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Factorization of the radial Schrödinger equation and four kinds of raising and lowering operators of 3D and 2D hydrogen atoms

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Abstract

It was shown that only for the Coulomb potential and isotropic harmonic oscillator the radial Schrödinger equation can be factorized. Four kinds of raising and lowering operators of 3D and 2D hydrogen atoms were constructed and the corresponding selection rules and conserved quantum numbers were discussed.

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I. INTRODUCTION

The factorization method introduced by Schrödinger [1] and extended by Infeld and Hull [2] was used in the previous paper [3] to treat the radial Schrödinger equation of 3-dimensional (3D) and 2-dimensional (2D) isotropic harmonic oscillator. It was found that two kinds of raising and lowering operators can be introduced directly from the factorization of the radial Schrödinger equation, and in terms of the two kinds of operator the other two kinds of raising and lowering operators can be constructed. In this paper we will investigate the possibility of factorization of the radial Schrödinger equation of a particle in a central potential. In Sect. II, it will be shown that only for the Coulomb potential and isotropic harmonic oscillator, the radial Schrödinger equation can be factorized. In Sect. III, four kinds of raising and lowering operators of a 3D hydrogen atom are introduced by using the factorization method and the recurrence relations of the confluent hypergeometric function, and the corresponding selection rules and conserved quantum numbers will be discussed. Similar investigation of a 2D hydrogen atom is given in Sect. III.

II. FACTORIZATION OF THE RADIAL SCHRÖDINGER EQUATION

For a particle in the central field $V(r)$, the energy eigenfunction may be chosen as the simultaneous eigenstate of the complete set of conserved observables $(H, \hat{\mathbf{l}}^2, \hat{l}_z)$, where $\hat{\mathbf{l}}$ is the angular momentum, i.e.,

$$\Psi = R_l(r)Y_{lm}(\theta,\phi) = \frac{\chi_l(r)}{r}Y_{lm}(\theta,\phi), \quad l = 0,1,2,\cdots, \quad m = l, l-1,\cdots,-l. \tag{1}$$

where $Y_{lm}(\theta,\phi)$ is the spherical harmonic function, and $\chi_l(r)$ satisfies the radial equation,

$$\chi_l''(r) + \left[\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]\chi_l(r) = 0, \qquad \chi_l(0) = 0. \tag{2}$$

Putting $\hbar = \mu = 1$, $-2E = \lambda_l$, (2) may be recast into the following form

$$\begin{aligned} D(l)\chi_l(r) &= \lambda_l\chi_l(r), \\ D(l) &\equiv \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - 2V(r). \end{aligned} \tag{3}$$

To factorize $D(l)$, let us define the l-dependent raising and lowering operators

$$\begin{aligned} A_+(l) &= \frac{d}{dr} - \frac{l+1}{r} + f(l,r), \\ A_-(l) &= \frac{d}{dr} + \frac{l}{r} + g(l,r), \end{aligned} \tag{4}$$

$f(l,r)$ and $g(l,r)$ are to be determined to meet the requirement of factorization of the Hamiltonian (or $D(l)$); i.e.,

$$\begin{aligned} A_-(l+1)A_+(l) &= D(l) + c_1(l), \\ A_+(l-1)A_-(l) &= D(l) + c_2(l), \end{aligned} \tag{5}$$

where $c_1(l)$ and $c_2(l)$ are r-independent constants.

Using (4), it can be shown that

$$\begin{aligned} A_-(l+1)A_+(l) &= \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + [f(l,r) + g(l+1,r)]\frac{d}{dr} + \frac{df(l,r)}{dr} \\ &+ [f(l,r) - g(l+1,r)]\frac{(l+1)}{r} + g(l+1,r)f(l,r), \\ A_+(l-1)A_-(l) &= \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + [g(l,r) + f(l-1,r)]\frac{d}{dr} + \frac{dg(l,r)}{dr} \\ &+ [f(l-1,r) - g(l,r)]\frac{l}{r} + f(l-1,r)g(l,r). \end{aligned} \tag{6}$$

Compairing (5) and (6), one gets

$$f(l,r) = -g(l+1,r), \tag{7}$$

$$-2V(r) + c_1(l) = \frac{df(l,r)}{dr} + 2f(l,r)\frac{l+1}{r} - f(l,r)^2, \tag{8}$$

$$-2V(r) + c_2(l) = -\frac{df(l-1,r)}{dr} + 2f(l-1,r)\frac{l}{r} - f(l-1,r)^2. \tag{9}$$

We assume $V(r)$ is l-independent. Replacing l by $(l+1)$ in (9), and substracting from (8), one gets

$$\frac{df(l,r)}{dr} = a(l), \quad a(l) = \frac{1}{2}\left[c_1(l) - c_2(l+1)\right]. \tag{10}$$

Thus

$$f(l,r) = a(l)r + b(l), \tag{11}$$

where $b(l)$ is an integrating constant. Substituting (11) into (8) or (9), we obtain

$$2V(r) + 2b(l)\frac{l+1}{r} - a(l)^2r^2 - 2a(l)b(l)r = \frac{1}{2}[c_1(l) + c_2(l+1)] - 2a(l) + b(l)^2. \tag{12}$$

(12) should hold at arbitrary point r and for any value of l, which implies that the form of the l-independent $V(r)$ must be one of the following three types of central potential:

$$(i)\ V(r) \propto \frac{1}{r}, \quad (ii)\ V(r) \propto r^2, \quad (iii)\ V(r) \propto r. \tag{13}$$

Now we investigate the three cases separately.

Case (i), $V(r) \propto \frac{1}{r}$. From (12), we have $a(l) = 0$ and $f(l,r) = b(l) = 1/(l+1)$. In this case, apart from an additive constant, $V(r) = -1/r$ (Coulomb potential). From (10) and (11), we obtain $c_1(l) = c_2(l+1)$, and $c_1(l) = -1/(l+1)^2$, $c_2(l) = -1/l^2$, $(l > 0)$. Thus, from (4), we get

$$\begin{aligned} A_+(l) &= \frac{d}{dr} - \frac{l+1}{r} + \frac{1}{l+1}, \\ A_-(l) &= \frac{d}{dr} + \frac{l}{r} - \frac{1}{l}. \end{aligned} \tag{14}$$

and (5) is reduced to

$$\begin{aligned} A_-(l+1)A_+(l) &= D(l) + \frac{1}{l+1}, \\ A_+(l-1)A_-(l) &= D(l) - \frac{1}{l^2}, \end{aligned} \tag{15}$$

In fact, (15) can be verified by straightforward calculation using (14). Using (15) and (3), it is easily shown that,

$$D(l)[A_+(l-1)\chi_{l-1}] = \lambda_{l-1}A_+(l-1)\chi_{l-1}, \tag{16}$$

$$D(l)[A_-(l+1)\chi_{l+1}] = \lambda_{l+1}A_-(l+1)\chi_{l+1}. \tag{17}$$

From (16) it is seen that if χ_{l-1} is an eigenstate of $D(l-1)$ with eigenvalue λ_{l-1}, then $A_+(l-1)\chi_{l-1}$ is an eigenstate of $D(l)$ with eigenvalue λ_{l-1}. Thus, the effect of A_+ is to increase the angular momentum by 1, but keep the energy eigenvalue unchanged. Similarly, the effect of A_- is to decrease the angular momentum by 1, but keep the energy eigenvalue unchanged. Therefore, A_+ and A_- are the angular momentum raising and lowering operators respectively, which have been investigated in [4-6].

Case (ii), $V(r) \propto r^2$. From (12) we have $b(l) = 0$, $a(l)$ =constant. Choosing natural units, $a(l)^2 = 1$, we get $V(r) = \frac{1}{2}r^2$ (isotropic harmonic oscillator), and $a(l) = \pm 1$.

For $a(l) = +1$, the raising and lowering operators are labelled by $A_\pm$,

$$A_+(l) = \frac{d}{dr} - \frac{l+1}{r} + r, \quad A_-(l) = \frac{d}{dr} + \frac{l}{r} - r, \tag{18}$$

For $a(l) = -1$, the raising and lowering operators are labelled by $B_\pm$,

$$B_+(l) = \frac{d}{dr} - \frac{l+1}{r} - r, \quad B_-(l) = \frac{d}{dr} + \frac{l}{r} + r. \tag{19}$$

In this case, (5) is reduced to

$$A_-(l+1)A_+(l) = D(l) + (2l+3), \qquad A_+(l-1)A_-(l) = D(l) + (2l-1), \tag{20}$$

$$B_-(l+1)B_+(l) = D(l) - (2l+3), \qquad B_+(l-1)B_-(l) = D(l) - (2l-1), \tag{21}$$

which can also be verified by straightforward calculation using (18) and (19). From (20), (21) and (3), it can be shown that

$$\begin{aligned} D(l)[A_+(l-1)\chi_{l-1}] &= (\lambda_{l-1}+2)A_+(l-1)\chi_{l-1}, \\ D(l)[A_-(l+1)\chi_{l+1}] &= (\lambda_{l+1}-2)A_-(l+1)\chi_{l+1}, \end{aligned} \tag{22}$$

$$\begin{aligned} D(l)[B_+(l-1)\chi_{l-1}] &= (\lambda_{l-1}-2)B_+(l-1)\chi_{l-1}, \\ D(l)[B_-(l+1)\chi_{l+1}] &= (\lambda_{l+1}+2)B_-(l+1)\chi_{l+1}. \end{aligned} \tag{23}$$

From (22) it is seen that the effect of A_+ (A_-) is to increase (decrease) the angular momentum by 1 and simultaneously decrease (increase) its energy eigenvalue by 1. Similarly, from (23) we see that the effect of B_+ (B_-) is to increase (decrease) the angular momentum by 1 and simultaneously increase (decrease) the energy eigenvalue by 1. In terms of $A_\pm$ and $B_\pm$ we may construct the other two bands of raising and lowering operators C and D of 3D isotropic harmonic oscillator (for details, see [3]).

Case (iii), $V(r) \propto r$. From (12) we have $a(l) = b(l) = 0$. But in this case, one gets $V(r)$ =constant. Thus, a linear potential $V(r) \propto r$ is excluded.

Therefore, we arrive at the conclusion that, *only for the Coulomb potential and isotropic harmonic oscillator the radial Schrödinger equation can be factorized.* It is worthwhile to note that, in addition to the O_3 geometric symmetry, there exists additional dynamical symmetry for both the Coulomb attractive potential (bound states, O_4 symmetry) and the isotropic harmonic oscillator (SU_3 symmetry), which results in the l-degeneracy in energy eigenvalues and additional constants of motion. The above conclusion holds both for the 3D and 2D hydrogen atoms and isotropic harmonic oscillators. Finally, the above conclusion reminds us of the famous Bertrand theorem in classical mechanics [7], which says: "The only central forces that result in closed orbits for all bound particles are the inverse square law and Hooke's law". The underlying physics concerning the connection between the Bertrand theorem in classical mechanics and the factorization of the radial Schrödinger equation needs further investigation.

III. FOUR KINDS OF RAISING AND LOWERING OPERATORS OF A 3D HYDROGEN ATOM

In Sect. II, using the factorization method, one kind of raising and lowering operators, $A_+(l)$ and $A_-(l)$ are derived. The other kinds of raising and lowering operator can be derived from the recurrence relations of the confluent hypergeometric functions.

For an electron in a 3D hydrogen atom, $V(r) = -e^2/r$, the radial wave equation reads ($\hbar = \mu = e = 1$)

$$\begin{aligned} \chi_l''(r) + \left[\frac{2}{r} - \frac{l(l+1)}{r^2}\right]\chi_l(r) &= \lambda_l \chi_l(r), \quad \lambda_l = -2E, \\ \chi_l(0) &= 0. \end{aligned} \tag{24}$$

Considering the asymptotic behavior of $\chi_l(r)$ as $r \to 0$ and $r \to \infty$, we may put

$$\chi_l(r) = r^{l+1} e^{-\sqrt{\lambda_l} r} u(r), \tag{25}$$

and $u(r)$ satisfies

$$r u'' + [2(l+1) - 2\sqrt{\lambda_l} r]u' - [(l+1)\sqrt{\lambda_l} - 1]u = 0. \tag{26}$$

Let $\xi = 2\sqrt{\lambda_l} r$, (26) is reduced to the confluent hypergeometric equation

$$\xi \frac{d^2u}{d\xi^2} + [2(l+1) - \xi]\frac{du}{d\xi} - \left[(l+1) - \frac{1}{\sqrt{\lambda_l}}\right]u = 0. \tag{27}$$

The solution u (satisfying $\chi_l(0) = 0$) may be expressed as the hypergeometric function $F(\alpha, \gamma, \xi)$,

$$\alpha = (l+1) - \frac{1}{\sqrt{\lambda_l}}, \quad \gamma = 2(l+1). \tag{28}$$

For bound states, the infinite series expansion of F must be terminated, thus it is required that $\alpha = -n_r$, $n_r = 0, 1, 2, \cdots$, where $n_r = 0, 1, 2, \cdots$ is the numbser of nodes of the radial wave function ($r = 0, \infty$ excluded). So $\sqrt{\lambda_l} = (l + n_r + 1)^{-1}$, and the energy eigenvalue is

$$E = E_n = -\frac{1}{2n^2} \text{ (natural unit,)} \quad n = l + n_r + 1 = 1, 2, 3, \cdots. \tag{29}$$

The corresponding radial wave function may be expressed as

$$\begin{aligned}\chi_{ln}(r) &\sim r^{l+1}e^{-r/n}F(-n_r, 2l+2, 2r/n) \\ &\propto \xi_n^{l+1}e^{-\xi_n/2}F(-n_r, 2l+2, \xi_n),\ \xi_n = 2r/n,\end{aligned} \tag{30}$$

For a given energy level E_n, $l = 0, 1, \cdots, n-1$, (l-degenerancy), and the degenercy is n^2.

Now, using the fundamental recurrence relations of confluent hypergeometric function (see Appendix of ref. [3]), one may prove

$$\left[(\gamma-\alpha-x)+x\frac{d}{dx}\right]F(\alpha,\gamma,x) = (\gamma-\alpha)F(\alpha-1,\gamma,x), \tag{31}$$

$$\left(\alpha+x\frac{d}{dx}\right)F(\alpha,\gamma,x) = \alpha F(\alpha+1,\gamma,x), \tag{32}$$

$$\left[(\alpha+x)-(\gamma+x)\frac{d}{dx}\right]F(\alpha,\gamma,x) = \frac{(\gamma-\alpha)(\gamma+1-\alpha)}{\gamma(\gamma+1)}xF(\alpha,\gamma+2,x), \tag{33}$$

$$\begin{aligned}&\left\{[(\gamma-1)(\gamma-2)+\alpha x]+x(\gamma-2+x)\frac{d}{dx}\right\}F(\alpha,\gamma,x) \\ &\quad = (\gamma-1)(\gamma-2)F(\alpha,\gamma-2,x),\end{aligned} \tag{34}$$

$$\begin{aligned}&\left[(\alpha-1)x+(\gamma-1-x)(\gamma-2-x)+(\gamma-2-x)x\frac{d}{dx}\right]F(\alpha,\gamma,x) \\ &\quad = (\gamma-1)(\gamma-2)F(\alpha-2,\gamma-2,x),\end{aligned} \tag{35}$$

$$\left[-\alpha+(\gamma-x)\frac{d}{dx}\right]F(\alpha,\gamma,x) = \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xF(\alpha+2,\gamma+2,x). \tag{36}$$

Defining operator $M(k)$,

$$M(k)f(x) = f(kx) \tag{37}$$

and using (31–36), we may derive other three kinds of raising and lowering operators, B, C and D given in (39)–(41), in addition to the operators $A_\pm(l)$ given in (14). To clearly indicate their effects, the operators $A_\pm(l)$ are relabelled as $A(l\uparrow,n)$, and $A(l\downarrow,n)$. The four kinds of raising and lowering operators of a 3D hydrogen atom are summarized in (38) through (41) and are graphically illustrated in Fig. 1. The corresponding selection rules and conserved quantum numbers are given in Table 1.

$$\begin{aligned}A(l\uparrow,n) &= \frac{d}{dr}-\frac{l+1}{r}+\frac{1}{l+1} \\ A(l\downarrow,n) &= \frac{d}{dr}+\frac{l}{r}-\frac{1}{l},\end{aligned} \tag{38}$$

$$\begin{aligned}B(l,n\uparrow) &= \left[r\frac{d}{dr}-\frac{r}{n+1}+n\right]M\left(\frac{n}{n+1}\right), \\ B(l,n\downarrow) &= \left[r\frac{d}{dr}+\frac{r}{n-1}-n\right]M\left(\frac{n}{n-1}\right),\end{aligned} \tag{39}$$

$$\begin{aligned}C(l\uparrow,n\uparrow) &= \left\{[(l+1)(n+1)+r]\frac{d}{dr}-\frac{r}{n+1}\right. \\ &\quad \left.-\frac{(l+1)^2(n+1)}{r}+(n-l-1)\right\}M\left(\frac{n}{n+1}\right), \\ C(l\downarrow,n\downarrow) &= \left\{[l(n-1)+r]\frac{d}{dr}+\frac{r}{n-1}\right. \\ &\quad \left.+\frac{l^2(n-1)}{r}-(n-l)\right\}M\left(\frac{n}{n-1}\right),\end{aligned} \tag{40}$$

$$\begin{aligned}D(l\downarrow,n\uparrow) &= \left\{[l(n+1)-r]\frac{d}{dr}+\frac{r}{n+1}\right. \\ &\quad \left.+\frac{l^2(n+1)}{r}-(n+l)\right\}M\left(\frac{n}{n+1}\right), \\ D(l\uparrow,n\downarrow) &= \left\{[(l+1)(n-1)-r]\frac{d}{dr}-\frac{r}{n-1}\right. \\ &\quad \left.-\frac{(l+1)^2(n-1)}{r}+(n+l+1)\right\}M\left(\frac{n}{n-1}\right).\end{aligned} \tag{41}$$

IV. FOUR KINDS OF RAISING AND LOWERING OPERATORS OF A 2D HYDROGEN ATOM

For an electron in a 2D hydrogen atom, $V(\rho) = -e^2/\rho$, the energy eigenfunction may be chosen as the simultaneous eigenstate of the complete set of conserved observables (H, $l_z = -i\hbar\frac{\partial}{\partial\phi}$),

$$\Psi = \frac{\chi_m(\rho)}{\sqrt{\rho}}e^{im\phi}, \quad m = 0, \pm1, \pm2, \cdots, \tag{42}$$

and $\chi_m(\rho)$ satisfies ($\hbar = \mu = e = 1$),

$$\begin{aligned}&\left(\frac{d^2}{d\rho^2}-\frac{m^2-1/4}{\rho^2}+\frac{2}{\rho}\right)\chi_m(\rho) \\ &= \left(\frac{d^2}{d\rho^2}-\frac{(m-1/2)(m+1/2)}{\rho^2}+\frac{2}{\rho}\right)\chi_m(\rho) \\ &= \lambda_m\chi_m(\rho),\end{aligned} \tag{43}$$

$$\chi_m(0) = 0, \quad \lambda_m = -2E. \tag{44}$$

Comparing (43) and (24), it is seen that there exist great similarity between 3D and 2D hydrogen atoms, and the correspondence between the parameters is

$$l(l+1) \leftrightarrow (m-1/2)(m+1/2) = (|m|-1/2)(|m|+1/2). \tag{45}$$

Considering the asymptotic behavior of $\chi_m(\rho)$ as $\rho \to 0$ and $\rho \to \infty$, we may put

$$\chi_m(\rho) = \rho^{|m|+1/2} e^{-\sqrt{\lambda_m}\rho} u(\rho), \tag{46}$$

$u(\rho)$ satisfies

$$\rho u'' + [2|m| + 1 - 2\sqrt{\lambda_m}\rho]u' - [(2|m|+1)\sqrt{\lambda_m} - 2]u = 0. \tag{47}$$

Let $\xi = 2\sqrt{\lambda_m}\rho$, (47) is reduced to the confluent hypergeometric equation

$$\xi\frac{d^2u}{d\xi^2} + (2|m| + 1 - \xi)\frac{du}{d\xi} - \left(|m| + 1/2 - \frac{1}{\sqrt{\lambda_m}}\right)u = 0. \tag{48}$$

The solution u (satisfying $\chi_m(0) = 0$) may be expressed as the hypergeometric function $F(\alpha, \gamma, \xi)$,

$$\alpha = |m| + 1/2 - \frac{1}{\sqrt{\lambda_m}}, \quad \gamma = 2|m| + 1. \tag{49}$$

For bound states, it is required that $\alpha = -n_\rho$, $n_\rho = 0, 1, 2, \cdots$. (number of the radial wave function $\chi_m(\rho)$). Thus, $\sqrt{\lambda_m} = (|m| + n_\rho + 1/2)^{-1}$, and the energy eigenvalue is

$$E = E_n = -\frac{1}{2n^2}, \quad n = |m| + n_\rho + 1/2 = 1/2, 3/2, 5/2, \cdots. \tag{50}$$

The corresponding radial wave function is

$$\begin{aligned}\chi_{|m|n}(r) &\sim \rho^{|m|}e^{-\rho/n}F(-n_\rho, 2|m|+1, 2\rho/n)\\ &\propto \xi_n^{|m|}e^{-\xi_n/2}F(-n_\rho, 2|m|+1, \xi_n), \quad \xi_n = 2\rho/n,\end{aligned} \tag{51}$$

and the degenercy of the level E_n is $2n = 1, 3, 5, \cdots$.

Using the factorization method similar to 3D hydrogen atom, we may construct the raising and lowering operators of a 2D hydrogen atom, for $m \geq 0$,

$$\begin{aligned}A_+(m) &= \frac{d}{d\rho} - \frac{m+1/2}{\rho} + \frac{1}{m+1/2},\\ A_-(m) &= \frac{d}{d\rho} + \frac{m-1/2}{\rho} - \frac{1}{m-1/2}.\end{aligned} \tag{52}$$

It is easily verified that

$$\begin{aligned}A_-(m+1)A_+(m) &= D(m) - 1/(m+1/2)^2,\\ A_+(m-1)A_-(m) &= D(m) - 1/(m-1/2)^2,\end{aligned} \tag{53}$$

and

$$\begin{aligned}D(m)\left[A_+(m-1)\chi_{m-1}\right] &= \lambda_{m-1}\left[A_+(m-1)\chi_{m-1}\right],\\ D(m)\left[A_-(m+1)\chi_{m+1}\right] &= \lambda_{m+1}\left[A_+(m+1)\chi_{m+1}\right].\end{aligned} \tag{54}$$

It is seen that the effect of A_+ (A_-) is to increase (decrease) the magnetic quantum number by 1, but keep the energy eigenvalue unchanged. Thus $A_\pm$ are the angular momentum raising and lowering operators and may be relabelled as $A(m\uparrow, n)$, and $A(m\downarrow, n)$.

Using the radial wave function $\chi_{|m|n}(\rho)$, (see (51)) and the recurrence relations of the confluent hypergeometric function (31)–(36), one may derived the other three kinds of raising and lowering operators in addition to $A_\pm$, which are summarized as follows ($m \geq 0$):

$$\begin{aligned}A(m\uparrow, n) &= \frac{d}{d\rho} - \frac{m+1/2}{\rho} + \frac{1}{m+1/2},\\ A(m\downarrow, n) &= \frac{d}{d\rho} + \frac{m-1/2}{\rho} - \frac{1}{m-1/2},\end{aligned} \tag{55}$$

$$\begin{aligned}B(m, n\uparrow) &= \left[\rho\frac{d}{d\rho} - \frac{\rho}{n+1} + n\right]M\left(\frac{n}{n+1}\right),\\ B(m, n\downarrow) &= \left[\rho\frac{d}{d\rho} + \frac{\rho}{n-1} + n\right]M\left(\frac{n}{n-1}\right),\end{aligned} \tag{56}$$

$$\begin{aligned}C(m\uparrow, n\uparrow) &= \Big\{[(|m|+1/2)(n+1) + \rho]\frac{d}{d\rho} - \frac{\rho}{n+1}\\ &- \frac{(|m|+1/2)^2(n+1)}{\rho} + (n - m - 1/2)\Big\}M\left(\frac{n}{n+1}\right),\\ C(m\downarrow, n\downarrow) &= \Big\{[(m-1/2)(n-1) + \rho]\frac{d}{d\rho} + \frac{\rho}{n-1}\\ &+ \frac{(m-1/2)^2(n-1)}{\rho} - n + m - 1/2\Big\}M\left(\frac{n}{n-1}\right),\end{aligned} \tag{57}$$

$$\begin{aligned}D(m\downarrow, n\uparrow) &= \Big\{[(m-1/2)(n+1) - \rho]\frac{d}{d\rho} + \frac{\rho}{n+1}\\ &+ \frac{(m-1/2)^2(n+1)}{\rho} - n - m + 1/2\Big\}M\left(\frac{n}{n+1}\right),\\ D(m\uparrow, n\downarrow) &= \Big\{[(m+1/2)(n-1) - \rho]\frac{d}{d\rho} - \frac{\rho}{n-1}\\ &- \frac{(m+1/2)^2(n-1)}{\rho} + n + m + 1/2\Big\}M\left(\frac{n}{n-1}\right).\end{aligned} \tag{58}$$

Because only the absolute value of m is involved in the energy eigenequation (43), we have $\lambda_{-m} = \lambda_m$, $\chi_{-m,n} = \chi_{m,n}$. Therefore, the expressions of these operators for $m \geq 0$ may be easily extended to $m \leq 0$ cases. Because m is not involved in the expression of B,

(56) remains unchanged for $m \leq 0$. As for the operators C and D, we have

$$C((-m)\uparrow, n\uparrow) = -D(m\downarrow, n\uparrow), \quad B((-m)\downarrow, n\downarrow) = -D(m\uparrow, n\downarrow),$$
$$D((-m)\downarrow, n\uparrow) = -C(m\uparrow, n\uparrow), \quad D((-m)\uparrow, n\downarrow) = -C(m\downarrow, n\downarrow). \tag{59}$$

For the operator A, we have

$$A((-m)\uparrow, n) = A(m\downarrow, n), \quad A((-m)\downarrow, n) = A(m\uparrow, n). \tag{60}$$

The selection rules and the conserved quantum numbsers of the operators A, B, C and D for a 2D hydrogen atom are summarized in Table 2.

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Table Captions

Table 1 The selection rules and conserved quantum numbers of four kinds of raising and lowering operators of a 3D hydrogen atom.

Table 2 The same as Table 1, but for a 2D hydrogen atom.

Table 1

raising and lowering operators	l	n_r	$n = l + n_r + 1$	conserved quantum number
$A(l\uparrow, n)$	$l \to l+1$	$n_r \to n_r - 1$	$n \to n$	$n,\ l + n_r$
$A(l\downarrow, n)$	$l \to l-1$	$n_r \to n_r + 1$	$n \to n$	
$B(l, n\uparrow)$	$l \to l$	$n_r \to n_r + 1$	$n \to n+1$	l
$B(l, n\downarrow)$	$l \to l$	$n_r \to n_r - 1$	$n \to n-1$	
$C(l\uparrow, n\uparrow)$	$l \to l+1$	$n_r \to n_r$	$n \to n+1$	n_r
$C(l\downarrow, n\downarrow)$	$l \to l-1$	$n_r \to n_r$	$n \to n-1$	
$D(l\downarrow, n\downarrow)$	$l \to l-1$	$n_r \to n_r + 2$	$n \to n+1$	$n + l,\ 2l + n_r$
$D(l\uparrow, n\uparrow)$	$l \to l+1$	$n_r \to n_r - 2$	$n \to n-1$	

Table 2

raising and lowering operators	m	n_ρ	$n = \lvert m\rvert + n_\rho + 1/2$	conserved quantum number
$A(m\uparrow, n)$	$m \to m+1$	$n_\rho \to n_\rho - 1$	$n \to n$	$n,\ \lvert m\rvert + n_\rho$
$A(m\downarrow, n)$	$m \to m-1$	$n_\rho \to n_\rho + 1$	$n \to n$	
$B(m, n\uparrow)$	$m \to m$	$n_\rho \to n_\rho + 1$	$n \to n+1$	m
$B(m, n\downarrow)$	$m \to m$	$n_\rho \to n_\rho - 1$	$n \to n-1$	
$C(m\uparrow, n\uparrow)$	$m \to m+1$	$n_\rho \to n_\rho$	$n \to n+1$	n_ρ
$C(m\downarrow, n\downarrow)$	$m \to m-1$	$n_\rho \to n_\rho$	$n \to n-1$	
$D(m\downarrow, n\uparrow)$	$m \to m-1$	$n_\rho \to n_\rho + 2$	$n \to n+1$	$n + \lvert m\rvert,\ 2\lvert m\rvert + n_\rho$
$D(m\uparrow, n\downarrow)$	$m \to m+1$	$n_\rho \to n_\rho - 2$	$n \to n-1$	

Figure Captions

Fig. 1 The energy levels of a 3D hydrogen atom and four kinds of raising and lowering operators. Operator A connects states with the same energy but different angular momenta. B connects states with the same angular momentum but different energy. C connects states with the same radial quantum number n_r and D connects states with the same $n+l$ (or $2l+n_r$).

Fig. 2 The same as Fig. 1, but for a 2D hydrogen atom.

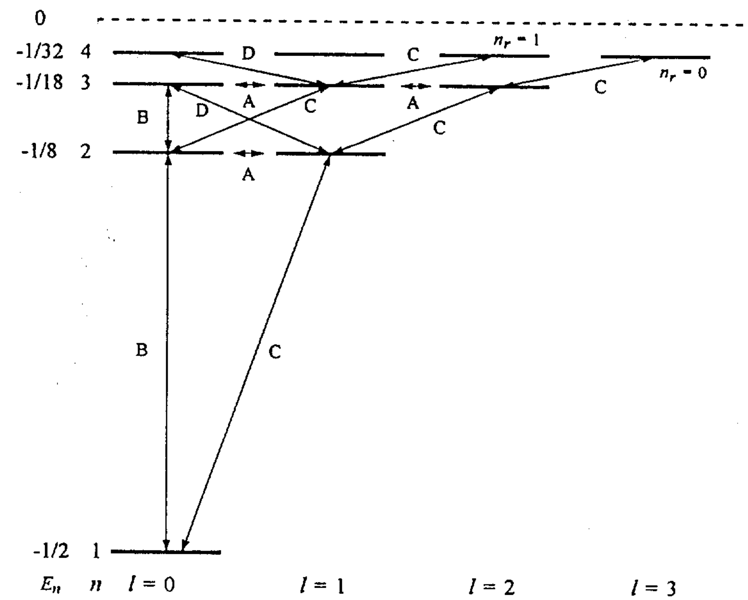

Fig. 1

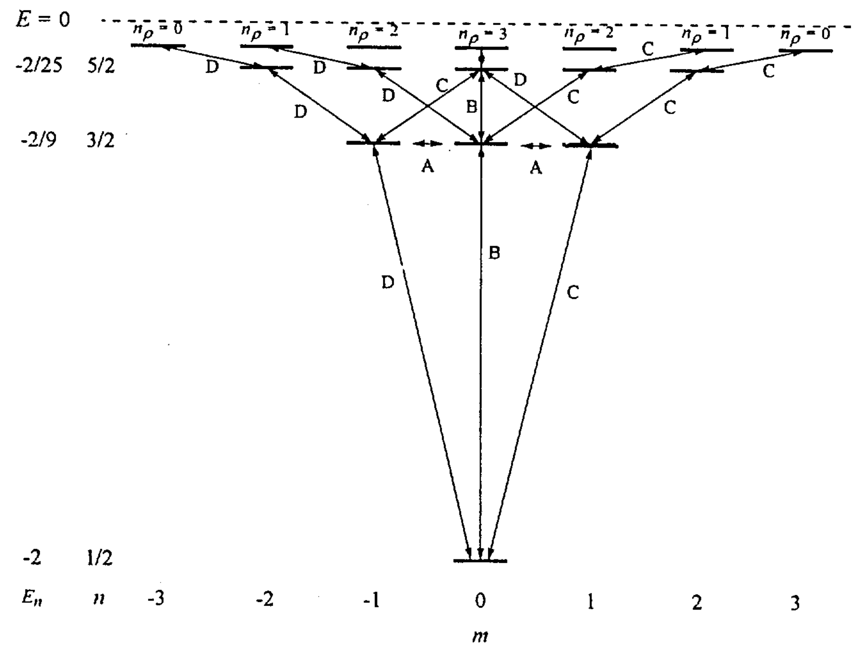

Fig. 2