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# Lectures on Affine Knizhnik-Zamolodchikov Equations, Quantum Many-Body Problems, Hecke Algebras, and Macdonald Theory

By

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# LECTURES ON AFFINE KNIZHNIK-ZAMOLODCHIKOV EQUATIONS, QUANTUM MANY-BODY PROBLEMS, HECKE ALGEBRAS, AND MACDONALD THEORY

Lectured * by
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ABSTRACT. This paper is a mini-course of lectures delivered by the first author at IIAS and prepared for publication by the others. We tried to follow closely the notes of the lectures not yielding the temptation of giving more examples and names. The focus is on the equivalence of the affine Knizhnik-Zamolodchikov equations and quantum many-body problems. In fact the course is an introduction to the new theory of spherical and hypergeometric functions based on affine and double affine Hecke algebras. It can be considered as a difference generalization of the Harish-Chandra theory as well as an important special case of the conformal field theory and the theory of quantum groups. Here mathematics and physics are closer to each other than Siamese twins. We did not try to separate them, but the course turned out to be mainly about the mathematical issues. However we hope that the paper will be understandable for both physicists and mathematicians, for those who want to master the new Hecke algebra technique.

# Contents

## 1. Introduction : Hecke algebras in representation theory

Before a systematic exposition, I will try to outline the main directions of the representation theory and harmonic analysis connected with the Macdonald theory.

A couple of remarks about the growth of Mathematics. It can be illustrated (with all buts and ifs) by the following diagram.

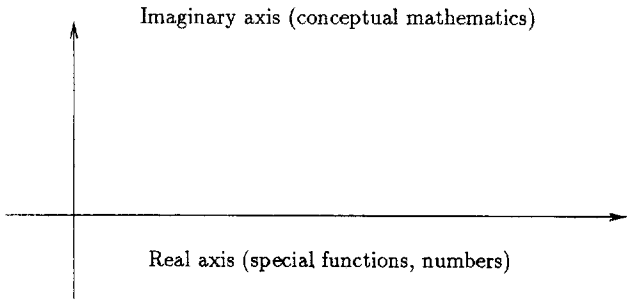


FIGURE .1. Real and Imaginary

It is extremely fast in the imaginary (conceptual) direction but very slow in the real direction. Mainly I mean modern mathematics, but it may be more general. For instance, ancient Greeks created a highly conceptual axiomatic geometry with a modest 'real output'. I do not think that the ratio *Real/Imaginary* is much higher now. There are many theories and a very limited number of functions which are really special. Let us try to project the representation theory on the real axis (Fig.1). We focus on Lie groups(algebras) and Kac-Moody algebras, ignoring the arithmetic direction (adèles and automorphic forms). Look at Fig.2.

1): By this I mean the zonal spherical functions on $K\backslash G/K$ for maximal compact $K$ in a semi-simple Lie group $G$. The theory was started by Gelfand et al. in the early 50's and completed by Harish-Chandra and many others. It generalized quite a few classical special functions. Lie groups helped a lot to elaborate a systematic approach, although much can be done without them, as we will see below.

2): The characters of Kac-Moody algebras can also be introduced without any representation theory (Looijenga, Saito). They are not too far from the products of classical one-dimensional $\theta$-functions. However it is a new and very important class of special functions with various applications. The representation theory explains well some of their properties (but not all).

3): This construction gives a lot of remarkable combinatorial formulas, and generating functions. Decomposing tensor products of finite dimensional representations of compact Lie groups was in the focus of representation theory in the 70's and early 80's, as well as various restriction problems. This direction is still very important, but the representation theory moved towards infinite-dimensional objects.

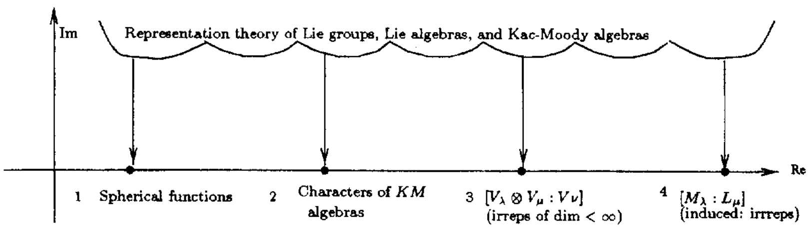


FIGURE .2. Representation Theory

4): Here the problem is to calculate the multiplicities of irreducible representations of Lie algebras in the Verma modules or other induced representations. It is complicated. It took time to realize that these multiplicities are 'real'.

Let us update the picture adding the results which were obtained in the 80's and 90's.

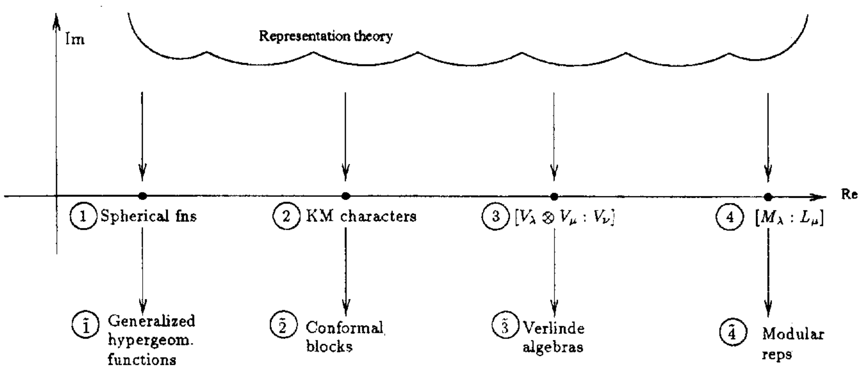


FIGURE .3. New Vintage

$\tilde{1}$): These functions will be the subject of my mini-course. We will study them in the differential and difference cases. It was an old question of how to introduce and generalize them using the representation theory. Now we have an answer.

$\tilde{2}$): Actually conformal blocks belong to the imaginary axis (conceptual mathematics). Only some of them can be considered as 'real' functions. Mostly it happens in the case of KZ-Bernard equation (a sort of elliptic KZ).

$\tilde{3}$): By Verlinde algebras, we mean the category of integrable representations of Kac-Moody algebras of given level with the fusion instead of tensoring. They can be also defined using quantum groups at roots of unity (Kazhdan-Lusztig).
$\tilde{4}$): Whatever you think about the 'reality' of $[M_\lambda : L_\mu]$, these multiplicities are connected with modular representations including the representations of the symmetric group over fields of finite characteristic. Nothing can be more real.

CONJECTURE. *The real projection of the representation theory goes through Hecke-type algebras.*

As to the examples under discussion the picture is as follows:

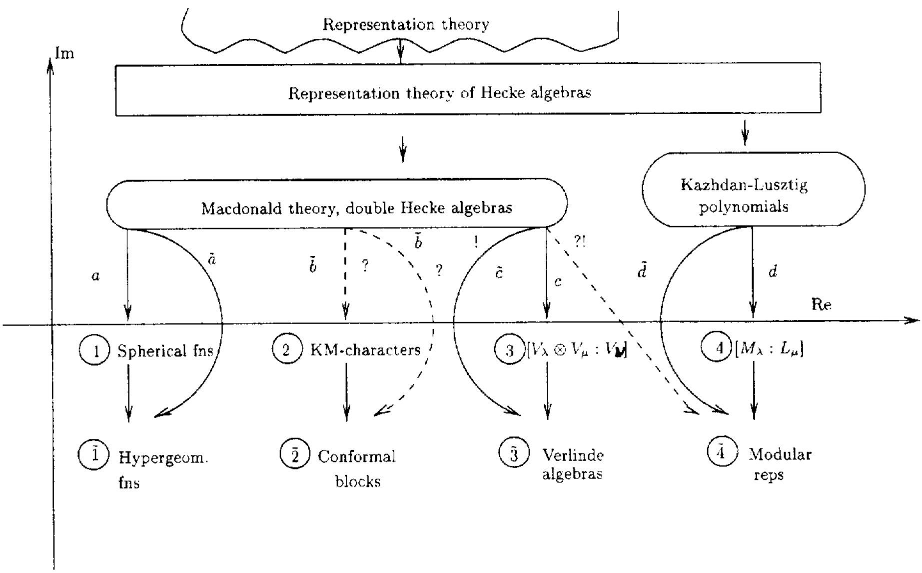


FIGURE .4. Hecke Algebras

$a$): This arrow seems the most recognized now. Several questions in the Harish-Chandra theory (the zonal case) were covered by the representation theory of the degenerate (graded) affine Hecke algebras defined by Lusztig [33]. For instance, the operators from [8, 13] give a very simple approach to the radial parts of Laplace operators on symmetric spaces and the Harish-Chandra isomorphism. The hypergeometric functions (the arrow ($\tilde{a}$)) appear naturally in this way.
Here the main expectations are connected with the difference theory. It was demonstrated in [14] that the difference Fourier transform is self-dual (it is not in the differential case). At least it holds for certain classes of functions. It must simplify and generalize the Harish-Chandra theory. The same program was started in the $p$-adic representation theory (see [15, 18]). The coincidence

of some difference spherical functions with proper Macdonald polynomials can be established using quantum groups (Noumi and others- see[38]). However at the moment the Hecke algebra technique is more efficient to deal with these polynomials (especially for arbitrary root systems).

$b$): The double Hecke algebras lead to a certain elliptic generalization of the Macdonald polynomials [16, 17, 18]. In the differential case there is also the so-called parabolic operator (see [21] and [16]). Still it is not what one could expect. As to ($\tilde{b}$), the conformal blocks of type $GL_n$ (i.e. over the products of curves with the action of the symmetric group) are much more general than the characters. Obviously Hecke algebras are not enough to get all of them. On the other hand, there is almost no theory of the conformal blocks for the configuration spaces connected with other root systems. Double affine Hecke algebras work well for all root systems.

$c$): Here one can rediscover the same combinatorial formulas (mostly based on the so-called Kostant partition function). I do not expect anything brand new. However if you switch to the spherical functions (instead of the characters) then the new theory results in the formulas for the products of spherical functions, which cannot be obtained in the classical theory (they require the difference setting). The multiplicities $[V_\lambda \otimes V_\mu : V_\nu]$ govern the products of the characters, which are the same in the differential and difference theory.
Concerning ($\tilde{c}$), the Macdonald theory at roots of unity gave a simple approach to the Verlinde algebras. All the resultss about the inner product and the action of $SL_2(\mathbb{Z})$ were generalized a lot. I mean [28], and my two papers [14, 15]. A. Kirillov Jr. was the first to find a one-parametric deformation of the Verlinde algebra in the case of $GL_n$. He used quantum groups at roots of unity. My technique is applicable to all root systems. The proofs are much simpler than those based on Kac-Moody algebras or quantum groups. It works even better for the non-symmetric Macdonald polynomials (the conformal blocks and Kac-Moody characters are symmetric in contrast to the main classical elliptic functions).

$d$): This arrow is the Kazhdan-Lusztig conjecture proved by Brylinski-Kashiwara and Beilinson-Bernstein and then generalized to the Kac-Moody case by Kashiwara-Tanisaki. By ($\tilde{b}$), I mean the modular Lusztig conjecture (partially) proved by Anderson, Jantzen, and Soergel. The arrow from the Macdonald theory to modular representations is marked by '!'. It seems the most challenging now. I hope to continue my results on the Macdonald polynomials at roots of unity from the restricted case (alcove) to arbitrary weights (parallelogram). If might give a one-parametric generalization of the classic theory, formulas for the modular characters (not only those for the multiplicities), and a description of modular representations of arbitrary Weyl groups. However now it looks very difficult.

To conclude, let me say a little something about the Verlinde algebras. I think now it is the most convincing demonstration of new methods based on double Hecke algebras. I also have certain personal reasons to be very interested in them. The conformal fusion procedure appeared in my paper 'Functional realization of basic representations of factorizable groups and Lie algebras' (Funct. Anal. Appl., **19** (1985), 36-52). Given an integrable representation of the $n$-th power of a Kac-Moody algebra and two sets of points on a Riemann surface ($n$ and $m$ points), I constructed an integrable representation of the $m$-th power of the same Kac-Moody algebra. The central charge here remains fixed. I missed that in the special case when $n = 2$, $m = 1$ the multiplicities of irreducibles in the resulting representation are structural constants of a certain commutative algebra, the Verlinde algebra. It was nice to know that these multiplicities (and much more) can be extracted from the simplest representation of the double affine Hecke algebra at roots of unity.

I should add one more remark. In fact I borrowed the 'fusion procedure' from arithmetics. I had known Ihara's papers 'On congruence monodromy problem' very well. A similar procedure was the key stone of his theory. Of course I changed something and added something (central charge), but the procedure is basically the same. Can we go back and define Verlinde algebras in arithmetics?

## 2. The affine Knizhnik-Zamolodchikov equation

We introduce the degenerate affine Hecke algebra and the corresponding affine Knizhnik-Zamolodchikov equation. We show how the former appear as the consistency and invariance conditions for the latter.

**2.1. The algebra $\mathcal{H}'_{A_1}$ and the hypergeometric equation.** In this section, we introduce the affine Knizhnik-Zamolodchikov (AKZ) equation associated with the root system of type $A_1$. It is a first-order differential equation for $\Phi$, where $\Phi$ depends on a single variable $u$ and takes values in an infinite-dimensional algebra called the degenerate affine Hecke algebra.

The equation is as follows:

$$\frac{\partial \Phi}{\partial u} = \left(k \frac{s}{e^u - 1} + x\right) \Phi. \tag{2.1}$$

Here $k \in \mathbb{C}$ is a parameter, and $s$ and $x$ are operators acting on a vector space where $\Phi$ takes its value. We impose the following two relations.

$$s^2 = 1, \tag{2.2}$$

$$sx + xs = k. \tag{2.3}$$

These relations make (2.1) invariant. Namely, if $\Phi$ solves (2.1), then

$$\tilde{\Phi}(u) = s\Phi(-u) \tag{2.4}$$

also is a solution of (2.1). We claim that (2.1) is integrable in terms of the classical hypergeometric functions. At least, this statement is valid under a certain irreducibility condition.

The $AKZ$ is the equation (2.1) with values in the *degenerate affine Hecke algebra* $\mathcal{H}'_{A_1}$ of type $A_1$ generated by the elements $s$ and $x$ satisfying the defining relations (2.2) and (2.3):

$$\mathcal{H}'_{A_1} = \langle s, x \rangle / \{(2.2), (2.3)\}. \tag{2.5}$$

Let $\Phi(u)$ be a function of $u$ with values in $\mathcal{H}'_{A_1}$. Note that one can multiply $\Phi(u)$ by an arbitrary constant element on the right, i.e. if $\Phi(u)$ is a solution, then $\Phi(u)a$ $(a \in \mathcal{H}'_{A_1})$ is also a solution. Let us check the invariance of AKZ (see (2.4)).

We plug in

$$\begin{aligned} -\frac{\partial \tilde{\Phi}(u)}{\partial u} &= s\left(k \frac{s}{e^{-u} - 1} + x\right) \Phi(-u) \\ &= \left(k \frac{s}{e^{-u} - 1} + sxs\right) \tilde{\Phi}(u) \end{aligned}$$

and use

$$\frac{1}{1 - e^{-u}} = \frac{1}{e^u - 1} + 1.$$

Finally,

$$\frac{\partial \tilde{\Phi}(u)}{\partial u} = \left(k \frac{s}{e^u - 1} + ks - sxs\right) \tilde{\Phi}(u),$$

where $ks - sxs = x$.

Now we will integrate (2.1). More generally, let us first consider the equation

$$\frac{\partial\Phi}{\partial z} = \left(\frac{A}{1-z} + \frac{B}{z}\right)\Phi. \tag{2.6}$$

It is (2.1) for $z = e^{-u}$, $A = -ks$, and $B = -x$. The equation (2.6) is much more complicated than the AKZ. However, if $A$, $B$ are $2\times 2$ matrices acting on the 2-component vector $\Phi$, this equation is nothing but the hypergeometric differential equation. It readily gives the formulas when $\Phi$ takes values in irreducible representations of $\mathcal{H}'_{A_1}$, because the latter exist only in dimensions 1 or 2.

Indeed, a generic 2-dimensional representation $\rho$ of $\mathcal{H}'_{A_1}$ is given by

$$\rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(x) = k\left(\frac{s}{2} + \begin{pmatrix} 0 & \zeta \\ \xi & 0 \end{pmatrix}\right). \tag{2.7}$$

Because of the gauge transformation $\zeta \to c\zeta$, $\xi \to c^{-1}\xi$, it is characterized by $\zeta\xi$, or by

$$\mu = \left(\zeta\xi + \frac{1}{4}\right)^{1/2}. \tag{2.8}$$

Then a solution $\Phi = \binom{\Phi_1}{\Phi_2}$ for $A = -k\rho(s)$, $B = -\rho(x)$ is given in terms of the Gauss hypergeometric function. The first component is

$$\Phi_1(u) = z^{-k\mu}(1-z)^k F(k(1-2\mu), k, 1-2k\mu; z), \tag{2.9}$$

where $z = e^{-u}$ and $F(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n$ with $(x)_n = x(x+1)\cdots(x+n-1)$.

If $\zeta\xi = 0$, then the representation $\rho$ (2.7) is reducible. In this case, solutions are in terms of elementary functions. We note that the parameters $\alpha, \beta, \gamma$ in (2.9) are not arbitrary but obey the constraint $\alpha + 1 = \beta + \gamma$.

**2.2. The AKZ equation for $GL_n$.** In this section, we introduce the AKZ equation of type $GL_n$. It can be obtained as a specialization of the standard Knizhnik-Zamolodchikov (KZ) equation from the conformal field theory. The consistency and invariance conditions give rise to the defining relations of the degenerate affine Hecke algebra $\mathcal{H}'_{GL_n}$ introduced by Drinfeld.

Recall that the KZ equation reads

$$\frac{\partial\Phi}{\partial z_i} = k\left(\sum_{\substack{0\le j\le n \\ j\ne i}} \frac{\Omega_{ij}}{z_i - z_j}\right)\Phi \quad (0 \le i \le n). \tag{2.10}$$

In the less sophisticated case $\Omega_{ij}$ are the permutation matrices [30]. Let us assume that $\Omega_{ij}$ are any constant elements (operators) and $\Omega_{ij} = \Omega_{ji}$. We consider $\Phi(z)$ $(z = (z_0, \ldots, z_n))$ taking values in the abstract algebra generated by the elements $\Omega_{ij}$. The self-consistency of the system of equations (2.10) means that

$$\frac{\partial A_j}{\partial z_i} - \frac{\partial A_i}{\partial z_j} = [A_i, A_j], \tag{2.11}$$

where

$$A_i = k\sum_{j\ne i} \frac{\Omega_{ij}}{z_i - z_j}. \tag{2.12}$$

It holds for all values of the complex parameter $k$ if and only if

$$[\Omega_{ij}, \Omega_{kl}] = 0, \tag{2.13}$$

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0, \tag{2.14}$$

where the indices $i, j, k, l$ are pairwise distinct. The KZ in this form is due to Aomoto [1] (it was also studied by Kohno [31]).

The trigonometric KZ (and the elliptic ones) where introduced for the first time in [4]. To be more exact, in this paper the equations which I called *the r-matrix KZ* were defined, the connections with Kac-Moody algebras were established, and the reduction procedure was applied to the monodromy (see below). The equations corresponding to the simplest trigonometric $r$-matrces (there are many of them due to Belavin and Drinfeld) are closely connected with the AKZ. There were doubts about the importance of my trigonometric KZ in physics when they appeared. Now they are quite common for both mathematicians and physicists.

Consider the group algebra $\mathbb{C}[\mathbb{S}_n]$ of the permutation group $\mathbb{S}_n$ of the set $\{1, \ldots, n\}$. We denote by $s_{ij}$ the transposition of $i$ and $j$. If we set $\Omega_{ij} = s_{ij}$ $(0 \le i, j \le n)$, the relations (2.13)-(2.14) are satisfied.

Setting

$$z_0 = 0, \tag{2.15}$$

$$\Omega_{ij} = s_{ij} \quad (i, j \neq 0), \tag{2.16}$$

$$\Omega_{0i} = k^{-1}\Omega_i, \tag{2.17}$$

the equation (2.10) turns into

$$\frac{\partial \Phi}{\partial z_i} = \left[ k \left( \sum_{\substack{1 \le j \le n \\ j \neq i}} \frac{s_{ij}}{z_i - z_j} \right) + \frac{\Omega_i}{z_i} \right] \Phi \quad (1 \le i \le n), \tag{2.18}$$

and the relations (2.13),(2.14) read as follows:

$$[s_{ij}, \Omega_i + \Omega_j] = 0, \tag{2.19}$$

$$[k s_{ij} + \Omega_i, \Omega_j] = 0, \tag{2.20}$$

$$[s_{ij}, \Omega_l] = 0, \tag{2.21}$$

where the indices $i, j, l$ are pairwise distinct.

Substituting

$$z_i = e^{v_i}, \tag{2.22}$$

we come to

$$\frac{\partial \Phi}{\partial v_i} = \left( k \sum_{j \neq i} \frac{s_{ij}}{1 - e^{v_j - v_i}} + \Omega_i \right) \Phi. \tag{2.23}$$

Using the elements

$$y_i = \Omega_i + k \sum_{j > i} s_{ij}, \tag{2.24}$$

$$\frac{\partial\Phi}{\partial v_i} = \left(k\sum_{j>i}\frac{s_{ij}}{e^{v_i-v_j}-1} - k\sum_{j<i}\frac{s_{ij}}{e^{v_j-v_i}-1} + y_i\right)\Phi. \tag{2.25}$$

The elements $\{y\}$ are convenient since in the limit

$$v_1 \gg v_2 \gg \cdots \gg v_n,$$

we get the system

$$\frac{\partial\Phi}{\partial v_i} = y_i\Phi. \tag{2.26}$$

The consistency of these equations is equivalent to the commutativity

$$[y_i, y_j] = 0. \tag{2.27}$$

We claim that (2.27), a 'limiting self-consistency', together with the relations

$$[s_i, y_j] = 0 \quad \text{if } j \neq i, i+1, \tag{2.28}$$

$$s_i y_i - y_{i+1} s_i = k, \tag{2.29}$$

where $s_i = s_{i\,i+1}$ $(1 \le i \le n-1)$, ensure (2.19)-(2.21).

It can be put in the following way. Let us introduce the *degenerate affine Hecke algebra of type $GL_n$* as an algebraic span of $\mathbb{C}[\mathbb{S}_n]$ and $y_i$ $(1 \le i \le n)$ with the relations (2.27), (2.28) and (2.29), denoting it by $\mathcal{H}'_{GL_n}$, or simply by $\mathcal{H}'_n$. We call the system (2.25) with the values in $\mathcal{H}'_n$ the *AKZ of type $GL_n$*. It is well-defined, i.e. self-consistent.

**2.3. The $\mathbb{S}_n$-invariance.** In this section, we discuss the $\mathbb{S}_n$-symmetry of the AKZ equation introduced in the previous section and clarify in full the definition of $\mathcal{H}'_n$.

The group $\mathbb{S}_n$ acts on $\mathbb{C}^n$ naturally by

$$v = (v_1, \dots, v_n) \in \mathbb{C}^n \mapsto w(v) = (v_{i_1}, \dots, v_{i_n}) \in \mathbb{C}^n$$

for $w^{-1} = (i_1, i_2, \dots, i_n) \in \mathbb{S}_n$. Given a function $\Phi(v)$ of $v \in \mathbb{C}^n$ with values in $\mathcal{H}'_n$, we define the action of $w \in \mathbb{C}[\mathbb{S}_n]$ on $\Phi(v)$ by

$$(w(\Phi))(v) = w \cdot \Phi\Big(w^{-1}(v)\Big). \tag{2.30}$$

Here the dot means the product in $\mathcal{H}'_n$. It follows from (2.28) and (2.29) that if $\Phi$ solves (2.25), then so does $w(\Phi)$. Just conjugate the equations by $\{s_i\}$. Moreover, the invariance is exactly equivalent to the relations (2.28) and (2.29). Thus the invariance and the limiting self-consistency (2.27) give the self-consistency of our system for all $k$.

**2.4. Degenerate affine Hecke algebras.** In this section, we fix notations for root systems and define the degenerate affine Hecke algebra for an arbitrary root system.

Let $\Sigma$ be a root system in $\mathbb{R}^n$ with the inner product $(\ ,\ )$. Choose a system of simple roots $\alpha_1, \dots, \alpha_n$ of $\Sigma$ and denote by $\Sigma_+$ the set of positive roots. For a root $\alpha \in \Sigma$, define the coroot $\alpha^\vee$ by

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

and the reflection $s_\alpha$ by

$$s_\alpha(u) = u - (\alpha^\vee, u)\alpha \quad (u \in \mathbb{R}^n).$$

We will denote $s_{\alpha_i}$ simply by $s_i$. The fundamental coweights $b_i$ are as follows:

$$(b_i, \alpha_j) = \delta_{ij}.$$

We also use the notation $a_i = \alpha_i^\vee$. For $u \in \mathbb{R}^n$, the coordinates will be $u_i = (u, \alpha_i)$. We also set $u_\alpha = (u, \alpha)$ for $\alpha \in \Sigma$. Check that

$$\frac{\partial u_\alpha}{\partial u_i} = \nu_\alpha^i = (b_i, \alpha) = \text{the multiplicity of } \alpha_i \text{ in } \alpha.$$

Let $W$ be the Weyl group of $\Sigma$: $W = \langle s_\alpha, \alpha \in \Sigma \rangle = \langle s_1, \dots, s_n \rangle$. Define the action of $W$ on functions on $\mathbb{R}^n$ by

$$^w f(u) = f(w^{-1}(u)) \quad (u \in \mathbb{R}^n). \tag{2.31}$$

Then we have

$$^w u_\alpha = (w^{-1}(u), \alpha) = u_{w(\alpha)}.$$

Now we can define the degenerate affine Hecke algebra $\mathcal{H}'_\Sigma$ associated with $\Sigma$. This definition is due to Lusztig [33] (he calls it the graded affine Hecke algebra, considering $k$ as a formal parameter). Drinfeld introduced this algebra in the $GL_n$-case in [19] prior to Lusztig. These algebras are natural degenerations of the corresponding $p$-adic ones.

DEFINITION 2.1. Let $\mathcal{H}'_\Sigma$ be the associative algebra generated by $\mathbb{C}[W]$ and $x_1, \dots, x_n$ with the following relations

$$[x_i, x_j] = 0, \quad \forall i, j, \tag{2.32}$$
$$[s_i, x_j] = 0, \quad \text{if} \quad i \neq j, \tag{2.33}$$
$$s_i x_i - \hat{x}_i s_i = k. \tag{2.34}$$

Here $k$ is a complex number and

$$\hat{x}_i = x_i - \sum_{j=1}^{n} (\alpha_i^\vee, \alpha_j) x_j. \tag{2.35}$$

Introducing

$$x_b = \sum_{i=1}^{n} (b, \alpha_i) x_i = \sum_{i=1}^{n} k_i x_i \quad \text{for} \quad b = \sum_{i=1}^{n} k_i b_i, \tag{2.36}$$

we can express the right hand side of (2.35) as $x_{s_i(b_i)} = x_{b_i} - x_{a_i}$. More generally,

$$s_i x_b - x_{s_i(b)} s_i = x_b s_i - s_i x_{s_i(b)} = k(b, \alpha_i). \tag{2.37}$$

Later we will use the following partial derivatives

$$\partial_b(u_\alpha) = (\alpha, b). \tag{2.38}$$

For instance,

$$\frac{\partial}{\partial u_i} = \partial_i = \partial_{b_i}$$

**2.5. The AKZ equation associated with $\mathcal{H}'_\Sigma$.** In this section we introduce the AKZ equation associated with the root system $\Sigma$, and give several examples.

Let us consider the following system of partial differential equations

$$\frac{\partial \Phi}{\partial u_i} = \left( k \sum_{\alpha \in \Sigma_+} \nu^i_\alpha \frac{s_\alpha}{e^{u_\alpha} - 1} + x_i \right) \Phi \qquad (1 \leq i \leq n). \tag{2.39}$$

Here $k$ is a complex number. We denote the right hand side of (2.39) as $A_i\Phi$. First we assume that $\Phi$ takes values in an associative algebra generated by $\mathbb{C}[W]$ and $x_1, \ldots, x_n$. We say that the system (2.39) is *self-consistent* provided that

$$[\frac{\partial}{\partial u_i} - A_i, \frac{\partial}{\partial u_j} - A_j] = 0. \tag{2.40}$$

It is called *invariant* if, for any solution $\Phi$ of (2.39) and any element $w$ of $W$, $w(\Phi)$ (see 2.30) is again a solution of (2.39).

THEOREM 2.1. *The system* (2.39) *is self-consistent and invariant if and only if* $s_1, \ldots, s_n, x_1, \ldots, x_n$ *satisfy the relations* (2.32), (2.33), (2.34) *defining* $\mathcal{H}'_\Sigma$.

We introduce the AKZ equation associated with $\Sigma$ to be the system (2.39) for functions $\Phi$ with values in $\mathcal{H}'_\Sigma$.

Using the notation $x_b$ and $\partial_b$ from (2.36) and (2.38), the system (2.39) can be expressed as

$$\partial_b \Phi = \left( k \sum_{\alpha \in \Sigma_+} (b, \alpha) \frac{s_\alpha}{e^{u_\alpha} - 1} + x_b \right) \Phi. \tag{2.41}$$

REMARK 2.1. The parameter $k$ may depend on the lengths of roots. Generally speaking the AKZ equation is as follows:

$$\frac{\partial \Phi}{\partial u_i} = \left( \sum_{\alpha \in \Sigma_+} k_{|\alpha|} \nu^i_\alpha \frac{s_\alpha}{e^{u_\alpha} - 1} + x_i \right) \Phi. \tag{2.42}$$

Let us write down the explicit forms of the AKZ equation in the simplest cases.

EXAMPLE 2.1. When $\Sigma = A_1$, the AKZ equation is exactly (2.1).

EXAMPLE 2.2. For $A_2$, the AKZ equation is

$$\begin{aligned} \frac{\partial \Phi}{\partial u_1} &= \left\{ k \left( \frac{s_{12}}{e^{u_1} - 1} + \frac{s_{13}}{e^{u_1 + u_2} - 1} \right) + x_1 \right\} \Phi, \\ \frac{\partial \Phi}{\partial u_2} &= \left\{ k \left( \frac{s_{23}}{e^{u_2} - 1} + \frac{s_{13}}{e^{u_1 + u_2} - 1} \right) + x_2 \right\} \Phi, \end{aligned}$$

where $s_{ij}$ denotes the transposition of $i$ and $j$. In this case $\hat{x}_1 = x_2 - x_1, \quad \hat{x}_2 = x_1 - x_2$.

EXAMPLE 2.3. The root system $B_2$ is realized in the following way. Let $\epsilon_1$ and $\epsilon_2$ form an orthonormal basis of $\mathbb{R}^2$. Then the set of positive roots consists of the following vectors:

$$\alpha_1 = \epsilon_1 - \epsilon_2,$$
$$\alpha_2 = \epsilon_2,$$
$$\alpha_1 + \alpha_2 = \epsilon_1,$$
$$\alpha_1 + 2\alpha_2 = \epsilon_1 + \epsilon_2.$$

Let $s = s_1$ and $t = s_2$. Then $s$ and $t$ satisfy $tsts = stst$ (the Coxeter relation for $W_{B_2} = W_{C_2}$) and $s^2 = 1$, $t^2 = 1$. In this case $\hat{x_1} = x_2 - x_1$, $\hat{x_2} = 2x_1 - x_2$. The AKZ equation reads as follows:

$$\frac{\partial\Phi}{\partial u_1} = \left\{k\left(\frac{s}{e^{u_1}-1} + \frac{sts}{e^{u_1+u_2}-1} + \frac{tst}{e^{u_1+2u_2}-1}\right) + x_1\right\}\Phi$$
$$\frac{\partial\Phi}{\partial u_2} = \left\{k\left(\frac{t}{e^{u_2}-1} + \frac{sts}{e^{u_1+u_2}-1} + 2\frac{tst}{e^{u_1+2u_2}-1}\right) + x_2\right\}\Phi.$$

Note the appearance of the coefficient 2 in the latter. In the case of $E_8$ the coefficients are from 1 to 6 (otherwise they are less than 6).

**2.6. The $A_{n-1}$ case.** In this section, we will show that the AKZ equation of type $GL_n$ discussed in §2.2 reduces to the AKZ equation for the root system $\Sigma \subset \mathbb{R}^{n-1}$ of type $A_{n-1}$.

First note that

$$x = y_1 + \ldots + y_n \tag{2.43}$$

is central in the algebra $\mathcal{H}'_n$. By setting

$$x_i = y_1 + \ldots + y_i - \frac{i}{n}x, \tag{2.44}$$

we have an embedding of $\mathcal{H}'_\Sigma$, where $\Sigma$ is the root system of type $A_{n-1}$, into $\mathcal{H}'_n$. We put

$$u_i = v_i - v_{i+1} \quad (1 \le i \le n-1). \tag{2.45}$$

The space $\mathbb{R}^{n-1}$ is identified with the quotient space of $\mathbb{R}^n = \{\sum_{i=1}^n v_i\epsilon_i \mid v_i \in \mathbb{R}\}$

$$\mathbb{R}^{n-1} \simeq \oplus_{i=1}^n \mathbb{R}\epsilon_i/\mathbb{R}\epsilon$$

where $\{\epsilon_i\}_{1\le i\le n}$ is the orthonormal basis and $\epsilon = \epsilon_1 + \ldots + \epsilon_n$. From (2.25) we have

$$\sum_{i=1}^n \frac{\partial\Phi}{\partial v_i} = x\Phi.$$

Therefore, the function

$$\Phi'(v) = e^{-x\cdot\frac{1}{n}(v_1+\cdots+v_n)}\Phi(v)$$

is well-defined on the quotient space $\mathbb{R}^{n-1}$. Now it is straightforward to see that (2.25) reduces to the AKZ equation of type $A_{n-1}$ for $\Phi'(v)$

$$\frac{\partial\Phi'}{\partial u_i} = \left(k\sum_{j\le i<l}\frac{s_{jl}}{e^{u_j+\cdots+u_{l-1}}-1} + x_i\right)\Phi' \qquad (1 \le i \le n-1).$$

## 3. Isomorphism theorems for the AKZ equation

We introduce the affine Hecke algebra $\mathcal{H}^{t}_{\Sigma}$ and connect them with the degenerate affine Hecke algebra $\mathcal{H}'_{\Sigma}$ using the monodromy of the AKZ equation. We also establish an isomorphism between the solution space of the AKZ equation and that of a quantum many body problem.

**3.1. Representations of $\mathcal{H}'_{\Sigma}$.** In this section we define induced representations of $\mathcal{H}'_{\Sigma}$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the character of $\mathbb{C}[x_1, \ldots, x_n]$ (i.e. a ring homomorphism $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$) is an assignment $x_i \mapsto \lambda_i$. We denote it by $\lambda$.

DEFINITION 3.1. We define an $\mathcal{H}'_{\Sigma}$-module $I_\lambda$ as the representation induced from $\lambda$ :

$$I_\lambda = \mathrm{Ind}^{\mathcal{H}'_{\Sigma}}_{\mathbb{C}[x_1,\ldots,x_n]}(\lambda) = \mathcal{H}'_{\Sigma} \otimes_{\mathbb{C}[x_1,\ldots,x_n]} \mathbb{C}_\lambda. \tag{3.1}$$

Here $\mathbb{C}_\lambda$ is endowed with the $\mathbb{C}[x_1, \ldots, x_n]$-module structure by the character $\lambda$.

We have the Poincaré-Birkhoff-Witt type theorem for $\mathcal{H}'_{\Sigma}$. Namely any $h \in \mathcal{H}'_{\Sigma}$ is expressed uniquely in either of the following ways:

$$h = \sum_{w \in W} p_w(x) w = \sum_{w \in W} w q_w(x) \tag{3.2}$$

with $p_w, q_w \in \mathbb{C}[x_1, \ldots, x_n]$. The existence results from the relations (2.32)–(2.34) in $\mathcal{H}'_{\Sigma}$. Hence

$$I_\lambda = \mathbb{C}[W] = \oplus_{w \in W} \mathbb{C} w. \tag{3.3}$$

Thus $I_\lambda$ is $\mathbb{C}[W]$ as a $W$-module, where the action of $x_i$ is determined by $x_i(e) = \lambda_i e$ for the identity $e \in W$. The action of $x_i$ on other elements of $\mathbb{C}[W]$ have to be determined using the defining relation (similar to the calculations in the Fock representation).

We also need another construction. Let $J$ be induced from the trivial character $+ : W \longrightarrow \mathbb{C},\ w \to 1$. Then

$$J = \mathrm{Ind}^{\mathcal{H}'_{\Sigma}}_{\mathbb{C}[W]}(+), \tag{3.4}$$

is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]$ as a vector space and moreover as a $\mathbb{C}[x_1, \ldots, x_n]$-module. To get finite-dimensional representations from $J$, we use the coincidence of the center of $\mathcal{H}'_{\Sigma}$ with the algebra of $W$-invariant polynomials in $x_i$. This theorem is due to Bernstein. The procedure is as follows. Let us fix an element $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and introduce the ideal $L_\lambda$ in $\mathbb{C}[x_1, \ldots, x_n]$ generated by $p(x) - p(\lambda)$ for all $W$-invariant polynomials $p$. Set $J_\lambda = J/L_\lambda$. Then $J_\lambda$ has a structure of $\mathcal{H}'_{\Sigma}$-module by virtue of the Bernstein theorem.

We will also use the anti-involution $^\circ$ on $\mathcal{H}'_{\Sigma}$:

$$x_i^\circ = x_i, \quad s_i^\circ = s_i,\ (ab)^\circ = b^\circ a^\circ,\ k^\circ = k. \tag{3.5}$$

Since the relations of $\mathcal{H}'_{\Sigma}$ are self-dual it is well-defined. For an $\mathcal{H}'_{\Sigma}$-module $V$, we consider its dual $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The dual has an anti-action (a right action) of $\mathcal{H}'_{\Sigma}$. Composing it with the anti-automorphism $^\circ$, we get a natural (left) action of $\mathcal{H}'_{\Sigma}$. We denote the resulting module by $V^\circ$.

We write $\lambda_b = \sum k_i \lambda_i$ for $b = \sum k_i b_i$.

THEOREM 3.1. (a) *$I_\lambda$ is irreducible if and only if $\lambda_{\alpha^\vee} \neq \pm k$ for any $\alpha \in \Sigma_+$.*

(b) *There exists a permutation $\lambda'$ of $\lambda$ (i.e. $\lambda' = w(\lambda)$ for $w \in W$) such that $\lambda'_{\alpha^\vee} \neq -k$ for any $\alpha \in \Sigma_+$. Then*

$$J_\lambda \simeq I_{\lambda'}. \tag{3.6}$$

(c) *For the longest element $w_0$ in $W$,*

$$I_\lambda^\circ = I_{w_0(\lambda)} \tag{3.7}$$

A key lemma in proving Theorem 3.1 is

LEMMA 3.2. *$I_{(0,\dots,0)}$ is irreducible.*

The proof from [14] is based on the intertwining operators of degenerate affine Hecke algebras (to be defined below). See also [29, 27, 41] and the references therein (the non-degenerate case).

DEFINITION 3.2. For $1 \le i \le n$ we set

$$f_i = f_{s_i} = s_i - \frac{k}{x_{a_i}}. \tag{3.8}$$

For $w \in W$ with a reduced decomposition $w = s_{i_n} \cdots s_{i_1}$, $f_w = f_{i_n} \cdots f_{i_1}$. We call the elements $f_w$ *intertwiners.*

The elements $f_w$ belong to the localization of the degenerate affine Hecke algebra $\mathcal{H}'_\Sigma$ by the $W$-invariant polynomials. They give a certain 'baxterization' of $w$, and are closely related to the Yang's $R$-matrix. Let us show that $f_w$ does not depend on the choice of the reduced decomposition of $w$.

We have

$$f_{s_i} x_b = x_{s_i(b)} f_{s_i}, \tag{3.9}$$

$$f_w x_b = x_{w(b)} f_w. \tag{3.10}$$

Indeed,

$$\left(s_i - \frac{k}{x_{a_i}}\right) x_b = x_{s_i(b)} \left(s_i - \frac{k}{x_{a_i}}\right), \tag{3.11}$$

which can be rewritten as follows:

$$s_i x_b - x_{s_i(b)} s_i = -k \frac{x_{s_i(b)} - x_b}{x_{a_i}}. \tag{3.12}$$

Using the definition of $x_b$ (2.36), the right hand side of (3.12) is $k(b, \alpha_i)$. So we come to (2.37). The relations (3.10) fix $f_w$ uniquely up to the multiplication on the right by functions in $x$. The leading terms of $f_w$ being $w$, they coincide for any reduced decompositions.

To demonstrate the role of intertwiners, let us check the irreducibility of $I_\lambda$ for generic $\lambda$. First note that the vectors $\{f_w(e) \in I_\lambda\}$ are common eigenvectors of $x_b$., because

$$x_b f_w(e) = f_w x_{w^{-1}(b)}(e) = \lambda_{w^{-1}(b)} f_w(e),$$

For a generic $\lambda$ the cigenvalues are simple, hence these vectors are linearly independent. Now, any nonzero $\mathcal{H}'_\Sigma$-submodule $A$ of $I_\lambda$ contains at least one eigenvector of $x_b$. By the simplicity of eigenvalues, such an eigenvector must be in the form $f_w(e)$ for some $w \in W$. On the other hand, $f_w$ are invertible elements. Indeed,

$$f_i^{-1} = (1 - \frac{k^2}{x_{a_i}{}^2})^{-1} f_i.$$

Therefore $e \in A$. Since $I_\lambda$ is generated by $e$, we conclude that $A = I_\lambda$.

Actually this very reasoning leads to the proof of the Theorem (a),(b). However if $\lambda$ is arbitrary one must operate with the intertwiners much more carefully. It is necessary to multiply them by the denominators and remember that the invertibility does not hold for special $\lambda$.

REMARK 3.1. The $\mathcal{H}'$-quotients $A$ of $J_\lambda^\circ$ will be interpreted below as certain quotients of the $D$-module representing the quantum many-body eigenvalue problem. A solution of the AKZ in $J_\lambda^\circ$ induces solutions in any of its $\mathcal{H}'$-quotients (if $I$ is reducible). It gives a one-to-one correspondence between the $\mathcal{H}'$-submodules (quotients, constituents) of $J$ and those of the $D$-modules representing the quantum many-body eigenvalue problem. The description of the latter is an analytical problem. The classification of the former is a difficult question in the representation theory of Hecke algebras. For instance, the multiplicities of the irreducible constituents are described in terms of the Kazhdan-Lusztig polynomials. It is very interesting to combine the two approaches together.

**3.2. The monodromy of the AKZ equation.** In this section we discuss the monodromy of the AKZ equation, which is a key ingredient in establishing the isomorphism between the AKZ equation in the representation $J_\lambda^\circ$ and the *quantum many-body problem* (QMBP) with the eigenvalue $\lambda$.

Let $U'$ be the open subset of $\mathbb{C}^n$ given by

$$U' = \{u \in \mathbb{C}^n | \prod_{\alpha \in \Sigma_+} (e^{u_\alpha} - 1) \neq 0\}. \tag{3.13}$$

The lattice generated by $b_1, \ldots, b_n$ will be denoted by $B$. It is isomorphic to $\mathbb{Z}^n$. and acts on $\mathbb{C}^n$ by translations. Namely, $b(u) = u + 2\pi\sqrt{-1}b$, where $b \in B$. The semi-direct product $\widetilde{W} = W \ltimes B$ is the so-called extended affine Weyl group, acting on $\mathbb{C}^n$ and leaving $U'$ invariant. Picking $u^0 \in U'$, we set

$$\pi_1 = \pi_1(U'/\widetilde{W}, u^0).$$

The group structure of $\pi_1$ is described as follows. Given an element $w \in \widetilde{W}$, let $\gamma_w$ be a path from $u^0$ to $w^{-1}(u^0)$ in $U'$. For elements $w_1$, $w_2 \in \widetilde{W}$, we define the composition $\gamma_{w_2} \circ \gamma_{w_1}$ of $\gamma_{w_1}$ and $\gamma_{w_2}$ as the path composed of $\gamma_{w_1}$ and the path $\gamma_{w_2}$ mapped by $w_1^{-1}$ (see Fig.5). The class of $\gamma$ will be denoted by $\bar{\gamma}$. The map $\bar{\gamma}_w \to w$ is a homomorphism onto $\widetilde{W}$.

It is convenient to choose $u^0$ and the generators of $\pi_1$ as follows. Set $\Re = \mathrm{Re}, \Im = \mathrm{Im}$,

$$C = (\sqrt{-1}\mathbb{R})^n \setminus \{u \in (\sqrt{-1}\mathbb{R})^n \,|\, 0 < \Im u_\alpha < 2\pi \text{ for every } \alpha \in \Sigma_+\}.$$

Then $\mathbb{C}^n \setminus C$ is a simply connected open subset of $U'$. Let us take $u^0 \in \mathbb{C}^n$ such that $\Re u_\alpha^0 \gg 0$ for $\alpha \in \Sigma_+$. For any element $w \in \widetilde{W}$, we denote a path from $u^0$ to $w^{-1}(u^0)$ in $\mathbb{C}^n \setminus C$ by $\gamma_w$. This condition simply means that whenever $u_\alpha \in \imath\mathbb{R}$ intersects the imaginary axis it must go through the 'window' $0 < \Im u_\alpha < 2\pi$.

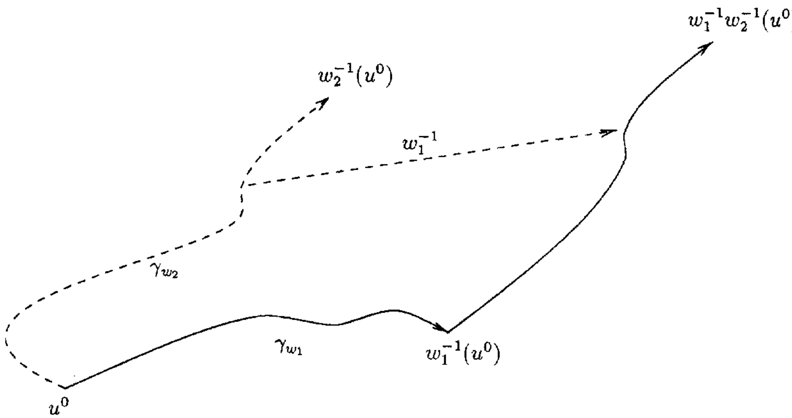


FIGURE .5. Composition of paths

For any element $w \in \widetilde{W}$ we define an element $\bar{\gamma}_w$ of $\pi_1$ to be the image of $\gamma_w$. Since $\mathbb{C}^n \setminus C$ is simply connected, $\bar{\gamma}_w$ depends only on $w$. We set $\tau_i = \gamma_{s_i}$ and choose $\chi_i$ to be a path from $u^0$ to the point $u'$ with the same coordinates $u'_j = u^0_j$ for $j \neq i$ and $u'_i = u^0_i + 2\pi\sqrt{-1}$. The structure of $\pi_1$ is described in the following theorem from [32].

THEOREM 3.3.

$$\pi_1 = \langle \bar{\tau}_1, \dots, \bar{\tau}_n, \bar{\chi}_1, \dots, \bar{\chi}_n \rangle, \tag{3.14}$$

$$\bar{\tau}_i \textit{ satisfy the Coxeter relations,} \tag{3.15}$$

$$[\bar{\chi}_i, \bar{\chi}_j] = [\bar{\tau}_i, \bar{\chi}_j] = 0 \quad (i \neq j), \tag{3.16}$$

$$\bar{\tau}_i^{-1}\bar{\chi}_i\bar{\tau}_i^{-1} = \bar{\chi}_{s_i(b_i)}. \tag{3.17}$$

*Here for* $b = \sum_{i=1}^n k_i b_i$ *we put*

$$\bar{\chi}_b = \prod_{i=1}^n \bar{\chi}_i^{k_i}. \tag{3.18}$$

Fig.6 proves the relation (3.17). It showes the $u_i$-coordinate only, which is sufficient for this relation.

Let us introduce the affine Hecke algebra $\mathcal{H}^t_\Sigma$ associated with a root system $\Sigma$ as a quotient of the group algebra of $\pi_1$ by the quadratic relations.

DEFINITION 3.3. The affine Hecke algebra associated with a root system $\Sigma$ is an associative $\mathbb{C}$-algebra generated by $1, T_1, \dots, T_n, X_1, \dots, X_n$ with the following relations:

$$T_i \text{ satisfy the Coxeter relations,} \tag{3.19}$$

$$[X_i, X_j] = [T_i, X_j] = 0 \quad i \neq j, \tag{3.20}$$

$$T_i^{-1}X_iT_i^{-1} = X_{s_i(b_i)}, \tag{3.21}$$

$$(T_i - t)(T_i + t^{-1}) = 0. \tag{3.22}$$

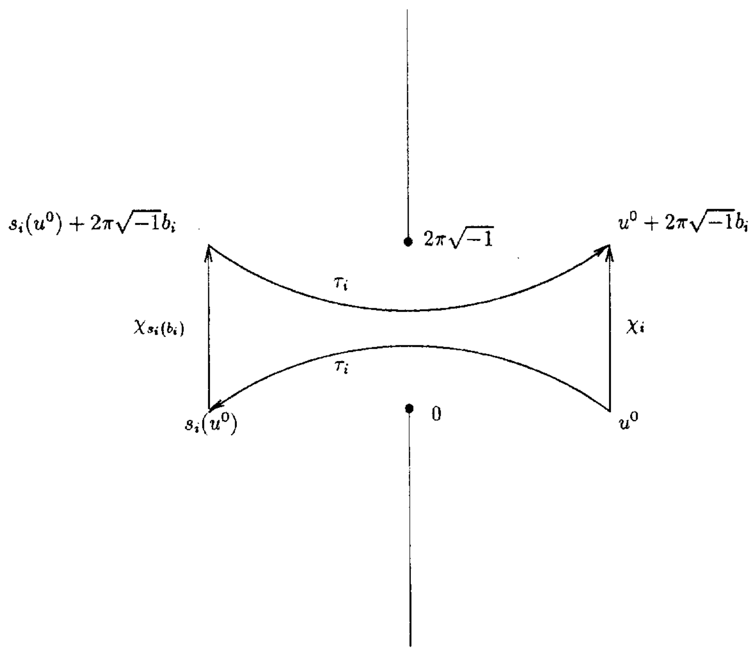


FIGURE .6. Proof of the relation (3.17)

The monomials $X_b$ are defined as in (3.18), $t \in \mathbb{C}^*$. Here and above we mean the homogeneous Coxeter relations: $T_iT_jT_i... = T_jT_iT_j...$, $m_{ij}$ factors on each side, where $m_{ij} = 2, 3, 4$ whenever the corresponding vertices in the Dynkin diagram are connected by 0,1,2 laces.

Let $\Phi$ be an invertible solution of the AKZ equation associated with $\Sigma$, defined in a neighborhood of $u^0$. Then, for $w \in \widetilde{W}$, $w^{-1}(\Phi)$ is defined near $w^{-1}(u^0)$ (see (2.30)). Let $\gamma$ be a path in $U'$ from $u^0$ to $w^{-1}(u^0)$. Denote by $(w^{-1}(\Phi))_{\bar{\gamma}}$ the analytic continuation of $w^{-1}(\Phi)$ back to $u^0$ along the path $\gamma$, where $\bar{\gamma}$ denotes the class of $\gamma$ in the fundamental group $\pi_1$. We will also use the projection homomorphism $\widetilde{W} \to W$ sending $w = \bar{w}b$ to $\bar{w}$ for $b \in B, \bar{w} \in W$. Using this homomorphism we can extend the action of $W$ from (2.30) to $\widetilde{W}$, multiplying $\Phi$ on the left by $\bar{w}$.

Let us define the monodromy $T_{\bar{\gamma}}$ to be the ratio

$$T_{\bar{\gamma}} = \left(w^{-1}(\Phi)\right)_{\bar{\gamma}}^{-1} \Phi = (\Phi(w(u)))_{\bar{\gamma}}^{-1} \cdot \bar{w} \cdot \Phi. \tag{3.23}$$

Here dot means the product in $\mathcal{H}'_{\Sigma}$. Since $\Phi$ and $w^{-1}(\Phi)$ both satisfy the same AKZ equation, $T_{\bar{\gamma}}$ does not depend on $u$. So it is an invariant of the homotopy class of $\gamma$ and is always invertible. If we choose $u^0$ and the paths $\gamma_w$ in $\mathbb{C}^n \setminus C$ as above, then $T_w$ for $w \in \widetilde{W}$ are well-defined. The monodromy is a homomorphism from $\pi_1$ (but not from $\widetilde{W}$), which readily results from the definition.

As a preparation for an explicit computation of $\{T_w\}$ in the next section, we shall introduce a special class of solutions $\Phi$.

PROPOSITION 3.4. *For generic $\lambda$, there exists a unique solution $\Phi_{as}(u)$ of the AKZ equation such that*

$$\Phi_{as}(u) = \hat{\Phi}(u)e^{\sum_{i=1}^n u_ix_i} \quad for \tag{3.24}$$

$$\hat{\Phi}(u) = 1 + \sum_{m=(m_1,\dots,m_n), m_i\geq 0, m\neq 0} \Phi_m e^{-\sum_{i=1}^n m_iu_i}, \tag{3.25}$$

*where $\Re u \longrightarrow \infty$, and $\Phi_m$ are independent of $u$.*

We call the solution in the proposition the *asymptotically free solution.* To be more exact, we need either to complete $\mathcal{H}'_{\Sigma}$, or restrict ourselves with finite-dimensional representations of this algebra. Then establishing the the (local) convergence is easy. In these notes we will follow the second way. We give general formulas, which are quite rigorous in finite-dimensional representations (say, in the induced representations).

Let us examine the condition necessary for the existence of the asymptotically free solutions in the case of $A_1$. A general consideration follows the same lines. In this case,

$$\Phi_{as}(u) = \left(1 + \sum_{m>0} \Phi_m e^{-mu}\right) e^{ux} = \hat{\Phi}(u)e^{ux}. \tag{3.26}$$

The equation (2.39) leads to

$$\frac{\partial\hat{\Phi}(u)}{\partial u} = k\frac{s}{e^u - 1}\hat{\Phi}(u) + [x, \hat{\Phi}(u)]. \tag{3.27}$$

Comparing the coefficients of $e^{-mu}$:

$$-m\Phi_m = [x, \Phi_m] + (terms\ with\ \Phi_j, j < m). \tag{3.28}$$

Given a representation of $\mathcal{H}'_\Sigma$, we find $\Phi_m$ assuming that $m + \mathrm{ad}(x)$ is invertible for any $m > 0$ in this representation. Therefore, setting $\mathrm{Spec}(x) = \{\mu_j\}$, the conditions $m + \mu_i - \mu_j \neq 0,\ m = 1, 2, \dots$, ensure the existence of the asymptotically free solutions. The convergence estimates are straightforward. These conditions are fulfilled in generic induced representations.

**3.3. Lusztig's isomorphisms via the monodromy.** In this section we establish an isomorphism between $\mathcal{H}_\Sigma^t$ and $\mathcal{H}_\Sigma'$ using the monodromy of the AKZ equation.

Let us fix an invertible solution $\Phi(u)$ of the AKZ system in a neighborhood of $u^0 \in U^* = \mathbb{C}^n \setminus C \subset U'$. The functions $\Phi(w(u))$ will be extended to $u^0$ through $U^*$. Since $\mathcal{H}_\Sigma$ is infinite-dimensional, we have to consider all formulas in finite dimensional representations. Once we get the final expressions it is not difficult to find a proper completion of the degenerate Hecke algebra for them.

THEOREM 3.5 ([7]). *There exists a homomorphism from $\mathcal{H}_\Sigma^t$ to $\mathcal{H}_\Sigma'$ given by*

$$T_j \longmapsto T_j', \qquad X_j \longmapsto X_j',$$

*where*

$$T_j' = \Phi(s_j(u))^{-1} s_j \Phi(u), \qquad X_j' = \Phi(u - 2\pi\sqrt{-1}b_j)^{-1}\Phi(u).$$

*If $t = \exp(\pi\sqrt{-1}k)$ is sufficiently general (say, not a root of unity), then it is an isomorphism at the level of finite dimensional representations or after a proper completion.*

Under the notation (3.23), $T_j' = T_{\tilde{\tau}_j}$ and $X_j' = T_{\tilde{\chi}_j}$. Hence the relations (3.19)-(3.21) result from Theorem 3.3, and only the quadratic relations (3.22) need to be proved. We skip a simple direct proof since these relations follow from the exact formulas below.

Let us find the formulas for $T_j'$ and $X_j'$ for the asymptotically free solution $\Phi_{as}(u)$. Given $b \in B$, we set

$$X_b = \prod_{j=1}^{n} X_j^{k_j} \qquad \text{for } b = \sum_{j=1}^{n} k_j b_j,$$

and define $X_b'$ analogously.

THEOREM 3.6 ([6]). *Let us choose the asymptotically free solution $\Phi_{as}(u)$ as $\Phi(u)$. Then*

(a) $X_j' = \exp(2\pi\sqrt{-1}x_j)$,

(b) $s_i - \dfrac{k}{x_{a_i}} = g(x_{a_i})\left(T_i' + \dfrac{t - t^{-1}}{X_{a_i}'^{-1} - 1}\right)$,

*where the function $g(v)$ is defined by*

$$g(v) = \frac{\Gamma(1+v)^2}{\Gamma(1+k+v)\Gamma(1-k+v)},$$

*and (b) is in fact a formula for $T_i'$ in terms of $\{s, x\}$.*

We will give a sketch of the proof of Theorem 3.6. The statement (a) is immediate, since

$$\Phi_{as}(u - 2\pi\sqrt{-1}b_j) = \hat{\Phi}(u)e^{\sum u_i x_i - 2\pi\sqrt{-1}x_j} = \Phi_{as}(u)e^{-2\pi\sqrt{-1}x_j}.$$

To prove the statement (b), we reduce the problem to the $A_1$ case. Let us fix the index $i$ $(1 \le i \le n)$. Set $E(u) = e^{\sum_{i=1}^n u_i x_i}$, so that $\Phi_{as}(u) = \hat{\Phi}(u)E(u)$. Let us define $\Phi^{(i)}(u)$ as follows:

$$\Phi^{(i)}(u) = \Phi^{\infty(i)}(u_i)E(u),$$

where $\Phi^{\infty(i)}(u_i) = \lim_{\Re u_j \to +\infty (j\neq i)} \hat{\Phi}(u)$. The AKZ system for $\Phi^{(i)}(u)$ reads:

$$\frac{\partial \Phi^{(i)}}{\partial u_i} = \left(k\frac{s_i}{e^{u_i}-1} + x_i\right)\Phi^{(i)}, \tag{3.29}$$

$$\frac{\partial \Phi^{(i)}}{\partial u_j} = x_j \Phi^{(i)} \qquad (j \neq i). \tag{3.30}$$

**Reduction procedure.** Since the monodromy $T_i'$ does not depend on $u$, the point $u^0$ and the path connecting $u^0$ and $s_i(u^0)$ may be replaced by any deformations in $U'$ or their limits. Provided the existence, the resulting monodromy coincides with $T_i'$. For instance, $T_i'$ equals

$$T_i^{(i)} = (\Phi^{(i)}(s_i(u)))^{-1} s_i \Phi^{(i)}(u).$$

Indeed, the latter is the limiting monodromy for a path with $\Re u_j (j \neq i)$ approaching the infinity. We note that $\Re u_j(s_i(u^0)) \to +\infty$ if $\Re u_j(u^0)$ does.

In the reduced equations (3.29) and (3.30), we may diminish the values, considering the subalgebra of $\mathcal{H}'_\Sigma$ generated by $x_j$ $(1 \leq j \leq n)$, and $s_i$. In this algebra, the following elements are central:

$$x_j (j \neq i), \quad x_i - \frac{1}{2} x_{a_i}.$$

Hence, if we define $E^{(i)}(u)$ by

$$E^{(i)}(u) = e^{\sum_{j=1}^n u_j x_j - u_i x_{a_i}/2},$$

it enjoys the following properties:

(i) $E^{(i)}(u)$ commutes with $s_i$,

(ii) $E^{(i)}(s_i(u)) = E^{(i)}(u)$.

The second property can be verified directly:

$$\sum_j (s_i(u))_j x_j - \frac{1}{2}(s_i(u))_i x_{a_i} = \sum_j (u_j - (a_i, \alpha_j) u_i) x_j + \frac{1}{2} u_i x_{a_i} = \sum_j u_j x_j - \frac{1}{2} u_i x_{a_i}.$$

We have used that $(s_i(u))_j = (\alpha_j, s_i(u))$ and $\sum_j (a_i, \alpha_j) x_j = x_{a_i}$. Setting

$$\tilde{\Phi}^{(i)}(u) = \Phi^{(i)}(u) E^{(i)}(u)^{-1},$$

the system of equations (3.29),(3.30) becomes precisely the AKZ equation for $\tilde{\Phi}^{(i)}(u)$ in the $A_1$ case:

$$\frac{\partial \tilde{\Phi}^{(i)}(u)}{\partial u_i} = \left(k\frac{s_i}{e^{u_i}-1} + \frac{1}{2} x_{a_i}\right)\tilde{\Phi}^{(i)}(u), \tag{3.31}$$

$$\frac{\partial \tilde{\Phi}^{(i)}(u)}{\partial u_j} = 0 \qquad (j \neq i). \tag{3.32}$$

Because of the above properties of $E^{(i)}(u)$, the monodromy of $\tilde{\Phi}^{(i)}(u)$ coincides with $T_i'$. However $\tilde{\Phi}^{(i)}(u)$ can be expressed in terms of the hypergeometric function, which conclude the proof up to a straightforward calculation.

To explain the structure of the formula for $T'$, let us involve the intertwiners of $\mathcal{H}^t_\Sigma$. They are defind similar to those in the degenerate case:

$$f_i = s_i - \frac{k}{x_{a_i}} \text{ for } \mathcal{H}'_\Sigma, \quad F_i = T_i + \frac{t - t^{-1}}{X_{a_i}^{-1} - 1} \text{ for } \mathcal{H}^t_\Sigma.$$

LEMMA 3.7. $F_i X_b = X_{s_i(b)} F_i$.

It readily results from the definition of $\mathcal{H}^t_\Sigma$ (cf. 3.10).

The image $F'_i$ of $F_i$ in $\mathcal{H}'_\Sigma$ with respect to the homomorphism constructed in Theorem 3.5 can be represented as $F'_i = g_i(x) f_i$ for a function $g_i$ of $x$. Indeed, $f_i X'_b = X'_{s_i(b)} f_i$, which gives the proportionality. Recall that $X'_b = \exp(2\pi\sqrt{-1}x_b)$. Here $g_i(x)$ must be of the form $g(x_{a_i})$ for a function $g$ in one variable, and can be calculated using the hypergeometric equation (3.31). We omit the details (see [6]).

We note, that the quadratic relations for $T'_i$ can be made quite obvious using the same reduction (the exact formulas above are not necessary). Set $i = 1$ to simplify the indices. We switch from (3.31) to (2.18) with two variables $z_1, z_2$ and a parameter $z_0$:

$$\frac{\partial \Phi'}{\partial z_j} = \left[ k \left( \frac{s_1}{z_j - z_k} \right) + \frac{\Omega_j}{z_j - z_0} \right] \Phi' \quad (j = 1, 2, k = 3 - j). \tag{3.33}$$

When $z_0 = 0$ the substitutions are as follows

$$2x_1 = \Omega_1 - \Omega_2 + ks_1, \ u_1 = \log(z_1/z_2), \ \Phi' = \Phi^{(1)}(u_1)(z_1 z_2)^{-1/2(\Omega_1+\Omega_2+ks_1)}.$$

The monodromy corresponding to the transposition of $z_1$ and $z_2$ for $z_0 = 0$ coincides with $T'_1$. It does not depend on $z_0$ up to a conjugation (the same reduction argument applied to the KZ-equation with three variables). Sending $z_0$ to infinity we eliminate the $\Omega$-terms. The monodromy of the resulting equation can be calculated immediately. Since it is conjugated to $T'_1$ we get the desired quadratic relations.

Heckman in [25] used a similar reduction approach when calculating the monodromy of the quantum many-body problem (also called the Heckman-Opdam system). Our next aim is establishing an isomorphism of AKZ and the latter. Combining Heckman's formulas and mine for the AKZ, which coincide since the representation of $\mathcal{H}^t$ is the same, we readily conclude that these equations are isomorphic for generic $\lambda$. This will be made much more constructive below. We will also consider any $\lambda$.

REMARK 3.2. Let us apply Theorem 3.6 to the standard rational KZ equation in the $GL_n$ case. We calculated the monodromy of

$$\frac{\partial \Phi}{\partial v_i} = \left( k \sum_{j>i} \frac{s_{ij}}{e^{v_i - v_j} - 1} - k \sum_{j<i} \frac{s_{ij}}{e^{v_j - v_i} - 1} + y_i \right) \Phi \quad (1 \le i \le n).$$

Taking special $y_i = k \sum_{j=i+1}^n s_{ij}$ and substituting $z_i = e^{v_i}$, we come to

$$\frac{\partial \Phi}{\partial z_i} = k \sum_{j \ne i} \frac{s_{ij}}{z_i - z_j} \Phi \quad (1 \le i \le n).$$

It corresponds to the simplest $\Omega_{ij} = 0$ in (2.18). By the way, these $\{y\}$ induce a homomorphism from $\mathcal{H}'_n$ to $\mathbb{C}\mathbb{S}_{n+1}$ due to Drinfeld. Diagonalizing the commuting elements $\sum_{j>i} s_{ij}$ we recover the monodromy computed by Tsuchiya-Kanie [45]. It also gives an explict example of the general results on the monodromy of the rational KZ over Lie algebras due to Drinfeld and Kohno (see [31]).

REMARK 3.3. In Theorems 3.5 and 3.6, we established the isomorphism

$$\mathcal{H}^t_\Sigma \simeq \mathcal{H}'_\Sigma, \qquad X_j \longmapsto t^{2x_j},$$

where $t = e^{\pi\sqrt{-1}k}$ and represented it as a relation between the intertwiners of the degenerate and non-degenerate affine Hecke algebras:

$$F_j = T_j + \frac{t - t^{-1}}{X_{a_j}^{-1} - 1} \longmapsto g(x_{a_j})\left(s_j - \frac{k}{x_{a_j}}\right).$$

This construction can be naturally generalized. In fact we need only a very mild restriction on $g(x)$ to get such a homomorphism. Normalizing the intertwiners to make them 'unitary' ($f^2 = 1 = F^2$), we come to the simplest possible map:

$$X_j \longmapsto t^{2x_j}, \quad \frac{F_j}{t + \frac{t-t^{-1}}{X_{a_j}^{-1}-1}} \longmapsto \frac{s_j - \frac{k}{x_{a_j}}}{1 - \frac{k}{x_{a_j}}}.$$

Actually here we have four formulas in one since we can put the denominators on the right and on the left. One of them was found by Lusztig in [33].

**3.4. The isomorphism of AKZ and QMBP.** Here we present the isomorphism between the AKZ equation and the quantum many-body problem (QMBP). The latter will appear as a 'trace' of the first.

We will need a variant of the general notion of monodromy by A. Grothendieck. Let us fix the notations:

$${}^w\Phi(u) = \Phi(w^{-1}(u)), \; w = \bar{w}b \in \widetilde{W} = W \ltimes B, u \in \mathbb{C}^n.$$

Given a finite union $C$ of affine real closed half-hyperplanes, we set $U = \mathbb{C}^n \backslash C$ assuming that

(i) $U$ does not contain 'bad hyperplanes' $\prod_{\alpha\in\Sigma_+}(e^{u_\alpha} - 1) = 0$,
(ii) $U$ is simply connected,
(iii) $\left(\mathbb{C}^n \backslash \cup_{w\in\widetilde{W}} w(C)\right)/\widetilde{W}$ is connected.

We shall refer to such $C$ as a *system of cutoffs*. In §3.2, a special system of cutoffs ($U^*$) has been already used in order to compute the monodromy.

Let us fix a system of cutoffs $C$ and $U$. Then for each $w \in \widetilde{W}$ there is a path $\gamma_w$ (unique up to homotopy) joining $u^0$ and $w^{-1}(u^0)$. So the choice of $C$ implies a choice of representatives $\bar{\gamma}_w$ in the fundamental group $\pi_1(U'/\widetilde{W}, u^0)$. Here $U'$ is the complement of the union of 'bad hyperplanes' (3.13).

We pick a solution $\Phi$ of the AKZ equation in $U$ and define the monodromy function $\mathcal{T}_w$ ($w \in \widetilde{W}$)

$$\bar{w}\Phi = {}^{w^{-1}}\Phi \cdot \mathcal{T}_w \quad w = \bar{w}b \in \widetilde{W}. \tag{3.34}$$

Here $\Phi$ is invertible at least at one point and is extended analitically to the whole $U$. The values are in the endomorphisms of any finite-dimensional represenation of $\mathcal{H}'_\Sigma$ (we will apply the construction to the induced representations).

The monodromy $\{\mathcal{T}_w\}_{w\in\widetilde{W}}$ satisfies the following:

(a) (1-cocycle condition) $\quad {}^{v^{-1}}(\mathcal{T}_w)\mathcal{T}_v = \mathcal{T}_{wv} \quad \forall w, v \in \widetilde{W}$,

(b) $\dfrac{\partial}{\partial u_i}\mathcal{T}_w = 0$, and hence $\mathcal{T}_w$ is locally constant.

The property (b) holds since both $\Phi$ and $w(\Phi) = \bar{w}^w\Phi$ satisfy the same differential equation of the first order (the AKZ equation). It readily results in the invertibility of $\mathcal{T}_w$ on $\mathbb{C}^n - \cup_{w\in\widetilde{W}} w(C)$. The latter set is not connected, so $\mathcal{T}$ is not just a constant.

Next let us introduce the operators $\sigma_w, \sigma'_w$ ($w \in \widetilde{W}$), acting on functions $F$ on $U$:

$$(\sigma_w F)(u) = ({}^{w^{-1}}F)(u) = F(w(u)), \quad \sigma_i = \sigma_{s_i},$$
$$(\sigma'_w F)(u) = ({}^{w^{-1}}F)(u)\mathcal{T}_w, \quad \sigma'_i = \sigma'_{s_i}.$$

The relations for the operators $\sigma'_w$ are the same as for the permutations $\sigma_w$:

$$\begin{aligned} &\text{a)} \qquad \sigma'_w\sigma'_v = \sigma'_{vw},\\ &\text{b)} \qquad \sigma'_w u_b = u_{w^{-1}(b)}\sigma'_w,\\ &\text{c)} \qquad \sigma'_w\partial_b = \partial_{w^{-1}(b)}\sigma'_w, \quad \partial_b(u_\alpha) = (b,\alpha). \end{aligned} \tag{3.35}$$

Note that the property a) follows from the 1-cocycle condition for $\{\mathcal{T}_w\}_{w\in\widetilde{W}}$. Indeed,

$$\begin{aligned} (\sigma'_w\sigma'_v)(F) &= \sigma'_w(\sigma'_v(F))\\ &= \sigma'_w({}^{v^{-1}}F\mathcal{T}_v)\\ &= {}^{w^{-1}}({}^{v^{-1}}F\mathcal{T}_v)\mathcal{T}_w\\ &= {}^{w^{-1}v^{-1}}F({}^{w^{-1}}\mathcal{T}_v)\mathcal{T}_w\\ &= {}^{(vw)^{-1}}F\mathcal{T}_{vw}\\ &= \sigma'_{vw}(F). \end{aligned}$$

Let $Sol_{AKZ}$ be the space of solutions of the AKZ equation with values in $\mathcal{H}'_\Sigma$. When we consider the AKZ equation on a finite-dimensional $\mathcal{H}'_\Sigma$-module $V$, we will denote the space of its solutions by $Sol_{AKZ}(V)$. Starting with AKZ let us go to QMBP. In what follows, $\Phi \in Sol_{AKZ}$ or $\Phi \in Sol_{AKZ}(\mathrm{End}(V))$. In the latter case all operators act on $\mathrm{End}(V)$-valued functions.

(1) Using $s_\alpha\Phi = \sigma'_{s_\alpha}\Phi$, we rewrite the AKZ equation:

$$\begin{aligned} x_i\Phi &= \left(\frac{\partial}{\partial u_i} - k\sum_{\alpha\in\Sigma_+}\nu^i_\alpha\frac{s_\alpha}{e^{u_\alpha}-1}\right)\Phi\\ &= \left(\frac{\partial}{\partial u_i} - k\sum_{\alpha\in\Sigma_+}\nu^i_\alpha(e^{u_\alpha}-1)^{-1}\sigma'_{s_\alpha}\right)\Phi \qquad (1\le i\le n). \end{aligned}$$

Let us denote:

$$\mathcal{D}'_i = \frac{\partial}{\partial u_i} - k\sum_{\alpha\in\Sigma_+}\nu^i_\alpha(e^{u_\alpha}-1)^{-1}\sigma'_{s_\alpha}.$$

The local invertibility of $\Phi$ and the relations $\mathcal{D}'_i\Phi = x_i\Phi$ result in the commutativity

$$[\mathcal{D}'_i, \mathcal{D}'_j] = 0 \quad \forall i, j.$$

Here one can use that the commutators do not contain the derivatives, which readily results from the relations for $\sigma'$. Moreover, the commutativity follows trom these relations algebraically. It was proved in [10] (see [13] for a more conceptual proof based on the induced representations). It also follows from the corresponding difference theory, where this and similar statements are much simpler (and completely conceptual).

(2) Since the multiplication by $x_i$ commutes with $\mathcal{D}'_j$, we get

$$p(x_1,\dots,x_n)\Phi = p(\mathcal{D}'_1,\dots,\mathcal{D}'_n)\Phi$$

for any polynomial $p \in \mathbb{C}[x_1,\dots,x_n]$.

(3) For $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$ let us take an $\mathcal{H}'_\Sigma$-module $V_\lambda$ with the following properties:

(i) $p(x_1,\dots x_n) = p(\lambda_1,\dots,\lambda_n)$ on $V_\lambda$ for any $p \in \mathbb{C}[x_1,\dots,x_n]^W$ (3.36)

(ii) there exists a linear map $\mathrm{tr} : V_\lambda \longrightarrow \mathbb{C}$ satisfying

$$\mathrm{tr}(wa) = \mathrm{tr}(a) \quad \forall w \in W, a \in V_\lambda. \tag{3.37}$$

Let $p(x_1,\dots,x_n)$ be a polynomial. Using the commutation relations (3.35), we can write

$$p(\mathcal{D}'_1,\dots,\mathcal{D}'_n) = \sum_{w\in W} \mathcal{D}'^{(p)}_w \sigma'_w,$$

where $\mathcal{D}'^{(p)}_w$ are differential operators (they do not contain $\sigma'$). They are scalar and commute with $\mathcal{H}'_\Sigma$. Thus

$$p(x_1,\dots,x_n)\Phi = \sum_{w\in W} \mathcal{D}'^{(p)}_w \sigma'_w \Phi = \sum_{w\in W} \mathcal{D}'^{(p)}_w w\Phi.$$

Now, we assume that $p$ is $W$-invariant. Applying tr (see (3.36) and (3.37)), we come to

$$p(\lambda_1,\dots,\lambda_n)\psi = L'_p\psi \text{ for } L'_p = \sum_{w\in W} \mathcal{D}'^{(p)}_w,$$

where

$$\psi(u) = \mathrm{tr}(\Phi(u))$$

is a $\mathbb{C}$-valued function. The differential operators $L'_p$ are $W$-invariant, which follows from the same construction (we will reprove this algebraically below).

Let us introduce the *trigonometric Dunkl operators* $\mathcal{D}_i$ $(1 \le i \le n)$ replacing $\sigma'$ by $\sigma$:

$$\mathcal{D}_i = \frac{\partial}{\partial u_i} - k \sum_{\alpha\in\Sigma_+} \nu^i_\alpha (e^{u_\alpha} - 1)^{-1}\sigma_{s_\alpha}.$$

Repeating the above construction, define $\mathcal{D}^{(p)}$ for a $W$-invariant polynomial $p$ by

$$p(\mathcal{D}_1,\dots \mathcal{D}_n) = \sum_{w\in W} \mathcal{D}^{(p)}_w \sigma_w.$$

Since in the construction of $L'_p$ and $L_p$ we use only the commutation relations (3.35) for $\sigma'_w$ and $\sigma_i$ these operators just coincide. The trigonometric Dunkl operators are from [8]. Dunkl introduced their rational counterparts (see also [13] and references therein). When defining my operators I also used [26]. Heckman's 'global Dunkl operators' are sufficient to introduce QMBP, but do not commute.

We are now in a position to introduce the QMBP with the eigenvalue $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. It is the following system of differential equations for a $\mathbb{C}$-valued function $\psi$.

$$L_p\psi = p(\lambda_1, \dots, \lambda_n)\psi \qquad (p \in \mathbb{C}[x_1, \dots, x_n]^W).$$

It is known [24] (and easy to see by looking at the leading terms of $L_p$) that the dimension of the space of solutions $\psi$ is $|W|$.

Summarizing, we come to the theorem.

THEOREM 3.8. *Applying* tr *we get a homomorphism*

$$\mathrm{tr} : Sol_{AKZ}(V_\lambda) \longrightarrow Sol_{QMBP}(\lambda).$$

*Here $Sol_{QMBP}(\lambda)$ denotes the space of solutions to QMBP with the eigenvalue $\lambda$.*

We can say more for concrete represenatations, especially for the induced representations $J_\lambda^\circ$ (see (3.4)). We define the 'trace'

$$\mathrm{tr} : J_\lambda^\circ \longrightarrow \mathbb{C}$$

as the map dual to the embedding

$$\mathbb{C} \longrightarrow J_\lambda = \mathbb{C}[x_1, \dots, x_n]/L_\lambda$$

sending 1 to $1 \in \mathbb{C}[x_1, \dots, x_n]$. Here $L_\lambda$ denotes the ideal generated by $p(x) - p(\lambda)$, $p \in \mathbb{C}[x_1, \dots, x_n]^W$. One easily checks that tr satisfies the conditions (3.37).

THEOREM 3.9 ([7]). *For any $\lambda \in \mathbb{C}^n$,* tr *gives an isomorphism*

$$\mathrm{tr} : Sol_{AKZ}(J_\lambda^\circ) \xrightarrow{\sim} Sol_{QMBP}(\lambda).$$

PROOF. The key observation:

$$\text{for any } \mathcal{H}'_\Sigma\text{- submodule } 0 \neq M \subset J_\lambda^\circ, \text{ we have } \mathrm{tr}\,|_M \neq 0. \tag{3.38}$$

Indeed, if $0 \neq f \in M$, then there exists a polynomial $p(x) \in \mathbb{C}[x_1, \dots, x_n]$ such that $f(p) \neq 0$. However $f(p) = tr(p(f)) \in \mathrm{tr}(M)$.

To prove Theorem 3.9, it is enough to show the injectivity of tr, since the surjectivity will then follow by comparing the dimensions of the solution spaces (both of them are $|W|$). So let us suppose that for $\varphi(u) \in Sol_{AKZ}(J_\lambda^\circ)$ identically

$$\mathrm{tr}(\varphi) = 0. \tag{3.39}$$

We will show that

$$\mathrm{tr}\,(\mathcal{H}'_\Sigma\varphi) = 0. \tag{3.40}$$

Differentiating (3.39),

$$0 = \mathrm{tr}(\frac{\partial\varphi}{\partial u_i}) = k \sum_{\alpha\in\Sigma_+} \nu_\alpha^i \,\mathrm{tr}\left(\frac{s_\alpha}{e^{u_\alpha} - 1}\varphi\right) + \mathrm{tr}(x_i\varphi).$$

By the $W$-invariance of tr, $\mathrm{tr}(s_\alpha\varphi) = \mathrm{tr}(\varphi) = 0$. Hence

$$\mathrm{tr}(x_i\varphi) = 0. \tag{3.41}$$

Differentiating this equation by $u_j$ we have

$$0 = k \sum_{\alpha\in\Sigma_+} \nu_\alpha^j \,\mathrm{tr}\left(x_i\frac{s_\alpha}{e^{u_\alpha} - 1}\varphi\right) + \mathrm{tr}(x_ix_j\varphi).$$

Using the commutation relations of $x_j$ and $s_\alpha$, we deduce from (3.39), (3.41) that

$$\operatorname{tr}(x_i x_j \varphi) = 0.$$

Proceeding in the same way, we establish that

$$\operatorname{tr}(x_{i_1} \cdots x_{i_l} \varphi) = 0$$

for any $i_1, \ldots, i_l$. Combining this with the $W$-invariance of tr, we get (3.40).

For each $u^0$, consider the submodule $M = \mathcal{H}'_\Sigma \varphi(u^0) \subset J^\circ_\lambda$. Then $\operatorname{tr}|_M = 0$, and from the key observation above, we deduce that $M = 0$. This completes the proof of Theorem 3.9. □

The map from Theorem 3.9 was found by Matsuo [37] for induced representations $I_\lambda$. He proved his theorem algebraically (without the passage through the trigonometric Dunkl operators discussed above) using an explicit presentation for AKZ in $I_\lambda$. The isomorphism for $J^\circ_\lambda$ (or for $I_\lambda$ with properly ordered $\lambda$ - (3.6)) was established independently and simultaneously by Matsuo and the author in [13]. He proved that a certain determinant is non-zero for properly ordered $\lambda$. I used the modules $J$. Matsuo was the first to conjecture that the QMBP (the Heckman-Opdam system) and a certain specialization of the trigonometric KZ from [4] are isomorphic. The affine KZ were defined in full generality a bit later (in [6]).

Let us give the formula for the simplest $L_p$.

EXAMPLE 3.1. Let $p_2(x_1, \ldots, x_n) = \sum_{i=1}^n x_{\alpha_i} x_i$. Then we have

$$L_2 = L_{p_2} = \sum_{i=1}^n \partial_{\alpha_i} \partial_i + \sum_{\alpha \in \Sigma_+} (\alpha, \alpha) \frac{k(1-k)}{(e^{u_\alpha} - e^{-u_\alpha})^2} + \text{const}.$$

It was studied in [39].

REMARK 3.4. More generally, let $A$ be a $\mathbb{C}[W]$-module and

$$V_{A,\lambda} = \left(\operatorname{Ind}_{\mathbb{C}[W]}^{\mathcal{H}'_\Sigma}(A)/L_\lambda\right)^\circ.$$

As before, $L_\lambda$ is the ideal generated by $p(x) - p(\lambda)$, $p \in \mathbb{C}[x_1, \ldots, x_n]^W$. Then the following holds

$$Sol_{AKZ}(V_{A,\lambda}) \xrightarrow{\sim} Sol_{QMBP_A}(\lambda)$$

where now the right hand side means a matrix version of QMBP (sometimes it is called spin-QMBP). It was introduced in [13] for the first time. It is a ceratin unification of the Haldane-Shastry model and that by Calogero-Sutherland.

For example, the $L$-operator corresponding to $p_2$ above reads

$$L_2 = \sum_{i=1}^n \partial_{\alpha_i} \partial_i + \sum_{\alpha \in \Sigma_+} (\alpha, \alpha) \frac{k(s^*_\alpha - k)}{(e^{u_\alpha} - e^{-u_\alpha})^2} + \text{const},$$

where by $s^*_\alpha$ we mean the image of $s_\alpha$ in $\operatorname{Aut}(A)$.

**3.5. The $GL_n$ case.** Let us describe AKZ and QMBP in the $GL_n$ case.

In §2.4, we introduced the degenerate affine Hecke algebra of type $GL_n$. It is the algebra

$$\mathcal{H}'_n = \langle \mathbb{C}\mathbb{S}_n, y_1, \dots, y_n \rangle$$

subject to the following relations:

$$s_i y_i - y_{i+1} s_i = k, \qquad s_i y_j = y_j s_i \quad (i \neq j, j+1),$$
$$y_i y_j = y_j y_i \quad (1 \leq i, j \leq n).$$

As in §2.4, we will use the coordinates $v_i$.

To prepare the passage to the difference case, we conjugate the AKZ for $GL_n$ by the function $\Delta^k$ for $\Delta = \prod_{i<j}(e^{v_i} - e^{v_j})$. The equation becomes as follows:

$$\frac{\partial \Phi}{\partial v_i} = \left( k \Big( \sum_{j(>i)} \frac{s_{ij} - 1}{e^{v_i - v_j} - 1} - \sum_{j(<i)} \frac{s_{ij} - 1}{e^{v_j - v_i} - 1} \Big) + y_i + k \Big( i - \frac{n+1}{2} \Big) \right) \Phi. \qquad (3.42)$$

Only in this form it can be quantized (see §4.2). The system is consistent and $\mathbb{S}_n$-invariant.

The corresponding Dunkl operators are given by the formula

$$\mathcal{D}_i = \frac{\partial}{\partial v_i} - k \left( \sum_{j(>i)} (e^{v_i - v_j} - 1)^{-1} (\sigma_{ij} - 1) - \sum_{j(<i)} (e^{v_j - v_i} - 1)^{-1} (\sigma_{ij} - 1) + i - \frac{n+1}{2} \right).$$

Here $\sigma_{ij}$ stands for the transpositions of the coordinates:

$$\sigma_{ij} v_i = v_j \sigma_{ij}.$$

Similary, $\sigma_w$ means the permutation of the coordinates corresponding to $w^{-1}$.

The main point of the theory is that they satisfy the relations from the degenerate Hecke algebra:

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 = [\mathcal{D}_i, y_j], \ i \neq j, \qquad \sigma_{i\,i+1} \mathcal{D}_i - \mathcal{D}_{i+1} \sigma_{i\,i+1} = k.$$

It holds for any root systems. This statement is from [13]. In these notes we will deduce these relations from the difference theory (where they are almost obvious). These relations readily give that $p(\mathcal{D}_1, \dots, \mathcal{D}_n)$ and the corresponding $L_p$ are $W$-invariant for the $W$-invariant polynomilas. Use the description of the center of $\mathcal{H}'$ to see this.

In the case of $GL_n$, given symmetric $p \in \mathbb{C}[x_1, \dots, x_n]^{\mathbb{S}_n}$,

$$p(\mathcal{D}_1, \dots, \mathcal{D}_n) = \sum_{w \in \mathbb{S}_n} D_w^{(p)} \sigma_w,$$

where $D_w^{(p)}$ are scalar differential operators,

$$L_p = p(\mathcal{D}_1, \dots, \mathcal{D}_n)\Big|_{symm.poly.} = \sum_{w \in \mathbb{S}_n} D_w^{(p)}.$$

Let us take the elementary symmetric polynomials:

$$e_m(x) = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m},$$

as $p$, setting $L_m = L_{e_m}$. Clearly

$$L_1 = \sum_{i=1}^{n} \frac{\partial}{\partial v_i}.$$

The next operator is:

$$L_2 = \sum_{i<j} \frac{\partial^2}{\partial v_i \partial v_j} - \frac{k}{2} \sum_{i<j} \coth\left(\frac{v_i - v_j}{2}\right) \left(\frac{\partial}{\partial v_i} - \frac{\partial}{\partial v_j}\right) - \frac{k^2}{4} \binom{n+1}{3}.$$

When we replace $e_2$ by $p = \sum_i x_i^2$, the corresponding $L$-operator is conjugated (by $\Delta^k$) to the original Sutherland operator up to a constant term [44]. For special values of the parameter $k$, these operators are the radial parts of the Laplace operators on the symmetric spaces. A particular case was considered by Koornwinder. The rational counterpart is due to Calogero. It is equivalent to a rational variant of the AKZ (an extension of the rational $W$-valued KZ from [4] by the $x$-s). Here the $J_\lambda$-modules cannot be represented as $I_\lambda$, the theorem holds in terms of $J$ only (see [13]). See also [24] and [40] where the algebraic and analytic theory was developed systematically.

## 4. Isomorphism theorems for the QAKZ equation

Let us now turn to the $q$-deformations. We introduce the quantum affine Knizhnik–Zamolodchikov (QAKZ) equation, and show that there is an isomorphism between solutions of the QAKZ equation and solutions of the generalized Macdonald eigenvalue problem.

**4.1. Affine Hecke algebras and intertwiners.** In this section we recall the definition of the affine Hecke algebra $\mathcal{H}_n^t$ in the case of $GL_n$.

Let $t \in \mathbb{C}^*$ be a parameter. Then $\mathcal{H}_n^t$ is the algebra defined over $\mathbb{C}$ by the following set of generators and relations:

$$\begin{aligned}
\text{generators} &: T_1, \dots, T_{n-1}, Y_1, \dots, Y_n, \\
\text{relations} &: (T_i - t)(T_i + t^{-1}) = 0, \quad (1 \le i \le n-1) && (4.1)\\
& \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad (1 \le i \le n-2) && (4.2)\\
& \quad T_i T_j = T_j T_i, \quad (|i-j| > 1) && (4.3)\\
& \quad Y_i Y_j = Y_j Y_i, \quad (1 \le i, j \le n) && (4.4)\\
& \quad Y_i T_j = T_j Y_i, \quad (j \ne i, i-1), && (4.5)\\
& \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1}. \quad (1 \le i \le n-1) && (4.6)
\end{aligned}$$

The relations (4.1) are called the quadratic relations, (4.2)-(4.3) the Coxeter relations, (4.4) the commutativity, and (4.5),(4.6) the cross relations.

Set

$$P = T_1 \cdots T_{i-1} Y_i T_i^{-1} \cdots T_{n-1}^{-1}.$$

It follows from the defining relations (4.1)-(4.6) that the right hand side is independent of $i$ ($1 \le i \le n$) and therefore equals to

$$P = T_1 \cdots T_{n-1} Y_n = Y_1 T_1^{-1} \cdots T_{n-1}^{-1}. \qquad (4.7)$$

LEMMA 4.1. *The algebra $\mathcal{H}_n^t$ can be presented as*

$$\mathcal{H}_n^t = \langle T_1, \dots, T_{n-1}, P \rangle / \sim, \qquad (4.8)$$

*where the quotient is by the quadratic relations* (4.1), *the Coxeter relations* (4.2)-(4.3) *and the following:*

(a) $PT_{i-1} = T_i P$ $(1 < i < n)$,
(b) $P^n$ *is central.*

PROOF. Notice that in terms of $Y_i$'s we have $P^n = Y_1 \cdots Y_n$. The relations (a) and (b) readily follow from (4.7) and the defining relations (4.1)-(4.6). For instance,

$$PT_1P^{-1} = Y_1 T_1^{-1}(T_2^{-1} T_1 T_2) T_1 Y_1^{-1} = Y_1 T_1^{-1}(T_1 T_2^{-1} T_1^{-1}) T_1 Y_1^{-1} = T_2.$$

To establish (4.8), we start with $T_1, \dots, T_{n-1}, P$ and introduce the elements $Y_1, \dots, Y_n$ by

$$Y_1 = PT_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \dots.$$

We must check the commutativity $Y_1 Y_2 = Y_2 Y_1$, $T_j Y_1 = Y_1 T_j$ $(j > 1)$, etc. using (a), (b). The first reads

$$Y_1 T_1^{-1} Y_1 T_1^{-1} = T_1^{-1} Y_1 T_1^{-1} Y_1.$$

We plug in the above formula for $Y_1$ and move $P$ to the left. The commuativity with 'distant' $T$ is obvious. The other relations formaly follows from these ones. We leave the verifications to the reader as an exercise. □

**4.2. The QAKZ equation.** In this section, we introduce the QAKZ equation.

DEFINITION 4.1. For $u \in \mathbb{C}$, we define the intertwiners by

$$F_i(u) = \frac{T_i + \dfrac{t - t^{-1}}{e^u - 1}}{t + \dfrac{t - t^{-1}}{e^u - 1}}. \tag{4.9}$$

They satisfy

$$F_i(u)F_i(-u) = 1, \tag{4.10}$$
$$F_i(u)F_{i+1}(u+v)F_i(v) = F_{i+1}(v)F_i(u+v)F_{i+1}(u). \tag{4.11}$$

The second relation can be deduced from Lemma 3.7 as we did for the degenerate Hecke algebra.

The *quantum affine Knizhnik-Zamolodchikov* (QAKZ) equation is the following system of difference equations for a function $\Phi(v)$ that takes values in $\mathcal{H}_n^t$ (or any $\mathcal{H}_n^t$-module).

$$\begin{aligned} &\Phi(v_1, \ldots, v_i + h, \ldots, v_n) \\ &\quad = F_{i-1}(v_i - v_{i+1} + h) \ldots F_1(v_i - v_1 + h) T_1 \cdots T_{i-1} Y_i \\ &\times T_i^{-1} \cdots T_{n-1}^{-1} F_{n-1}(v_i - v_n) \cdots F_i(v_i - v_{i+1}) \\ &\quad \times \Phi(v_1, \ldots, v_i, \ldots, v_n) \quad (i = 1, \ldots, n). \end{aligned} \tag{4.12}$$

Here $h$ is a new parameter.

THEOREM 4.2. *The QAKZ system* (4.12) *is self-consistent. It is invariant in the following sense: if $\Phi(v)$ is a solution, then so is*

$$F_i(v_{i+1} - v_i)\, {}^{s_i}\Phi(v) = {}^{s_i}\left(F_i(v_i - v_{i+1})\Phi(v)\right).$$

This follows from (4.10), (4.11). Later we will make it quite obvious.

Let us discuss the quasi-classical limit of the QAKZ system. Setting

$$t = e^{kh/2} = q^k, \quad q = e^h,$$

let $h \to 0$. The generators $T_i$, $Y_i$ are supposed to have the form

$$\begin{aligned} T_i &= s_i + \frac{kh}{2} + \cdots \qquad (s_i^2 = 1), \\ Y_i &= 1 + hy_i + \cdots, \end{aligned}$$

where by $\cdots$ we mean terms of order $h^2$. The relations of the degenerate affine Hecke algebra for $s_i, y_i$ can be readily verified. Using the formula

$$tT_i^{-1}F_i(u) = 1 + \frac{kh}{e^u - 1}(s_i - 1) + \cdots,$$

we find that

$$h^{-1}\left(\Phi(\ldots,v_i+h,\ldots)-\Phi(\ldots,v_i,\ldots)\right)$$
$$=\left\{y_i+k\left(\sum_{j(>i)}\frac{s_{ij}-1}{e^{v_i-v_j}-1}-\sum_{j(<i)}\frac{s_{ij}-1}{e^{v_j-v_i}-1}+i-\frac{n+1}{2}\right)\right\}\Phi(\ldots,v_i,\ldots)+\cdots.$$

Hence the AKZ equation (3.42) is a semi-classical limit ($h\to 0$) of the QAKZ equation.

To make the QAKZ equations more transparent, let us discuss the action of the affine Weyl group. The affine Weyl group of type $GL_n$ is the semi-direct product

$$\widetilde{\mathbb{S}}_n=\mathbb{S}_n\ltimes\mathbb{Z}^n,$$

where

$$\mathbb{Z}^n=\bigoplus_{i=1}^{n}\mathbb{Z}\gamma_i$$

is a free abelian group of rank $n$. Define the action of $\widetilde{\mathbb{S}}_n$ on a vector $v=(v_1,\ldots,v_n)\in\mathbb{R}^n$ by

$$s_{ij}v=(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_n)=s_{ji}v,\qquad i<j,$$
$$\gamma_i v=(v_1,\ldots,v_i+h,\ldots,v_n),\ {}^{\gamma_i}(v_j)=v_j-h\delta_{ij}.$$

We also introduce

$$\pi=\gamma_1 s_1\cdots s_{n-1}=s_1\cdots s_{n-1}\gamma_n.$$

Its action on $\mathbb{R}^n$ and the coordinates reads as

$$\pi v=(v_n+h,v_1,\ldots,v_{n-1}),\ {}^{\pi}v_n=v_1-h,\ {}^{\pi}v_1=v_2,\cdots.$$

LEMMA 4.3. *$\widetilde{\mathbb{S}}_n$ can be presented as*

$$\widetilde{\mathbb{S}}_n=\langle s_1,\ldots,s_{n-1},\pi\rangle/\sim,$$

*where the relations are*

$$s_i^2=1,\quad s_is_j=s_js_i\quad(|i-j|>1),\quad s_is_{i+1}s_i=s_{i+1}s_is_{i+1},$$

*and*

(a) $\pi s_{i-1}\pi^{-1}=s_i\ (1<i<n)$,
(b) $\pi^n$ *is central.*

It is convenient to represent the elements $\gamma_i,\pi$ graphically.

Fig.7 showes a reduced decomposition of $\gamma_i$:

$$\gamma_i=s_{i-1}\cdots s_1\pi s_{n-1}\cdots s_i.$$

For a function $\Psi(v)$ with values in $\mathcal{H}_n^t$, let

$$\tilde{s}_i(\Psi)=F_i(v_{i+1}-v_i)\,{}^{s_i}\Psi,\tag{4.13}$$
$$\tilde{\pi}(\Psi)=P\,{}^{\pi}\Psi.\tag{4.14}$$

THEOREM 4.4 ([9]). *The formulas (4.13), (4.14) can be extended to an action of $\widetilde{\mathbb{S}}_n$.*

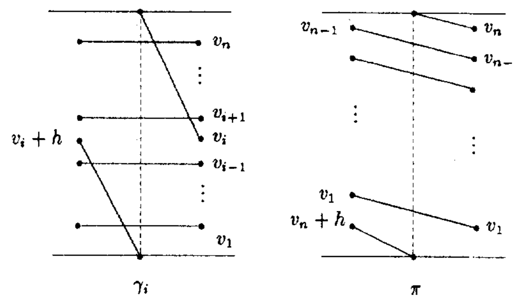


FIGURE .7. Graphs for $\gamma, \pi$

We denote this action by $\tilde{\mathbb{S}}_n \ni w : \Psi \mapsto \tilde{w}(\Psi)$. For instance,

$$\begin{aligned}&\tilde{\gamma}_i(\Psi)(v_1, \dots, v_n)\\ &= F_{i-1}(v_{i-1} - v_i)^{-1} \cdots F_1(v_1 - v_i)^{-1} P F_{n-1}(v_i - v_n - h) \cdots F_i(v_i - v_{i+1} - h)\\ &\quad \times \Psi(v_1, \dots, v_i - h, \dots, v_n).\end{aligned}$$

Hence the QAKZ equation simply means the invariance of $\Phi(v)$ with respect to the pairwise commuting elements $\gamma_i$:

$$\text{QAKZ} \iff \tilde{\gamma}_i(\Phi) = \Phi \quad (i = 1, \dots, n). \tag{4.15}$$

Let us connect QAKZ with the q-KZ introduced by Smirnov and Frenkel-Reshetikhin [43, 23]. We fix an $N$-dimensional complex vector space $V$ and introduce $T \in \mathrm{End}(V \otimes V)$ by

$$T = (t - t^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + t \sum_{i=1}^{N} E_{ii} \otimes E_{ii}$$

due to Baxter and Jimbo. The algebra $\mathcal{H}_n^t$ acts on $V^{\otimes n}$ by

$$T_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes T(a_i \otimes a_{i+1}) \otimes \cdots \otimes a_n, \tag{4.16}$$
$$P(a_1 \otimes \cdots \otimes a_n) = C a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}, \tag{4.17}$$

where $a_i \in V$ and $C = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. One can check that this action is well-defined by a direct calculation.

For $N = n$, let

$$\left(V^{\otimes n}\right)_0 = \mathrm{span}\{e_{w(1)} \otimes \cdots \otimes e_{w(n)} \mid w \in \mathbb{S}_n\}$$

be the 0-weight subspace. Here $e_1, \dots, e_n$ denote the standard basis of $V$. It is easy to see that this subspace is closed under the action of $\mathcal{H}_n^t$. We state the next proposition without proof.

PROPOSITION 4.5. *If $N = n$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ is generic, then the 0-weight space $(V^{\otimes n})_0$ is isomorphic to $I_\lambda = \mathrm{Ind}_{\mathbb{C}[Y_1, \dots, Y_n]}^{\mathcal{H}_n^t}(\lambda)$.*

Writing down AQKZ in $(V^{\otimes n})_0$ we get the q-KZ (for $GL_n$ and in the fundamental representation). Combining this observation with the isomorphism with the Macdonald eigenvalue problem (our next aim) we can explain why the Macdonald polynomials appear in many calculations involving the vertex operators.

**4.3. The monodromy cocycle.** Let $\Phi$ be a solution of the QAKZ equation. Thanks to (4.15), $\tilde{w}(\Phi)$ is also a solution of the QAKZ equation for any $w \in \tilde{\mathbb{S}}_n$. We define $\mathcal{T}_w \in \mathcal{H}_n^t$ by

$$^{w}\mathcal{T}_w = \Phi^{-1}\tilde{w}(\Phi) \quad \text{for } w \in \tilde{\mathbb{S}}_n$$

and call it the *monodromy cocycle*. It follows From (4.13) and (4.14) that

$$F_i(v_i - v_{i+1})\Phi = {}^{s_i}\Phi\mathcal{T}_i \tag{4.18}$$

and

$$P\Phi = {}^{\pi^{-1}}\Phi\mathcal{T}_\pi. \tag{4.19}$$

Here $\mathcal{T}_i$ stands for $\mathcal{T}_{s_i}$.

LEMMA 4.6.

$$^{w_2^{-1}}(\mathcal{T}_{w_1})\mathcal{T}_{w_2} = \mathcal{T}_{w_1 w_2} \quad \textit{for } w_1, w_2 \in \tilde{\mathbb{S}}_n.$$

Indeed,

$$\Phi^{w_1 w_2}\mathcal{T}_{w_1 w_2} = \widetilde{w_1 w_2}(\Phi) = \tilde{w}_1(\tilde{w}_2\Phi) = \tilde{w}_1(\Phi^{w_2}\mathcal{T}_{w_2}) = \Phi^{w_1}\mathcal{T}_{w_1}{}^{w_1 w_2}\mathcal{T}_{w_2}.$$

The QAKZ equation implies that $\mathcal{T}_{\gamma_i} = 1$. Hence $\mathcal{T}_w$ depends only on the image $\bar{w}$ of $w$ in $\mathbb{S}_n$.

Let $\mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$ be the set of $\mathcal{H}_n^t$-valued function on $\mathbb{C}^n$. Next we define two anti-actions of $\tilde{\mathbb{S}}_n$:

$$\sigma_w(\Psi) = {}^{w^{-1}}\Psi, \tag{4.20}$$

$$\sigma'_w(\Psi) = {}^{w^{-1}}\Psi\mathcal{T}_w, \tag{4.21}$$

where $w \in \tilde{\mathbb{S}}_n$ and $\Psi \in \mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$. Lemma 4.6 means exactly that $\sigma'$ is an anti-action (i.e. $\sigma'_{w_1 w_2} = \sigma'_{w_2}\sigma'_{w_1}$). For instance, $\sigma_{\gamma_i}(v_i) = v_i + h = \sigma'_{\gamma_i}(v_i)$.

We note that in the difference theory the monodromy can be always made trivial. Indeed, the 1-cocycle $\{\mathcal{T}_w, w \in W\}$ is always a co-boundary because of the Hilbert 90 theorem. Hence conjugating solutions of AQKZ we can always get rid of the monodromy. So the above actions $\sigma, \sigma'$ are not too much different in contrast to the differential theory.

This argument can be applied to the AQKZ itself, although the group $\mathbb{Z}^n$ is infinite. We can formally solve the QAKZ equation as follows. Let $\Psi \in \mathcal{F}(\mathbb{C}^n, \mathcal{H}_n^t)$. Then the infinite sum

$$\sum_{b \in B} \tilde{b}(\Psi), \tag{4.22}$$

where $B = \oplus_{i=1}^n \mathbb{Z}\gamma_i \subset \tilde{\mathbb{S}}_n$, satisfies the AQKZ, provided the convergence. For example, if $\Psi$ is rapidly decreasing, then one can check that $\sum_{b\in B}\tilde{b}(\Psi)$ is convergent.

We see, that constructing End$(V)$-valued solution $\Phi$ to QAKZ for finite-dimensional $\mathcal{H}_n^t$-modules $V$ poses no problem. What is more difficult is to ensure a proper asymptotical behavior.

**4.4. Isomorphism of QAKZ and the Macdonald eigenvalue problem.** In this subsection, we introduce the Macdonald eigenvalue problem and prove its equivalence to the QAKZ equation. This is a $q$-analogue of the relation between AKZ and QMBP discussed in §3.4.

Let $\Phi$ be a solution of the QAKZ equation with values in $\mathrm{End}(V)$ for a $\mathcal{H}_n^t$-module $V$. We assume that it is invertible for sufficiently general $v$. Setting $\sigma_i' = \sigma_{s_i}'$, we get from (4.18) and (4.9):

$$F_i(v_i - v_{i+1})\Phi = \sigma_i'(\Phi),$$
$$T_i\Phi = \left(t\sigma_i' + \frac{t-t^{-1}}{e^{v_i-v_{i+1}}-1}(\sigma_i'-1)\right)\Phi.$$

Let us introduce the operator $\hat{T}_i{}'$ $(1 \le i \le n)$ by

$$\hat{T}_i{}' = t\sigma_i' + \frac{t-t^{-1}}{e^{v_i-v_{i+1}}-1}(\sigma_i'-1). \tag{4.23}$$

Then $\hat{T}_i{}'\Phi = T_i\Phi$ and $\sigma_\pi'\Phi = P\Phi$ (see (4.19)). The operators $\hat{T}_i{}'$ and $\sigma_\pi'$ commute with the left multiplication by $T_j$, $P$ and any elements from $\mathcal{H}_n^t$. Using all these:

$$\begin{aligned} Y_i\Phi &= T_{i-1}^{-1}\cdots T_1^{-1}PT_{n-1}\cdots T_{i+1}T_i\Phi \\ &= T_{i-1}^{-1}\cdots T_1^{-1}PT_{n-1}\cdots T_{i+1}\hat{T}_i'\Phi \\ &= \hat{T}_i'T_{i-1}^{-1}\cdots T_1^{-1}PT_{n-1}\cdots T_{i+1}\Phi \\ &\cdots \\ &= \hat{T}_i'\cdots\hat{T}_{n-1}'\sigma_\pi'(\hat{T}_1')^{-1}\cdots(\hat{T}_{i-1}')^{-1}\Phi. \end{aligned}$$

We come to the following definition:

$$\Delta_i' = \hat{T}_i'\cdots\hat{T}_{n-1}'\sigma_\pi'(\hat{T}_1')^{-1}\cdots(\hat{T}_{i-1}')^{-1},\ 1 \le i \le n. \tag{4.24}$$

Since $Y_i\Phi = \Delta_i'\Phi$ and $Y_i$ commute with each other,

$$[\Delta_i', \Delta_j'] = 0.$$

By the construction, the operators $\Delta_i'$ act in $\mathrm{End}(V)$-valued functions. However if we understand them formaly the commutativity can be deduced from the relations

$$\sigma_i'v_i = v_{i+1}\sigma_i', \tag{4.25}$$
$$\sigma_i'\sigma_{\gamma_i} = \sigma_{\gamma_{i+1}}\sigma_i'. \tag{4.26}$$
$$\sigma_{\gamma_i}' = \sigma_{\gamma_i} \tag{4.27}$$

The latter means that $\mathcal{T}_{\gamma_i} = 1$.

Let $Q$ be a polynomial in $n$ variables. Then

$$Q(Y_1,\dots,Y_n)\Phi = Q(\Delta_1',\dots,\Delta_n')\Phi$$

and we can represent

$$Q(\Delta_1',\dots,\Delta_n') = \sum_{w\in\mathbb{S}_n} D_w^{(Q)}\sigma_w', \tag{4.28}$$

where $D_w^{(Q)}$ are pure difference operators, which do not contain $\sigma_w'$ $(w \in \mathbb{S}_n)$.

For symmetric $Q$, we introduce a difference operator of Macdonald type $M_Q$ by

$$M_Q = \sum_{w \in \mathbb{S}_n} D_w^{(Q)}.$$

Let $\varphi$ be a $\mathbb{C}$-valued function on $\mathbb{C}^n$. The system

$$M_Q \varphi = Q(\lambda_1, \dots, \lambda_n)\varphi. \tag{4.29}$$

will be called the Macdonald eigenvalue problem. The operators $M_Q$ can be calculated for $\sigma$ instead of $\sigma'$. As in the differential case, the result will be the same.

Fix $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. We take a left $\mathcal{H}_n^t$-module $V_\lambda$ with the following properties:

(1) for any symmetric polynomial $Q$ in $n$ variables and all $a \in V_\lambda$,

$$Q(Y_1, \dots, Y_n)a = Q(\lambda_1, \dots, \lambda_n)a,$$

(2) there exists a $\mathbb{C}$-linear map $\mathrm{tr} : V_\lambda \to \mathbb{C}$ such that

$$\mathrm{tr}((T_i - t)a) = 0$$

for all $i$ and $a \in V_\lambda$.

As always, we fix a (local) invertible solution $\Phi(v)$ of the QAKZ equation with values in $\mathrm{End}(V_\lambda)$. Note that all $V_\lambda$-valued solutions of the QAKZ equation can be written in the form $\varphi(v) = \Phi(v)a(v)$ for $B$-periodic $V_\lambda$-valued function $a(v)$:

$$a(\dots, v_i + h, \dots) = a(v) \text{ for } i = 1, \dots, n.$$

THEOREM 4.7. *Let $V_\lambda$ be an $\mathcal{H}_n^t$-module with the above properties, $Sol_{QAKZ}(V_\lambda)$ be the space of solutions of the QAKZ equation with values in $V_\lambda$, and $Sol_{Mac}(\lambda)$ the space of solutions of the Macdonald eigenvalue problem* (4.29). *Then*

$$Sol_{QAKZ}(V_\lambda) \xrightarrow{\mathrm{tr}} Sol_{Mac}(\lambda).$$

PROOF. Let $\varphi(v) = \Phi(v)a \in Sol_{QAKZ}(V_\lambda)$. Then

$$(\sigma_i' - 1)\Phi = \left(t + \frac{t - t^{-1}}{e^{v_i - v_{i+1}} - 1}\right)^{-1} (T_i - t)\Phi.$$

For a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ of $w \in \mathbb{S}_n$,

$$\begin{aligned}
\sigma_w' - 1 &= \sigma_{i_l}' \cdots \sigma_{i_1}' - 1 \\
&= \sigma_{s_{i_l}}' \cdots \sigma_{i_2}'(\sigma_{i_1}' - 1) + \sigma_{i_l}' \cdots \sigma_{i_2}' - 1 \\
&\cdots \\
&= \sum_{k=1}^{l} \sigma_{i_l}' \cdots \sigma_{i_{k+1}}'(\sigma_{i_k}' - 1).
\end{aligned}$$

Since $\sigma'_i$ commutes with the left action of $\{T\}$, we have

$$\begin{aligned}(\sigma'_w - 1)\Phi &= \sum_{k=1}^{l} \sigma'_{i_1} \cdots \sigma'_{i_{k+1}} (\sigma'_{i_k} - 1)\Phi \\ &= \sum_{k=1}^{l} \sigma'_{i_1} \cdots \sigma'_{i_{k+1}} (\text{a scalar function})(T_{i_k} - t)\Phi \\ &= \sum_{k=1}^{l} (\text{a scalar function})(T_{i_k} - t)\sigma'_{i_1} \cdots \sigma'_{i_{k+1}}\Phi.\end{aligned}$$

Using the commutativity of $D_w^{(Q)}$ with $T_i - t$ we represent $D_w^{(Q)}(\sigma'_w - 1)\Phi$ as a sum $\sum(T_i - t)\Psi_i$ for some $\mathcal{H}_n^t$-valued functions $\Psi_i$. Finally

$$\begin{aligned}Q(\lambda_1, \dots, \lambda_n)\Phi &= Q(\Delta'_1, \dots, \Delta'_n)\Phi \\ &= \sum_{w \in \mathbf{S}_n} D_w^{(Q)} \sigma'_w \Phi \\ &= \sum_{w \in \mathbf{S}_n} D_w^{(Q)} \Phi + \sum_{w \in \mathbf{S}_n} D_w^{(Q)} (\sigma'_w - 1)\Phi \\ &= M_Q \Phi + \sum (T_i - t)\Psi_i.\end{aligned}$$

Applying this relation to $a \in V_\lambda$ and taking tr, we conclude:

$$Q(\lambda_1, \dots, \lambda_n)\,\mathrm{tr}(\varphi) = M_Q\,\mathrm{tr}(\varphi).$$

□

Let us now consider $V_\lambda = J_\lambda^\circ$. The definition is quite similar to the differential case. We start with

$$J_\lambda = \mathrm{Ind}_{H_n^t}^{\mathcal{H}_n^t}(+)/L_\lambda.$$

Here $H_n^t = \langle T_1, \dots, T_{n-1} \rangle \subset \mathcal{H}_n^t$, $+ : H_n^t \longrightarrow \mathbb{C}$ is the one-dimensional representation sending $T_i$ to $t$, and $L_\lambda$ is the ideal generated by $p(Y_1, \dots, Y_n) - p(\lambda)$ ($p \in \mathbb{C}[x_1, \dots, x_n]^{\mathbf{S}_n}$). As in §3.1, $J_\lambda^\circ$ stands for the dual module defined via the anti-involution $\circ$ of $\mathcal{H}_n^t$:

$$Y_i^\circ = Y_i, \qquad T_i^\circ = T_i.$$

The main result of this subsection is the following theorem from [10, 12].

THEOREM 4.8. *If $V_\lambda = J_\lambda^\circ$, then the map from $Sol_{QAKZ}(V_\lambda)$ to $Sol_{Mac}(\lambda)$ is injective.*

The theorem results from the following two lemmas.

LEMMA 4.9. *Let $K$ be a $\mathcal{H}_n^t$-submodule of $J_\lambda^\circ$. Then* $\mathrm{tr}(K) = 0$ *implies* $K = 0$.

The proof repeats that in the differential case (see ((3.38))).

LEMMA 4.10. *Let $\varphi$ be a $V_\lambda$-valued solution of the QAKZ equation. Assume* $\mathrm{tr}(\varphi) = 0$. *Then* $\mathrm{tr}(\mathcal{H}_n^t \varphi) = 0$.

PROOF. First

$$\operatorname{tr}(T_i\varphi) = t\operatorname{tr}(\varphi) = 0$$

for all $i$. Then $\pi = s_1 s_2 \cdots s_{n-1}\gamma_n$,

$$\begin{aligned}\sigma'_\pi - 1 &= \sigma_{\gamma_n}\sigma'_{n-1}\cdots\sigma'_1 - 1\\ &= \sigma_{\gamma_n}\left(\sum_{k=1}^{n-1}\sigma'_{n-1}\cdots\sigma'_{k+1}(\sigma'_k - 1)\right) + \sigma_{\gamma_n} - 1,\end{aligned}$$

and $\operatorname{tr}(\sigma_{\gamma_n}\varphi) = \operatorname{tr}(\varphi) = 0$. Therefore, representing $\varphi = \Phi a$ ($a \in V_\lambda$), we have

$$\begin{aligned}\operatorname{tr}(P\varphi) - \operatorname{tr}(\varphi) &= \operatorname{tr}\left((\sigma'_\pi - 1)\Phi a\right)\\ &= \operatorname{tr}\left(\sigma_{\gamma_n}(\sum_{k=1}^{n-1}\sigma'_{n-1}\cdots\sigma'_{k+1}(\sigma'_k - 1)\Phi)a\right)\\ &= \sum_{k=1}^{n-1}\operatorname{tr}\left(\sigma_{\gamma_n}\sigma'_{n-1}\cdots\sigma'_{k+1}f_i(v)(T_k - t)\Phi a\right)\\ &= \sum_{k=1}^{n-1}\operatorname{tr}\left((T_k - t)\sigma_{\gamma_n}\sigma'_{n-1}\cdots\sigma'_{k+1}f_i(v)\Phi a\right)\\ &= 0,\end{aligned}$$

where $f_i(v)$ are $\mathbb{C}$-valued function. Hence $\operatorname{tr}(P\varphi) = 0$ and

$$\begin{aligned}\operatorname{tr}(Y_n\varphi) &= \operatorname{tr}(T_{n-1}^{-1}\cdots T_1^{-1}P\varphi)\\ &= t^{1-n}\operatorname{tr}(P\varphi)\\ &= 0.\end{aligned}$$

Now we shall prove that $\operatorname{tr}(Y_i\varphi) = 0$ for all $i$ by induction. Assume that $\operatorname{tr}(Y_i\varphi) = 0$ for $k+1 \le i \le n$. Since $Y_k = T_{k-1}^{-1}\cdots T_1^{-1}PT_{n-1}\cdots T_k$ it is enough to see that $\operatorname{tr}(PT_{n-1}\cdots T_k\varphi) = 0$. Since $\varphi$ is a solution of the QAKZ equation we have

$$\begin{aligned}\operatorname{tr}(F_{k-1}^{-1}\cdots F_1^{-1}PF_{n-1}\cdots F_k\varphi) &= \operatorname{tr}({\gamma_k}^{-1}\varphi)\\ &= {\gamma_k}^{-1}\operatorname{tr}(\varphi)\\ &= 0.\end{aligned}$$

On the other hand,

$$F_i(v) = c_i(v)(T_i + f_i(v))$$

where $c_i(v)$ and $f_i(v)$ are some scalar functions. Therefore

$$\begin{aligned}0 = \operatorname{tr}(PF_{n-1}\cdots F_k\varphi) &= \operatorname{tr}(c_{n-1}\cdots c_k P(T_{n-1} + f_{n-1}(v))\cdots(T_k + f_k(v))\varphi)\\ &= \sum_{I=(i_1,\dots,i_l)}\operatorname{tr}(c_I(v)PT_{i_l}\cdots T_{i_1}\varphi)\end{aligned}$$

where $I = (i_1, \dots, i_l)$ is a sequence of integers such that $k \le i_1 < i_2 < \cdots < i_l \le n-1$, and $c_I(v)$ is some scalar function. If $I \ne I_0 = (k, k+1, \dots, n-1)$ then there are the following possibilities:

(1) $i_l \ne n-1$,

(2) $i_l = n-1$ and there exists an $m$ $(1 \leq m \leq l)$ such that $i_j - i_{j-1} = 1$ for any $j = m+1, m+2, \ldots, l$ and $i_m - i_{m-1} > 1$,

(3) otherwise.

case (1): As $i_l < n-1$, we have

$$\begin{aligned}\operatorname{tr}(PT_{i_l} \cdots T_{i_1}\varphi) &= \operatorname{tr}(T_{i_l+1} \cdots T_{i_1+1}P\varphi) \\ &= t^l \operatorname{tr}(P\varphi) \\ &= 0.\end{aligned}$$

case (2): Since$[T_i, T_j] = 0$ for $|i-j| > 1$,

$$\begin{aligned}\operatorname{tr}(P(T_{i_l} \cdots T_{i_m})(T_{i_{m-1}} \cdots T_{i_1})\varphi) &= \operatorname{tr}(P(T_{i_{m-1}} \cdots T_{i_1})(T_{i_l} \cdots T_{i_m})\varphi) \\ &= \operatorname{tr}(T_{i_{m-1}+1} \cdots T_{i_1+1}PT_{i_l} \cdots T_{i_m}\varphi) \\ &= t^{m-1} \operatorname{tr}(PT_{i_l} \cdots T_{i_m}\varphi).\end{aligned}$$

By the induction hypothesis, $\operatorname{tr}(PT_{i_l} \cdots T_{i_m}\varphi) = 0$. Hence

$$\operatorname{tr}(PT_{i_l} \cdots T_{i_1}\varphi) = 0.$$

case (3): In this case $I = (i_1, \ldots, i_l)$ must be of the form $i_l = n-1,\ i_{l-1} = n-2, \ldots, i_1 = n-l > k$. By induction, $\operatorname{tr}(PT_{i_l} \cdots T_{i_1}\varphi) = 0$. So $\operatorname{tr}(Y_i\varphi) = 0$ for all $i$.

Because of the relations between $T$ and $Y$, it remains to check that $\operatorname{tr}(Y_{i_1} \cdots Y_{i_l}\varphi) = 0$ for any $l$. One can show this by induction on $l$. □

**4.5. Macdonald operators.** We set

$$\hat{T}_i = t\sigma_i + \frac{t-t^{-1}}{e^{v_i - v_{i+1}} - 1}(\sigma_i - 1), \qquad (1 \le i \le n-1), \tag{4.30}$$

$$G_{ij} = t + \frac{t-t^{-1}}{e^{v_i - v_j} - 1}(1 - \sigma_{ij}), \qquad (1 \le i, j \le n), \tag{4.31}$$

$$\Delta_i = \hat{T}_i \cdots \hat{T}_{n-1}\sigma_\pi \hat{T}_1^{-1} \cdots \hat{T}_{i-1}^{-1}. \tag{4.32}$$

Here $\sigma_w$ are from (4.20), $\sigma_{ij} = \sigma_{s_{ij}}$.

Switching from $\{T\}$ to $\{G\}$:

$$\begin{aligned} \hat{T}_i\sigma_i &= G_{ii+1}, \\ G_{ij}^{-1} &= t^{-1} - \frac{t-t^{-1}}{e^{v_i - v_j} - 1}(1 - \sigma_{ij}), \\ \Delta_i &= G_{ii+1} \cdots G_{in}\sigma_{\gamma_i} G_{1i}^{-1} \cdots G_{i-1i}^{-1}. \end{aligned}$$

Let $e_m$ be the $m$-th elementary symmetric polynomial in $n$ variables. We represent

$$e_m(\Delta_1, \ldots, \Delta_n) = \sum_{w \in S_n} D_w^{(m)} \sigma_w, \tag{4.33}$$

for difference operators $D_w^{(m)}$, and define

$$M_m = M_{e_m} = \sum_{w \in S_n} D_w^{(m)}.$$

All these operators are $W$-invariant, which results from the following lemmas.

LEMMA 4.11. *Consider the algebra $\hat{\mathcal{H}}$ generated by $\hat{T}_i$ $(1 \le i \le n-1)$, $\Delta_j$ $(1 \le j \le n)$. Then $T_i \mapsto \hat{T}_i$, $Y_j \mapsto \Delta_j$ extends to an algebra isomorphism $\mathcal{H}_n^t \tilde{\to} \hat{\mathcal{H}}$. Moreover, if $Q$ is a symmetric polynomial in $n$ variables, then $Q(\Delta_1, \ldots, \Delta_n)$ is a central element in $\hat{\mathcal{H}}$.*

Actually this observation is the key point (it can be checked directly or with some representation theory). We note that the formulas for $T$ generalize the so-called Demazure operations and the Bernstein-Gelfand-Gelfand operations. They were also studied by Lusztig and in a paper by Kostant-Kumar.

Form now on we identify $\hat{\mathcal{H}}$ with $\mathcal{H}_n^t$.

LEMMA 4.12. *Let $f(v_1, \ldots, v_n)$ be a function on $\mathbb{C}^n$. Then $f$ is symmetric if and only if $(\hat{T}_i - t)f = 0$ for all $i$.*

LEMMA 4.13. *Let $Q$ be a symmetric polynomial in $n$ variables. Then $Q(\Delta_1, \ldots, \Delta_n)$ acts on the space of the symmetric polynomials in $e^{v_i}$ $(1 \le i \le n)$.*

PROOF. This follows immediately from Lemma 4.11 and 4.12. □

Let us calculate $M_1$. Since $M_1$ is symmetric, it is enough to find the coefficient of $\sigma_{\gamma_1}$. Using the $G$-representation it is easy to see that $\sigma_{\gamma_1}$ does not appear in $\Delta_2, \ldots, \Delta_n$. The $\sigma_{\gamma_1}$-factor of $\Delta_1$ is equal to $\prod_{i=2}^n \frac{te^{v_1 - v_i} - t^{-1}}{e^{v_1 - v_i} - 1}\sigma_{\gamma_1}$.

After the symmetrization we get the formula:

$$M_1 = \sum_{i=1}^n \prod_{j \ne i} \frac{te^{v_i} - t^{-1}e^{v_j}}{e^{v_i} - e^{v_j}} \sigma_{\gamma_i}.$$

Similarly,

$$M_m = \sum_{I=(i_1,\dots,i_m)} \prod_{i\in I\ j\notin I} \frac{te^{v_i} - t^{-1}e^{v_j}}{e^{v_i} - e^{v_j}} \sigma_{\gamma_{i_1}} \cdots \sigma_{\gamma_{i_m}}$$

where $I = (i_1, \dots, i_m)$ is a sequence of integers such that $1 \le i_1 < \cdots < i_m < n$.

To recapitulate, let us consider the classical limit of the Macdonald operators. Setting $q = e^h$ and $t = q^{k/2}$, $h \to 0$, we have

$$\begin{aligned} \Delta_i &= 1 + h\mathcal{D}_i + O(h^2), \\ M_1 - n &= h \sum \frac{\partial}{\partial v_i} + O(h^2), \end{aligned}$$

$$M_2 - (n-1)M_1 + \frac{n(n-1)}{2} = h^2 L_2 + O(h^3).$$

**4.6. Comments.**

REMARK 4.1. Take a solution $\Phi = \Phi(v)$ of the QAKZ equation in an $\mathcal{H}^t$-module $V$, assuming that $\Phi$ has the *trivial monodromy*. Then, for any polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$, we have

$$p(Y_1, \dots, Y_n)\Phi = p(\Delta_1, \dots, \Delta_n)\Phi, \tag{4.34}$$

where $\Delta_i$ are the difference Dunkl operators defined before. Note that $\Delta_i'$ can be replaced by $\Delta_i$ because the monodromy of $\Phi$ is trivial. We also need a linear functional $\mathrm{pr} : V_\lambda \to \mathbb{C}$ for a vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that

$$\mathrm{pr}(Y_i b) = \lambda_i \,\mathrm{pr}(b) \quad (i = 1, \dots, n) \tag{4.35}$$

for any $b \in V$. Given any element $a \in V$, let us define a scalar-valued function $\varphi = \varphi(v)$ setting

$$\varphi(v) = \mathrm{pr}(\Phi(v)a) \in \mathbb{C}. \tag{4.36}$$

Then the formula (4.34) implies

$$p(\lambda_1, \dots, \lambda_n)\varphi = p(\Delta_1, \dots, \Delta_n)\varphi. \tag{4.37}$$

Thus, the scalar-valued function $\varphi = \varphi(v)$ solves the *Dunkl eigenvalue problem.*

REMARK 4.2. *Arbitrary root systems.* Let $\Sigma = \{\alpha\} \in \mathbb{R}^n$ be any reduced root system of rank $n$ (of type $A, B, C, D, E, F$ or $G$), and

$$\mathcal{H}^t = \langle T_1, \dots, T_n, X_1, \dots, X_n \rangle \tag{4.38}$$

the corresponding affine Hecke algebra. The *baxterization* (a parametric deformation satisfying the Yang-Baxter relations) of $T_i$ will be given by

$$F_i = T_i + \frac{t - t^{-1}}{e^{u_i} - 1} \quad \text{with} \quad u_i = (u, \alpha_i) \tag{4.39}$$

for each $i = 1, \dots, n$. We also have to use the element

$$T_0 = X_{\theta^\vee} T_\theta^{-1} \tag{4.40}$$

corresponding to the simple affine root $\alpha_0 = \delta - \theta$ for $\theta$ being the highest root. Its baxterization is quite similar:

$$F_0 = T_0 + \frac{t - t^{-1}}{e^{h-u_\theta} - 1}, \tag{4.41}$$

where $u_\theta = (u, \theta)$. The functions $F_0, F_1, \ldots, F_n$ satisfy the Yang-Baxter equations associated with the extended Dynkin diagram. For example, in the case of $\circ^1 \Rightarrow \circ^2$, we have

$$F_1(v)F_2(u+v)F_1(2u+v)F_2(u) = F_2(u)F_1(2u+v)F_2(u+v)F_1(v). \tag{4.42}$$

The arguments of $F_i$ can also be determined graphically by means of the equivalent pictures of the reflection of two particles (see [9]).

Using $T_0$, the affine Hecke algebra $\mathcal{H}^t$ has an alternative representation

$$\mathcal{H}^t = \langle T_0, T_1, \ldots, T_n; \Pi \rangle, \tag{4.43}$$

where $\Pi$ is a certain finite abelian group. The group $\Pi$ is isomorphic to $P^\vee/Q^\vee$. It is the set of all elements of the extended affine Weyl group

$$\widetilde{W} = W \ltimes B, \quad B = \bigoplus_{i=1}^{n} \mathbb{Z}\, b_i, \tag{4.44}$$

preserving the set $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ of the simple affine roots. It gives the embedding of $\Pi$ into the automorphism group of the extended Dynkin diagram. The action of $\widetilde{W}$ on $\mathbb{R}^n \bigoplus \mathbb{R}\delta$ is by the affine reflections and the corresponding shifts in the $\delta$-direction for $B$:

$$b(z + \zeta\delta) = z + (\zeta - (b, z))\delta.$$

LEMMA 4.14. $\widetilde{W} = \langle s_0, s_1, \ldots, s_n; \Pi \rangle$ *with* $s_0 = (\theta^\vee) \cdot s_\theta$.

The group $\Pi$ can be embedded into the affine Hecke algebra. The images $P_\pi$ of the elements $\pi \in \Pi$ permute $\{T_i\}$ in the same way as $\pi$ do in $\widetilde{W}$ with $\{s_i\}$. The baxterization of the elements in $\Pi$ is trivial: $F_\pi = P_\pi$ for each $\pi \in \Pi$.

Keeping the notations of the previous sections, we have the following theorem.

THEOREM 4.15. *Given any* $\mathcal{H}^t$*-valued function* $\Psi = \Psi(u)$, *the formulas*

$$\widetilde{s}_i(\Psi) = {}^{s_i}(F_i\Psi) \tag{4.45}$$

*for all* $i = 0, 1, \ldots, n$, *and*

$$\widetilde{\pi}(\Psi) = P_\pi\, {}^{\pi}\Psi \tag{4.46}$$

*for all* $\pi \in \Pi$ *induce a representation of* $\widetilde{W}$.

The QAKZ equation for $\Sigma$ is the invariance condition $\widetilde{b}(\Phi) = \Phi$ for all $b \in B$. It can be shown that this equation is equivalent to the difference QMBP associated with the root system $\Sigma$ defined via similar Dunkl operators. A conceptual proof of this isomorphism theorem is given by means of the intertwiners of double affine Hecke algebras (see [10, 12]).

## 5. Double affine Hecke algebras and Macdonald polynomials

**5.1. Macdonald polynomials : the $A_1$ case.** The subject of this section is to show how the Hecke algebra technique is applied to the Macdonald polynomials. We will concentrate on the duality and the recurrence relations. The key notion will be the double affine Hecke. Let us start with $A_1$.

The corresponding $L$-operator in the differential case reads as follows

$$L^{(k)} = \frac{\partial^2}{\partial u^2} + 2k\frac{e^u + e^{-u}}{e^u - e^{-u}}\frac{\partial}{\partial u} + k^2, \tag{5.1}$$

where $k$ is a complex parameter. There are two special values of $k$ when the operator $L^{(k)}$ is very simple. For $k = 0$ we have $L^{(0)} = \partial^2/\partial u^2$. When $k = 1$,

$$L^{(1)} = d^{-1}\frac{\partial^2}{\partial u^2}d, \quad \text{with} \quad d = e^u - e^{-u}. \tag{5.2}$$

Similarly, we can conjugate by $d^k$ for any $k$:

$$d^k L^{(k)} d^{-k} = \frac{\partial^2}{\partial u^2} - \frac{4k(k-1)}{(e^u - e^{-u})^2}. \tag{5.3}$$

Sometimes $L$ is more convenient to deal with in this form.

Let us now consider the eigenvalue problem for the operator $L^{(k)}$:

$$L^{(k)}\varphi = \lambda^2\varphi. \tag{5.4}$$

If $k = 1$, the solution of this equation is immediate:

$$\varphi(u;\lambda) = \frac{\sinh(u\lambda)}{\sinh(u)\sinh(\lambda)}. \tag{5.5}$$

In this normalization it is symmetric with respect to $u$ and $\lambda$. Without $\sinh(\lambda)$ it generalizes the characters of finite-dimensional representations of $SL_2(\mathbb{C})$ ($k$=1).

If $k = 1/2$, this operator is the *radial part* of the Casimir operator for the symmetric space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. It is the restriction of the Casimir operator $C$ on the double coset space $SO_2\backslash SL_2/SO_2$ wich is identified with a domain in $\mathbb{R}^*/S_2$. If $k = 1, 2$, then $L^{(k)}$ corresponds in the same way to $SL_2(\mathbb{C})/SU(2)$ and $SL_2(\mathcal{K})/SU_2(\mathcal{K})$ for the quaternions $\mathcal{K}$.

For any $k$, one can find a family of even ($u \to -u$) solutions of the form

$$p_n = e^{nu} + e^{-nu} + \text{lower integral exponents}, \tag{5.6}$$

such that

$$L^{(k)}p_n = (n+k)^2 p_n \tag{5.7}$$

for $n = 0, 1, 2, \ldots$. This family of hyperbolic polynomials satisfy the orthogonality relations

$$\text{Constant Term } (p_n p_m d^{2k}) = c_n \delta_{nm}. \tag{5.8}$$

They are called the *ultraspherical polynomials.*

We can also consider the rational limit

$$\ell^{(k)} = \frac{\partial^2}{\partial u^2} + \frac{2k}{u}\frac{\partial}{\partial u} \tag{5.9}$$

of the operator $L^{(k)}$, switching from the Sutherland model to the Calogero model. The solutions of the rational eigenvalue problem are expressed in terms of the Bessel function. In this case, the solutions can be normalized to ensure the symmetry between the variable and the eigenvalue. In the trigonometric case it is possible only for two special values $k = 0, 1$. It is one of the main demerits of the harmonic analysis on the symmetric spaces.

In the difference theory, this symmetry holds for any root systems. This discovery is expected to renew the Harish-Chandra theory. The so-called group case ($k$=1) is an intersection point of the differential (classical) and difference (new) theories.

We now turn to the difference version. We set $x = e^u$ and introduce the 'multiplicative difference' $\Gamma_q$ acting as $\Gamma_q(f(x) = f(qx)$ and satisfying the commutation relation $\Gamma_q x = qx\Gamma_q$. The Macdonald operator $L$ is expressed as follows:

$$L = \frac{tx - t^{-1}x^{-1}}{x - x^{-1}}\Gamma_q + \frac{tx^{-1} - t^{-1}x}{x^{-1} - x}\Gamma_q^{-1}. \tag{5.10}$$

The parameter $k$ in the difference setup is determined from the relation $t = q^k$. When $q = t$ (or $k = 1$), the operator $L$ is simple:

$$L = \frac{1}{x - x^{-1}}(\Gamma_q + \Gamma_q^{-1})(x - x^{-1}). \tag{5.11}$$

Compare this formula with (5.2) in the differential case and notice that (5.11) is easier to check than (5.2).

The eigenvalue problem

$$L\varphi = (\Lambda + \Lambda^{-1})\varphi \tag{5.12}$$

always has a self-dual family of solutions. When $n = 0, 1, 2, \ldots$, there exists a unique family of the so-called $q, t$-*ultraspherical* (or Roger-Askey-Ismail) Laurent polynomials

$$p_n = x^n + x^{-n} + \text{lower terms}, \tag{5.13}$$

which are symmetric with respect to the transformation $x \to x^{-1}$, and satisfy the equation

$$Lp_n = (tq^n + t^{-1}q^{-n})p_n. \tag{5.14}$$

The following duality theorem is proved in the next section by using the double affine Hecke algebra.

THEOREM 5.1 (DUALITY). *$p_n(tq^m)p_m(t) = p_m(tq^n)p_n(t)$ for any $m, n = 0, 1, 2, \ldots$.*

If we set

$$\pi_n(x) = \frac{p_n(x)}{p_n(t)}, \tag{5.15}$$

the duality can be rewritten as follows:

$$\pi_n(tq^m) = \pi_m(tq^n) \quad (m, n = 0, 1, 2, \ldots). \tag{5.16}$$

The Askey-Ismail polynomials are nothing but the Macdonald polynomials of type $A_1$. There are three main Macdonald's conjectures for the Macdonald polynomials associated with root systems (see [34, 35]):

(1) the scalar product conjecture,

(2) the evaluation conjecture,

(3) the duality conjecture.

One may also add the Pieri rules to the list. These conjectures were justified recently using the double affine Hecke algebras in [11, 14].

**5.2. A modern approach to $q,t$-ultraspherical polynomials.** The duality from Theorem 5.1 can be rephrased as the symmetry of a certain scalar product. Actually this product is a diffence counterpart of the spherical Fourier transform. For any symmetric Laurent polynomials $f, g \in \mathbb{C}[x+x^{-1}]$, we set

$$\{f,g\} = (a(L)g)(t), \tag{5.17}$$

where $a$ is a polynomial such that $f(x) = a(x+x^{-1})$. So we apply the operator $a(L)$ to $g$ and then evaluate the result at $x = t$.

THEOREM 5.2. *$\{f,g\} = \{g,f\}$ for any $f, g \in \mathbb{C}[x+x^{-1}]$.*

Theorem 5.1 follows from Theorem 5.2. Indeed, if $f = \pi_m$ and $g = \pi_n$, we can compute the scalar product as follows:

$$\{\pi_m, \pi_n\} = (a(L)\pi_n)(t) = a(tq^n + t^{-1}q^{-n})\pi_n(t) = \pi_m(qt^n). \tag{5.18}$$

Use $L\pi_n = (tq^n + t^{-1}q^{-n})\pi_n$ and the normalization $\pi_n(t) = 1$. Hence Theorem 5.2 implies $\pi_m(qt^n) = \pi_n(qt^m)$. Actually the theorems are equivalent, since $p_n$ form a basis in the space of all symmetric Laurent polynomials.

DEFINITION 5.1. The double affine Hecke algebra $\mathcal{HH}^{q,t}$ of type $A_1$ is the quotient

$$\mathcal{HH}^{q,t} = \langle X, Y, T\rangle / \sim, \tag{5.19}$$

by the relations for the generators $X, Y, T$

$$TXT = X^{-1}, \qquad T^{-1}YT^{-1} = Y^{-1}, \tag{5.20}$$

$$Y^{-1}X^{-1}YXT^2 = q^{-1}, \quad (T-t)(T+t^{-1}) = 0. \tag{5.21}$$

Here we consider $q, t$ as numbers or parameters. The first point of the theory is the following statement of PBW type.

Any element of $H \in \mathcal{HH}$ can be uniquely expressed in the form

$$H = \sum_{\substack{i,j\in\mathbb{Z}\\ \epsilon=0,1}} c_{i\epsilon j} X^i T^\epsilon Y^j \quad (c_{i\epsilon j} \in \mathbb{C}). \tag{5.22}$$

The second imortant fact is the symmetry of $\mathcal{HH}^{q,t}$ with respect to $X$ and $Y$.

THEOREM 5.3. *There exists an anti-involution $\phi : \mathcal{HH} \to \mathcal{HH}$ such that $\phi(X) = Y^{-1}$, $\phi(Y) = X^{-1}$ and $\phi(T) = T$.*

Indeed, $\phi$ transposes the first two relations and leaves the remaining invariant.

Next, we introduce the expectation value $\{H\}_0 \in \mathbb{C}$ of an element $H \in \mathcal{HH}^{q,t}$ by

$$\{H\}_0 = \sum_{\substack{i,j\in\mathbb{Z}\\ \epsilon=0,1}} c_{i\epsilon j} t^{-i} t^\epsilon t^j, \tag{5.23}$$

using the expression (5.22). The definitions of $\phi$ and $\{\ \}_0$ give that

$$\{\phi(H)\}_0 = \{H\}_0 \quad \text{for any } H \in \mathcal{H}^{q,t}. \tag{5.24}$$

Now we can introduce the operator counterpart of the pairing $\{f,g\} = \{g,f\}$ on $\mathcal{H}^{q,t} \times \mathcal{H}^{q,t}$: setting

$$\{A,B\}_0 = \{\phi(A)B\}_0 \tag{5.25}$$

for any $A, B \in \mathcal{H}^{q,t}$.

The $\phi$-invariance of the expectation value (5.24) ensures that it is symmetric

$$\{A,B\}_0 = \{B,A\}_0.$$

We also remark that this pairing is non-degenerate for generic $q, t$.

Theorem 5.2 readily follows from

LEMMA 5.4. *For any symmetric Laurent polynomials* $f(x)$,$g(x) \in \mathbb{C}[x+x^{-1}]$,

$$\{f(X), g(X)\}_0 = \{f,g\}.$$

To prove the lemma we need to introduce the *basic representation* of the double affine Hecke algebra $\mathcal{H}^{q,t}$. Consider the one-dimensional representation of the Hecke algebra $\mathcal{H}_Y = \langle T, Y\rangle$ sending $T \mapsto t$ and $Y \mapsto t$. We denote this representation simply by $+$. Then take the induced representation

$$V = \mathrm{Ind}_{\mathcal{H}_Y}^{\mathcal{H}}(+) = \mathcal{H}/\{\mathcal{H}(T-t) + \mathcal{H}(Y-t)\} \simeq \mathbb{C}[x, x^{-1}], \tag{5.26}$$

where the last isomorphism is $x^n \leftrightarrow X^n \mod \mathcal{H}(T-t) + \mathcal{H}(Y-t)$. Under this identification of $V$ with the ring $\mathbb{C}[x,x^{-1}]$ of Laurent polynomials, the element $X$ acts on $\mathbb{C}[x,x^{-1}]$ as the multiplication by $x$, while $T$ and $Y$ act by the operators

$$\widehat{T} = ts + \frac{t-t^{-1}}{x^2-1}(s-1) \quad \text{and} \quad \widehat{Y} = s\Gamma_q \widehat{T}, \tag{5.27}$$

respectively. Here $s(f)(x) = f(x^{-1})$, the equality $Hf(x) = g(x)$ in $V$ means that $Hf(X) - g(X) \in \mathcal{H}(T-t) + \mathcal{H}(Y-t)$. The latter readily gives the desired formulas for $\hat{T}, \hat{Y}$.

The expectation value is the composition

$$\mathcal{H} \xrightarrow{\alpha} V \cong \mathbb{C}[x,x^{-1}] \xrightarrow{\beta} \mathbb{C}, \tag{5.28}$$

where $\alpha$ is a residue mod $\mathcal{H}(T-t) + \mathcal{H}(Y-t)$ and $\beta(f) = f(t^{-1})$ is the evaluation map at $t^{-1}$. Take any $f, g \in \mathbb{C}[X, X^{-1}]$. Then

$$\{f(X), g(X)\}_0 = \{\phi(f(X))g(X)\}_0 = \{f(Y^{-1})g(X)\}_0 = f(\hat{Y}^{-1})(g)(t^{-1}). \tag{5.29}$$

The last equality follows from (5.28). If $f$ and $g$ are symmetric and $f(X) = a(X + X^{-1})$, then

$$\{f(X), g(X)\}_0 = a(L)(g)(t) = \{f,g\}, \tag{5.30}$$

since the operator $\widehat{Y} + \widehat{Y}^{-1}$ acts on symmetric Laurent polynomials as $L$. It is straightforward. The duality is established.

This method of proving of the duality theorem can be generalized to any root system.

We now discuss the application of the duality to the *Pieri rules*, the recurrence formulas for $\pi_n$'s with respect to the index $n$. First we will *discretize* functions and operators.

Recall that the renormalized $q,t$-ultraspherical polynomials $\pi_n(x)$ are characterized by the conditions

$$L\pi_n = (tq^n + t^{-1}q^{-n})\pi_n, \quad \pi_n(t) = 1, \tag{5.31}$$

where

$$L = \frac{tx - t^{-1}x^{-1}}{x - x^{-1}}\Gamma + \frac{tx^{-1} - t^{-1}x}{x^{-1} - x}\Gamma^{-1}. \tag{5.32}$$

As always, $\Gamma x = qx\Gamma$. Denote the set of $\mathbb{C}$-valued functions on $\mathbb{Z}$ by Funct$(\mathbb{Z},\mathbb{C})$. For any Laurent polynomial $f \in \mathbb{C}[x, x^{-1}]$ or more general rational function, we define $\widehat{f} \in$ Funct$(\mathbb{Z},\mathbb{C})$ by setting

$$\widehat{f}(m) = f(tq^m) \quad \text{for all} \quad m \in \mathbb{Z}. \tag{5.33}$$

Considering $\mathcal{A} = \langle \mathbb{C}(x), \Gamma\rangle$ as an abstract algebra with the fundamental relation $\Gamma x = qx\Gamma$, the action of $\mathcal{A}$ on $\varphi \in$ Funct$(\mathbb{Z},\mathbb{C})$ is as follows:

$$\widehat{x}\varphi(m) = tq^m\varphi(m), \quad \widehat{\Gamma}\varphi(m) = \varphi(m+1). \tag{5.34}$$

The correspondence $f \mapsto \widehat{f}$, whenever it is well-defined (the functions $f$ may have denominators), is an $\mathcal{A}$-homomorphism $\mathbb{C}(x) \to$ Funct$(\mathbb{Z},\mathbb{C})$.

Due to (5.31):

$$\widehat{L}\widehat{\pi}_n(m) = (tq^{-n} + t^{-1}q^n)\widehat{\pi}_n(m). \tag{5.35}$$

The Pieri rules result directly from this equality. Indeed, the duality $\widehat{\pi}_n(m) = \widehat{\pi}_m(n)$ implies:

$$\mathcal{L}\widehat{\pi}_m(n) = (tq^n + t^{-1}q^{-n})\widehat{\pi}_m(n) = (\widehat{x} + \widehat{x}^{-1})\widehat{\pi}_m(n). \tag{5.36}$$

Here $\mathcal{L}$ is $\widehat{L}$ acting on the indices $m$ instead of $n$. Explicitly,

$$\frac{t^2q^m - t^{-2}q^{-m}}{tq^m - t^{-1}q^{-m}}\widehat{\pi}_{m+1}(n) + \frac{q^{-m} - q^m}{t^{-1}q^{-m} - tq^m}\widehat{\pi}_{m-1}(n) = (\widehat{x} + \widehat{x}^{-1})\widehat{\pi}_m(n). \tag{5.37}$$

For generic $q,t$, the mapping $f \to \widehat{f}$ is injective. Hence one can pull (5.37) back, removing the hats:

$$(x + x^{-1})\pi_m = \frac{t^2q^m - t^{-2}q^{-m}}{tq^m - t^{-1}q^{-m}}\pi_{m+1} + \frac{q^m - q^{-m}}{tq^m - t^{-1}q^{-m}}\pi_{m-1}. \tag{5.38}$$

This is the *Pieri formula* in the case of $A_1$. See [2]. We remark that this formula makes sense when $m = 0$, since the coefficient of $\pi_{m-1}$ vanishes at $m = 0$. Generally speaking the 'vanishing conditions' are much less obvious.

The Pieri rules obtained above can be used to prove the so-called *evaluation conjecture* describing the values of $p_n$ at $x = t$. Applying (5.38) repeatedly, we get the formula

$$(x + x^{-1})^\ell\pi_m = c_{\ell,m}\pi_{m+\ell} + \text{lower terms} \tag{5.39}$$

for each $\ell = 0, 1, 2, \ldots$. The leading coefficient $c_{\ell,m}$ can be readily calculated:

$$c_{\ell,m} = \prod_{i=0}^{\ell-1} \frac{t^2 q^{m+i} - t^{-2} q^{-m-i}}{t q^{m+i} - t^{-1} q^{-m-i}}. \tag{5.40}$$

Let us look at (5.39) for $m = 0$:

$$(x + x^{-1})^\ell = c_{\ell,0} \pi_\ell + \text{lower terms}. \tag{5.41}$$

Comparing the coefficients of $x^l + x^{-l}$, we have $1 = c_{\ell,0}/p_\ell(t)$, since $p_\ell = x^\ell + x^{-\ell} + \cdots$. Hence

$$p_\ell(t) = c_{\ell,0} = \prod_{i=0}^{\ell-1} \frac{t^2 q^i - t^{-2} q^{-i}}{t q^i - t^{-1} q^{-i}}. \tag{5.42}$$

This value is easy to calculate directly (the formulas for $p_n$ are known). However the method descibed in this section is applicable to arbitrary root systems. We need only the duality, which is the main advantage of the difference harmonic analysis in contrast to the classical Harish-Chandra theory.

**5.3. The $GL_n$ case.** In this last subsection, we will discuss the double affine Hecke algebra and applications for $GL_n$. Since we have already clarified the $A_1$ case in full detail, we will try to get concentrated on the main points only.

In the $GL_n$ case, the Macdonald operators $M_0 = 1, M_1, \ldots, M_n$ are as follows:

$$M_m = \sum_{I=\{i_1<\ldots<i_m\}} \prod_{\substack{i\in I\\ j\notin I}} \frac{t x_i - t^{-1} x_j}{x_i - x_j} \Gamma_{i_1} \cdots \Gamma_{i_m}. \tag{5.43}$$

In this normalization, $t = q^{k/2}$ (cf. the differential case). For instance, the so-called group case is for $k = 1/2$ (in contrast to $SL_2$ considered above).

The *Macdonald polynomials* $p_\lambda$ for $GL_n$ satisfy the Macdonald eigenvalue problem:

$$M_m p_\lambda = e_m(t^{n-1} q^{\lambda_1}, \ldots, t^{-n+1} q^{\lambda_n}) p_\lambda \quad (m = 0, 1, \ldots, n), \tag{5.44}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are partitions, i.e., sequences of integers $\lambda_i \in \mathbb{Z}$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$). Here $e_m$ is the elementary symmetric function of degree $m$. Civen $\lambda$, $p_\lambda = p_\lambda(x)$ is a symmetric polynomial in $x = (x_1, \ldots, x_n)$ of degree $|\lambda| = \sum_{i=1}^n \lambda_i$ in the form

$$p_\lambda(x) = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \text{lower order terms}. \tag{5.45}$$

The lower order terms are understood in the sense of the dominance ordering. Namely, a partition obtained from $\lambda$ by subtracting simple roots $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$ is lower than $\lambda$. For instance,

$$(\lambda_1, \lambda_2, \ldots) > (\lambda_1 - 1, \lambda_2 + 1, \ldots) > (\lambda_1 - 1, \lambda_2, \lambda_3 + 1 \ldots) > \cdots. \tag{5.46}$$

We will use the abbreviation

$$t^{2\rho} q^\lambda = (t^{n-1} q^{\lambda_1}, \ldots, t^{-n+1} q^{\lambda_n}), \tag{5.47}$$

where $2\rho = (n-1, n-3, \ldots, -n+1)$. So $M_m p_\lambda = e_m(t^{2\rho} q^\lambda) p_\lambda$. Using $k$: $t = q^{k/2}$, and $t^{2\rho} q^\lambda = q^{k\rho+\lambda}$.

Given a partition $\lambda$, we set

$$\pi_\lambda(x) = \frac{p_\lambda(x)}{p_\lambda(t^{2\rho})} = \frac{p_\lambda(x_1, \dots, x_n)}{p_\lambda(t^{n-1}, t^{n-3}, \dots, t^{-n+1})}. \tag{5.48}$$

THEOREM 5.5 (DUALITY). *For any partitions $\lambda$ and $\mu$, we have*

$$\pi_\lambda(t^{2\rho}q^\mu) = \pi_\mu(t^{2\rho}q^\lambda). \tag{5.49}$$

This duality theorem implies the following *Pieri formula.*

THEOREM 5.6.

$$e_m(x)\pi_\lambda(x) = \sum_{|I|=m} \prod_{\substack{i\in I\\ j\notin I}} \frac{t^{2(j-i)+1}q^{\lambda_i-\lambda_j} - t^{-1}}{t^{2(j-i)}q^{\lambda_i-\lambda_j} - 1} \pi_{\lambda+\epsilon_I}(x), \tag{5.50}$$

*where $\epsilon_I = \sum_{i\in I}\epsilon_i$ (sum of unit vectors).*

Here the summation is taken only over subsets $I \subset \{1, 2, \dots, n\}$ ($|I| = m$) such that $\lambda + \epsilon_I$ remain partitions (generally speaking, dominant). It happens automatically, since the coefficient of $\pi_{\lambda+\epsilon_I}(x)$ on the right vanishes unless $\lambda + \epsilon_I$ is dominant.

We can also determine the value of the Macdonald polynomial $p_\lambda(x)$ at $x = t^{2\rho}$ exactly by the method used in the $A_1$ case. The formula was conjectured by Macdonald and prooved by Koornwinder. The above theorems (for $GL_n$) are also due to Macdonald and Koornwinder. See also [22]. For arbitrary roots they were established in my recent papers.

Our approach is based on the double Hecke algebras. The operators $M_m$ appear naturally using the operators $\Delta_i$ from (4.32). The latter describe the action of the generators $Y_i$ in the induced representation $\mathrm{Ind}_{\mathcal{H}_Y}^{\mathcal{H\!H}}(+)$ isomorphic to the algebra of Laurent polynomials $\mathbb{C}[X_1^{\pm1}, \dots, X_n^{\pm1}]$. So the analogy with (5.26) is complete.

The *double affine Hecke algebra* (DAHA) $\mathcal{H\!H} = \mathcal{H\!H}^{q,t}$ for $GL_n$ is the algebra generated by the following two commutative algebras of Laurent polynomials in $n$ variables:

$$\mathbb{C}[X_1^{\pm1}, \dots, X_n^{\pm1}] \quad \text{and} \quad \mathbb{C}[Y_1^{\pm1}, \dots, Y_n^{\pm1}], \tag{5.51}$$

and the Hecke algebra of type $A_{n-1}$:

$$\mathcal{H} = \langle T_1, \dots, T_{n-1}\rangle \tag{5.52}$$

with the standard braid and quadratic relations. The remaining relations are as follows:

$$T_iX_iT_i = X_{i+1}\ (i = 1, \dots, n-1), \quad T_iX_j = X_jT_i\ (j \neq i, i+1), \tag{5.53}$$

$$T_i^{-1}Y_iT_i^{-1} = Y_{i+1}\ (i = 1, \dots, n-1), \quad T_iY_j = Y_jT_i\ (j \neq i, i+1), \tag{5.54}$$

$$Y_2^{-1}X_1Y_2X_1^{-1} = T_1^2, \tag{5.55}$$

$$\widetilde{Y}X_j = qX_j\widetilde{Y} \quad \text{and} \quad \widetilde{X}Y_j = q^{-1}Y_j\widetilde{X}. \tag{5.56}$$

Here $\widetilde{X} = \prod_{i=1}^n X_i$ and $\widetilde{Y} = \prod_{i=1}^n Y_i$. They commute with $\{T_1, \dots, T_{n-1}\}$ thanks to 5.53 and 5.55.

When $q = 1, t = 1$ we come to the *elliptic* or 2-extended *Weyl group* of type $GL_n$ due to Saito [42]. If $q = 1$ and there are no quadratic relations, the corresponding group is the *elliptic braid group* ($\pi_1$ of the product of $n$ elliptic curves without the diagonals divided by $\mathbb{S}_n$). It was calculated by Birman [3] and Scott.

Establishing the connection with (4.32), $X_i = e^{v_i}$, $T_i = \hat{T}_i$ and $Y_i = \Delta_i$ give the so-called polynomial (or basic) representation of $\mathcal{HH}$.

There is another version of this definition, using the element $\pi$. It is introduced from the formula

$$Y_1 = T_1 \cdots T_{n-1}\pi^{-1} \tag{5.57}$$

and has the following commutation relations with $X_i$ and $T_i$:

$$\pi X_i = X_{i+1}\pi \quad (i = 1, \dots, n-1), \quad \pi X_n = q^{-1}X_1\pi \tag{5.58}$$

and

$$\pi T_i = T_{i+1}\pi \quad (i = 1, \dots, n-2). \tag{5.59}$$

In the polynomial representation this element coincides with $\pi$ from Lemma 4.3. Note that it acts on the functions $X_i = e^{v_i}$ through the action of $\pi^{-1}$ on vectors $v$. Considered formally, $\pi$ is the image of the element $P$ from Lemma 4.1 with respect to the Kazhdan-Lusztig automorphism, sending $T \to T^{-1}$, $Y \to Y^{-1}$, $t \to t^{-1}$.

Since

$$T_{i-1} \cdots T_1 X_1 T_1 \cdots T_{i-1} = X_i, \quad T_1 \cdots T_{i-1} Y_i T_{i-1} \cdots T_1 = Y_1, \tag{5.60}$$

we can reduce the list of generators. Namely,

$$\mathcal{HH} = \langle X_1, Y_1, T_1, \dots, T_{n-1} \rangle \tag{5.61}$$

or

$$\mathcal{HH} = \langle X_1, \pi, T_1, \dots, T_{n-1} \rangle. \tag{5.62}$$

In terms of $\{T, \pi, X\}$, the list of defining relations of $\mathcal{HH}$ is as follows:

(a) $X_i X_j = X_j X_i$ $(1 \le i, j \le n)$,

(b) the braid relations and quadratic relations for $T_1, \dots, T_{n-1}$,

(c) $\pi X_i = X_{i+1}\pi$ $(i = 1, \dots n-1)$ and $\pi^n X_i = q^{-1} X_i \pi^n$ $(i = 1, \dots, n)$,

(d) $\pi T_i = T_{i+1}\pi($ $i = 1, \dots n-2)$ and $\pi^n T_i = T_i \pi^n$ $(i = 1, \dots, n-1)$.

For instance, let us deduce (5.55) from these formulas. Substituting, the left hand side equals:

$$\begin{aligned}
&(T_1\pi T_{n-1}^{-1} \cdots T_2^{-1}) X_1 (T_2 \cdots T_{n-1}\pi^{-1}T_1^{-1}) X_1^{-1} \\
&= T_1\pi T_{n-1}^{-1} \cdots T_2^{-1}(T_2 \cdots T_{n-1}) X_1 \pi^{-1}(T_1^{-1}X_1^{-1}T_1^{-1})T_1 \\
&= T_1(\pi X_1 \pi^{-1} X_2^{-1})T_1 = T_1^2.
\end{aligned}$$

This representation, however, is not convenient from the viewpoint of the symmetry between $X_i$ and $Y_i$, which will be discussed next. It is better to use $\{Y\}$ instead of $\pi$.

The algebra $\mathcal{HH}$ contains the following two affine Hecke algebras:

$$\mathcal{H}_X^t = \langle X_1, \dots, X_n, T_1, \dots, T_{n-1} \rangle,$$

$$\mathcal{H}_Y^t = \langle Y_1, \dots, Y_n, T_1, \dots, T_{n-1} \rangle. \tag{5.63}$$

They are isomorphic to each other by the correspondence $X_i \leftrightarrow Y_i^{-1}$. This map can be extended to an anti-involution of $\mathcal{H}$. It is a general statement which holds for any root systems.

THEOREM 5.7. *There exists an anti-involution* $\phi : \mathcal{H}^{q,t} \to \mathcal{H}^{q,t}$ *such that* $\phi(X_i) = Y_i^{-1}$, $\phi(Y_i) = X_i^{-1}$ *for* $i = 1, \ldots, n$, *and* $\phi(T_i) = T_i$ $(i = 1, \ldots, n-1)$. *It preserves* $q, t$.

PROOF. We need to check that the relation (5.55) is self-dual with respect to $\phi$. The other relations are obviously $\phi$-invariant. One has:

$$1 = T_1^{-2} Y_2^{-1} X_1 Y_2 X_1^{-1} == T_1^{-1}\{Y_1^{-1}(T_1 X_1 T_1^{-1}) Y_1 T_1^{-1} X_1^{-1}\}$$
$$= Y_1^{-1} X_2 T_1^{-2} Y_1 (T_1^{-1} X_1^{-1} T_1^{-1}) = Y_1^{-1} X_2 T_1^{-2} Y_1 X_2^{-1}.$$

The latter can be rewritten as $Y_1 X_2^{-1} Y_1^{-1} X_2 = T_1^2$, which is the $\phi$-image of (5.55). □

Using this involution we can establish the duality theorem for the $GL_n$ case in the same way as we did in the $A_1$ case. Generalizing the theory to the case of arbitrary roots we can prove the Macdonald conjectures and much more. It gives a very convincing example of the power of the modern difference-operator methods.

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