ANALYTIC PROPERTIES OF KAON-PION SCATTERING AMPLITUDE

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A B S T R A C T

Some rigorous analytic properties of the kaon-pion scattering amplitude as a function of energy and momentum transfer are proved, following the method of Mandelstam. The partial wave amplitudes turn out to be analytic in the neighbourhood of the elastic cut and the analytic continuation into the second Riemann sheet is easily carried out.

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In this paper we derive some analytic properties of the kaon-pion scattering amplitude as a function of energy and momentum transfer in the axiomatic formulation by making use of the method developed by Mandelstam [1] for the scattering of the lightest equal mass particles. Though the kaon-pion scattering process is rather far from being of immediate physical interest, except for its appearance as an intermediate transition in some interesting processes [2], at least from a theoretical viewpoint it is still worthwhile to consider a possible extension of the Mandelstam procedure to unequal mass scattering.

In fact, $K\bar{K} \to \pi\pi$ is the only annihilation process, for which ordinary dispersion relation can be proved in the framework of the axiomatic field theory. First of all, we give the proof of the ordinary dispersion relations for $K\pi \to K\pi$ and $K\bar{K} \to \pi\pi$ processes and determine the Lehmann ellipses [3] associated with each of the three channels. Then, a function $\varphi(s,t)$ is constructed in terms of the Bergmann-Weil integrals which involve the absorptive parts of the transition amplitudes analytically continued to complex values of momentum transfer associated with each channel which are contained in the large Lehmann ellipse in an exactly same manner as in Ref. [1]. The function so defined can be shown to be analytic in a domain of the form $|(s-s_0)t(u-u_0)| < a$, except for the cuts along the real axis in each variable starting from the corresponding normal threshold. Using the proved dispersion relations in s and t channels, the function $\varphi(s,t)$ turns out to be identical with the scattering amplitude $T(s,t)$ in the physical regions of each channel, having chosen appropriate subtraction terms.

Extension of the analyticity domain through the unitarity is impossible in our case until we could analyse production processes, since the most stringent restriction on the validity of the dispersion relation comes from the inelastic region. However, the partial wave amplitudes turn out to be analytic in the neighbourhood of the elastic cut, and the analytic continuation into the second Riemann sheet is easily carried out.

1. Proof of dispersion relations and Lehmann ellipses

Let the four momenta of the incoming and outgoing kaons be k_1 and k_2 respectively, while those of the pions be p_1 and p_2. We neglect the spin and isospin of the particles, since its inclusion does not affect our analyticity considerations. As usual we introduce

$$s=(p_1+k_1)^2, \quad t=(p_1-p_2)^2, \quad u=(p_1-k_2)^2 \tag{1}$$

with

$$s+t+u=2m^2+2\mu^2, \quad p_i^2=\mu^2, \quad k_i^2=m^2$$

As it is well known, s is square of total energy in c.m.s. for the channel $K_1+\pi_1 \to K_2+\pi_2$ and $-t$ is the momentum transfer in this channel. For the channel $K_1\bar{K}_2 \to \pi_1\pi_2$, t is square of the total energy in c.m.s., while $-s$ is the momentum transfer.

a) $K_1+\pi_1 \to K_2+\pi_2$ (s-channel)

According to Lehmann [3] we can derive dispersion relation provided that

$$0<-t<4\left\{\frac{(s+m^2-\mu^2)^2}{4s}-m^2+\frac{32\mu^3(m+\mu)}{s-(m-\mu)^2}\right\}\equiv 4x_s^2 \tag{2}$$

for $s>(m+\mu)^2$, where we used $\langle 0|j_K(0)|n\rangle=0$ for $p_n^2<(m+2\mu)^2$. Unfortunately, the minimum value of the r.h.s. cannot be found algebraically, but its lower bound turns out to be $>27\mu^2$. At this point it should be noted that the r.h.s. takes its minimum value in the interval $m+2\mu<s<m+3\mu$,

and for this reason use of the unitarity below the inelastic threshold does not lead us to any extension of analyticity, until we have some means to analyse the three-particle intermediate state.

Now the large Lehmann ellipse associated with this channel, in which the corresponding absorptive part is regular, is given by

$$t = 2\,\underline{K}^2(s) - 2\,y_s \cos\alpha + 2i\sqrt{y_s^2 - \underline{K}^4(s)}\,\sin\alpha \quad , \tag{3}$$

with

$$K^2(s) \equiv \frac{(s+m^2-\mu^2)^2}{4s} - m^2$$

$$y_s \equiv 2\,x_s^2 - \underline{K}^2(s) \,. \tag{4}$$

As long as the value of t is within the large Lehmann ellipse given by Ref. [3], the absorptive part can be found by analytic continuation from the physical region by using the partial wave expansion. As to the u-channel $K_1 + \pi_2 \rightarrow K_2 + \pi_1$, (2) and (3) turn out to be also true, except for the interchange $s \leftrightarrow u$ due to the crossing symmetry.

b) $\underline{K_1 + \overline{K}_2 \rightarrow \pi_1 + \pi_2}$

s being momentum transfer in this channel, dispersion relation can be proved if [4].

$$0 < -s < 2\left(x_1 x_2 + \sqrt{x_1^2 - \underline{q}_1^2}\,\sqrt{x_2^2 - \underline{q}_2^2}\,\right) + \underline{q}_1^2 + \underline{q}_2^2 \quad , \tag{5}$$

with

$$x_1^2 \equiv \frac{t}{4} - m^2 + \frac{16(m+\mu)^2\mu^2}{t}, \quad x_2^2 \equiv \frac{t}{4} - \mu^2 + \frac{64\mu^4}{t}$$

$$\underset{\sim}{q_1}^2 = \frac{t}{4} - m^2, \qquad \underset{\sim}{q_2}^2 = \frac{t}{4} - \mu^2. \tag{6}$$

Though we cannot find the exact minimum of the r.h.s. of (5), its lower bound can be easily found as follows. First of all, we have

$$\mathrm{Min}\ x_1^2(t) = 4(m+\mu)\mu - m^2 \qquad \text{at} \quad t = 8(m+\mu)\mu$$

$$\mathrm{Min}\ x_2^2(t) = 17\mu^2 \qquad \text{at} \quad t = 16\mu^2. \tag{7}$$

(From this it is obvious that dispersion relation can be derived for $K+\bar{K} \rightarrow K+\bar{K}$ process up to the momentum transfer $4\Delta^2 < 4(m+\mu)\mu$.) Thus we have

$$\begin{aligned}(\text{r.h.s.})\ \text{of}\ (5) &> 2\sqrt{(\mathrm{Min}\ x_1^2)(\mathrm{Min}\ x_2^2)} \\ &\quad + 2\,\mathrm{Min}\left(\sqrt{x_1^2-\underset{\sim}{q_1}^2}\sqrt{x_2^2-\underset{\sim}{q_2}^2} + \underset{\sim}{q_1}^2 + \underset{\sim}{q_2}^2\right) \\ &= 2\sqrt{4(m+\mu)\mu - m^2}\sqrt{17\mu^2} + 8\sqrt{2(m+\mu)\mu} - m^2 - \mu^2.\end{aligned} \tag{8}$$

In the physical annihilation process the momentum transfer can never vanish and we must show that the value given by (8) is well above the momentum transfer at the physical threshold, i.e., at $t=4m^2$. It is in fact true, since

$$-s = m^2 - \mu^2 \equiv -s_0 \qquad \text{at} \quad t = 4m^2. \tag{9}$$

In the proof of dispersion relation we must also require that the absorptive part for the crossing process (in this case s-channel) can be analytically continued with respect to the external mass variable up to the physical mass under the same restriction on the momentum transfer. As it was proved in a) the absorptive part for the s-channel allows the required continuation, so long as the momentum transfer is $27\,\mu^2$, which is well above $-s_0$.

The large Lehmann ellipse for the absorptive part in this channel turns out to be

$$s = \underset{\sim}{q_1}^2 + \underset{\sim}{q_2}^2 - 2y_t \cos\alpha - 2i\sqrt{y_t^2 - \underset{\sim}{q_1}^2 \underset{\sim}{q_2}^2}\,\sin\alpha \tag{10}$$

with

$$y_t \equiv x_1 x_2 + \sqrt{x_1^2 - \underset{\sim}{q_1}^2}\,\sqrt{x_2^2 - \underset{\sim}{q_2}^2} \; .$$

It should be mentioned at this point that the absorptive part in this channel can be found through partial wave expansion for any values of s in the ellipse (10) only if $t > 4m^2$. As to the interval $4\mu^2 < t < 4m^2$, however, we do not have direct means for its physical interpretation, though it is a <u>mathematically</u> well defined quantity. At least for the elastic region, i.e., $4\mu^2 < t < 9\mu^2$, it may be possible to give a certain physical meaning through the procedure of Streater [5].

Summarizing this section, we have now

$$T_s(s,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_1(s',t)}{s'-s}\,ds' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_2(u',t)}{u'-u}\,du' \tag{11a}$$

$$T_t(s,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_2(s,t'(u'))}{u'-u}\,du' + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{A_3(s,t')}{t'-t}\,dt' \tag{11b}$$

with

$$\begin{aligned} A_1(s,t) &\sim \sum_n \langle K(k_1)\pi(p_1)|n\rangle\langle n|K(k_2)\pi(p_2)\rangle \\ A_2(s,t) &\sim \sum_n \langle K(k_1)\pi(-p_2)|n\rangle\langle n|K(k_2)\pi(-p_1)\rangle \\ A_3(s,t) &\sim \sum_n \langle K(k_1)\bar{K}(-k_2)|n\rangle\langle n|\pi(-p_1)\pi(p_2)\rangle, \end{aligned} \tag{12}$$

where we denoted the transition amplitudes by $T_s(s,t)$ and $T_t(s,t)$ for each channel. Besides, $A_i(s,t)$ (i=s or t) is analytic in the corresponding large Lehmann ellipse given by (3) or (10), which will be denoted by $\mathcal{L}_s$ and $\mathcal{L}_t$, respectively. The crossing symmetry reads in our case as

$$A_1(s,t) = A_2(s,t) = A_1(u,t) \tag{13}$$

2. Construction of $\varphi(s,t)$

Following Mandelstam [1] we introduce the function

$$2\pi^2 i\,\varphi(s,t) = \int_{(m+\mu)^2}^{\infty} ds' \int_{\partial\mathcal{M}} dt' \, \frac{-(s'-s_0)(t+t'-u-u'+2u_0)\,A_1(s',t')}{2(s-s')\{(s'-s_0)t'(u'-u_0)-(s-s_0)t(u-u_0)\}}$$

$$+ \int_{(m+\mu)^2}^{\infty} du' \int_{\partial\mathcal{M}} dt' \, \frac{-(u'-u_0)(s+s'-t-t'-2s_0)\,A_2(u',t')}{2(u'-u)\{(s'-s_0)t'(u'-u_0)-(s-s_0)t(u-u_0)\}} \qquad (14)$$

$$+ \int_{4\mu^2}^{\infty} dt' \int_{\partial\mathcal{M}} ds' \, \frac{-t'(s+s'-u-u')\,A_3(s',t')}{2(t'-t)\{(s'-s_0)t'(u'-u_0)-(s-s_0)t(u-u_0)\}} \, ,$$

where a will be chosen such that the region defined by

$$\mathcal{M} \equiv \{ s, t, u \; : \; |(s-s_0)\,t\,(u-u_0)| < a \} \qquad (15)$$

($u_0 = s_0$ was introduced only to put the crossing symmetry $s \longleftrightarrow u$ into evidence) lies always in the corresponding $\mathcal{L}$, if one of s, t and u is real and bigger than the associated normal threshold, which is $(m+\mu)^2$ for s- and u-channel and $4\mu^2$ for t-channel. Furthermore, we require that for some physical value of s, t or u, $\mathcal{M}$ is contained even in the small Lehmann ellipse associated with that channel.

Now we proceed to determine the maximum value of a. As we have no longer the complete symmetry among s, t and u as in the case of equal mass scattering, we shall again consider s- and t-channels separately.

a) s-channel

For a fixed $s > (m+\mu)^2$ the domain $\mathfrak{M}$ is characterized by

$$t = \frac{2m^2+2\mu^2-s-u_0}{2} \pm \sqrt{\frac{(2m^2+2\mu^2-s-u_0)^2}{4}+\frac{\alpha}{s-s_0}}\,, \qquad (16)$$

$$|\alpha| < a$$

and we require that $\mathfrak{M} \subset \mathcal{L}_s$. From (3) we know that $\mathcal{L}_s$ intersects the real axis of the t-plane at

$$t_1 = -\left\{ 4\underset{\sim}{K}^2(s) + \frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2} \right\}$$

$$t_2 = \frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2}\,. \qquad (17)$$

It is then necessary that these points lie outside $\mathfrak{M}_s$, namely

$$(s-s_0)\left\{ 4\underset{\sim}{K}^2(s) + \frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2} \right\}\left\{ \frac{(m^2-\mu^2)^2}{s} + \frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2} - u_0 \right\} > a\,, \qquad (18a)$$

and

$$(s-s_0)\,\frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2}\left\{ s-2m^2-2\mu^2+\frac{128\,(m+\mu)\,\mu^3}{s-(m-\mu)^2}+u_0 \right\} > a\,. \qquad (18b)$$

For the equal mass case these two inequalities turn out to be identical, however, it is not the case for the unequal mass case. But we have

$$s-2m^2-2\mu^2+u_0 < \frac{(s+m^2-\mu^2)^2}{s} - 4m^2$$

since

$$u_0 = \mu^2 - m^2 < 0$$

thus it suffices to consider (18b) only. Majorizing the product of the first two factors by $128(m+\mu)\mu^3$, one gets immediately [see (2)]

$$a < \mathrm{Min}(\text{l. h. s. of (18b)}) = 128(m+\mu)\mu^3 - 27\mu^2 \approx 9500\,\mu^6 . \tag{19}$$

Now we proceed to show that under the condition (19) $\mathcal{M}$ is in fact contained in $\mathcal{L}_s$, i.e., $\partial\mathcal{L}_s \cap \partial\mathcal{M} = 0$. To this end, we must prove that [see (3) and (16)]

$$\left| \left(2y_s\cos\alpha + 2i\sqrt{y_s^2 - \underset{\sim}{K}^4(s)}\,\sin\alpha - 2\underset{\sim}{K}^2(s) + \frac{s-2m^2-2\mu^2+u_0}{2} \right)^2 - \frac{(s-2m^2-2\mu^2+u_0)^2}{4} \right| > \frac{a}{|s-s_0|} \tag{20}$$

$$0 \le \alpha \le 2\pi .$$

Should it not be the case, we would have

$$\left(2y_s\cos\alpha + 2i\sqrt{y_s^2 - \underset{\sim}{K}^4(s)}\,\sin\alpha - 2\underset{\sim}{K}^2(s) + \frac{s-2m^2-2\mu^2+u_0}{2} \right)^2 - \frac{(s-2m^2-2\mu^2+u_0)^2}{4} = \frac{a\,e^{i\theta}}{s-s_0} \tag{21}$$

$$0 \le \theta \le 2\pi .$$

Then it would clearly imply that

$$\{\mathrm{Re}(\text{l. h. s.})\}^2 + \{\mathrm{Im}(\text{l. h. s.})\}^2 = \frac{a^2}{(s-s_0)^2} . \tag{22}$$

It is easy to see that the l.h.s. of (22) takes its minimum value at $\alpha = 0$. But for $\alpha = 0$, (22) reads

$$\left(2y_s - 2\underset{\sim}{K}^2(s) + \frac{s-2m^2-2\mu^2+u_0}{2} \right)^2 - \frac{(s-2m^2-2\mu^2+u_0)^2}{4} = \frac{a}{s-s_0} \tag{23}$$

and it gives a contradiction to (19).

b) t-channel

For a fixed real $t > 4\mu^2$, $\mathfrak{M}$ is now given by

$$s = \frac{2m^2+2\mu^2-t}{2} \pm \sqrt{\left(\frac{2m^2+2\mu^2-t-2u_0}{2}\right)^2 + \frac{\alpha}{t}}\,, \quad |\alpha| < a\,. \tag{24}$$

$\mathcal{L}_t$ intersects the real axis in the s-plane at

$$s_{1,2} = -\frac{t}{2} + m^2 + \mu^2 \mp \sqrt{\left(\frac{t}{4} - m^2 + \frac{16(m+\mu)^2\mu^2}{t}\right)\left(\frac{t}{4} - \mu^2 + \frac{64\mu^4}{t}\right)} \mp \frac{64(m+\mu)\mu^3}{t}\,. \tag{25}$$

In order that $\mathfrak{M} \subset \mathcal{L}_t$, we must then require that these points be lying outside $\mathfrak{M}$, namely

$$\left|(s_{1,2} - s_0)\, t\, (2m^2 + 2\mu^2 - s_{1,2} - t - u_0)\right| > a\,. \tag{26}$$

In virtue of the crossing symmetry $s \leftrightarrow u$ either sign leads us to the same inequality. As in the case of s-channel, we can easily find the lower bound to the l.h.s. of this inequality, namely

$$\begin{aligned} \text{l.h.s.} > \; & \mathrm{Min}\left\{ 2\sqrt{\left(\frac{t^2}{4} - m^2 t + 16(m+\mu)^2\mu^2\right)\left(\frac{t^2}{4} - \mu^2 + 64\mu^4\right)} \right. \\ & \left. + 64(m+\mu)\mu^3 - \frac{t^2}{2} + (m^2+\mu^2)t - u_0 t \right\} \\ & \cdot \mathrm{Min}\left\{ 2\sqrt{\left(\frac{t}{4} - m^2 + \frac{16(m+\mu)^2\mu^2}{t}\right)\left(\frac{t}{4} - \mu^2 + \frac{64}{t}\mu^4\right)} \right. \\ & \left. + \frac{64(m+\mu)\mu^3}{t} + \frac{t}{2} - m^2 - \mu^2 + s_0 \right\} \\ > \; & \left(4m^2\mu^2 + 16(m+\mu)^2\mu^2 + 64\mu^4 - (m^2+\mu^2)^2 + 64(m+\mu)\mu^3 - u_0 t\right) \\ & \cdot \mathrm{Min}\,(\text{2nd term}) \approx 4524\,\mu^6\,. \end{aligned} \tag{27}$$

It is also seen immediately as above that we have in fact $\mathcal{L}_t \supset \mathfrak{M}$, under the condition (27). Taking the smaller among (19) and (27) we obtain

$$\text{lower bound to } a = 4524\,\mu^6 . \tag{28}$$

It is clear from the definition of the domain $\mathfrak{M}$ that it is included in the small Lehmann ellipse associated with the corresponding channel, for some physical value of s, t or u. For instance, take sufficiently large value of s, then from $t = a/(s-s_0)(u-u_0)$, t becomes sufficiently small to guarantee $\mathfrak{M}$ to be within the small Lehmann ellipse.

3. Analyticity of $\varphi(s,t)$

From the definition (14) it is evident that the singularities of $\varphi(s,t)$ occur if and only if any factor of the denominators vanishes, i.e., there is no singularity in $\mathfrak{M}$, except for the real cuts starting from the corresponding normal threshold in each channel, namely

$$s > (m+\mu)^2, \qquad t > 4\mu^2, \qquad u > (m+\mu)^2.$$

In particular, for t fixed at t = 0, $\varphi(s,0)$ is analytic in the entire s-plane except for the real cut. It will also be the case with its finite derivative with respect to t, i.e., $\frac{\partial^n}{\partial t^n}\varphi(s,t=0)$. Furthermore, the discontinuity across the real axis turns out to be

$$\frac{1}{2i}\frac{\partial^n}{\partial t^n}\left\{\varphi(s+i\epsilon,t=0)-\varphi(s-i\epsilon,t=0)\right\} = \frac{\partial^n}{\partial t^n}A_1(s,t=0) \equiv A_1^{(0,n)}(s,0)$$
$$\text{for} \quad s > (m+\mu)^2$$

$$\frac{1}{2i}\frac{\partial^n}{\partial t^n}\{\varphi(u+i\epsilon, t=0) - \varphi(u-i\epsilon, t=0)\} = \frac{\partial^n}{\partial t^n}A_2(u, t=0) \equiv A_2^{(0,n)}(u, 0)$$

$$\text{for} \quad u > (m+\mu)^2.$$

Thus we arrive at the dispersion relations

$$\varphi^{(0,n)}(s,0) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{A_1^{(0,n)}(s',0)}{s'-s}ds' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{A_2^{(0,n)}(u',0)}{u'-u}du' \qquad (29a)$$

if we suppress the subtraction terms. Though the treatment of the subtraction terms is not trivial, the Mandelstam procedure can be straightforwardly translated into an unequal mass case and we shall skip its discussion. Comparing (29a) with (11a), we obtain

$$\varphi^{(0,n)}(s, t=0) = T_s^{(0,n)}(s, t=0).$$

As to the t-channel, $\varphi(s,t)$ is analytic in the entire t-plane except for the real cut, since then $|(s-s_0)t(u-u_0)| < a$ is satisfied by any values of t at s = s Therefore, $\varphi(s,t)$ and $\frac{\partial^n}{\partial s^n}(s,t)$ satisfy the corresponding dispersion relations. The discontinuity across the real t-axis now turns out to be

$$\frac{1}{2i}\frac{\partial^n}{\partial s^n}\{\varphi(s_0, t+i\epsilon) - \varphi(s_0, t-i\epsilon)\} = A_3^{(n,0)}(s_0, t)$$

$$\text{for} \quad t > 4\mu^2$$

$$\frac{1}{2i}\frac{\partial^n}{\partial s^n}\{\varphi(s_0, t+i\epsilon) - \varphi(s_0, t-i\epsilon)\} = A_2^{(n,0)}(s_0, t)$$

$$\text{for} \quad t < (m-\mu)^2 - s$$

$$(\text{i.e.} \quad u > (m+\mu)^2).$$

With it we proved the dispersion relations

$$\varphi^{(n,0)}(s_0,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_2^{(n,0)}(s,t'(u'))}{u'-u}\,du' + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{A_3^{(n,0)}(s,t')}{t'-t}\,dt'. \tag{29b}$$

Comparing it with (11b) we have

$$\varphi^{(n,0)}(s_0,t) = T_t^{(n,0)}(s_0,t).$$

Summarizing our results, there exists a function $\varphi(s,t)$ which is analytic in $\mathcal{M}$, except for the real cuts starting from the normal thresholds of each channel, and

$$\varphi(s,t) = T_s(s,t)$$

around $t = 0$ and

$$\varphi(s,t) = T_t(s,t)$$

around $s = s_0$.

4. Analyticity of the partial wave

The partial wave amplitude $T_\ell(s)$ is defined by

$$T_\ell(s) = \frac{1}{2}\int_{-1}^{1} dz\; T_s(s, -2\underset{\sim}{k}^2(s)(1-z))\, P_\ell(z) \tag{30}$$

with

$$z = \cos\theta\,, \quad t = -2\underset{\sim}{K}^2(1-\cos\theta)\,, \quad u = \frac{(m^2-\mu^2)^2}{s} - 2\underset{\sim}{K}^2(1+\cos\theta).$$

Defining

$$\varphi_\ell(s) = \frac{1}{2}\int_{-1}^{1} dz \quad \varphi(s,t)\, P_\ell(z)$$

we obtain

$$\lim_{\epsilon\to 0} \varphi_\ell(s+i\epsilon) = T_\ell(s)\,. \tag{31}$$

Since $\varphi(s,t)$ is analytic in $\mathfrak{M}$ except for the real cut, $\varphi_\ell(s)$ turns out to be analytic in the region defined by

$$\overline{\mathfrak{M}} = \left\{ s : \left|(s-s_0)\left(\frac{(m^2-\mu^2)^2}{s} - 2\underset{\sim}{K}^2(1+\cos\theta) + m^2-\mu^2\right) 2\underset{\sim}{K}^2(s)(1-\cos\theta)\right| < a \right\} \tag{32}$$

$$(\, 0 \leq \theta \leq \pi \,)$$

except for the above-mentioned cut and the well-known kinematic cuts [6], which are given by

$$\begin{aligned} &(i) \quad (m+\mu)^2 \leq s \leq \infty \\ &(ii) \quad (m-\mu)^2 \geq s \geq -\infty \\ &(iii) \quad |s| = m^2-\mu^2 \,. \end{aligned} \tag{33}$$

We now proceed to show that the domain (27) includes the elastic region, i.e., $(m+\mu)^2 < s < (m+2\mu)^2$. Clearly, the point $s = (m+\mu)^2$ is included in $\overline{\mathfrak{M}}$. Besides,

$$4(s-s_0)\left\{\frac{(m^2-\mu^2)^2}{s} + m^2 - \mu^2\right\} \underset{\sim}{K}^2(s)$$

is monotonously increasing in s, since

$$\frac{\partial}{\partial s}\left\{(s-s_0)\left(\frac{(m^2-\mu^2)^2}{s}+m^2-\mu^2\right)\right\} > 0$$

Therefore, it suffices now to prove

$$4\left\{(m+2\mu)^2+m^2-\mu^2\right\}\left\{\frac{(m^2-\mu^2)^2}{(m+2\mu)^2}+m^2-\mu^2\right\}\underset{\sim}{K}^2(s=m+2\mu) < a .$$

Putting the physical mass value $(m = 3.5\mu)$, we have

$$l.h.s \simeq 2092\,\mu^6 < a .$$

With it we proved that the neighbourhood of the elastic region is included in $\mathfrak{M}$, and that $\varphi_\ell(s)$ is in fact analytic around the interval $(m+\mu)^2 \leq s \leq (m+2\mu)^2$.

It is now straightforward to use this analyticity together with the unitarity below the production threshold to carry out the analytic continuation of $T_1(s)$ to the second Riemann sheet around the zero kinetic energy which was performed by Zimmermann [6] for equal mass particle scattering. Following Zimmermann we introduce the two-particle irreducible amplitude

$$\varphi_\ell(s) = \frac{\mathcal{f}_\ell(s)}{1+\frac{i\pi\rho(s)}{4s}\,\mathcal{f}_\ell(s)} \qquad (34)$$

where $\rho(s) \equiv \sqrt{(s-(m-\mu)^2)(s-(m+\mu)^2}$ stands for that branch which is analytic in the complex s-plane with the cuts $s > (m+\mu)^2$, $s < (m-\mu)^2$, and is obtained by continuing $\rho(s)$ from the upper side of the right hand cut.

Then by making use of the unitarity

$$\mathrm{Im}\; T_\ell(s) = \frac{\pi \rho(s)}{4s} \left| T_\ell(s) \right|^2 \qquad \text{for } (m+\mu)^2 < s < (m+2\mu)^2, \tag{35}$$

it turns out that $\varphi_\ell^{ir}(s)$ is analytic in $\overline{\mathfrak{M}}$ and the right hand cut actually starts at the inelastic threshold $s = (m+2\mu)^2$, except for possible existence of poles. Now the analytic properties of the partial wave amplitude $\varphi_\ell(s)$ can be obtained by solving (34) with respect to $\varphi_\ell^{ir}(s)$

$$\varphi_\ell(s) = F_\ell(s) + \frac{i\pi \rho(s) G_\ell(s)}{4s} \tag{36}$$

with

$$F_\ell(s) \equiv \frac{\varphi_\ell^{ir}(s)}{1 + \frac{\pi^2}{16 s^2} \rho^2(s)\, \varphi_\ell^{ir}(s)^2} \tag{37}$$

and

$$G_\ell(s) \equiv \frac{\varphi_\ell^{ir}(s)^2}{1 + \frac{\pi^2}{16 s^2} \rho^2(s)\, \varphi_\ell^{ir}(s)^2} . \tag{38}$$

From this expression it is clear that $\varphi_\ell(s)$ has a twofold branch point at $s = (m+\mu)^2$ and can be continued to everywhere in the domain $\mathfrak{M}$ on the Riemann surface of $\rho(s)$ with cut $s \geq (m+2\mu)^2$, except for poles. Denoting the values of $\varphi_\ell(s)$ on the second sheet by the index 2

$$\varphi_\ell^{(2)}(s) = F_\ell(s) - \frac{i\pi \rho(s)}{4s} G_\ell(s) \tag{39}$$

By taking the limit of (36) we obtain

$$\left.\begin{aligned} \operatorname{Re} T_\ell(s) &= F_\ell(s) \\ \operatorname{Im} T_\ell(s) &= \frac{\pi \rho(s)}{4s} G_\ell(s) \end{aligned}\right\} \quad \text{for } (m+\mu)^2 \leq s \leq (m+2\mu)^2 \tag{40}$$

and

$$\lim_{\epsilon \to 0} \varphi_\ell^{(2)}(s+i\epsilon) = F_\ell(s) - \frac{i\pi\rho(s)}{4s} G_\ell(s) = T_\ell^*(s) . \tag{41}$$

Thus we are able to continue the unitarity condition (35) analytically into the domain $\mathfrak{M}$ of the complex s-plane to obtain

$$\varphi_\ell(s) - \varphi_\ell^{(2)}(s) = \frac{i\pi\rho(s)}{4s} \varphi_\ell(s) \varphi_\ell^{(2)}(s) . \tag{42}$$

Though the treatment of each partial wave given above is very simple, the determination of the Lehmann ellipse in the cos Θ-plane, in which the partial wave expansion converges, is rather involved and depends very much on the value a, for which we could not algebraically find the maximum possible value.

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R E F E R E N C E S

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