}_d + \bar{\psi}_a)(\varphi_d + \varphi_a)$ and we see that the matrix element is built up from five parts :

$$M = \frac{ig}{\sqrt{V}}\{I_1 + I_2 + I_3 + I_4 + I_5\}$$

with

$$I_1 = \int d_4x\, \bar{\psi}_c\, \gamma^{\mu}(1+\gamma^5)u^n(\underline{q})e^{i\underline{qx}}\,\varphi_c$$

$$I_2 = \int d_4x\, \bar{\psi}_c\, \gamma^{\mu}(1+\gamma^5)u^n(\underline{q})e^{i\underline{qx}}\,\varphi_d$$

$$I_3 = \int d_4x\, \bar{\psi}_d\, \gamma^{\mu}(1+\gamma^5)u^n(\underline{q})e^{i\underline{qx}}\,\varphi_c \qquad (5.2)$$

$$I_4 = \int d_4x\, \bar{\psi}_c\, \gamma^{\mu}(1+\gamma^5)u^n(\underline{q})e^{i\underline{qx}}\,\varphi_a$$

$$I_5 = \int d_4x\, \bar{\psi}_a\, \gamma^{\mu}(1+\gamma^5)u^n(\underline{q})e^{i\underline{qx}}\,\varphi_c$$

The integrals $I_1 - I_3$ are of the same type as those calculated by Nordsieck [9] and Bethe-Maximon. Essentially they can all be derived from Nordsieck's result :

$$I(\lambda') = \int d_3r \frac{e^{i\vec{Q}\vec{r} - \lambda' r}}{r} {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr)\,{}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) =$$

$$= \frac{4\pi}{Q^2+\lambda'^2}\left(\frac{2\vec{k}\vec{Q}-2ik\lambda'+Q^2+\lambda'^2}{Q^2+\lambda'^2}\right)^{ia_1}\left(\frac{2\vec{p}\vec{Q}-2ip\lambda'+Q^2+\lambda'^2}{Q^2+\lambda'^2}\right)^{-ia_2}.$$

$$. \; F(-ia_1,ia_2;1;1-x(Q,\lambda')) \tag{5.3}$$

$$x(Q,\lambda') = \frac{(Q^2+\lambda'^2)(Q^2+2\vec{k}\vec{Q}+2\vec{p}\vec{Q}+2\vec{p}\vec{k}-2pk+\lambda'^2-2i\lambda'k-2i\lambda'p)}{(Q^2+2\vec{k}\vec{Q}+\lambda'^2-2i\lambda'k)(Q^2+2\vec{p}\vec{Q}+\lambda'^2-2i\lambda'p)}$$

F is a normal hypergeometric function, see (5.10) below. We have put $\vec{Q} = \vec{q} - \vec{p} - \vec{k}$. The integral I_1 can be obtained by differentiation of $I(\lambda')$ with respect to λ', while the integrals I_2 and I_3 can be reduced to derivatives of $I(0)$ with respect to $\vec{p}$ and $\vec{k}$ for fixed Q with the help of identities of the type

$$\frac{\partial}{\partial x_i}\,{}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr) = \frac{p}{r}\frac{\partial}{\partial p_i}\,{}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr), \quad i = 1,2,3$$

For completeness we write down explicitly the three types of integrals involved in (5.2) :

$$\int d_3r\; e^{i\vec{Q}\vec{r}}\,{}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr)\,{}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) =$$

$$= \left(-\frac{d}{d\lambda'} I(\lambda')\right)_{\lambda'=0}$$

$$= \left(\frac{2\vec{k}\vec{Q}+Q^2}{Q^2}\right)^{ia_1}\left(\frac{2\vec{p}\vec{Q}+Q^2}{Q^2}\right)^{-ia_2}\left[\left\{\frac{a\varepsilon}{D_2} - \frac{a\omega}{D_1}\right\}F(-ia_1,ia_2;1;1-x)\right.$$

$$\left. - 2i\left\{\frac{Q^2(k+p)}{D_1D_2} - \frac{(kD_2+pD_1)x'}{D_1D_2}\right\}F'(-ia_1,ia_2;1;1-x)\right] \tag{5.4}$$

$$\int d_3r\, e^{i\vec{Q}\vec{r}} \left\{ \frac{\partial}{\partial x_i}\, {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr) \right\} {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

$$= p\left(\frac{\partial}{\partial p_i}\, I(o)\right)_{\vec{Q}=\text{const.}}$$

$$= \left(\frac{2\vec{k}\vec{Q}+Q^2}{Q^2}\right)^{ia_1} \left(\frac{2\vec{p}\vec{Q}+Q^2}{Q^2}\right)^{-ia_2} \left[\frac{-2ia\varepsilon}{D_2}\, F(-ia_1, ia_2;1;1-x) \right.$$

$$\left. - 2p\left\{ \frac{Q^2(Q_i+k_i-\frac{k}{p}p_i)}{D_1D_2} - \frac{xQ_i}{D_2} \right\} F'(-ia_1, ia_2;1;1-x) \right]$$

$$\int d_3r\, e^{i\vec{Q}\vec{r}}\, {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr) \left\{ \frac{\partial}{\partial x_i}\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) \right\}$$

$$= k\left(\frac{\partial}{\partial k_i}\, I(o)\right)_{\vec{Q}=\text{const.}}$$

$$= \left(\frac{2\vec{k}\vec{Q}+Q^2}{Q^2}\right)^{ia_1} \left(\frac{2\vec{p}\vec{Q}+Q^2}{Q^2}\right)^{-ia_2} \left[\frac{2ia\omega}{D_1}\, F(-ia_1, ia_2;1;1-x) \right.$$

$$\left. - 2k\left\{ \frac{Q^2(Q_i+p_i-\frac{p}{k}k_i)}{} - \frac{xQ_i}{D_1} \right\} F'(-ia_1, ia_2;1;1-x) \right]$$

where now

$$x = x(\vec{Q}, o) = \frac{Q^2(Q^2+2\vec{k}\vec{Q}+2\vec{p}\vec{Q}+2\vec{p}\vec{k}-2pk)}{D_1D_2}$$

$$D_1 = Q^2 + 2\vec{Q}\vec{k}$$

$$D_2 = Q^2 + 2\vec{Q}\vec{p}$$

One observes that $x \to o$ for $Q \to o$. (Remember that the angle between $\vec{p}$ and $\vec{k}$ is of the order Q/k or Q/p).

We consider next I_4. The integral to be calculated is:

$$\int d_3r \int d_3k' \frac{e^{i\vec{\alpha}\vec{r}}}{(k'^2-k^2)|\vec{k}-\vec{k}'|}\, {}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr)\, {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr) \qquad (5.5)$$

with $\vec{\alpha} = \vec{Q} + \vec{k} - \vec{k}'$. This integral cannot be evaluated exactly and we must make some further approximation. Let us consider first the $\vec{r}$ integration :

$$\int d_3r \; e^{i\vec{\alpha}\vec{r}} \; {}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr) \; {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr) \tag{5.6}$$

In our approximation we need only take the leading part of this for small values of α. Now for small α (5.6) has a leading term of the $\delta(\vec{\alpha})$ type [10]. This allows us to change the factor $|\vec{k} - \vec{k}'|^{-1}$ in (5.5) so that the value of the $\vec{k}'$ integrand is not changed in the point $\vec{k}' = \vec{Q} + \vec{k}$ (or $\vec{\alpha} = 0$). A change that fulfils this condition is :

$$\frac{1}{|\vec{k}-\vec{k}'|} \rightarrow \frac{(\vec{Q},\vec{k}'-\vec{k})}{Q\,|\vec{k}'-\vec{k}|^2}$$

This change has the further advantage that the new integrand has the same derivative with respect to $\vec{k}'$ in the point $\vec{k}' = \vec{Q} + \vec{k}$ as the old one. This ensures that this term gets the same phase factor as all other terms in the matrix element, which simplifies our final expression. The change just described makes the integral (5.5) of the type $\frac{\partial}{\partial k_i} I(\lambda')$. In the further treatment of (5.5) one can either do the $\vec{r}$ integration and the subsequent convolution over $\vec{k}'$, or one can prove that :

$$\int d_3k' \frac{e^{-i\vec{k}'\vec{r}}}{(k'^2-k^2-io)} \frac{(\vec{k}'-\vec{k})_i}{|\vec{k}'-\vec{k}|^2} \; {}_1F_1(-ia_1;1;i\vec{k}'\vec{r}+ikr) =$$

$$= \frac{\pi^2}{ia_1k} e^{-i\vec{k}\vec{r}} \frac{\partial}{\partial x_i} \; {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

We chose the latter way, the proof is given in Appendix B.

At this place we want to clarify a little the above procedure. Adopting the same reasoning as given above one would expect a $\delta(\vec{Q})$ function for I_1. Indeed, on performing the differentiation with respect to λ' of $I(\lambda')$ one also gets a term of the form $\lambda' / (\lambda'^2+Q^2)^2$ which is of the δ- type (see Appendix C, Eq. (C.1)). However, we consider only cases where $Q \neq 0$, so that this term does not contribute. Thus, in the case of I_1, the terms here neglected play a major role.

We can now write down the matrix element in terms of $I(\lambda')$ and $1^t_\mu = \frac{1}{2}(\bar{u}^m_t(p)\gamma^\mu \ (1+\gamma^5)u^n(q))$:

$$
\begin{aligned}
M(\underline{q},\underline{p},\underline{k},t,s) = {} & 4\pi i g\delta(E-\varepsilon-\omega)\frac{M(\omega)N(\varepsilon)}{V^{3/2}\sqrt{2\omega}}\Big[- 1^t_\mu e^s_\mu(k)\Big(\frac{dI(\lambda')}{d\lambda'}\Big)_{\lambda'=0} + \\
& + 1^t_j e^s_i(k)\frac{(2-\lambda)}{2M^2}\Big\{ik_j k\frac{\partial}{\partial k_i}I(0) + ikQ_j\frac{\partial}{\partial k_i}I(0) + kp\frac{\partial^2}{\partial p_j\partial k_i}I_2 \\
& + 1^t_j e^s_4(k)\frac{(1-\lambda)a_1 k}{M^2\omega}\Big\{- ik_j I(0) - iQ_j I(0) - p\frac{\partial}{\partial p_j}I_2\Big\} \\
& + 1^t_j e^s_4(k)\frac{\lambda}{2\omega M^2}\Big\{- (\vec{k}\vec{Q})k\frac{\partial}{\partial k_j}I(0) - kpk_i\frac{\partial^2}{\partial k_j\partial p_i}I_2 - kk_j k_i\frac{\partial}{\partial k_i}I(0) \\
& + M^2 k\frac{\partial}{\partial k_j}I(0)\Big\} \\
& + 1^t_4 e^s_i(k)\Big(-\frac{2\omega^2-\lambda k^2}{2M^2\omega}\Big)k\frac{\partial}{\partial k_i}I(0) \\
& + 1^t_4 e^s_4(k)\Big\{\frac{(1-\lambda)a_1 k}{M^2}I(0) - \frac{i\lambda k}{2M^2}k_i\frac{\partial}{\partial k_i}I(0)\Big\} \\
& + \Big\{\delta_{4i}1^t_\mu + \delta_{\mu i}1^t_4 - \delta_{\mu 4}1^t_i + \varepsilon_{k4i\mu}1^t_k\Big\}e^s_\mu(k)\frac{p}{2\varepsilon}\frac{\partial}{\partial p_i}I(0) \\
& + 1^t_\mu e^s_\mu(k)\frac{a^2}{4\pi}\Big(\frac{\pi^2}{ia_1}\frac{Q_i}{Q}\frac{\partial}{\partial k_i} - \frac{\pi^2}{ia_2}\frac{Q_i}{Q}\frac{\partial}{\partial p_i}\Big)I(0)\Big]
\end{aligned}
\qquad (5.7)
$$

where we have used the following relations :

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu} = \delta_{\alpha\mu}\gamma^{\nu} + \delta_{\alpha\nu}\gamma^{\mu} - \delta_{\mu\nu}\gamma^{\alpha} + \varepsilon_{\kappa\mu\alpha\nu}\gamma^{\kappa}\gamma^{5}$$

$\varepsilon_{\lambda\mu\alpha\nu}$ is the completely antisymmetry tensor, $\varepsilon_{1234} = 1$.

$$\int d_3r \; e^{i\vec{Q}\vec{r}} \, {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr)\frac{\partial}{\partial x_i}\, \frac{k}{r}\, \frac{\partial}{\partial k_j}\; {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr) =$$

$$= - ik\, Q_i \frac{\partial}{\partial k_j} I(0) - kp\frac{\partial^2}{\partial p_i \partial k_j} I_2 \tag{5.8}$$

$$I_2 = \int d_3r \; \frac{e^{i\vec{Q}\vec{r}}}{r^2}\, {}_1F_1(ia_2;1;i\vec{p}\vec{r}+ipr)\, {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

We look now into questions of order of magnitude. I_1 (Eq. (5.2)) contains the two leading parts Ψ_c and φ_c of muon and W wave functions. However, if $\vec{k}$, $\vec{p}$ and $\vec{q}$ point in the same direction and if $k/\omega \simeq p/\varepsilon$, we have

$$\frac{\varepsilon}{D_2} \sim \frac{\omega}{D_1}$$

which, as can be seen in (5.4) gives rise to a strong cancellation in I_1. Because of this cancellation, and because of the fact that the part from (5.4) containing a F' function is of lower order in the momentum transfer the integral I_1 becomes of the same order as I_2 to I_4. It is this fact *)

---

*) A further enhancement of this effect comes about through the spinor factors in all these integrals.

which makes necessary the consideration of quantities of lower order, i.e., of quantities of the order of magnitude $I(0)$. It will further be clear from this that the term $(dI(\lambda') / d\lambda')_{\lambda'=0}$ must be manipulated without further approximations. We drop all terms containing I_2 because they are one order of magnitude smaller in Q than the terms $I(0)$. One may well ask why we did not drop directly the terms $\partial_i \partial_j$ and $r^{-1}\partial_j$ from $\Omega_{\mu\nu}$. The reason is that we want to be correct also up to first order in the coupling constant a and these terms contain still first order contributions, that are extracted by the procedure (5.8). We insert now $I(\lambda')$ from (5.3) into (5.7) to get :

$$M(\underline{q},\underline{p},\underline{k},t,s) = \underline{\frac{8\sqrt{2}\,\pi^2 iga}{V^{3/2}\sqrt{\omega}}}\, M(\omega)N(\varepsilon)\underline{\delta(E-\varepsilon-\omega)}\,.$$

$$\cdot\left(\frac{2\vec{k}\vec{Q}+Q^2}{Q^2}\right)^{ia_1}\left(\frac{2\vec{p}\vec{Q}+Q^2}{Q^2}\right)^{-ia_2}\frac{1}{Q^2}\Bigg[V(x)\Bigg\{\underline{\underline{1^t_\mu e^s_\mu\left(\frac{2\varepsilon}{D_2}-\frac{2\omega}{D_1}\right.}}+$$

$$+\frac{\pi a}{2}\frac{Q}{D_1}+\frac{\pi a}{2}\frac{Q}{D_2}\Big)-\underline{\underline{1^t_\mu e^s_\nu\frac{A_{\mu\nu}}{D_1}}}+$$

$$\underline{\underline{-\left(\delta_{4\nu}1^t_\mu+\delta_{\mu\nu}1^t_4-\delta_{\mu 4}1^t_\nu+\varepsilon_{k4\nu\mu}1^t_k\right)e^s_\mu\frac{iQ_\nu}{D_2}}}\Bigg\}+$$

$$+\frac{ia_1a_2}{a}W(x)\Bigg\{1^t_\mu e^s_\mu\left(2x\left(\frac{k}{D_1}+\frac{p}{D_2}\right)-2Q^2\left(\frac{k+p}{D_1D_2}\right)\right.+$$

$$+\frac{\pi ak}{\omega}\left(Q\frac{Q^2+\vec{p}\vec{Q}-\frac{p}{k}(\vec{k}\vec{Q})}{D_1D_2}-Q\frac{x}{D_1}\right)$$

$$+\frac{\pi ap}{\varepsilon}\left(Q\frac{Q^2+\vec{k}\vec{Q}-\frac{k}{p}(\vec{p}\vec{Q})}{D_1D_2}-Q\frac{x}{D_2}\right)\Big)+1_\mu e_\nu B_{\mu\nu}/D_1 \tag{5.9}$$

$$+\left(\delta_{4i}1^t_\mu+\delta_{\mu i}1^t_4-\delta_{\mu 4}1^t_i+\varepsilon_{k4i\mu}1^t_k\right)e^s_\mu\left(\frac{-i}{2\varepsilon D_2}\right)\left(2xpQ_i-\frac{2Q^2p}{D_1}\left(Q_i+k_i-\frac{k}{p}P_i\right)\right)\Bigg\}\Bigg]$$

with $Q_\mu = (\vec{Q};0)$ and

$$A_{\mu\nu} = (k+Q)_\mu \left\{ \frac{(2-\lambda)\omega}{M^2} Q_\nu + \frac{i(2-\lambda)(\vec{k}\vec{Q}) + i(1-\lambda)Q^2}{M^2}\delta_{\nu 4} \right\} +$$

$$- i\lambda Q_\mu \delta_{\nu 4} + i\lambda \delta_{\mu 4} Q_\nu$$

$$B_{\mu\nu} = (k_\mu + Q_\mu)\left[\frac{(2-\lambda)k}{M^2}\beta_\nu - \frac{(2-\lambda)k}{M^2}\delta_{\nu 4}\beta_4 + \right.$$

$$\left. - \frac{2i\lambda k}{\omega M^2}\delta_{\nu 4} Q^2 \left\{ (\vec{Q}\vec{p})(\vec{Q}\vec{k}) - Q^2(\vec{p}\vec{k} - pk) \right\} \right]$$

$$- \frac{i\lambda k}{\omega}\delta_{\nu 4}\beta_\mu + \frac{i\lambda k}{\omega}\delta_{\mu 4}\beta_\nu + \frac{i\lambda k}{\omega M^2}\delta_{\nu 4} Q_\mu \frac{Q^2}{D_2}(\vec{p}\vec{k} - pk)$$

$$- \frac{i\lambda k}{\omega M^2}\frac{Q^2}{D_2}(\vec{k}\vec{Q})\delta_{\nu 4}(p_\mu - \frac{p}{k}k_\mu) + \frac{i\lambda k}{\omega M^2}\frac{Q^2}{D_2}(\vec{k}\vec{Q})\delta_{4\nu}\delta_{4\mu}(p_4 - \frac{p}{k}k_4)$$

$$\beta_\mu = xQ_\mu - \frac{Q^2}{D_2}(Q_\mu + p_\mu - \frac{p}{k}k_\mu)$$

Because $q_\mu l_\mu = 0$ one may eventually write $-p_\mu$ for $(k+Q)_\mu$. Further :

$$V(x) = F(-ia_1; ia_2; 1; 1-x)$$

$$F(a,b;c;z) = 1 + \frac{ab}{1c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2 + \ldots \qquad (5.10)$$

$$a_1 a_2 W(x) = F'(-ia_1, ia_2; 1; 1-x)$$

$$x = \frac{Q^2\{q^2 - (p+k)^2\}}{D_1 D_2}$$

$$D_1 = Q^2 + 2\vec{Q}\vec{k} = (\vec{k}+\vec{Q})^2 - k^2 \qquad D_2 = Q^2 + 2\vec{Q}\vec{p} = (\vec{p}+\vec{Q})^2 - k^2$$

In first order in a this formula agrees with the formula given in Ref. [1] up to the cut-off function in Q describing nuclear effects (see Section 7). To simplify the comparison we underlined the parts which contribute in first order in a. If $a_1 = a_2$, $V(x)$ and $W(x)$ are real functions of x. Davies, Bethe and Maximon [11] investigated the behaviour of these functions in the neighbourhood of $x = 0$ with the result

$$V(o) = \frac{1}{\Gamma(1-ia)\Gamma(1+ia)} \qquad W(x) = -V(o)\log x \qquad o < x \ll 1 \tag{5.11}$$

Formula (5.9) is correct up to terms of order $aQ\omega^{-1}$ or $aQ\varepsilon^{-1}$, i.e., it contains all terms in first order in a irrespective of their order in Q/ω or Q/ε, and all terms in first order in Q/ω or Q/ε irrespective of their order in a.

6. CORRECTIONS FOR LOW ENERGY MUONS

The present treatment breaks down for low energy muons, because of the fact that the second and third terms of the right-hand side of Eq. (4.2), which we treated in Born approximation, cannot be considered as small with respect to the Coulomb term if $\mathcal{E} \simeq Q$. For the energies of interest with the present accelerators, as one can see from the energy distribution of the outgoing muons given in Ref. [1], a considerable fraction of muons come out with relatively low energy. It is, therefore, highly desirable that we extend the region where our formula is valid into the low muon-energy direction. A very lucky circumstance is that the matrix element (5.9) deviates in structure really very little from what one would get by treating in (3.2) and (4.2) all right-hand side terms in Born approximation with respect to plane waves instead of Coulomb wave functions. This makes it possible to change the matrix element so that in the region of small p it is still correct up to second order in a. We want to stress the fact that no change has to be made in the normalization factor $N(\mathcal{E})$ given in Section 4, because it is related to the asymptotic behaviour of the Coulomb wave function which is determined by the Coulomb term alone.

Let us now try to find the necessary second order correction. Consider (4.2). Because of the fact that the Coulomb term is treated exactly, and because of the fact that the a^2/r^2 term has already been treated we need to solve (4.2) where in the right-hand side only the term x_i/r^3 is retained and ψ is replaced by ψ_d from (4.3):

$$(\nabla^2 + p^2)\psi(x) = a\gamma^4\gamma^j \frac{x_j}{r^3} \frac{N^*(\mathcal{E})}{2} e^{i\vec{p}\vec{r}} \gamma^4\gamma^i u(p)\frac{\partial}{\partial x_i} \cdot {}_1F_1(-ia_2;1;-i\vec{p}\vec{r}-ipr) \qquad (6.1)$$

where we take from ${}_1F_1$ in the right-hand side only the lowest order part in a. Using a representation of the form (3.13) for ${}_1F_1$ we get :

$$\frac{\partial}{\partial x_i}\ {}_1F_1(-ia_2;1;-i\vec{p}\vec{r}-ipr) =$$

$$= (-ip_i - ip\frac{x_i}{r})\frac{1}{2\pi i}\int^{o^+,1^+} dt\ t^{-ia_2}(t-1)^{ia_2}\ e^{(-i\vec{p}\vec{r}-ipr)t}$$

After partial integration we get in lowest order in a_2 :

$$\frac{\partial}{\partial x_i}\ {}_1F_1(-ia_2;1;-i\vec{p}\vec{r}-ipr) \simeq$$

$$\simeq ia_2\ \frac{(-ip_i - ip\frac{x_i}{r})}{(-i\vec{p}\vec{r}-ipr)}\ (1 - e^{-i\vec{p}\vec{r}-ipr})$$

We use the identity

$$\gamma^j\gamma^i = \delta_{ij} - \varepsilon_{ji\nu 4}\gamma^\nu\gamma^4\gamma^5 \equiv \delta_{ji} + f_{ji}$$

so that (6.1) transforms into :

$$(\nabla^2 + p^2)\psi(x) = -\frac{ia^2}{2}N^*(\varepsilon)e^{i\vec{p}\vec{r}}\left\{\frac{1}{pr^3} + f_{ji}\frac{x_j p_i}{pr^3}\cdot\right. \qquad (6.2)$$

$$\left.\cdot\frac{1}{\vec{p}\vec{r}+pr}\right\}(1 - e^{-i\vec{p}\vec{r}-ipr})$$

In using the solution of (6.2) in the matrix element no attempt will be made to incorporate further Coulomb effects of either muon or vector boson and this means that we need to calculate only the Fourier transform of the right-hand side of (6.2). This Fourier transform is given in Appendix C, where also some other Fourier transforms pertinent to our problem are listed. The result (Eqs. (C.9) and (C.10)) is exactly the same as what one would get by treating a diagram with two vertices of the type of the second term in the right-hand side of (4.2). We could have taken Dalitz'[12] results for the second order Born approximation for electron scattering were it not for the fact that our initial muon is a virtual one, not being on the mass shell.

7. THE CUT-OFF FUNCTION FOR HIGH MOMENTUM TRANSFER

The process under consideration requires high energy incident neutrinos and with the present accelerators most neutrinos will be at best in the threshold region for coherent W production. This means that the structure of the nucleus plays a very important if not decisive role. Of course, we are not able to treat the Coulomb potential of any realistic nuclear charge distribution consistently in the same way as we treated above the point charge Coulomb potential, and the only thing we can do is to make some reasonable guess on how to improve our formula for the higher momentum transfer where the nuclear structure is important.

Again, as in Section 6, we will try to correct our matrix element so that in the momentum transfer region just mentioned it is still correct in first order in a and as far as possible also in second order in a. Let us start by writing down the potential as used in Ref. [1], together with its Fourier transform (μ = constant related to the nuclear dimensions) :

$$V(r) = \frac{a}{r}\left\{(1 - e^{-\mu r}) - \frac{\mu}{2} e^{-\mu r}\right\} \tag{7.1}$$

$$V(Q) = \int d_3 r \, e^{i\vec{Q}\vec{r}} \, V(r) = \frac{4\pi a}{Q^2(1+\frac{Q^2}{\mu^2})^2}$$

The effect of the extra terms as compared with the point Coulomb field is, of course, to suppress the higher momentum transfer part. As a consequence, certain asymptotic properties, i.e., normalization constants and low momentum transfer behaviour, are not affected by the change from a point charge potential to the potential (7.1).

In order now to see how the potential appears in our matrix element we try to understand (5.9) in terms of a calculation with Feynman diagrams. Schematically we write :

$$M(\underline{q},\underline{p},\underline{k},t,s) = C\ V(x)(F_1 + F_2) + iF_3$$

with:

$$F_1 = \frac{a}{Q^2}\left(\frac{2\varepsilon}{D_2} - \frac{2\omega}{D_1}\right) l_\mu e_\mu - \frac{a}{Q^2}\frac{1}{D_1} A_{\mu\nu}\, l_\mu\, e_\nu$$

$$- \frac{1}{Q^2}\frac{iQ_\nu}{D_2}(\delta_{4\nu} l_\mu + \delta_{\mu\nu} l_4 - \delta_{\mu 4} l_\nu + \varepsilon_{k4\nu\mu} l_k) e_\mu$$

$$F_2 = \frac{\pi a^2}{2Q}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) l_\mu e_\mu - \frac{\pi a^2}{4pD_2}\Big\{\Theta(p - |\vec{Q}+\vec{p}|) +$$

$$- \frac{p}{|\vec{Q}+\vec{p}|}\,\Theta(|\vec{Q}+\vec{p}| - p)\Big\} l_\mu e_\mu$$

F_3 stands for all other terms, i.e., all terms containing $W(x)$ and the imaginary part of the second order muon correction. F_1 is precisely what one would get in a lowest order plane wave calculation (see Fig. 2) and we see the usual effect that the higher order diagrams involving the Coulomb potential have a relatively small effect, through the function $V(x)$ (whose lowest order contribution contains a^2 and is real). F_2 can be seen as arising from a second order calculation (see Fig. 3), leaving aside second order effects of the Coulomb potential $2a\omega/r^{-1}$. Clearly, in order to make the matrix element correct in first order in a we must make in F_1 the change

$$\frac{1}{Q^2} \rightarrow \frac{1}{Q^2\left(1+\frac{Q^2}{\mu^2}\right)^2}$$

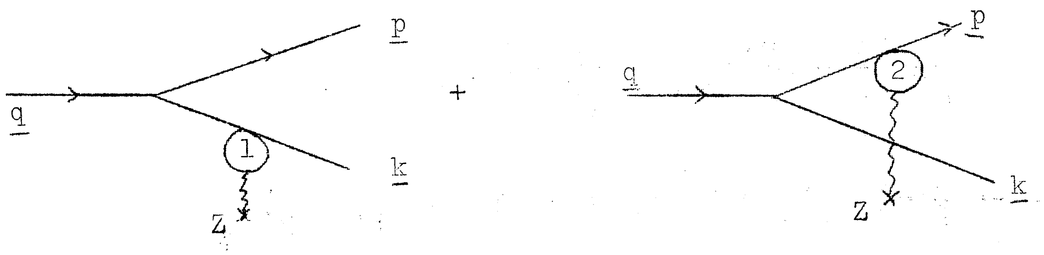

Fig. 2

The diagrams contributing to F_1

1 = Interaction given by right-hand side of Eq. (3.2) in first order in a.

2 = Interaction given by right-hand side of Eq. (4.2) in first order in a.

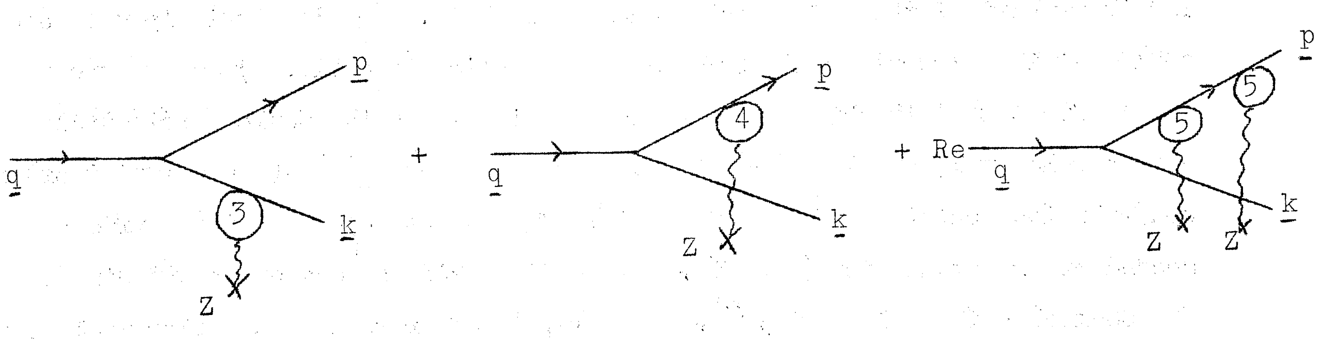

Fig. 3

The diagrams contributing to F_2

3 = a^2/r^2 interaction arising from the seventh term on the right-hand side of Eq. (3.2).

4 = a^2/r^2 interaction arising from the third term in round brackets on the right-hand side of Eq. (4.2).

Re = Real part of diagram whereby phase is fixed so that the first two diagrams are real.

5 = Interaction arising from the second term in round brackets on the right-hand side of Eq. (4.2).

In second order only the first term of F_2 (first and second diagrams of Fig. 3) can be calculated. For this part we need to change Q^{-1}, being the Fourier transform of r^{-2}, into the Fourier transform $V'(Q)$ of $V(r)^2$ which can be calculated :

$$V'(Q) = \frac{1}{Q} - \frac{4}{\pi Q} \operatorname{arctg} \frac{Q}{\mu} + \frac{2}{\pi Q} \operatorname{arctg} \frac{Q}{2\mu} + \frac{2\mu^3}{\pi(Q^2+4\mu^2)^2}$$

$$- \frac{2\mu}{\pi(Q^2+\mu^2)} + \frac{2\mu}{\pi(Q^2+4\mu^2)} \qquad (7.2)$$

This is a rather clumsy function and we replace it by :

$$\frac{1}{Q}\, g(Q) \equiv \frac{1}{Q} \left(\frac{1}{1+\frac{Q^2}{\mu^2}}\right)^3 \left(A + BQ + \frac{C}{(1+\frac{Q^2}{\mu^2})^6}\right) \qquad (7.3)$$

with $A = 0.244$, $B = 0.790$ and $C = 0.529$. A comparison of the two functions in the range of interest is given in Table 1.

We are not able to calculate the changes involved in the other second order terms. One can only say that for higher Q some cut-off should appear, and one can establish the limiting behaviour in Q for $Q \rightarrow \infty$. It seems reasonable to take (7.2) as a model for the other terms, and we take therefore as cut-off function for the second and higher order terms the same function which multiplies Q^{-1} in (7.2) or (7.3).

At this point we want to remark that the extra terms in the equations of motion due to the difference between $V(r)$ and ar^{-1} can be treated in Born approximation in the way we treated the r^{-2} term, with an analogous ansatz (the Eq. (3.12) above), which is seen to be correct if $\mu r \ll \vec{k}\vec{r} + kr$ or $\vec{p}\vec{r} + pr$, or $\mu \ll k$ or p and, of course, $kr \gg 1$, $pr \gg 1$.

However, at least one arrives at the same results as above and one is still left with the objection that such an approach is essentially inconsistent because one should treat the changes in the Coulomb term r^{-1} exactly. We think it therefore more realistic to introduce the cut-off as suggested by perturbation theory with respect to the coupling constant a.

T A B L E 1

Percentage deviation between
(7.2) and (7.3) with $Q' = Q/\mu$

Q'	$100(V'(Q')/\frac{1}{Q'}\, g(Q') - 1)$
0.1	8.09
0.2	-0.00
0.3	-1.37
0.4	-0.06
0.5	1.34
0.6	1.77
0.7	1.35
0.8	0.61
0.9	-0.07
1.0	-0.49
1.1	-0.64
1.2	-0.56
1.3	-0.35
1.4	-0.06
1.5	0.25
1.6	0.53
1.7	0.75
1.8	0.91
1.9	0.99
2.0	0.98
2.1	0.90
2.2	0.75
2.3	0.54
2.4	0.27
2.5	-0.05
2.6	-0.41
2.7	-0.80
2.8	-1.22
2.9	-1.66
3.0	-2.11
3.1	-2.58
3.2	-3.05
3.3	-3.52
3.4	-3.99
3.5	-4.46
3.6	-4.92
3.7	-5.37
3.8	-5.82
3.9	-6.26
4.0	-6.69

8. FINAL RESULT

For completeness we write down the whole matrix-element with the corrections indicated in Sections 6 and 7. We remark that $|\vec{p}+\vec{Q}| - p > o$, as $2\vec{Q}\vec{p}+Q^2 > o$

$$M(\underline{q},\underline{p},\underline{k},t,s) = \frac{8\sqrt{2}\,\pi^2 iga}{V^{3/2}\sqrt{\omega}} M(\omega)N(\varepsilon)\delta(E-\varepsilon-\omega)\left(\frac{D_1}{Q^2}\right)^{ia_1}\left(\frac{D_2}{Q_2}\right)^{-ia_2} \cdot$$

$$\cdot \frac{1}{Q^2(1+\frac{Q^2}{\mu^2})^2}\Bigg[V(y)\Big\{1^t_\mu e^s_\mu\Big(\frac{2\varepsilon}{D_2} - \frac{2\omega}{D_1} + \frac{\pi a}{2}\frac{Q}{D_1}h(Q) + \frac{\pi a}{2}\frac{Q}{D_2}h(Q)$$

$$- \frac{\pi a}{4D_2}\frac{Q^2}{\sqrt{Q^2+2\vec{Q}\vec{p}+p^2}}h(Q)\Big) - 1^t_\mu e^s_\nu \frac{A_{\mu\nu}}{D_1}$$

$$- (\delta_{4\nu}1^t_\mu + \delta_{\mu\nu}1^t_4 - \delta_{\mu 4}1^t_4 + \varepsilon_{k4\nu\mu}1^t_k)e^s_\mu \frac{iQ_\nu}{D_2} +$$

$$+ (\delta_{4\mu}1^t_\nu - \delta_{\nu\mu}1^t_4 + \varepsilon_{k\nu 4\mu}1^t_k)e^s_\mu \varepsilon_{ji\nu 4}\frac{p_j Q_i}{D_2}\frac{a\pi}{4} \cdot$$

$$\cdot \frac{Q^2}{Q^2p^2-(\vec{Q}\vec{p})^2}\Big(Q - \frac{Q^2+\vec{p}\vec{Q}}{\sqrt{p^2+2\vec{p}\vec{Q}+Q^2}}\Big)h(Q)\Big\} +$$

$$+ \frac{ia_1a_2}{a}W(x)h(Q)\Big\{1^t_\mu e^s_\mu\Big[2x\Big(\frac{k}{D_1} + \frac{p}{D_2}\Big) - 2Q^2\Big(\frac{k+p}{D_1D_2}\Big) +$$

$$+ \frac{\pi a k}{\omega}\Big(Q\frac{Q^2+\vec{p}\vec{Q}-\frac{p}{k}(\vec{k}\vec{Q})}{D_1D_2} - Q\frac{x}{D_1}\Big)$$

$$+ \frac{\pi a p}{\varepsilon}\Big(Q\frac{Q^2+\vec{k}\vec{Q}-\frac{k}{p}(\vec{p}\vec{Q})}{D_1D_2} - Q\frac{x}{D_2}\Big)\Big] + 1_\mu e_\nu \frac{B_{\mu\nu}}{D_1}$$

$$+ (\delta_{4i}1^t_\mu + \delta_{\mu i}1^t_4 - \delta_{\mu 4}1^t_i + \varepsilon_{k4i\mu}1^t_k)e^s_\mu\Big(\frac{-i}{2\varepsilon D_2}\Big) \cdot$$

$$\cdot \Big(2xpQ_i - \frac{2Qp^2}{D_1}\Big(Q_i + k_i - \frac{k}{p}p_i\Big)\Big)\Big\}$$

$$+ \frac{ia}{4}(\delta_{4\mu} l^{t}_{\nu} - \delta_{\nu\mu} l_{4} + \varepsilon_{\kappa\nu 4\mu} l_{\kappa}) e^{s}_{\mu} \varepsilon_{ji\nu 4} \frac{p_j Q_i}{D_2} \cdot h(Q)$$

$$\cdot \frac{Q^2}{Q^2 p^2 - (\vec{Q}\vec{p})^2} \left\{ \frac{Q^2 + \vec{p}\vec{Q}}{|\vec{Q}+\vec{p}|} \log \frac{\sqrt{Q^2 + 2\vec{Q}\vec{p} + p^2} - p}{\sqrt{Q^2 + 2\vec{Q}\vec{p} + p^2} + p} + \frac{\vec{Q}\vec{p}}{2p} \log\left(\frac{2\vec{p}\vec{Q} + Q^2}{Q^2}\right) \right\}$$

$$- \frac{ia}{4} h(Q) l^{t}_{\mu} e^{s}_{\mu} \frac{1}{D_2} \left\{ \frac{1}{p} \log\left(\frac{Q^2 + 2\vec{Q}\vec{p}}{Q^2 + 2\vec{Q}\vec{p} + p^2}\right) + \frac{1}{\sqrt{Q^2 + 2\vec{Q}\vec{p} + p^2}} \log \frac{\sqrt{Q^2 + 2\vec{Q}\vec{p} + p^2} + p}{\sqrt{Q^2 + 2\vec{Q}\vec{p} + p^2} - p} \right\} \Bigg] \qquad (8.1)$$

All notations are as in (5.9) with the additions :

$$h(Q) = \frac{1}{(1 + \frac{Q^2}{\mu^2})} \left\{ A + BQ + \frac{C}{(1 + \frac{Q^2}{\mu^2})^6} \right\}$$

$A = 0.244$, $B = 0.790$, $C = 0.529$

Finally, the relation between μ and nuclear dimension as given in Ref. [1] :

$$\mu = 3.440 \cdot A^{-1/3}\ 10^{13}\ \text{cm}^{-1}$$

$$= 679 \cdot A^{-1/3}\ \text{MeV}$$

where A is the mass number of the nucleus.

If we now consider (8.1) we see the following corrections to the lowest order calculation of Ref. [1] :

i) the Sommerfeld factor $M(\omega)N(\varepsilon)$,
ii) the four a^2 terms multiplied by $V(y)$,
iii) a factor $V(y)$,
iv) a group of terms multiplied by $W(x)$,
v) second order terms containing logarithms.

We remark that the imaginary parts of $V(x)$ and $W(x)$ are mostly very small (see limiting formulas (5.11)). However, due to the appearance of γ^5 in the l_μ this does not imply that there is no interference of the terms multiplied by $V(x)$ and the other terms.

ACKNOWLEDGEMENTS

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APPENDIX A

We show in this appendix that the second order diagrams belonging to (4.1) can be reformed into second order diagrams belonging to (4.2). The lepton part of this second order diagram (Fig.4) is:

$$a^2\bar{u}(p)\frac{\gamma^4}{Q_2^2}\,\frac{i\gamma p'-m}{p'^2+m^2}\,\frac{\gamma^4}{Q_1^2}\,\frac{i\gamma p''-m}{p''^2+m^2}\gamma^\mu(1+\gamma^5)u(q) \qquad (A.1)$$

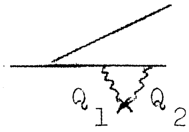

Fig.4

$$p' = p + Q_2,\quad p'' = p' + Q_1 = p + Q_1 + Q_2 \qquad Q = Q_1 + Q_2$$

There is one integration over a closed loop.
In this appendix p, p', p'', Q_1, Q_2, Q are four vectors. Of course the fourth component of the Q's is zero. The three-vector part of a vector p will be denoted by $\vec{p}$. $\bar{u}(p)$ satisfies:

$$\bar{u}(p)(i\gamma^\nu p_\nu + m) = o \qquad (A.2)$$

We take the $\vec{\gamma}\vec{p}$ part of $\gamma p'$ in (A.1), commutate it with γ^4 to the left and apply (A.2). We get:

$$a^2\bar{u}(p)\frac{\gamma^4}{Q_1^2}\,\frac{i\vec{\gamma}\vec{Q}_2+2i\gamma^4 p_4}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{i\gamma p''-m}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q) =$$

$$= a^2\bar{u}(p)\left(\frac{i\gamma^4(\vec{\gamma}\vec{Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{(i\gamma p''-m)}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q)$$

We repeat this manipulation with $\vec{\gamma}\vec{p}$ embodied in $\gamma p''$:

$$= a^2 u(p)\left(\frac{i\gamma^4(\vec{\gamma}\vec{Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^3}{Q_1^2}\,\frac{(i\vec{\gamma}\vec{Q}_1+i\vec{\gamma}\vec{Q}_2) + 2i\gamma^4 p_4}{Q^2 + 2\vec{Q}\vec{p}}\,\cdot$$

$$\gamma^\mu(1+\gamma^5)u(q) +$$

$$- a^2\bar{u}(p)\Big(\frac{[i\gamma^4(\vec{\gamma}\,\vec{Q}_2),i\vec{\gamma}\vec{p}]}{Q_2^2}\Big)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{1}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q)$$

$$+ a^2\bar{u}(p)\Big(\frac{[i\gamma^4 p_4,i\gamma^4(\vec{\gamma},\vec{Q}_2)]}{Q_2^2}\Big)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{1}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q)$$

$$= a^2\bar{u}(p)\Big(\frac{i\gamma^4(\vec{\gamma}\vec{Q}_2)}{Q_2^2}-\frac{2\varepsilon}{Q_2^2}\Big)\frac{1}{Q_2^2+2Q_2p}\Big(\frac{i\gamma^4\vec{\gamma}\vec{Q}_1}{Q_1^2}-\frac{2\varepsilon}{Q_1^2}\Big)\frac{\gamma^\mu(1+\gamma^5)}{Q^2+2Qp}u(q)$$

$$+ a^2\bar{u}(p)\Big(\frac{i\gamma^4(\vec{\gamma}\vec{Q}_2)}{Q_2^2}-\frac{2\varepsilon}{Q_2^2}\Big)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{i\vec{\gamma}\vec{Q}_2}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q)$$

$$+ a^2\bar{u}(p)\Big(\frac{+2\gamma^4\vec{Q}_2\vec{p}-2\vec{\gamma}\vec{Q}_2p_4}{Q_2^2}\Big)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{\gamma^\mu(1+\gamma^5)}{Q^2+2\vec{Q}\vec{p}}u(q)$$

$$= a^2\bar{u}(p)\Big(\frac{i\gamma^4(\vec{\gamma}\vec{Q}_2)}{Q_2^2}-\frac{2\varepsilon}{Q_2^2}\Big)\frac{1}{Q_2^2+2\vec{Q}_2\vec{p}}\Big(\frac{i\gamma^4\vec{\gamma}\vec{Q}_1}{Q_1^2}-\frac{2\varepsilon}{Q_1^2}\Big)\frac{\gamma^\mu(1+\gamma^5)}{Q^2+2\vec{Q}\vec{p}}u(q)$$

$$+ a^2\bar{u}(p)\frac{1}{Q_1^2Q_2^2}\,\frac{1}{Q^2+2\vec{Q}\vec{p}}\gamma^\mu(1+\gamma^5)u(q)$$

After performing the integration over the closed loop the factor $Q_1^{-2}Q_2^{-2}$ goes into $0.5\pi Q^{-1}$, i.e. the Fourier transform of r^{-2} and we have the desired result.

APPENDIX B

Equivalence of ansatz and derivative formula.

$$I = \int d_3k' \frac{e^{-i\vec{k}'\vec{r}-\lambda|\vec{k}'-\vec{k}|}}{(k'^2-k^2-io)^2} \frac{(\vec{k}'-\vec{k})_i}{(\vec{k}'-\vec{k})} {}_1F_1(-ia_1;1;ik'r+ikr) =$$

$$= \frac{e^{-i\vec{k}\vec{r}}}{2\pi i} \int^{o^+,1^+} dt\ t^{-ia_1-1}(t-1)^{ia_1} \int d_3q\ q_i \frac{e^{i\vec{q}\vec{r}(t-1)-\lambda q}}{(q^2-2\vec{q}\vec{k}-io)q^2} e^{(i\vec{k}\vec{r}+ikr)t} \tag{B.1}$$

We will set $\lambda = o$

Consider first the q-integration

$$I(q) = \int d_3q\ q_i \frac{e^{i\vec{q}\vec{r}(t-1)}}{(q^2-2\vec{q}\vec{k}-io)q^2} = \frac{1}{i(t-1)} \frac{\partial}{\partial x_i} \int d_3q \frac{e^{i\vec{q}\vec{r}(t-1)}}{(q^2-2\vec{q}\vec{k}-io)q^2}$$

We neglect difficulties around q = o as they are unreal in this case. Using Feynman's trick we get:

$$I(q) = \frac{1}{i(t-1)} \frac{\partial}{\partial x_i} \int_o^1 dz \int d_3q \frac{e^{i\vec{q}\vec{r}(t-1)}}{\{(\vec{q}-\vec{k}z)^2+(-ikz+o)^2\}^2}$$

Using (C.1) we get:

$$I(q) = \frac{\pi^2}{i(t-1)} \frac{\partial}{\partial x_i} \int_o^1 dz \frac{e^{(i\vec{k}\vec{r}+ikr)z(t-1)-or(t-1)}}{-ikz+o}$$

$$= \frac{\pi^2}{ik} \int_o^1 dz \left\{ -(k_i+k\frac{x_i}{r}) e^{(ikr+ikr)z(t-1)} \right\}$$

$$= \frac{-\pi^2}{ik} \frac{k_i+k\frac{x_i}{r}}{(i\vec{k}\vec{r}+ikr)(t-1)} \left\{ e^{(i\vec{k}\vec{r}+ikr)(t-1)} - 1 \right\}$$

There is no pole at t = 1, so $\lambda = o$ is justified. We insert now I(q) in (B.1), use the relation

$$t^{-ia_1-1}(t-1)^{ia_1-1} = \frac{1}{ia_1} \frac{d}{dt} t^{-ia_1}(t-1)^{ia_1}$$

and perform a partial integration (no "boundary" terms arise as the contour never goes through a cut):

$$I = \frac{e^{-i\vec{k}\vec{r}}}{2\pi i.} \frac{\pi^2}{ik.ia_1}(k_i+k\frac{x_i}{r}) \int^{o^+,1^+} dt\ t^{-ia_1}(t-1)^{ia_1}.\ e^{(i\vec{k}\vec{r}+ikr)t}$$

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$$= \frac{e^{-i\vec{k}\vec{r}}}{a_1 k} \cdot \frac{\pi^2}{i} \frac{\partial}{\partial x_i} {}_1F_1(-ia_1;1;i\vec{k}\vec{r}+ikr)$$

APPENDIX C

We list some 3 dim. Fourier transforms of interest to us. Everywhere $\beta = |\vec{\beta}|$, $\lambda > o$.

$$\int d_3r \quad e^{i\vec{\alpha}\vec{r}-\lambda r} = \frac{8\pi}{(\alpha^2+\lambda^2)^2}, \lim_{\lambda \to o} = \frac{2\pi^2}{\alpha^2}\delta(\alpha) \tag{C.1}$$

$$\int d_3r \; \frac{1}{r} \; e^{i\vec{\alpha}\vec{r}-\lambda r} = \frac{4\pi}{\alpha^2+\lambda^2} \tag{C.2}$$

$$\int d_3r \; \frac{1}{r^2} e^{i\vec{\alpha}\vec{r}-\lambda r} = \frac{4\pi}{\alpha} \operatorname{arc tg} \frac{\alpha}{\lambda} \tag{C.3}$$

$$\int d_3r \; \frac{x_i x_j}{r^4} e^{i\vec{\alpha}\vec{r}-\lambda r} = -\delta_{ij}\left\{\frac{2\pi\lambda}{\alpha^2} - 2\left(\frac{\pi}{\alpha} + \frac{\pi\lambda^2}{\alpha^3}\right)\operatorname{arc tg}\frac{\alpha}{\lambda}\right\}$$

$$+ \frac{\alpha_i\alpha_j}{\alpha^2}\left\{\frac{6\pi\lambda}{\alpha^2} - 2\left(\frac{\pi}{\alpha} + \frac{3\pi\lambda^2}{\alpha^3}\right)\operatorname{arc tg}\frac{\alpha}{\lambda}\right\} \tag{C.4}$$

$$\int d_3r \; \frac{x_i}{r^3} e^{i\vec{\alpha}\vec{r}-\lambda r} = \frac{4\pi i\alpha_i}{\alpha^2} - \frac{4\pi i\lambda\alpha_i}{\alpha^3}\operatorname{arc tg}\frac{\alpha}{\lambda} \tag{C.5}$$

$$\int d_3r \; \frac{x_i}{r^2} e^{i\vec{\alpha}\vec{r}-\lambda r} = -\frac{4\pi i\alpha_i\lambda}{\alpha^2(\lambda^2+\alpha^2)} + \frac{4\pi i\alpha_i}{\alpha^3}\operatorname{arc tg}\frac{\alpha}{\lambda} \tag{C.6}$$

$$\int d_3r \; \frac{x_i}{r} e^{i\vec{\alpha}\vec{r}- \; r} = \frac{8\pi i\alpha_i}{(\lambda^2+\alpha^2)^2} \tag{C.7}$$

$$\int d_3r \; \frac{x_i x_j}{r^3} e^{i\vec{\alpha}\vec{r}-\lambda r} = \delta_{ij}\left\{\frac{4\pi}{\alpha^2} - \frac{4\pi\lambda}{\alpha^3}\operatorname{arc tg}\frac{\alpha}{\lambda}\right\}$$

$$- \frac{\alpha_i\alpha_j}{\alpha^2}\left\{\frac{8\pi}{\alpha^2} + \frac{4\pi\lambda^2}{\alpha^2(\alpha^2+\lambda^2)} + \frac{12\pi\lambda}{\alpha^3}\operatorname{arc tg}\frac{\alpha}{\lambda}\right\} \tag{C.8}$$

Finally the Fourier transforms needed in Section 6.

$$\int d_3r \; \frac{1}{r^3}\left(1 - e^{-i\vec{p}\vec{r}-ipr}\right)e^{i\vec{\alpha}\vec{r}} = 2\pi\left\{\log\left|1-\beta^2\right| + \beta\log\left|\frac{1+\beta}{1-\beta}\right| + \right.$$

$$\left. + i\pi\Theta(\beta-1) + i\pi\beta\Theta(1-\beta)\right\} \tag{C.9}$$

with $\beta = \frac{P}{|\vec{\alpha}-\vec{p}|}$

$$f_{ij}P_j \int d_3r \frac{x_i}{r^3(\vec{p}\vec{r}+pr)}(1 - e^{-i\vec{p}\vec{r}-ipr})e^{i\vec{\alpha}\vec{r}} =$$

$$= 2\pi f_{ij}\alpha_i p_j \frac{1}{p^2\alpha^2-(\vec{p}\vec{\alpha})^2}\left\{\frac{p(\vec{\alpha}\vec{p}-\alpha^2)}{|\vec{p}-\vec{\alpha}|}\log\frac{p+|\vec{p}-\vec{\alpha}|}{p-|\vec{p}-\vec{\alpha}|-io} - \frac{(\vec{p}\vec{\alpha})}{2}\log(\frac{\alpha^2-2\vec{\alpha}\vec{p}}{\alpha^2})^2 + \right.$$

$$\left. + i\pi\alpha p \right\} \tag{C.10}$$

To evaluate the last integral we reject terms of the form $f_{ij}p_ip_j$ and make use of what is essentially Feynman's rule:

$$\frac{1}{\vec{p}\vec{r}+pr}(1 - e^{-i\vec{p}\vec{r}-ipr}) = i\int_0^1 dz\ e^{(-i\vec{p}\vec{r}-ipr)z}$$

We also added a factor $e^{-\lambda r}$, $\lambda \to o$ which is necessary to make the result unambigous in the region $p < |\vec{p}-\vec{\alpha}|$.

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$$\int d_3r \, e^{i\vec{\alpha}\vec{r} + ia\ln\left(\frac{\vec{p}\vec{r} + pr}{\vec{k}'\vec{r} + kr}\right)}$$

The logarithm has only angular dependence.

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