THE TOPOLOGY OF REDUCED THREE-BODY PROBLEM AND THE EXISTENCE OF PERIODIC ORBITS

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ABSTRACT. We consider the Three-Body Problem (3BP) with a Newtonian-like potential proportional $||x_i - x_j||^{-\alpha}$ with $\alpha \geq 2$ we consider also the Newtonian case $\alpha = 1$. We study the dynamics reduced on the manifold defined by the vanishing of the total angular momentum. Using variational methods and the topology of the reduced configuration space, we prove the existence of periodic solutions: For the Newtonian potential, we prove the existence of a weak periodic solution.

1. INTRODUCTION

In this paper we want to use the topology of the planar Newtonian-like Three-Body Problem (3BP) together with the variational methods to study existence of periodic solutions.
We study the 3BP with the following potential:

$$V(x_1, x_2, x_3) \doteq \sum_{i \neq j} \frac{m_i m_j}{||x_i - x_j||^{\alpha}}$$

(with $\alpha \geq 2$ and $\alpha = 1$).
The potential V is $O(2, \mathbb{R})$ invariant and we study the $SO(2, \mathbb{R})$ reduction of the system on the submanifold of the phase space defined by the vanishing of the total angular momentum J. The interest of this choice is due to Sundman's result [4] about total collision solutions. For the Newtonian potential, he proved that total collision solutions take place on $J = 0$, for a generic α one can refer to [5], [2], [3].
We study existence of the periodic orbits on $J = 0$. In order to apply the variational method, first we construct the Lagrangian function on the manifold defined by $J = 0$. This is obtained by the Routh's reduction method. The topology of the reduced configuration $\mathcal{M}_r$ space is then studied. It turns out that $\mathcal{M}_r$ is a double covering of the space of the relative distances of the bodies. On $\mathcal{M}_r$, with the reduced Lagrangian we construct the reduced Least Action principle. Then we apply variational methods to study the reduced Action. If the coincidence set is eliminated,the topology of the reduced configuration space leads to identify classes of non contractible trajectories: the reduced Action is positive and coercive on these sets and therefore attains its minima.
For $\alpha \geq 2$ critical points are strong solutions of the reduced equation of motion and different critical points lie into disjoint classes. For $\alpha = 1$ (the Newtonian potential) critical points are weak solution of the reduced equations of motion and we are not able to distinguish the homotopy class of these solutions.
These critical points are T-periodic solutions of the 3BP in the reduced configuration space. It is not possible to check whether these orbits are T-periodic in the unreduced configuration space since they are not explicitly constructed.
A general reference for the application of the variational methods to the N-Body Problem is the monograph [9].

2. Lagrangian reduction and reduced configuration space

The system is composed of 3 point-like particle of masses m_1, m_2, m_3 which lie on a plane and interact through a newtonian potential. The configuration space is taken to be $\mathbb{R}^6 = \{x_i \in \mathbb{R}^2 : i = 1,2,3\}$ with $x_i \doteq (x_i^1, x_i^2)$. The system has 6 degrees of freedom.

The dynamics of the system is described by the Lagrangian function $L : \mathbb{R}^{12} \to \mathbb{R}$.

$$L = \sum_{i=1}^{3} \frac{m_i}{2} \|\dot{x}_i\|^2 + \sum_{i \neq j}^{3} \frac{m_i m_j}{\|x_i - x_j\|^\alpha} \tag{1}$$

where $< .,. >$ $\|.\|$ are the scalar product and the norm in $\mathbb{R}^2$.
The Lagrangian L is defined outside the coincidence set:

$$K_c \doteq \{(x_1, \ldots, x_3) \in \mathbb{R}^6 \mid x_i = x_j,\ i \neq j\} \tag{2}$$

The linear momentum $P = (P_1, P_2)$ and the angular momentum J (a scalar)

$$P_k = \sum_{i=1}^{3} m_i \dot{x}_i^k \quad k = 1,2 \qquad J = \sum_{i=1}^{3} m_i [x_i^1 \dot{x}_i^2 - x_i^2 \dot{x}_i^1] \tag{3}$$

are integrals of motion.
For fixed values of J we describe the motion through a reduced Lagrangian obtained by the Routh's procedure (see [7]).

Let the center of mass be at rest in the origin, consider the class of frames defined by selecting one of the bodies, say the 3^{th} one, and setting:

$$\begin{cases} q_1 \doteq x_3 - x_1 \\ q_2 \doteq x_3 - x_2 \end{cases} \tag{4}$$

We denote by $\mathcal{M}$ the reduced configuration space.

The Lagrangian takes the following form

$$L = \sum_{i=1}^{3} \frac{1}{2} M_{ij}(\dot{q}_i, \dot{q}_j) + \sum_{i=1}^{3} \frac{m_3 m_i}{\|q_i\|^\alpha} + \frac{m_1 m_2}{\|q_1 - q_2\|^\alpha} \tag{5}$$

where $\|q\| = (q, q)$

$$M \doteq \begin{pmatrix} m_i \left(1 - \frac{m_i}{\mu}\right) & -\frac{m_i m_j}{\mu} \\ -\frac{m_i m_j}{\mu} & m_i \left(1 - \frac{m_i}{\mu}\right) \end{pmatrix} \tag{6}$$

and $\mu \doteq \sum_{i=1}^{2} m_i$ is the total mass of the system.
By the further change of variables

$$\begin{aligned} T_1 : \mathbb{R}^6 \backslash K_c &\longrightarrow \mathbb{R}^6 \backslash K_c \\ (q_1, q_2) &\longrightarrow (\rho_1, \theta_1, \rho_2, \theta_2) \end{aligned} \tag{7}$$

$$\begin{cases} q_i^1 = \rho_i \cos\theta_i \\ q_i^2 = \rho_i \sin\theta_i \end{cases} \tag{8}$$

the Lagrangian becomes:

$$L = \sum_{i,j=1}^{2} \frac{1}{2} A_{ij} \dot{\rho}_i \dot{\rho}_j + \sum_{i,j=1}^{2} \frac{1}{2} B_{ij} \dot{\theta}_i \dot{\theta}_j + \sum_{i,j=1}^{2} C_{ij} \dot{\rho}_i \dot{\theta}_j + V(\rho_1, \rho_2, \theta_1, \theta_2) \tag{9}$$

where:

(10) $$A \doteq \begin{pmatrix} M_{11} & -M_{12}\cos(\theta_1-\theta_2) \\ -M_{21}\cos(\theta_2-\theta_1) & M_{22} \end{pmatrix}$$

(11) $$B \doteq \begin{pmatrix} A_{11}\rho_1^2 & A_{12}\rho_1\rho_2 \\ A_{21}\rho_1\rho_2 & A_{22}\rho_2^2 \end{pmatrix}$$

and

(12) $$C \doteq \begin{pmatrix} 0 & M_{12}\rho_1\sin(\theta_1-\theta_2) \\ M_{21}\rho_2\sin(\theta_2-\theta_1) & 0 \end{pmatrix}$$

(13) $$V(\rho_1,\rho_2,\theta_1,\theta_2) \doteq \sum_{i=1}^{2}\frac{m_3 m_i}{\rho_i^\alpha} + \frac{m_1 m_2}{\left(\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)}\right)^\alpha}$$

The reduced system is defined on $\mathcal{M} \simeq \mathbb{R}_+^2 \times [0,2\pi]^2$, with $\mathbb{R}_+ \doteq \{\rho > 0\}$.
The matrices A, B, and C are functions of ρ_1,ρ_2 and $\theta_1-\theta_2$. The Lagrangian (9) is invariant under rotation

$$\theta_i \to \theta_i + \alpha, \ \alpha \in [0,2\pi]$$

One can then introduce a cyclic coordinate conjugated to the total angular momentum and apply Routh's construction of the reduced Lagrangian.

Setting:

$$\mathcal{T}_2 : \mathcal{M} \to \mathcal{M}$$

(14) $$\varphi_2 = \theta_2 - \theta_1$$

one finds the following form for L:

(15) $$\begin{aligned} L = {} & \textstyle\sum_{i,j=1}^{2} \frac{1}{2}A_{ij}\dot\rho_i\dot\rho_j + C_{12}\dot\rho_1\dot\varphi_2 + \frac{1}{2}B_{22}\dot\varphi_2^2 + \\ & + \textstyle\frac{1}{2}\dot\theta_1^2 \sum_{i,j}^{2} B_{ij} + \dot\theta_1\left[\left(\sum_{i\neq 1}^{2} B_{i2}\dot\varphi_2 + \sum_{i\neq j} C_{ij}\dot\rho_i\right)\right] + V(\rho_1,\rho_2,\varphi_2) \end{aligned}$$

θ_1 is the cyclic coordinate and the total angular momentum is given by

(16) $$J = \frac{\partial L}{\partial\dot\theta_1} = \left(\sum_{i}^{2} B_{i2}\dot\varphi_2 + \sum_{i\neq j} C_{ij}\dot\rho_i\right) + \dot\theta_1 \sum_{i,j=1}^{2} B_{ij}$$

Now Routh's prescription gives the reduced Lagrangian for $J = 0$:

(17) $$\begin{aligned} R = {} & \textstyle\sum_{i,j=1}^{2} \frac{1}{2}A_{ij}\dot\rho_i\dot\rho_j + C_{21}\dot\rho_1\dot\varphi_2 + \frac{1}{2}B_{22}\dot\varphi_2^2 + \\ & - \frac{\left[\sum_i (B_{i2}\dot\varphi_2 + C_{ij}\dot\rho_i)\right]^2}{2\sum_{i,j=1}^{2} B_{ij}} + V(\rho_1,\rho_2,\varphi_2) \end{aligned}$$

The reduced system has configuration space given by:

$$\mathcal{M}/S^1 \simeq \mathbb{R}_+^2 \times [0,2\pi]$$

Remark 2.1. *In the reduction we obtained a system defined on the quotient space $\mathcal{M}_r$ that is a quotient. In order to reconstruct the whole motion on $\mathbb{R}^2$ one needs to use the expression of $\dot\theta_1$ given by the condition $J = 0$. A periodic motion on $\mathcal{M}_r$ with period T leads to a periodic motion (in general with different period) of the original problem only if:*

(18) $$\frac{1}{2\pi}\int_0^T dt\dot\theta_1 \in \mathbb{Q}$$

3. REDUCED 3BP

Let us return to the formulation of (17) and describe the reduced system. We have seen that $\mathcal{M} \simeq \mathbb{R}^2_+ \times [0,2\pi] \times [0,2\pi]$ and then

$$(\mathcal{M}\backslash K_c)_r \simeq \mathbb{R}^2_+ \times [0,2\pi]$$

and the reduced Lagrangian $R : T(\mathcal{M}\backslash K_c)_r \to \mathbb{R}$ is written as follows:

$$R(\zeta,\dot\zeta) = \frac{1}{2}\sum_{i,j} M^{(1)}_{ij}(\zeta)\dot\rho_i\dot\rho_j + \frac{1}{2}\sum_{i} M^{(2)}_{ij}(\zeta)\dot\rho_i\dot\varphi + \tag{19}$$
$$+\frac{1}{2}M^{(3)}(\zeta)\dot\varphi^2 + V(\rho_1,\rho_2,\varphi)$$

the matrices $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$ forms a positive definite quadratic matrix on $\mathbb{R}^{2N-3}$.
We introduce another description of the reduced configuration space $(\mathcal{M}\backslash K_c)_r$. We now show that $(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to the algebraic manifold

$$\mathcal{N}\backslash K_c \doteq \{(\rho_1,\rho_2,\rho_3,z) \in \mathbb{R}^3_+ \times \mathbb{R}^1 \mid z^2\sum_i^3 \rho_i^2 = [A(\rho_1,\rho_2,\rho_3)]^2\} \tag{20}$$

where $A(\rho_1,\rho_2,\rho_3)$ is proportional to the oriented area of the triangle whose sides are ρ_1,ρ_2,ρ_3:

$$[A(\rho_1,\rho_2,\rho_3)]^2 \doteq \frac{1}{2}\left[\sum_i^3 \rho_i\right]\prod_{i,j,k}{}' (\rho_i+\rho_j-\rho_k)$$

$\prod'_{i,j,k}$ is the product cyclic in the indices i,j,k. From the definition of $\mathcal{N}_r$ one verifies

$$\lim_{(\rho_1,\rho_2,\rho_3)\to(0,0,0)} z(\rho_1,\rho_2,\rho_3) = 0$$

Proposition 3.1. *$(\mathcal{M}\backslash K_c)_r$ is diffeomorphic to $\mathcal{N}\backslash K_c$.*

Proof. In $(\mathcal{M}\backslash K_c)_r \simeq \mathbb{R}^2_+ \times S^1$ we take ζ and we describe it by the local coordinates (r_1,r_2,φ); the we define the map f as follows:

$$\begin{cases} \rho_i &= r_i \ i=1,2 \\ \rho_3 &= (r_1^2+r_2^2-r_1r_2\cos\varphi)^{1/2} \\ z &= \frac{r_1r_2\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} \end{cases}$$

At the coincidence set K_c the jacobian is not defined. One verifies that the rank of the jacobian of f equals three out of K_c. Indeed the jacobian has the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{r_1-r_2\cos\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} & \frac{r_2-r_1\cos\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} & \frac{r_1r_2\sin\varphi}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\varphi}} \\ \frac{r_2^3\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} & \frac{r_1^3\sin\varphi}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} & \frac{r_2r_1[(r_1^2+r_2^2)\cos\varphi-r_1r_2\cos 2\varphi]}{2\sqrt{2}\sqrt{r_1^2+r_2^2-r_1r_2\cos\varphi}} \end{pmatrix}$$

The transformation f can be inverted, f^{-1} is given by:

$$\begin{cases} r_i &= \rho_i \ i=1,2 \\ \varphi &= \begin{cases} -\arccos[(\rho_1^2+\rho_2^2-\rho_3^2)/(2\rho_1\rho_2)] & \text{if } z(\rho_1,\rho_2,\rho_3)<0 \\ +\arccos[(\rho_1^2+\rho_2^2-\rho_3^2)/(2\rho_1\rho_2)] & \text{if } z(\rho_1,\rho_2,\rho_3)\geq 0 \end{cases} \end{cases}$$

Therefore $(\mathcal{M}\backslash K_c)_r$ and $\mathcal{N}\backslash K_c$ are diffeomorphic. □

In the sequel we often consider $(\mathcal{M}\backslash K_c)_r \simeq \mathcal{N}\backslash K_c$.
About $\mathcal{N}\backslash K_c$ note that:
(i) The expression defining $[A(\rho_1, \rho_2, \rho_3)]^2$ must be positive, then $\rho_i + \rho_j \geq \rho_k$ for all permutation of i, j, k. These are the *triangular inequalities*.
(ii) If $z = 0$, $\rho_i > 0$, $i = 1, 2, 3$ then for some indices i, j, k $\rho_i + \rho_j = \rho_k$, this corresponds to a collinear configuration of the three bodies.
(iii) In the closure of $\mathcal{N}\backslash K_c$ there are $\rho_i = 0$ for some i then triangular inequalities imply that $\rho_j = \rho_k$ with $j, k \neq i$.
We term $\mathcal{N}$ the closure in $\mathbb{R}^4$ of $\mathcal{N}\backslash K_c$.
Now $\mathcal{N}$ can be embedded into $\mathcal{RD} \times \mathbb{R}^1$ where

(21) $$\mathcal{RD} \doteq \{r = (r_1, r_2, r_3) \in \bar{\mathbb{R}}^3_+ \mid r_i + r_j - r_k \geq 0 \text{ cyclic permutations of } i, j, k\}$$

where $\bar{\mathbb{R}}_+ \doteq \mathbb{R}_+ \cup \{0\}$.
$\mathcal{RD}$ is the set of the relative distances among the three bodies. Note that $\partial\mathcal{RD} \neq \emptyset$:

$$\partial\mathcal{RD} = \cup'_{i,j,k}\pi^i_{jk}$$

where π^i_{jk} are:

(22) $$\pi^i_{jk} \doteq \{r = (r_1, r_2, r_3) \in \bar{\mathbb{R}}^3_+ \mid r_i + r_j - r_k = 0\}$$

Using that $(\mathcal{M}\backslash K_c)_r \simeq \mathcal{N}\backslash K_c$ and local charts we can write the reduced Lagrangian in terms of the relative distances.

(23) $$R(\zeta, \dot{\zeta}) = \sum_{i,j=1}^{3} M_{ij}(\zeta)\dot{r}_i\dot{r}_j + \sum_{i,j,k}' \frac{m_i m_j}{r_k}$$

The 3×3 symmetric matrix M has entries smooth homogeneous functions of the r's (see [1]):

$$M^{-1} \doteq \begin{pmatrix} 1/2(1/m_2 + 1/m_3) & -(r_1^2 + r_2^2 - r_3^2)/2r_1r_2m_3 & -(r_1^2 + r_3^2 - r_2^2)/2r_1r_3m_2 \\ -(r_1^2 + r_2^2 - r_3^2)/2r_1r_2m_3 & 1/2(1/m_1 + 1/m_3) & -(r_3^2 + r_2^2 - r_1^2)/2r_3r_2m_1 \\ -(r_1^2 + r_3^2 - r_2^2)/2r_1r_3m_2 & -(r_3^2 + r_2^2 - r_1^2)/2r_3r_2m_1 & 1/2(1/m_1 + 1/m_2) \end{pmatrix}$$

In the application of variational methods we will need that the reduced Lagrangian written in term of local coordinates $z = (z_1, z_2, z_3) \in (\mathcal{M}\backslash K_c)_r$ hence:

(24) $$R(\zeta, \dot{\zeta}) = \sum_{i,j=1}^{3} M_{ij}(\zeta)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{\rho_{ij}(z)}$$

We now extend, at least formally, the Lagrangian on the space $\mathcal{N}$ that contains the coincidence set K_c. We will show that consider trajectories which have the tangent vector with a finite number of discontinuities.

3.1. **Geometry of the reduced configuration space.** The Lagrangian L is invariant under the lift on $T\mathcal{M}$ of the diagonal action of the group $O(2, \mathbb{R})$.
We denote this action as follows:

(25) $$\begin{aligned} \Phi : O(2, \mathbb{R}) \times \mathcal{M} &\to \mathcal{M} \\ (g, x) &\to \Phi_g(x) = (g \cdot x_1, g \cdot x_2) \end{aligned}$$

where $g\cdot$ denotes the standard action of $O(2, \mathbb{R})$ on the plane.
Recall the following properties of $O(2, \mathbb{R})$:

Proposition 3.2. *The group $O(2, \mathbb{R})$ is generated by the set $\mathcal{S}_2$ of all reflections with respect to independent lines in the plane.*

Proof. Here we consider the natural action of $O(2,\mathbb{R})$ on the vector space $\mathbb{R}^2$. The proof is elementary, it is given noticing that in the plane the product of two reflections is a transformation with unit determinant. This transformation is a rotation.
Now we want to give an explicit matrix construction:
In a chosen coordinate system we take a direction $[v] \doteq [v_1 : v_2] \in \mathbb{R}P^1$. One can show that the reflection $S_{[v_1:v_2]} \in \mathcal{S}_2$ w.r.t. the direction $[v_1 : v_2]$ takes the following matrix form:

$$S_{[v_1:v_2]} = \frac{1}{||\vec{v}||^2} \begin{pmatrix} v_1^2 - v_2^2 & 2v_1v_2 \\ 2v_1v_2 & v_2^2 - v_1^2 \end{pmatrix}$$

If one considers $[v_1 : v_2] = [\cos\alpha : \sin\alpha]$

$$S_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Now given two directions in plane defined by two angles, respectively α and β, by means of simple manipulations one finds that:

$$S_\alpha \cdot S_\beta = \begin{pmatrix} \cos 2(\beta-\alpha) & \sin 2(\beta-\alpha) \\ -\sin 2(\beta-\alpha) & \cos 2(\beta-\alpha) \end{pmatrix}$$

and so:

$$S_\alpha \cdot S_\beta = R_{2(\beta-\alpha)} \in SO(2,\mathbb{R})$$

□

The reduced configuration space is given by the quotient:

$$(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$$

We now consider the the geometry of reduction of the 3BP:
The symmetry $\mathcal{S}_2$ is not reduced. one can describe the reduction of the configuration space by the following diagram:

$$\begin{array}{ccc} (\mathcal{M}\backslash K_c) & \stackrel{F}{\rightarrow} & \mathbb{R}_+^2 \times [0,2\pi] \times [0,2\pi] \\ & \stackrel{p}{\searrow} & \downarrow \pi \\ & & \mathbb{R}_+^2 \times [0,2\pi] \end{array}$$

With F we denote the diffeomorphism describing the coordinate transformation from q_i to r_i, φ_i, θ_1. The map π describes the quotient of $(\mathcal{M}\backslash K_c)$ w.r.t. $SO(2,\mathbb{R})$ action. The map $p \doteq \pi \circ F$ provides the reduction and induces a map $\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R}) \rightarrow \mathcal{R}D\backslash K_c$.
The map $\tilde{p}$ is the transformation between the coordinates (r_1, r_2, φ) and (ρ_2, ρ_2, ρ_3):

$$\begin{cases} \rho_i & = r_i \ i = 1,2 \\ \rho_3 & = (r_1^2 + r_2^2 - r_1 r_2 \cos\varphi)^{1/2} \end{cases}$$

Note that:

$$\tilde{p}(r_1, r_2, \varphi) = \tilde{p}(r_1, r_2, 2\pi - \varphi)$$

If one studies the configuration space in terms of the $\mathcal{R}D$ it turns out that:

Proposition 3.3. *The map $p : (\mathcal{M}\backslash K_c) \rightarrow \mathbb{R}_+ \times [0,2\pi]$ induces a map $\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2.\mathbb{R}) \rightarrow \mathcal{R}D\backslash K_c$ which is a ramified covering with a monodromy group isomorphic to $\mathbb{Z}_2$.*

Recall the definition of *ramified covering*:

Definition 3.1. *A quadruple $(\bar{S}, S, \mathbb{Z}_n, p_S)$ where $p_S : \bar{S} \to S$ is a ramified covering if:*
(i) $\bar{S}$, S are manifolds.
(ii) There exists a closed subset $K \subset \bar{S}$ such that $(\bar{S}\backslash K, S\backslash p_S(K), Z_n, p_S)$ is a covering with monodromy group $\mathbb{Z}_n$,
(iii) For all $s \in S$ there exist a neighborhood $V(s)$ homeomorphic to a ball in S such that the connected components of $p_S^{-1}(V(s))$ are homeomorphic to a ball in $\bar{S}$

Proof. Consider the quotient manifold $(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$. Any class $[q] \in (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$ is composed of configurations which differ by a rotation.
Now chosen a direction $[v_1 : v_2] \in \mathbb{R}P^1$ any element of $\mathcal{S}_2$ can be written as the product of a reflection w.r.t. a fixed chosen direction times a rotation:

$$\mathcal{S}_2 \ni S_{[v_1:v_2]} \cdot R_\alpha = R_\beta \cdot S_{[v_1:v_2]}$$

for some $R_\alpha, R_\beta \in SO(2,\mathbb{R})$.
Therefore on $(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$ we define the action of $\mathcal{S}_2$ as follows:

$$\begin{aligned} \tilde{\Phi} : \mathcal{S}_2 \times (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R}) &\to (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R}) \\ (S_{[v_1:v_2]}, [q]) &\to \tilde{\Phi}_{S_{[v_1:v_2]}}([q]) \doteq [\Phi_{S_{[v_1:v_2]}}(q)] \end{aligned} \tag{26}$$

Then the action of $\mathcal{S}_2$ on $(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$ is equivalent to the action of only one reflection w.r.t. a *chosen* line $[v_1 : v_2] \in \mathbb{R}P^1$. The direction $[v_1 : v_2]$ corresponds to the classes $\lambda[v]$ with $\lambda \in \mathbb{R}\backslash\{0\}$. Consider the group $G = \{S_{[v_1:v_2]}, id\}$. The action of G on $(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\backslash\{\lambda[v]\}$ is proper and discontinuous without fixed points, then

$$(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\backslash\{\lambda[v]\} \to \{(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\backslash\{\lambda[v]\}\}/G$$

is a covering. In fact we can use the following result (see [8]):

Theorem. *Let X be a connected, locally arcwise-connected topological space and let G a properly discontinuous group of homeomorphisms of X. Let $\tilde{p} : X \to X/G$ the natural projection of X onto the quotient space. Then the couple $(X, \tilde{p})$ is a regular covering space of X/G.*

In our case $X = (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\backslash\{\lambda[v]\}$ and the rank of G is finite and equals two, then we have a ramified covering whose monodromy group is $\mathbb{Z}_2$. The map $\tilde{p}$ can be defined as

$$\tilde{p}([q]) \doteq p(q) = (\pi \circ F)(q) = \eta \in \mathcal{R}D$$

One verifies that $\tilde{p}$ is not a homeomorphism at $\{\lambda[v]\}$. One can also verifies that for any $[q] \in (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})/G$ there exist a neighborhood whose connected part, homeorphic to a disk, is mapped by $\tilde{p}^{-1}$ into open set in $(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})$ homeomorphic to a disk.
Therefore we can conclude that

$$\tilde{p} : (\mathcal{M}\backslash K_c)/SO(2,\mathbb{R}) \to \{(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\}/G \tag{27}$$

is a ramified covering whose branching points are $\lambda[v]$ with $\lambda \in \mathbb{R}$.
The thesis is obtained noticing that

$$\{(\mathcal{M}\backslash K_c)/SO(2,\mathbb{R})\}/G \simeq (\mathcal{M}\backslash K_c)/O(2,\mathbb{R}) \simeq \mathcal{R}D\backslash K_c$$

□

Now we want to extend the reduced configuration space adding the coincidence set K_c. Note that there is only one configuration where the quotient is singular, this is the total coincidence configuration K^*, i.e. the origin in $\mathcal{M}$. We define the reduced configuration space $\mathcal{M}_r$ as:

$$\mathcal{M}_r = \tilde{p}^{-1}(\mathcal{R}D)$$

where we define $\tilde{p}(K^*) = K^*$.
We give a geometric description of the ramified covering. $\mathcal{M}_r$ is embedded into $\mathbb{R}^4$ and $\mathcal{M}_r$ is an algebraic manifold: for any $(\rho_1, \rho_2, \rho_3) \in \mathcal{R}D\backslash\partial\mathcal{R}D$ we have two values for z, $z = \pm A(\rho_1, \rho_2, \rho_3)/\sqrt{\sum_i^3 \rho_i^2}$.

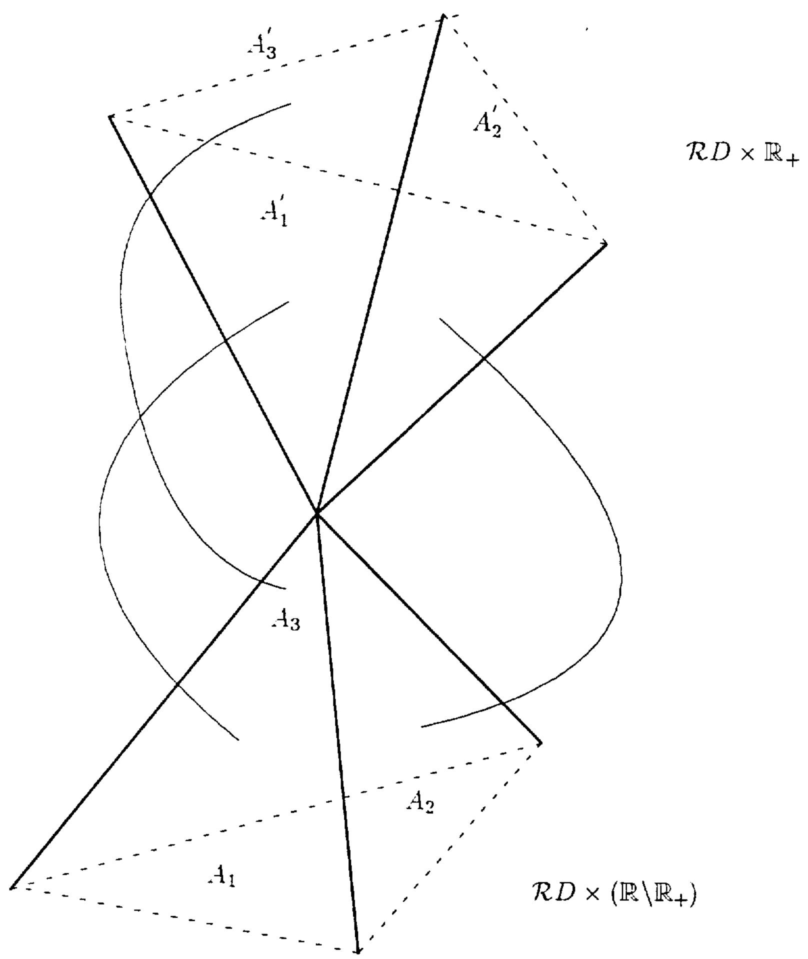

FIGURE 1. Reduced Configuration Space. The couple of surfaces (A_i, A_i'), $i = 1,2,3$ are identified (glued)

In the figure 1 we show that $\mathcal{M}_r$ can be thought as two copies of $\mathcal{R}D$ (two infinite dihedra) embedded in $\mathbb{R}^3$ with common vertex and common faces. They form the two sheets of the covering that are glued along the collinear configurations, (thin and dashed lines represent the gluing). The surfaces $A_i, i = 1,2,3$ and $A_i', i = 1,2,3$ are the two copies of the collinear configurations $z = 0$. Heavy lines correspond to the coincidence of two bodies, while the common vertex is coincidence of three bodies. Indeed, consider the two spaces $\mathcal{R}D \times \mathbb{R}_+$ and $\mathcal{R}D \times (\mathbb{R}\backslash\mathbb{R}_+)$. Define the map:

$$
\begin{aligned}
i : \partial\mathcal{R}D &\rightarrow \mathcal{R}D \\
\zeta &\rightarrow i(\zeta) \doteq \zeta
\end{aligned}
\tag{28}
$$

now take the disjoint union

$$\mathcal{R}D \times \mathbb{R}_+ \sqcup \mathcal{R}D \times (\mathbb{R}\backslash\mathbb{R}_+)$$

then the following equivalence relation is defined:

$$\zeta \sim \zeta' \text{ iff:}$$
$$\text{either} \quad \zeta = \zeta'$$
$$\text{or} \quad i(\zeta) = \zeta'$$

then, by i, we define the gluing

$$\mathcal{M}_r = (\mathcal{R}D \times \mathbb{R}_+ \sqcup \mathcal{R}D \times (\mathbb{R}\backslash\mathbb{R}_+))/\sim = \mathcal{R}D \times \mathbb{R}_+ \cup_i \mathcal{R}D \times (\mathbb{R}\backslash\mathbb{R}_+)$$

For $N = 3$ we define the following involution:

$$\begin{aligned} \sigma : \mathcal{M}_r &\rightarrow \mathcal{M}_r \\ \zeta = (r_1, r_2, \varphi) &\rightarrow \sigma[\zeta] = (r_1, r_2, 2\pi - \varphi) \end{aligned} \tag{29}$$

The action of σ on $\mathcal{M}_r$ corresponds to the action of $\mathcal{S}_2$ on $\mathcal{M}$ and in fact it has a manifold of fixed points corresponding to $\partial\mathcal{R}D$.

4. Reduced Least Action Principle

Consider $\mathcal{M}_r$ as submanifold of $\mathbb{R}^4$. We specify the functional spaces of trajectories on $\mathcal{M}_r$ in order to study the solutions of the Euler-Lagrange equations as critical points of the reduced Action functional.

Choose $T > 0$ (the period), on $\mathbb{R}^4$ and define the space of continuous functions $C^0_T(\mathbb{R}^4)$ as follows:

$$C^0_T(\mathbb{R}^4) \doteq \{\zeta(t) \in C^0([0,T], \mathbb{R}^4) \mid \zeta(t) = \zeta(t+T), \ ||\zeta||_\infty = \sup_t |\zeta| < \infty\} \tag{30}$$

with $|\zeta| \doteq \max_{i=1} |z_i|$.

We now define the fuctional space on trajectories on $\mathcal{M}_r$.

First we define the space of continuous periodic trajectories on $\mathcal{M}_r$:

$$C^0([0,T], \mathcal{M}_r) \doteq \{\zeta(.) \in C^0([0,T], \mathbb{R}^4) \ |\zeta(t) \in \mathcal{M}_r, \quad ||\zeta(t)||_\infty < \infty\} \tag{31}$$

We also define:

$$C^\infty([0,T], \mathcal{M}_r) \doteq \{\zeta(.) \in C^\infty([0,T], \mathbb{R}^4) \ |\zeta(t+T) = \zeta(t), \ \zeta(t) \in \mathcal{M}_r\} \tag{32}$$

In the construction of the reduced Lagrangian we found a positive definite quadratic form defined on $T\mathcal{M}_r$:

$$\langle M(\zeta)v, v\rangle \doteq \sum_{i,j} M_{ij}(\zeta) v^i v^j \tag{33}$$

with $\zeta \in \mathcal{M}_r$ and $v \in T\mathcal{M}_r$.

We describe a trajectory $\zeta : \mathbb{R} \rightarrow \mathcal{M}_r$ with r_1, r_2, φ, then using (33) we define $H^1_T(\mathcal{M}_r)$ the following Sobolev space:

$$H^1_T(\mathcal{M}_r) \doteq \{\zeta(.) \in C^\infty([0,T], \mathcal{M}_r) \mid \zeta(t) = \zeta(t+T), \ ||\zeta||^2_{H^1_T(\mathcal{M}_r)} < \infty\} \tag{34}$$

where $||\zeta||^2_{H^1_T(\mathcal{M}_r)} \doteq \int_0^T dt[\langle M(\zeta)\dot\zeta(t), \dot\zeta(t)\rangle + \sum_i r_i^2(t)]$, and

$$\langle M(\zeta)\dot\zeta(t), \dot\zeta(t)\rangle = \left[\sum_{i,j} [M^{(1)}_{ij}(\zeta)\dot r_i \dot r_j + M^{(2)}_i(\zeta)\dot r_i \dot\varphi] + M^{(3)}_{33}(\zeta)(\dot\varphi)^2\right]$$

By $H^1_T(\mathcal{M}_r)$ we define the following space:

$$\begin{aligned} \Lambda^a_T(\mathcal{M}_r) \doteq \{\zeta(.) \in H^1_T(\mathcal{M}_r) \mid \\ r_i(t+T/2) = r_i(t) \ i = 1,2,3 \ \varphi(t+T/2) = 2\pi - \varphi(t)\} \end{aligned} \tag{35}$$

The space $\Lambda_T^a(\mathcal{M}_r)$ can be described using the involution σ introduced in the preceding Chapter:

$$\Lambda_T^a(\mathcal{M}_r) \doteq \{\zeta \in H_T^1(\mathcal{M}_r) \mid \zeta(t+T/2) = \sigma(\zeta(t)),\ \zeta(t) \in \mathcal{M}_r\ \forall t \in [0,T]\} \tag{36}$$

Note that by the standard Sobolev embedding $\zeta \in H_T^1(\mathcal{M}_r)$ implies that $\zeta \in C_T^0(\mathcal{M}_r)$ therefore the condition $\zeta(t) \in \mathcal{M}_r$ for all $t \in [0,T]$ is well defined. On $H_T^1(\mathcal{M}_r)$ we define the following (Action) functional $\mathcal{A}_T[.]$

$$\mathcal{A}_T[\zeta] \doteq \int_0^T dt R(\zeta,\dot{\zeta}) \quad \text{with } \zeta \in H_T^1(\mathcal{M}_r) \tag{37}$$

The Action $\mathcal{A}_T[.]$ is continuous on the set:

$$D_{\mathcal{A}} \doteq \left\{ \zeta \in H_T^1(\mathcal{M}_r) \mid \int_0^T dt \sum_{i,j} \frac{m_i m_j}{(\rho_{ij}(r(t)))^\alpha} < \infty \right\} \tag{38}$$

Note that for $\alpha = 1$ $D_{\mathcal{A}}$ contains the collision solutions at which the Action is not differentiable. The Least Action Princple states that, in the domain of differentiability of $\mathcal{A}_T[.]$, the equation of motions are given by the first variation of $\mathcal{A}_T[.]$:

$$\langle D\mathcal{A}_T[\zeta], v\rangle = 0 \quad \text{for all } v \in T_\zeta H_T^1(\mathcal{M}_r) \simeq H_T^1(T_\zeta \mathcal{M}_r) \tag{39}$$

We can now define the Action functional for trajectories in $\mathcal{M}_r$. Let us recall the reduced Lagrangian is:

$$R(\zeta.\dot{\zeta}) = \frac{1}{2}\sum_{i,j=1}^{3} M_{ij}(z)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{\rho_k(z)^\alpha}$$

We take the following integral as a definition of the reduced Action functional:

$$\mathcal{A}_T[\zeta(t)] = \int_0^T dt \left\{ \frac{1}{2}\sum_{i,j=1}^{3} M_{ij}(z)\dot{z}_i\dot{z}_j + \sum_{i,j} \frac{m_i m_j}{(\rho_{ij}(z))^\alpha} \right\} \tag{40}$$

$\mathcal{A}_T[.]$ is defined in $H_T^1(\mathcal{M}_r) \cap D_{\mathcal{A}}$.

Any C^2 solution of (39) is termed *strong solution.*

In the next section, we will show that for the considered problem with $\alpha \geq 2$ we can prove the existence of *strong* T-periodic solution, while $\alpha = 1$ we can show the existence of *generalized* T-periodic solution.

5. T-PERIODIC SOLUTION FOR THE NEWTONIAN-LIKE REDUCED 3BP, AND GENERALIZED SOLUTIONS FOR THE REDUCED NEWTONIAN 3BP

In this final section we study of the Action principle for $\mathcal{A}_T[.]$ in two different cases:
i) $\alpha \geq 2$ Newtonian-like Potential,
ii) $\alpha = 1$ Newtonian Potential.
The case i) it is the case of *Strong Force* SF potential, in fact:

Definition 5.1. *The potential $V : \mathbb{R}^6\backslash\{0\} \to \mathbb{R}_+$ satisfies the SF condition if there exist $a, r > 0$ such that*

$$V(x_1, x_2, x_3) \geq a \sum_{i \neq j} \frac{1}{||x_i - x_j||^2}$$

for all $(x_1\ x_2\ x_3)$ such that $0 < ||x_i - x_j|| < r$.

The variational methods can be directly applied in the case i). In case ii) one studies just the modified problem.
In fact, for $\alpha = 1$ we consider the Action $\mathcal{A}_T[.]$ expressed in terms of the relative distances r's and we define the new Action

$$\mathcal{A}_T^\delta[\zeta] \doteq \mathcal{A}_T[\zeta] + \int_0^T dt \sum_{i=1}^{3} \frac{\delta}{r_i^2(z(t))} \tag{41}$$

The new Action (41) is of class C^1 wherever defined and it takes value $+\infty$ on collision solutions. In the domain of definition of the Action, using the geometry of the coincidence set K_c, we define classes of non-contractible trajectories, on which an inequality of Poincarè type holds and we can prove the coercivity of the $\mathcal{A}_T^\delta[.]$.
On each such class Γ the Action attains absolute minimum which is a strong T-periodic solutions of the reduced 3BP with SF. We prove that when $\delta \to 0$ the sequence ζ_Γ^δ converges weakly in H^1 to a trajectory ζ_Γ which is a weak T-periodic solution of the reduced 3BP. In general in our context we cannot prove that $\zeta_\Gamma \neq \zeta_{\Gamma'}$ for $\Gamma \neq \Gamma'$. We have to notice that this solution lives in the reduced configuration space $\mathcal{M}_r$. There is the problem of the lifting of the T-periodic orbit into the unreduced configuration space. The condition (18) cannot be directly verified since the solution is not explicit given.

5.1. **The "strong force" method.** We describe any trajectory on $\mathcal{M}_r$ using the local coordinates given by:

$$\begin{cases} r_i = z_i \quad i = 1,2 \\ r_3 = \sqrt{z_1^2 + z_2^2 - 2z_1 z_2 \cos z_3} \end{cases}$$

here z_3 is the angle between r_1 and r_2.
The the Action is now expressed in terms of $\zeta = (z_1, z_2, z_3)$. Now $\mathcal{A}_T[.]$ will be written as follows:

$$\mathcal{A}_T[\zeta] = \int_0^T dt \left\{ \sum_{i,j=1}^{3} \bar{M}_{ij}(z)\dot{z}_i\dot{z}_j + \sum_{i,j,k\neq 3}' \frac{m_i m_j}{z_k} + \frac{m_1 m_2}{r_3(z)} \right\} \tag{42}$$

The functional $A_T^\delta[.]$ is defined on $H_T^1(\mathcal{M}_r)$ by

$$\mathcal{A}_T^\delta[\zeta] \doteq \mathcal{A}_T[\zeta] + \mathcal{F}^\delta[\zeta]$$

$$\text{with } \mathcal{F}^\delta[x] \doteq \int_0^T dt \sum_{i=1}^{2} \frac{\delta}{z_i^2(t)} + \int_0^T \frac{\delta}{r_3^2(z_1, z_2, z_3)} \tag{43}$$

For every $\delta > 0$ the $\mathcal{A}_T^\delta[.]$ is of class C^1 on its domain of definition and it formally takes value $+\infty$ on collision solutions of the 3BP.
Then we study the sublevel sets of the Action $\mathcal{A}_T^\delta[.]$.

$$S_c = \{A_T^\delta[\zeta] \leq c\}$$

We will show that we can find a set of T periodic trajectories Γ such that $S_c \cap \Gamma$ is invariant under the gradient flow and $A_T^\delta[.]$ is coercive on $S_c \cap \Gamma$ (not empty for $c > 0$), i.e.

$$\lim_{n\to\infty} \mathcal{A}_T^\delta[\zeta_n] = +\infty \text{ if } \{\zeta_n\}_n \subset S_c \cap \Gamma \text{ and } ||\zeta_n||_{H_T^1(\mathcal{M}_r)} \to \infty$$

Then $S_c \cap \Gamma$ is compact in $C^0([0,T], \mathcal{M}_r)$ and one concludes that minima exist.
To obtain a solution for the 3BP without the SF additional term, we study the limit of ζ^δ when $\delta \to 0$. We prove that this limit exists, and it corresponds to a weak T-periodic solution of the problem. *Weak* solutions are defined as follows (see [9]):

Definition 5.2 (Weak solutions). *We term $\zeta^0(t)$ a weak solution of 3BP iff:*
(1) ζ^δ is a strong solution for $\mathcal{A}_T^\delta[.]$ for any $\delta > 0$
(2) $\lim_{\delta\to 0}\zeta^\delta = \zeta^0$ weakly in $\Lambda_T(\mathcal{M})$ and uniformly in $[0,T]$
(3) $\mathcal{A}_T^\delta[\zeta^\delta] < \infty$ for all $\delta > 0$.

Then we prove that ζ^0 is a *generalized* solution i.e. it fulfills the properties collected in the following definition:

Definition 5.3. *Let $I_c(\zeta^0)$ be the subset of $[0,T]$ such that*

$$I_c(\zeta^0) \doteq \{t \in [0,T] \mid \zeta^0(t) \in K_c\},$$

we term $\zeta_0(.)$ a T-periodic generalized solution of the Euler-Lagrange equation iff:
(0) $\zeta^0(t+T) = \zeta^0(t)$ for all $t \in [0,T]$
(1) $I_c(\zeta^0)$ has zero Lebesgue measure
(2) $\zeta^0 \in C^2([0,T]\backslash I_c)$ and satisfies the Euler-Lagrange equations.
(3) ζ^0 has for all t in $[0,T]\backslash I_c$ the same Energy.
(4) $\mathcal{A}_T[\zeta^0] < \infty$.

In particular we show that the set of collision times $I_c(\zeta_0)$ is discrete.

5.2. **Class of non-contractible trajectories.** For the modified Action $\mathcal{A}_T^\delta[.]$ the coincidence set K_c is a singularity. In fact one can prove that the Action increases without bound on any sequence of trajectories converging weakly in $H_T^1(\mathcal{M}_r)$ and uniformly in $[0,T]$ to a trajectory intersecting K_c. We now study the space of non contractible loops of $\mathcal{M}_r\backslash K_c$. We show that there exist classes of non-contractible loops on which the Action is finite and coercive.

The first homotopy group of $\mathcal{M}_r\backslash K_c$ can be computed. Consider the Fig. 2, $\mathcal{M}_r\backslash K_c$ is arcwise connected and it is homotopic to $\mathbb{R}^3$ minus three independent half-lines l_1, l_2, l_3 having a common origin. Denoting by γ_i a continuous loop around l_i and by $[\gamma_i]$ its homotopy class, one can prove that:

$$[\gamma_i] + [\gamma_j] = [\gamma_k] \text{ with } i,j,k \text{ cyclic permutation of } 1,2,3 \tag{44}$$

hence the presentation of $\pi_1(\mathbb{R}^3\backslash\{l_1, l_2, l_3\})$ is given by two of the cycles $[\gamma_i]$ $i = 1,2,3$ and one of the relations (44). Therefore we have:

Proposition 5.1. *The first homotopy group of the space $\mathcal{M}_r\backslash K_c$ is given by:*

$$\pi_1(\mathcal{M}_r\backslash K_c) \simeq \mathbb{Z}\oplus\mathbb{Z} \tag{45}$$

Proof. We know that:

$$\pi_1(\mathcal{M}_r\backslash K_c) \simeq \pi_1(\mathbb{R}^3\backslash\{l_1, l_2, l_3\})$$

Without loss of generality we identify the space $\mathbb{R}^3\backslash\{l_1, l_2, l_3\}$ with $\mathbb{R}^3$ with the negative half-axes removed.
Now we apply a corollary of the Siefert-Van Kampen theorem which states that if a space X can be covered by two open arcwise connected sets U and V such that

$$\pi_1(U\cap V) \simeq 0$$

then

$$\pi_1(X) \simeq \pi_1(U)\oplus\pi_1(V)$$

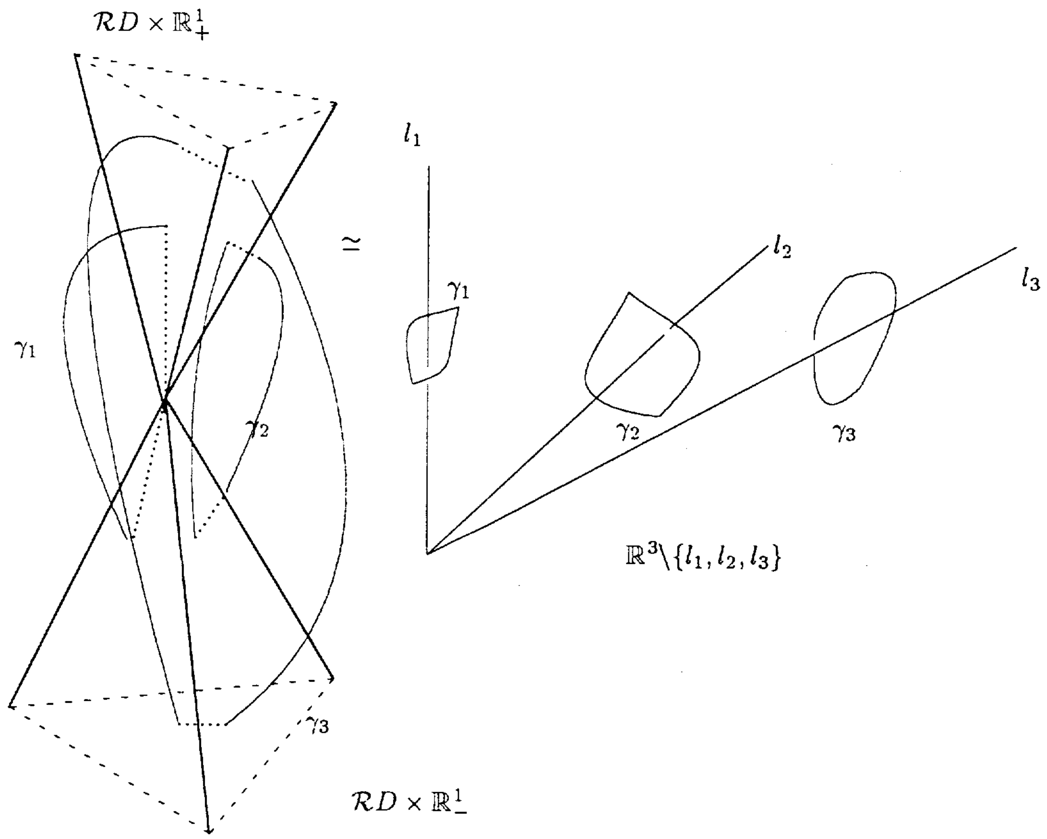

FIGURE 2. non-deformable loops γ_1, γ_2 and γ_3 with the "strong force"

We take $X = \mathcal{M}_r \backslash K_c$ and define

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y \neq 0, z \neq 0\}$$
$$V = \{(x, y, z) \in \mathbb{R}^3 \mid y > 0, x \neq 0, z \neq 0\} \cap \{\mathbb{R}^3 \backslash \{z > 0, x = y = 0\}\}$$

One verifies that:

$$\pi_1(U) \simeq \mathbb{Z} \ \pi_1(V) \simeq \mathbb{Z} \ \pi_1(U \cap V) \simeq 0$$

and this concludes the proof. □

Note that the classes $n[\gamma_i] - m[\gamma_j]$ with $i \neq j$, $n, m \in \mathbb{N}$ are not homotopic to one of the generators. On these classes we now evaluate the Action.
Now we can state the following Lemma:

Lemma 5.1. *For all $i = 1, 2, 3$ for any $\lambda \in [\gamma_i]$ there exist $t_1 < t_2$ (depending on λ) such that $\lambda(t_1)$ and $\lambda(t_2)$ are different collinear configurations.*

Proof. Indeed in $\mathcal{M}_r$ collinear configurations forms three planes. We define the varieties π^i_{jk} as the subset of $\mathcal{M}_r$ such that:

$$r_i = r_j + r_k \quad \text{cyclic permutation of } i, j, k$$

We have three π^i_{jk}, they have co-dimension one, $\mathcal{M}_r$ has three dimension. Now the union $\cup'_{i,j,k}\pi^i_{jk}$ (that is $\partial\mathcal{R}D$ see Chapter 1), disconnects $\mathcal{M}_r$. Now coincidence configurations are:

$$l_i = \pi^j_{ik} \cap \pi^k_{ij}$$

Any element $\lambda(t) \in [\gamma_i]$ is homotopic to a generator of $\pi_1(\mathcal{M}_r \backslash K_c)$ which does not intersect l_i and must have points in the two connected part of $\mathcal{M}_r \backslash \partial\mathcal{R}D$. by the continuity of $\lambda(t)$ we conclude that there exist two *different* times $t_1 \neq t_2$ such that

$$\lambda(t_1) \in \pi^k_{ij} \quad \lambda(t_2) \in \pi^j_{ik}$$

□

The Action $\mathcal{A}^\delta_T[.]$ is finite and C^1 on the open set

$$\Gamma_T(\mathcal{M}_r) \doteq H^1_T(\mathcal{M}_r) \cap \{\zeta(t) \notin K_c \ \forall t \in [0,T]\} \tag{46}$$

We now define the classes of trajectories where we study the Action.

Definition 5.4. *We term Σ_T the set of all smooth, closed trajectories $\gamma(t) = \gamma(t+T)$ in $\mathcal{M}_r \backslash K_c$ homotopic to an element of*

$$n[\gamma_i] - m[\gamma_j]$$

for some $i \neq j$ and $m, n \in \mathbb{N}\backslash\{0\}$.

Remark 5.1. *Using the preceding Lemma 5.1 we can always choose the parametrization of $\lambda : [0,T] \to \mathcal{M}_r \backslash K_c$, homotopic to a fundamental cycle, such that:*
if $\lambda \sim \gamma_i$ then $\lambda(t_1) \in \pi^k_{ij}$ and $\lambda(t_2) \in \pi^j_{ik}$ with $t_1 < t_2$,
if $\lambda \sim -\gamma_i$ then $\lambda(t_1) \in \pi^j_{ik}$ and $\lambda(t_2) \in \pi^k_{ij}$ with $t_1 < t_2$.

We now prove an important property of the elements of Σ_T:

Proposition 5.2. *For any $\gamma \in \Sigma_T$ there exist at least four times $0 < t_1 < t_2 < t_3 < t_4 < T$ such that:*

$$\begin{aligned} r_i(t_1) = r_j(t_1) + r_k(t_1), \ \ r_j(t_2) = r_i(t_2) + r_k(t_2) \\ r_k(t_3) = r_j(t_3) + r_i(t_3), \ \ r_j(t_4) = r_i(t_4) + r_k(t_4) \end{aligned} \tag{47}$$

for a sequence of the indices i, j, k.

Proof. Without loss of generality consider the class $[\gamma_i] - [\gamma_j]$. Take λ, one of its element. Now we can continuously deform λ in such a way that it becomes the union of $\lambda_i \in [\gamma_i]$ and $\lambda_j \in -[\gamma_j]$. Up to a reparametrization we can write that λ_i is defined in $[0, T/2)$ and λ_j is defined in $[T/2, T]$. Now we can apply the preceding Lemma 5.1 and Remark 5.1 to λ_i and λ_j and we conclude that

$$\lambda_i(t_1) \in \pi^k_{ij} \quad \lambda_i(t_2) \in \pi^j_{ik}$$

with $t_1 < t_2$ in $[0, T/2)$ and

$$\lambda_j(t_3) \in \pi^i_{jk} \quad \lambda(t_4) \in \pi^k_{ji}$$

with $t_3 < t_4$ in $[T/2, T]$. This concludes the proof. □

Definition 5.5. *We call Γ_4 the $||.||_\infty$-completion of Σ_T.*

In Γ_4 there are trajectories which enter the coincidence set K_c, these are the collision trajectories. We can define a subset of Γ_4 which does not contain collision trajectories, in fact we have:

Proposition 5.3. *If V satisfies SF then for any $c \in (0, +\infty)$ the set*

$$\Lambda_4 \doteq \Gamma_4 \cap \{\zeta \in \Gamma_T(\mathcal{M}_r) \mid \mathcal{A}^\delta_T[\zeta] \leq c\} \tag{48}$$

does not contain collision trajectories.

Proof. By contradiction we assume the there exists a sequence $\{\zeta(.)_n\} \subset \Lambda_4$ which converge in the $||.||_\infty$ to a trajectory $\zeta(.)$ which enters in K_c. Therefore we can consider that there exist $\tau \in [0, T]$ such that $\zeta(\tau) \in K_c$. Then there exists $\epsilon > 0$ such that $||\zeta_n(t)|| < r$ for $t \in [\tau - \epsilon, \tau + \epsilon]$ and n large. For all i

$$\int_0^T dt V(\zeta(t)) \geq \int_0^T dt \frac{1}{(r_i^{(n)}(t))^2}$$

The bound on the Action implies:

$$\int_0^T dt \frac{1}{(r_i^{(n)}(t))^2} \leq c \quad \int_0^T dt [\dot{r}_i^{(n)}(t)]^2 \leq \frac{c}{a_1} \quad \text{for } i = 1, 2$$

$$\int_0^T dt \frac{1}{(r_3(r_1^{(n)}(t)), r_2^{(n)}(t), z^{(n)}(t))^2} \leq c \quad \int_0^T dt [\dot{z}^{(n)}(t)]^2 \leq \frac{c}{a_1}$$

$$\int_\tau^{\tau+\epsilon} dt \frac{1}{(r_i^{(n)}(t))^2} \leq c$$

$$\int_0^T dt [\dot{r}_i^{(n)}(t)]^2 \leq \frac{c}{a_1}$$

but

$$(\ln(r_i^{(n)}(\tau + \epsilon)) - \ln(r_i^{(n)}(\tau)))^2 \leq ||r_i^{(n)}||_{L^2}^2 \int_0^T dt \frac{1}{(r_i^{(n)}(t))^2} \leq \frac{c^2}{a_1}$$

taking $n \to \infty$ we get a contradiction. In fact either $r_i(t) = 0$ for all $t \in [\tau, \tau+\epsilon]$ or $r_i^{(n)}(\tau) \to 0$. □

Note that this Proposition holds for the modified Action $\mathcal{A}_T^\delta[.]$ with $\delta > 0$.

For any trajectory in Σ_T a Poincaré's inequality holds. We now study the trajectories in Σ_T that have four collinear times. Since we consider with time intervals where the trajectories are not collinear we can use as coordinates the relative distances r_1, r_2, r_3.

Proposition 5.4. *For all* $\zeta \in \Sigma_T$*:*

$$\int_0^T dt \langle M(r)\dot{r}, \dot{r}\rangle \geq \frac{8a_1}{9T} \sup_{t\in[0,T]} \min_{l\in\{i,2,3\}} r_l^2(t) \tag{49}$$

Proof. Let us assume the following sequence of collinear configurations:

$$r_3(t_1) = r_2(t_1) + r_1(t_1), \; r_2(t_2) = r_3(t_2) + r_1(t_2)$$
$$r_1(t_3) = r_2(t_3) + r_3(t_3), \; r_2(t_4) = r_3(t_4) + r_1(t_4)$$

We have to estimate the kinetic energy of a trajectory which passes through at least four collinear configurations. Three collinear configurations are *different.*

For simplicity we put:

$$r_i(0) = x_i = \sup_{t\in[0,T]} r_i(t) \text{ with } i = 1, 2, 3 \text{ and } x_1 \leq x_2 + x_3$$
$$r_i(t_1) = \xi_i \text{ with } i = 1, 2, 3 \text{ and } \xi_3 = \xi_2 + \xi_1$$
$$r_i(t_2) = \eta_i \text{ with } i = 1, 2, 3 \text{ and } \eta_2 = \eta_1 + \eta_3$$
$$r_i(t_3) = \nu_i \text{ with } i = 1, 2, 3 \text{ and } \nu_1 = \nu_2 + \nu_3$$
$$r_i(t_4) = \chi_i \text{ with } i = 1, 2, 3 \text{ and } \chi_2 = \chi_1 + \chi_3$$

Now we have:

$$\int_0^T dt\langle M(r)\dot{r},\dot{r}\rangle \geq \sum_{l=0}^{3}\int_{t_l}^{t_{l+1}} dt\langle M(r)\dot{r},\dot{r}\rangle$$

then

$$\int_0^T dt\langle M(r)\dot{r},\dot{r}\rangle \geq \sum_{l=0}^{3}\frac{a_1}{t_{l+1}-t_l}\left[\sum_{i=1}^{3}(r_i(t_l)-r_i(t_{l+1}))^2\right]$$

Now in each interval $[t_l, t_{l+1}]$ we minimize the auxiliary functions:

$$f_{[t_l,t_{l+1}]} = \sum_{i=1}^{3}(r_i(t_l)-r_i(t_{l+1}))^2$$

Taking account of the constraints of collinearity one finds:

$$\min f_{[0,t_1]} = \frac{1}{3}(x_3-x_2-x_1)^2$$

$$\min f_{[t_1,t_2]} = \frac{2^2}{3^3}(x_1+x_3+x_1-x_2)^2$$

$$\min f_{[t_2,t_3]} = \frac{2^2}{3^5}(x_1+x_3+x_1-x_2)^2$$

$$\min f_{[t_3,t_4]} = \frac{2^2}{3^7}(3x_3+4x_2x_2+x_3-x_1)^2$$

Now taking account of the triangle inequalities and that $t_{l+1}-t_l < T$ one finds the thesis. □

5.3. **Periodic solutions of the 3BP.** Now we can prove the main Theorem.

Theorem 5.1. *In the set $\Lambda_4 = \{\mathcal{A}_T^\delta[\zeta] \leq c\} \cap \Gamma_4$ there exist ζ^δ strong T-periodic solution of 3BP reduced on $\mathcal{J}_0$ with SF. The solution ζ^δ converges uniformly in $[0,T]$ to ζ^0 that is a weak T-periodic solution of the reduced 3BP. The limit ζ^0 is a generalized solution of the reduced 3BP.*

Proof. On Λ_T we have that:

$$\mathcal{A}_T^\delta[\zeta] \geq \frac{8a_1}{9T}\min_i\left\{\sup_t r_1^2(t), \sup_t r_2^2(t), \sup_t r_3^2(r_1(t), r_2(t), z(t))\right\} +$$

(50)
$$+\sum_i^2\int_0^T dt\frac{\delta}{r_i^2(t)} + \int_0^T dt\frac{\delta}{(r_3(r_1(t), r_2(t), z(t)))^2}$$

For any sequence $\{\zeta_n\}_n \in \Lambda_4$ such that $||\zeta_n||_\infty \to \infty$ we have

$$\mathcal{A}_T^\delta[\zeta_n] \to +\infty$$

hence the Action is coercive. The Action is C^1 on Λ_4 since no trajectory has a collision. The Action is bounded from below and hence by standard argument:

$$\mathcal{A}_T^\delta[\zeta^\delta] = \min_{\zeta\in\Lambda_4}\mathcal{A}_T^\delta[\zeta]$$

therefore ζ^δ solves the Euler-Lagrange equation for $\mathcal{A}_T^\delta[.]$.

We now prove that when one removes the SF one obtains a weak solution for the 3BP. For all $\delta \in (0,1)$ we have:

$$\mathcal{A}_T^\delta[\zeta^\delta] \leq \mathcal{A}_T^1[\zeta^1] \doteq K < \infty$$

this implies that

$$||\zeta^\delta||_{H_T^1(\mathcal{M}_r)} \leq \frac{K}{a_1}$$

therefore ζ^δ converges weakly in $H^1_T(\mathcal{M}_r)$ and uniformly in $[0,T]$ to trajectory ζ^0.
ζ^0 is different from zero since

$$K > \int_0^T \sum_{ijk}{}' \frac{m_j m_k}{r_i^\delta}$$

(here $r_3 = r_3(r_1, r_2, z)$). The preceding expression would give a contradiction for $\delta \to 0$.

Now we prove that ζ^0 is a generalized solution of the 3BP.
From the previous inequalities we have that:

$$\liminf_{\delta\to 0} \int_0^T dt \left[\sum_{ijk}{}' \frac{m_i m_j}{r_k^\delta} + \sum_i \frac{\delta}{(r_i^\delta)^2} \right] < K$$

(here $r_3 = r_3(r_1, r_2, z)$). Using that $\zeta^\delta \to \zeta^0$ uniformly in $[0,T]$ and Fatou's lemma one finds that the set of collision times has zero Lebesgue measure.
The complement of the collision set is open and dense, we call it I. Take a smooth function w with support in $I \subset [0,T]$. Consider the equations of motion

$$\langle D\mathcal{A}_T^\delta[\zeta^\delta], w\rangle = 0$$

$$\int_I dt \sum_{ij} \bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{w}_i = -\int_I dt \sum ij \sum_k w_k \frac{\partial}{\partial z_k^\delta} \bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{z}_j^\delta +$$

$$-\int_I dt \left[\sum_{ij,k\neq 3}{}' w_k \frac{m_i m_j}{(z_k^\delta)^2} + \sum_{k\neq 3} w_k \frac{2\delta}{(r_k^\delta)^3} \right]$$

$$-\int_I dt \left[w_3 \frac{m_1 m_2}{(r_3(z_1^\delta, z_2^\delta, z_3^\delta))^2} + w_3 \frac{2\delta}{(r_3(z_1^\delta, z_2^\delta, z_3^\delta))^3} \right]$$

For all $t \in I$ and for $\delta \in [0,1]$ we have

$$\sup_{ijk} \left| \frac{\partial M^{ij}(z^\delta)}{\partial z_k^\delta} \right| < c_2$$

and

$$\left| \sum_{ijk\neq 3}{}' \frac{m_i m_j}{(z_k^\delta)^2} + \sum_{k\neq 3} \frac{2\delta}{(r_k^\delta)^3} + \frac{m_1 m_2}{(r_3^\delta)^2} + \frac{2\delta}{(r_3^\delta)^3} \right| < c_3$$

with $c_2, c_3 > 0$ by Lebesgue's dominate convergence theorem we can pass to the limit $\delta \to 0$ getting the weak form of the equations of motion. The strong form of the equations of motion out of the collision set is obtained using standard regularity arguments.
To prove that ζ^0 is a generalized solution we have to prove that the mechanical energy has the same value in all I. The energy is:

$$E_\delta = \frac{1}{2}\sum_{ij} \bar{M}_{ij}(z^\delta)\dot{z}_i^\delta \dot{z}_j^\delta - \sum_{ijk\neq 3}{}' \frac{m_i m_j}{z_k^\delta} - \sum_{k\neq 3} \frac{\delta}{(r_k^\delta)^2} - \frac{m_1 m_2}{r_3^\delta(z)} - \frac{\delta}{(r_3^\delta(z))^2}$$

then one finds

$$E_\delta \leq \frac{1}{T}\left[a_2 \int_0^T dt \sum_k (\dot{z}_k^\delta)^2 - \mathcal{A}_T^\delta[\zeta^\delta] \right]$$

therefore E_δ is bounded when $\delta \to 0$. Then for any t^* for which $\zeta^0(t)$ is generalized solution we have:

$$E_0 = \frac{1}{2}\sum_{ij} \bar{M}(z^0(t^*))\dot{z}_i^0(t^*)\dot{z}_j^0(t^*)) - \sum_{ijk\neq 3}' \frac{m_i m_j}{z_k^0(t^*)} - \frac{m_1 m_2}{r_3(z^0(t^*))}$$

where $E_\delta \to E_0$ (up to a sub-sequence). E_0 does not depend on t^*. □

Now we can prove:

Corollary 5.1. *The weak solution $\zeta^0(.)$ has at most a finite number of collisions.*

Proof. Note that $I_c(\zeta^0) \subset [0,T]$ is bounded and if there are not accumulation points then $I_c(\zeta^0)$ is finite and hence the collisions are isolated.
We now prove that there are no accumulation points in $I_c(\zeta^0)$.
We have seen that $\zeta^\delta(.)$ is a strong T-periodic solution. Let us define the function:

(51) $$\Delta^\delta(t) \doteq \frac{1}{2}\sum_i^3 (r_i(t))^2$$

We define the function (51) using the the relative distances because the finite dimensional metric defined by matrix M is equivalent to the Euclidean metric.
Let us assume that along $\zeta^d e$

(52) $$\frac{d^2}{dt^2}\Delta^\delta(t) > 0 \quad \text{for all } t, \delta \text{ such that } \Delta^\delta(t) < \mu \text{ with } \mu > 0$$

Then for any $t \in [0,T]\backslash I_c(\zeta^0)$ such that $\Delta^0(t) < \mu/2$ we get $\Delta^\delta(t) < \mu$ for δ small enough.
Now $\zeta^\delta \to \zeta^0$ uniformly in $[0,T]\backslash I_c(\zeta^0)$ we obtain

$$\frac{d^2}{dt^2}\Delta^0(t) > 0 \quad \text{for } t \in [0,T]\backslash I_c(\zeta^0) \text{ and } \Delta^0(t) < \mu/2$$

By contradiction, if $\bar{t}$ is an accumulation point of $I_c(\zeta^0)$ one can take a sequence $\{t_n\}_n$ with $t_n < t_{n+1}$ such that $t_n \to \bar{t}$. Then there exists $\bar{t}_n \in [t_n, t_{n+1}]$ where $\Delta^0(.)$ attains its maximum at $\bar{t}_n$. Now $\Delta^0(t)$ is convex then $\Delta^0(\bar{t}_n) = \mu/2$. Hence we get:

$$\mu/2 = \lim_{n\to\infty} \Delta^0(\bar{t}_n) = 0$$

and this is a contradiction.

Now to conclude the Corollary we have to prove (52).
We evaluate the second time derivative of (51) along ζ^δ we write Δ^δ in the coordinates r_1, r_2, r_3. This can be done because any strong solution $\zeta^\delta(.)$ is collinear at most on a discrete set of times (see [1]). We have:

(53) $$\frac{1}{2}\frac{d^2}{dt^2}\sum_i (r_i^\delta)^2 = \sum_i r_i^\delta \ddot{r}_i^\delta + \sum_i (\dot{r}_i^\delta)^2$$

Now the Euler-Lagrange equations are:

For $i = 1, 2$ and j, k are determined by the cyclic permutation

(54) $$2\frac{d}{dt}\sum_l M_{il}\dot{r}_l^\delta = \frac{\partial}{\partial r_i^\delta}\sum_{lm} M_{lm}\dot{r}_l^\delta \dot{r}_m^\delta + \\ -\frac{m_k m_j}{(r_i^\delta)^2} - \frac{2\delta}{(r_i^\delta)^3}$$

moreover the conservation of the energy gives

$$\frac{1}{2}\sum_{ij} M_{ij}(r^\delta)\dot{r}_i^\delta \dot{r}_j^\delta = E_\delta + \sum_{ijk}' \frac{m_i m_j}{r_k^\delta} + \sum_k \frac{\delta}{(r_k^\delta)^2}$$

Substituting into the expression (53) we obtain:

$$\frac{d^2}{dt^2}\Delta^\delta(t) = -\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{jm}}{\partial r_l^\delta}\dot r_l^\delta \dot r_m^\delta +$$

$$+\frac{1}{2}\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{lm}}{\partial r_i^\delta}\dot r_l^\delta \dot r_m^\delta + \frac{1}{a_1}\Delta_1(\zeta^\delta) \tag{55}$$

where

$$\Delta_1(\zeta^\delta) = 2E^\delta + \frac{1}{2}{\sum_{ijk}}' \frac{m_i m_j}{r_k} \tag{56}$$

Now one can evaluate the derivatives of the matrix M by the formula $\partial M = -M\cdot\partial M^{-1}\cdot M$. Matrix M has smooth entries. Using the explicit form of M^{-1} given in section 2 we find:

$$-\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{jm}}{\partial r_l^\delta}\dot r_l^\delta \dot r_m^\delta + \frac{1}{2}\sum_{ijlm} r_i^\delta M_{ij}^{-1}\frac{\partial M_{lm}}{\partial r_i^\delta}\dot r_l^\delta \dot r_m^\delta \geq$$

$$C{\sum_{ijk}}'\left(\frac{(r_k^\delta)^2-(r_i^\delta)^\delta}{r_k^\delta r_j^\delta}\right)\sum_i (\dot r_i^\delta)^2 \tag{57}$$

The constant C depends only on the masses. Considering the properties of regularity of the matrix M, we see that we can choose $z_i(t)$ so small that $\Delta_1(\zeta^\delta)$ is positive definite and it is the main contribution to (55). So there exists μ such that (52) holds. □

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