AN UNSUBTRACTED DISPERSION RELATION AND PION-NUCLEON FORWARD CHARGE-EXCHANGE SCATTERING

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ABSTRACT

An unsubtracted dispersion relation which relates the real part of the pion-nucleon charge-exchange forward amplitude to the total cross-sections for $\pi^{\pm}$-p scattering is written down. The validity of this equation is discussed and its relation to the once subtracted dispersion relations. It is used to calculate the real part of the forward charge-exchange amplitude : the results are presented and their comparison with experiment discussed.

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1. Introduction.

It is well known that dispersion relations for forward scattering amplitudes, and particularly those for the pion-nucleon forward scattering amplitudes, are uniquely suited for a direct comparison with experiment. This is a consequence of unitarity, in the form of the optical theorem, which relates the imaginary part of the forward amplitude to the total cross-section:

$$A_B^{\pm}(\omega_L) = \frac{q}{4\pi}\sigma^{\pm}(\omega_L). \qquad (1)$$

$A_B^{\pm}(\omega_L)$ are the imaginary parts of the forward scattering amplitudes in the centre of mass system, the superscripts $\pm$ refer to $\pi^{\pm}$-p scattering; $\sigma^{\pm}(\omega_L)$ are the total cross-sections; ω_L is the energy of the pion in the laboratory system and q is the magnitude of the pion momentum in the centre of mass system. We may use equation (1) to express the real part of the forward amplitude in terms of the total cross-section and the differential cross-section in the forward direction:

$$|D_B^{\pm}(\omega_L)| = \left\{ \frac{d\sigma^{\pm}(\omega_L)}{d\Omega}\bigg|_{\theta=0} - \left[\frac{q}{4\pi}\sigma^{\pm}(\omega_L)\right]^2 \right\}^{\frac{1}{2}} \qquad (2)$$

where $D_B^{\pm}(\omega_L)$ are the real parts of the forward amplitudes in the centre of mass system and $d\sigma^{\pm}(\omega_L)/d\Omega\big|_{\vartheta=0} \equiv |D_B^{\pm}(\omega_L)|^2 + |A_B^{\pm}(\omega_L)|^2$ is the differential cross-section for the forward direction in the centre of mass system. Because of equations (1) and (2) the dispersion relations for the pion nucleon forward amplitudes may be expressed, apart from the coupling constant and in the case of the usual once subtracted relations the scattering lengths, solely in terms of $\sigma^{\pm}(\omega_L)$ and $d\sigma^{\pm}(\omega_L)/d\Omega\big|_{\vartheta=0}$ at physical energies, which are directly measurable quantities experimentally.*)

*) These are the nuclear scattering cross-sections: at low energies a correction must be made for the Coulomb interaction.

The usual once subtracted dispersion relations for pion-nucleon forward scattering take the form

$$D_B^{\pm}(\omega_L) = \frac{1}{3}\left(1+\frac{\mu}{M}\right)\left[(a_1+2a_3) \mp \frac{\omega_L}{\mu}(a_1-a_3)\right]\frac{q}{q_L} \pm \frac{2f^2}{\mu^2}\frac{q\,q_L}{\omega_L \mp \mu^2/2M}$$
$$+ \frac{q\,q_L}{4\pi^2} P\int_{\mu}^{\infty}\frac{d\omega_L'}{q_L'}\left\{\frac{\sigma^{\pm}(\omega_L')}{\omega_L'-\omega_L} + \frac{\sigma^{\mp}(\omega_L')}{\omega_L'+\omega_L}\right\} \qquad (3)$$

where M and μ are the masses of the nucleon and the pion respectively; $q_L = \sqrt{\omega_L^2 - 1}$; a_1 and a_3 are the scattering lengths in the $T = \frac{1}{2}$ and $T = \frac{3}{2}$ states respectively and f^2 is the unrationalized renormalized vector coupling constant. P denotes a principal value Cauchy integral. Although early attempts to compare these with experimental data led to an apparent discrepancy[1], later comparisons using better data have shown that there is good agreement[2]. A recent analysis by Woolcock[3] finds a 98% probability that the relations (3) are in agreement with the experimental data up to 220 MeV. At higher energies there is good agreement with the data available, in particular with an accurate result by Foote et al.[4] for π^+-p scattering at 310 MeV.

An unsubtracted dispersion relation can be derived which relates $\frac{-1}{\sqrt{2}}\left[D_B^-(\omega_L) - D_B^+(\omega_L)\right]$ and $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$. The former of these quantities is equal to $Re.f^{C.E.}(0,\omega_L)$, the real part of the charge exchange forward amplitude, and is given in terms of the charge exchange forward differential cross-section in the centre of mass system by:

$$\left|\frac{-1}{\sqrt{2}}\left(D_B^-(\omega_L) - D_B^+(\omega_L)\right)\right| \equiv \left|Re.f(0,\omega_L)\right|$$
$$= \left\{\left.\frac{d\sigma^{C.E.}(\omega_L)}{d\Omega}\right|_{\theta=0} - \left[\frac{q}{4\pi}\frac{(\sigma^-(\omega_L)-\sigma^+(\omega_L))}{\sqrt{2}}\right]^2\right\}^{\frac{1}{2}} \qquad (4)$$

The purpose of this paper is to present calculated values for $\mathrm{Re}.f^{C.E.}(0,\omega_L)$ obtained from the unsubtracted dispersion relation.

In Section 2 the relationship between this unsubtracted dispersion relation and the subtracted relations (3) is discussed: it is shown that the unsubtracted relation can be derived from the subtracted ones by using a sum-rule, whose validity has been demonstrated within the experimental errors[5]. Although the verification of the sum-rule ensures the agreement with experiment of the unsubtracted relation as a consequence of the agreement of the subtracted relations, nevertheless a direct calculation of $\mathrm{Re}.f^{C.E.}(0,\omega_L)$ from the unsubtracted relation will have much smaller errors. This is discussed in Section 4 together with the results of the calculation. In Section 3 the sensitivity of the results to the values of $\sigma^{\pm}(\omega_L)$ at high energies is investigated and in Section 5 the comparison of the results with experimental data is discussed.

2. The Unsubtracted Dispersion Relation.

A sum-rule may be derived by taking the limit at infinite energy of a combination of the subtracted dispersion relations (3). This sum-rule may be written in the form:[6]

$$\lim_{\omega_L'\to\infty} \left\{ \left[D_B^-(\omega_L') - D_B^+(\omega_L') \right] / q' \right\} = \frac{2}{3\mu}\left(1+\frac{\mu}{M}\right)(a_1-a_3) - \frac{4f^2}{\mu^2} - \frac{1}{2\pi^2} P\int_{\mu}^{\infty} \frac{d\omega_L'}{q_L'} \left[\sigma^-(\omega_L') - \sigma^+(\omega_L')\right]. \qquad (5)$$

Although no 'proof' has been given that the left hand side of this equation should be zero this result follows from simple phenomenological arguments. In any model where the assumption is made of a finite interaction radius, the asymptotic behaviour of the forward amplitude is such that $D_B(\omega_L)$, $A_B(\omega_L)$ cannot increase more rapidly than $q \times$ (constant). The assumption that

3016/p the interaction becomes purely absorptive in the high energy limit, which follows from the relative phase space available to an infinite number of absorptive channels as opposed to one elastic channel, gives the result that $D_B(\omega_L)/A_B(\omega_L) \to 0$. It immediately follows from this argument that the term on the left hand side of equation (5) should vanish.

An evaluation of the integral with the experimental data for $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$, assuming $\sigma^-(\omega_L) = \sigma^+(\omega_L)$ above 2 GeV, showed that a value of zero for the right hand side of equation (5) is indeed consistent with the accepted values for f^2, a_1, a_3 [5]. If we now take the difference of the two equations (3), to obtain the **subtracted** dispersion relation for $\left[D_B^-(\omega_L)-D_B^+(\omega_L)\right]$, we may then substitute from equation (5) for $a_1 - a_3$. If we assume that the term on the left hand side of equation (5) vanishes we are left with the unsubtracted dispersion relation for $\left[D_B^-(\omega_L)-D_B^+(\omega_L)\right]$, which is:*)

$$\left[D_B^-(\omega_L) - D_B^+(\omega_L)\right] = \frac{4f^2 q}{\omega_L q_L} + \frac{\omega_L q}{2\pi^2 q_L} P \int_{\mu}^{\infty} \frac{d\omega_L'}{\omega_L'^2 - \omega_L^2}\, q_L' \left[\sigma^-(\omega_L') - \sigma^+(\omega_L')\right]. \tag{6}$$

We notice that when $\omega_L = \mu$ this is the sum-rule.

The convergence properties of the integral in equation (6) are the same as of that in the sum-rule: in the latter case comparison with the experimental data seems to justify the assumption that the integral converges and support the validity of a cut-off procedure in evaluating it.

3. Contributions from the Total Cross-Sections above 2 GeV.

The subtracted dispersion relations (3) are quite insensitive to the values of the total cross-sections at high energies. Calculations of these dispersion integrals have usually been made under the assumption that $\sigma^-(\omega_L) = \sigma^+(\omega_L) = 30\text{mb}$ above 2 GeV.

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*) In the Born term we have neglected the factor

$$\left(1-(\mu/2M)^2\right)/\left(1-(\mu^2/2\omega_L M)^2\right)$$

The present experimental situation is that $\sigma^-(\omega_L)$ decreases from about 30mb at 2 GeV to about 25 mb at 20 GeV and $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ decreases slowly from a value of about 2.5mb at 2 GeV to about 1.5mb at 20 GeV [7]. It may be shown that the effect of these deviations from constant values of $\sigma^\pm(\omega_L)$ on the values of $D_B^-(\omega_L)$ and $D_B^+(\omega_L)$, calculated for example for $\omega_L = 4$, is considerably smaller than the errors involved in the calculation. This is also true of the effect due to putting $\sigma^-(\omega_L) = \sigma^+(\omega_L) = 0$ above 20 GeV. So the good convergence properties of the once subtracted relations make them insensitive to the high energy data.

The effect of the results for the total cross-sections between 2 GeV and 20 GeV on the sum-rule (5) has been discussed by Hamilton [8]. Here again the calculation was originally made under the assumption that $\sigma^-(\omega_L) = \sigma^+(\omega_L) = 30$mb above 2 GeV. It was shown that in this case also the deviation of the total cross-sections from these values between 2 GeV and 20 GeV produced an effect which is less than the errors involved.

We have seen in Section 2 that the sum-rule (5) is simply the unsubtracted dispersion relation (6) evaluated at threshold ($\omega_L = \mu$). An evaluation of equation (6) at energies above threshold could be more sensitive to the high energy contributions because of the diminished role of the Born term and its associated error, and from the weighting factor $\omega_L q/q_L$ which enhances the integral term relative to $\left[D_B^-(\omega_L) - D_B^+(\omega_L)\right]$ as ω_L increases. Equation (6) has also the advantage that, as a dispersion relation, it can be evaluated for different values of ω_L at which experimental measurements of $d\sigma^{C.E.}/d\Omega|_{\vartheta=0}$ have been made, thus providing better statistics than the single evaluation of the sum-rule (5).

The experimental data are discussed in Section 5. It is seen that the measurement of $d\sigma^{C.E.}/d\Omega|_{\vartheta=0}$ at 371 MeV is well suited for a comparison with the theoretical result for $\left[D_B^-(\omega_L) - D_B^+(\omega_L)\right]$ and we shall see how sensitive this comparison is to the high energy data. The experimental result is

$$\left.\frac{d\sigma^{C.E.}}{d\Omega}\right|_{\theta=0} = 3.89 \pm 0.25 \text{ mb}$$

Taking the value $[\sigma^-(\omega_L) - \sigma^+(\omega_L)] = 13 \pm 2\text{mb}$ (Figure 1) gives the result $[D_B^-(\omega_L) - D_B^+(\omega_L)] = 0.611 \pm 0.02$, in units $\hbar = \mu = c = 1$. The calculated value (Figure 2) is 0.608. The approximate error in this value, based on the estimate given in equation (7) is

$$\frac{\omega_L q}{2\pi^2 q_L} \times 0.2 \simeq 0.025.$$

The contribution to the calculated result coming from the total cross-sections between 2 GeV and 20 GeV is $\doteqdot$ 0.06: this is larger than the combined errors from the calculated and experimental values.

The comparison at this energy can finally be used to estimate an upper limit on the value of the integral in equation (6) from 20 GeV to ∞. This inequality can be written approximately as

$$\frac{1}{2\pi^2}\int_{20\,\text{GeV}}^{\infty} \frac{d\omega_L'}{q_L'}\left[\sigma^-(\omega_L') - \sigma^+(\omega_L')\right] < 0.02$$

in units $\hbar = \mu = c = 1$.

4. Results and Discussion.

The values of $-\sqrt{2}\,\text{Re.}f^{C.E.}(0,\omega_L)$ $(\equiv [D_B^-(\omega_L) - D_B^+(\omega_L)])$ calculated from equation (6) are shown in Figure 2 together with the values of $-\sqrt{2}\,\text{Im.}f^{C.E.}(0,\omega_L)$ $(\equiv \frac{q}{4\pi}[\sigma^-(\omega_L) - \sigma^+(\omega_L)])$. The integral was evaluated using the values for $[\sigma^-(\omega_L) - \sigma^+(\omega_L)]$ shown in Figure 1. The data below 360 MeV are those listed by Noyes and Edwards[3] with the results of Kruse and Arnold[9] also taken into account. From 360 MeV to 1500 MeV the results of Brisson et al.[10] are used and these are continued
3016/p smoothly to give a constant value for $\sigma^-(\omega_L) - \sigma^+(\omega_L)$ of 2mb

between 2 GeV and 20 GeV [7]. A cut-off was inserted at 20 GeV. f^2 was assumed to have the value 0.081 ± 0.002. Figure 3 shows the values of $d\sigma^{C.E.}/d\Omega\Big|_{\vartheta=0} \equiv \left|\mathrm{Re}.f^{C.E.}(0,\omega_L)\right|^2 + \left|\mathrm{Im}.f^{C.E.}(0,\omega_L)\right|^2$ obtained from the results in Figure 2 [11].

It is clear from the discussion in Section 2 that these results are not independent of those obtained from the subtracted dispersion relations. However in using the unsubtracted dispersion relation we are automatically taking account of the cancellation, due to the sum-rule, between the large and increasing terms ($\sim\omega_L$), coming from the first two terms on the right hand side of equation (3), and the integral term which is also large. Thus a direct calculation of the unsubtracted dispersion relation, based on the assumption that the sum-rule holds exactly, avoids the large errors which arise in each of the terms on the right hand side of equation (3) for values of ω_L appreciably greater than 1.

Except near the threshold, when the errors in both of the terms on the right hand side of equation (6) are of a similar magnitude, the error in an evaluation of the unsubtracted dispersion relation comes chiefly from the principal value integral term. An estimate of this error is not easily made since this depends not only on the experimental errors in the values of the total cross-sections but also on the errors in the derivatives of these quantities with respect to ω_L. An estimated average value of this error is:

$$\delta\left\{ P\int_{\mu}^{\infty} \frac{d\omega_L'}{\omega_L'^2-\omega_L^2}\, q_L' \left[\sigma^-(\omega_L')-\sigma^+(\omega_L')\right]\right\} = 0.2 \qquad (7)$$

in units $\hbar = \mu = c = 1$; for values of ω_L between 1 and 12.5. Of course this error will not really be independent of ω_L: for some values of ω_L the derivative of $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ will be better determined than for others, giving a smaller error, and vice versa. However this is probably a reasonable average error. In estimating this value two sources of error have been ignored: (i) the contribution to the integral from energies above 20 GeV; (ii) possible deviations of $\left[\sigma^-(\omega_L)-\sigma^+(\omega_L)\right]$ from a smooth behaviour (Figure 1) between 1.5 GeV and 4 GeV, there being a lack of experimental information in this energy region. The first of these effects has already been discussed in Section 3: it would give a constant correction (independent of ω_L) over the energy region of interest. A possible additional 'structure' in $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ between 1.5 GeV and 4 GeV could introduce a correction which should be small for $\omega_L < 8$ but could show a quite strong energy dependence in the region of $\omega_L \simeq 12$. Evidence for either of these two effects could appear in a comparison of the results shown in Figures 2 and 3 with experimental data.

5. Comparison with Experimental Results.

It may be seen from equation (4) that the error in a calculation of $Re.f^{C.E.}(0,\omega_L)$ from the experimental values for $d\sigma^{C.E.}/d\Omega\big|_{\vartheta=0}$ and $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ is given by

$$\frac{\delta\left(Re\, f^{C.E.}(0,\omega_L)\right)}{Re.\, f^{C.E.}(0,\omega_L)} = \left[\frac{1}{2}\,\frac{\delta\left(\frac{d\sigma^{C.E.}}{d\Omega}\right)_{\theta=0}}{\left(\frac{d\sigma^{C.E.}}{d\Omega}\right)_{\theta=0}} + \frac{\delta\left[\sigma^-(\omega_L)-\sigma^+(\omega_L)\right]}{\left[\sigma^-(\omega_L)-\sigma^+(\omega_L)\right]}\,\frac{\left[Im.\, f^{C.E.}(0,\omega_L)\right]^2}{\frac{d\sigma^{C.E.}}{d\Omega}\Big|_{\theta=0}}\right]$$

$$\times\left(\frac{d\sigma^{C.E.}}{d\Omega}\right)_{\theta=0}\Big/\left[Re\, f^{C.E.}(0,\omega_L)\right]^2. \qquad (8)$$

It is clear from this that the most favourable energy regions in which to make an accurate determination of $Re.f^{C.E.}(0,\omega_L)$ will be those for which $\left(d\sigma^{C.E.}/d\Omega\big|_{\vartheta=0}\right)/\left[Re.f^{C.E.}(0,\omega_L)\right]^2$ is as near to 1 as possible. Since

$$\frac{d\sigma^{C.E.}}{d\Omega}\Big|_{\theta=0} = \left|f^{C.E.}(0,\omega_L)\right|^2$$

it follows that these will be the values of ω_L for which $\left|Im.f^{C.E.}(0,\omega_L)\right|^2/\left(d\sigma^{C.E.}/d\Omega\big|_{\vartheta=0}\right)$ has the smallest values. This is important, as the relative error in $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ is generally rather large.

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From these considerations we see that the energy regions suited for a comparison of the dispersion relation (6) with experiment are those in which $\left|Re.f^{C.E.}(0,\omega_L)\right| / \left|Im.f^{C.E.}(0,\omega_L)\right|$ is large and at any rate > 1. Figure 2 shows that this condition is satisfied in the energy regions 0 to 120 MeV, 250 to 600 MeV and 1050 to 1600 MeV [11].

The results of Caris et al.[12] between 230 MeV and 371 MeV are shown in Figures 2 and 3. In Figure 2 it is seen that these are in good agreement with the theoretical values for $Re.f^{C.E.}(0,\omega_L)$: the small errors attached to the experimental results show that these lie in one of the favourable energy regions for a determination of $Re.f^{C.E.}(0,\omega_L)$. That the experimental results at the lower energies lie consistently above the theoretical curve suggests a small modification in the shape of the 3-3 resonance shown in Figure 1, between 220 MeV and 350 MeV. This would also improve the agreement in Figure 3.

Preparations are at present being made at Saclay for an experiment which will measure $d\sigma^{C.E.}/d\Omega\big|_{\vartheta=0}$ in the forward direction at energies between 1000 MeV and 1400 MeV [13]. A comparison of these results with those of Figures 2 and 3 should show up any large deviations of $\left[\sigma^-(\omega_L) - \sigma^+(\omega_L)\right]$ from the values of Figure 1 between 1.5 GeV and 4 GeV.

It is clear that the energy region between 300 MeV and 600 MeV should provide the most precise comparison of the dispersion relation results with experiment. A verification of this dispersion relation provides, at the same time, a check on the validity of charge independence for pion-nucleon interactions at these energies [14]. The only results at present available in this region are at 317 MeV and 371 MeV: these results provide a reasonably stringent test of the unsubtracted dispersion relation.

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REFERENCES

1) G. Puppi and A. Stanghellini, Nuovo Cimento 5, 1305 (1957).

2) T.D. Spearman, Nuovo Cimento 15, 147 (1960).

3) W.S. Woolcock, to be published in the proceedings of the 1961 Aix-en-Provence International Conference on Elementary Particles.

4) J.H. Foote, O. Chamberlain, E.H. Rogers, H.M. Steiner, C.E. Wiegand and T. Ypsilantis, Phys. Rev. Letters 4, 30 (1960); and J.H. Foote, thesis (unpublished) UCRL-9191 (1960).

5) T.D. Spearman, Nuclear Physics 16, 402 (1960).

6) M.L. Goldberger, H.Miyazawa and R. Oehme, Phys, Rev. 99,986 (1955).

7) G. von Dardel, R. Mermod, P.A. Piroue, M. Vivargent, G. Weber and K. Winter, Phys. Rev. Letters 7, 127 (1961); and to be published; and S.J. Lindenbaum, W.A. Love, J.A. Niederer, S. Ozaki, J.J. Russell and L.C.L. Yuan, Phys. Rev. Letters 7, 352 (1961).

8) J. Hamilton, Proceedings of the CERN Conference on Theoretical Aspects of Very High Energy Phenomena, CERN, 1961.

9) U.E. Kruse and R.C. Arnold, Phys. Rev. 116, 1008 (1959).

10) J.C. Brisson, J.F. Detouef, P. Falk-Vairant, L.Van Rossum and G. Valladas, Nuovo Cimento 19, 210 (1961).

11) Similar results have been obtained by Cronin using the subtracted dispersion relations: J.W. Cronin, Phys. Rev. 118, 824 (1960).

12) J.C. Caris, R.W. Kenney, V. Perez-Mendez and W.A. Perkins, Phys. Rev. 121, 893 (1961).

13) Private communication from P. Sonderegger.

14) I am grateful to Professor J. Hamilton for this observation.

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Figure Captions

Figure 1: $\left[\sigma^{-}(\omega_L) - \sigma^{+}(\omega_L)\right]$

Figure 2: The calculated values for $\left[D_B^{-}(\omega_L) - D_B^{+}(\omega_L)\right]$, represented by the solid line; $\left[A_B^{-}(\omega_L) - A_B^{+}(\omega_L)\right]$ is represented by the broken line.

Figure 3: Calculated values for $\left.\frac{d\sigma}{d\Omega}^{C.E.}\right|_{\vartheta=0}$.

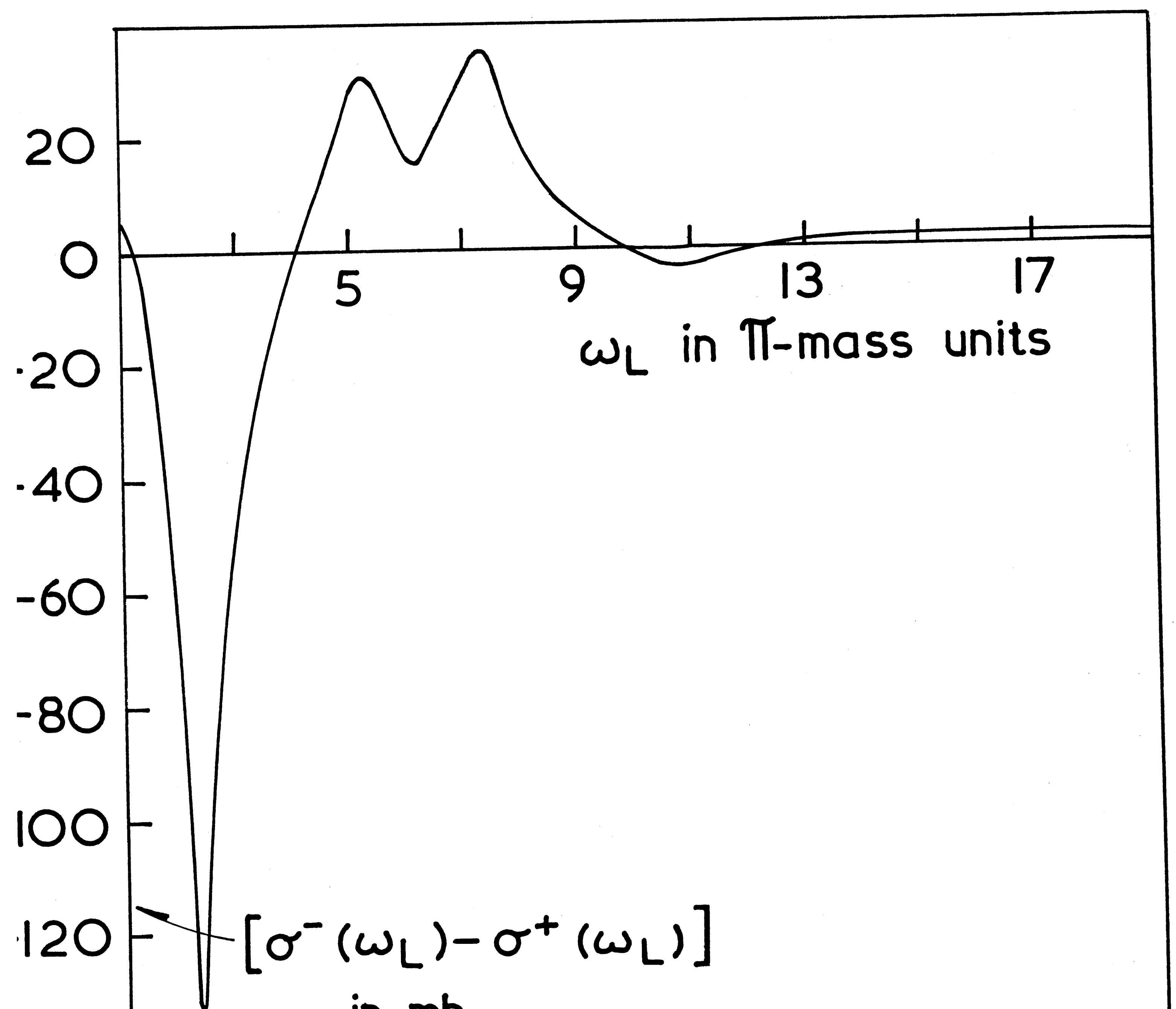

FIG. 1

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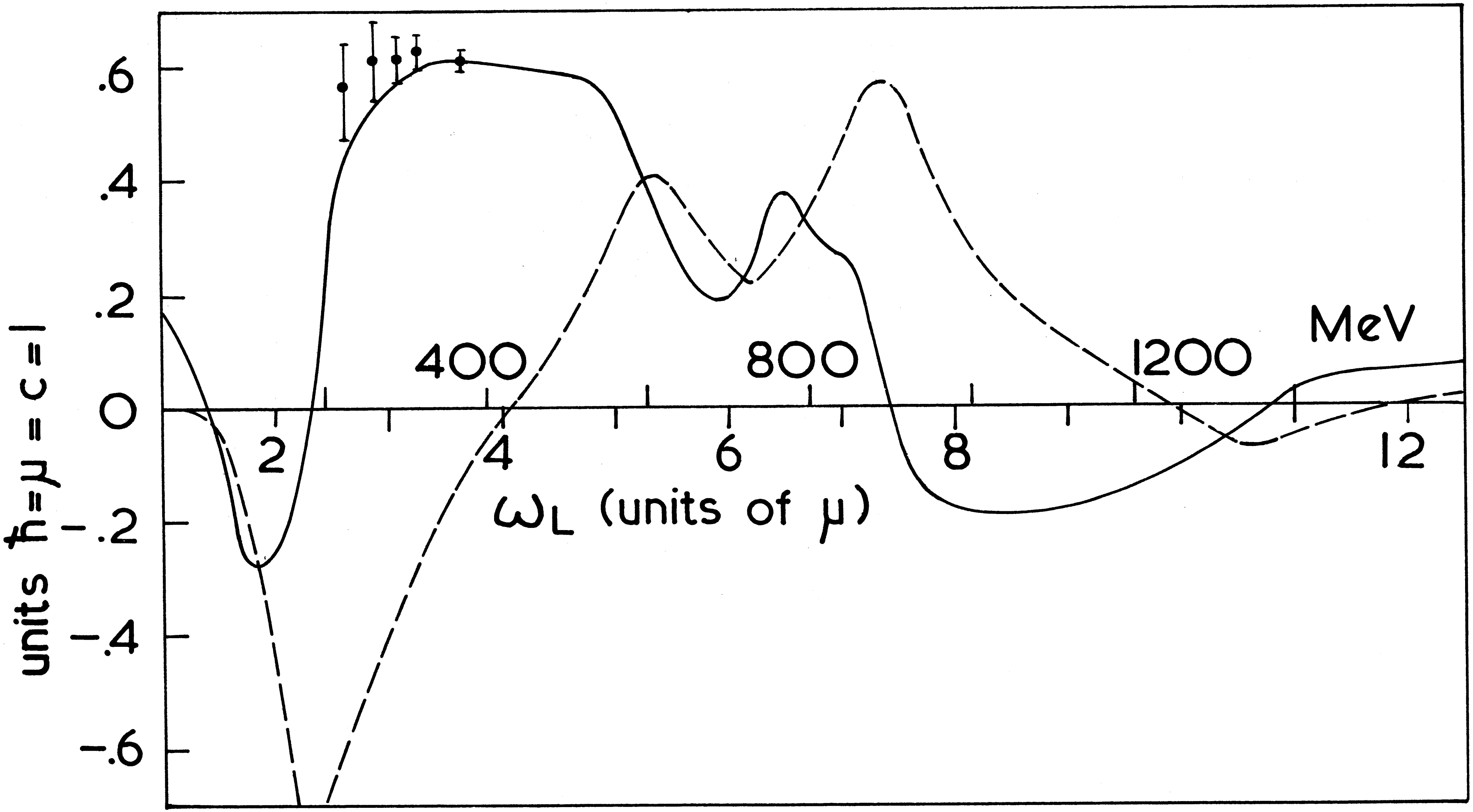
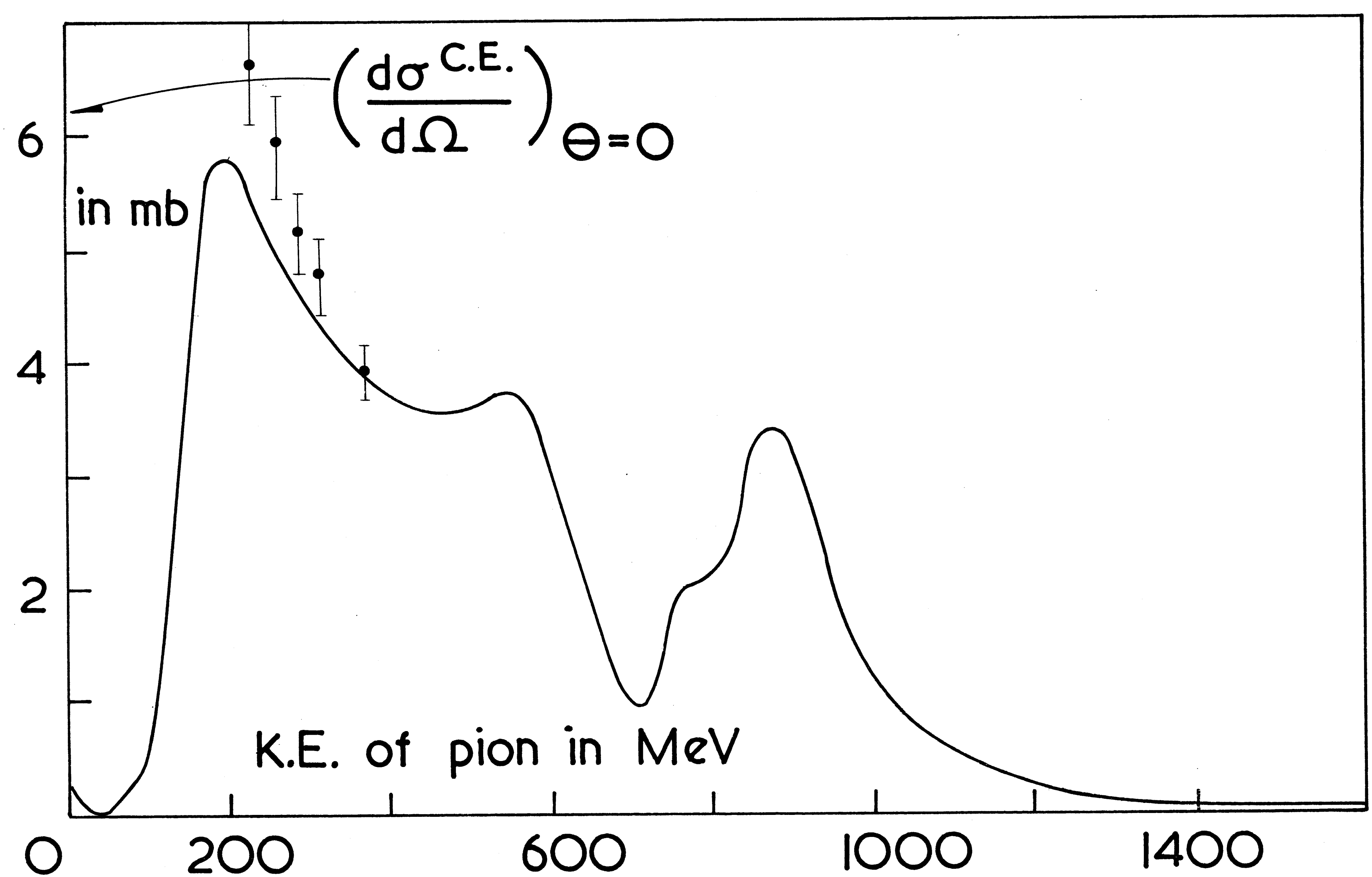


FIG. 2



$\left(\frac{d\sigma^{C.E.}}{d\Omega}\right)_{\theta=0}$
in mb
6
4
2
K.E. of pion in MeV
0
200
600
1000
1400

FIG. 3