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Four kinds of raising and lowering operators of 3D and 2D isotropic harmonic oscillators

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Abstract

Using the factorization method, four kinds of raising and lowering operators of three-dimensional (3D) and two-dimensional (2D) isotropic harmonic oscillators were derived, and the corresponding selection rules and conserved quantum numbers were discussed.

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I. INTRODUCTION

The factorization method for treating the energy eigenvalue of a one-dimensional (1D) harmonic oscillator and the concept of raising and lowering operators were introduced by Schrödinger [1] and extended by Infeld and Hull [2], which have been discussed in many textbooks of quantum mechanics [3]. The factorization of the Hamiltonian and the raising and lowering operators are extended further in the supersymmetric quantum mechanics [4–7], (for a recent comprehensive review see ref. [8]). The supersymmetric quantum mechanics is restricted to the one-dimensional (1D) system, and also applied to treat the radial equation of a particle in central field [9–11]. It is well known that the bound states of a particle in a regular 1D potential are nondegenerate, so there exist only one kind of raising and lowering operators. However, the bound states of a central potential are in general degenerate. In particular, for the potential having certain dynamical symmetry, e.g., the Coulomb field (O_4 Symmetry) and the isotropic 3D-harmonic oscillator (SU_3 symmetry), there exist additional degeneracy (l-degeneracy). Therefore, there may exist various kinds of raising and lowering operators. To our knowledge, no systematic investigation concerning this problem is reported. In this paper, we will use the factorization method to investigate the four kinds of raising and lowering operators of the 3-dimensional and 2-dimensional isotropic harmonic oscillators and the corresponding selection rules and conserved quantum numbers.

II. 3D ISOTROPIC HARMONIC OSCILLATOR

For a particle in 3D isotropic harmonic oscillator potential $V(r) = \frac{1}{2}M\omega^2 r^2$, the energy eigenequation (in spherical coordinates) is

$$H\Psi = \left[-\frac{\hbar^2}{2M}\frac{1}{r}\frac{d^2}{dr^2}r + \frac{\hat{\mathbf{l}}^2}{2Mr^2} + \frac{1}{2}M\omega^2 r^2\right]\Psi = E\Psi \qquad (1)$$

where $\hat{\mathbf{l}}$ is the angular momentum operator. Ψ may be chosen as the simultaneous eigenstate of the complete set of conserved observables $(H, \hat{\mathbf{l}}^2, l_z)$, i.e.,

$$\Psi = \frac{\chi_l(r)}{r}Y_{lm}(\theta,\phi), \qquad l = 0,1,2,\cdots, \quad m = l, l-1, \cdots, -l, \qquad (2)$$

where $Y_{lm}(\theta,\phi)$ is the spherical harmonic function, and $\chi_l(r)$ satisfies the radial equation (using the natural units $\hbar = M = \omega = 1$)

$$\begin{aligned} D(l)\chi_l(r) &= \lambda_l\chi_l(r), \quad \lambda_l = -2E \\ \chi_l(0) &= 0, \\ D(l) &\equiv \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - r^2. \end{aligned} \tag{3}$$

Now, we may define two kinds of l-dependent operators

$$A_+(l) = \frac{d}{dr} - \frac{l+1}{r} + r, \qquad A_-(l) = \frac{d}{dr} + \frac{l}{r} - r, \tag{4}$$

$$B_+(l) = \frac{d}{dr} - \frac{l+1}{r} - r, \qquad B_-(l) = \frac{d}{dr} + \frac{l}{r} + r. \tag{5}$$

It is easily verified that these operators satisfy the following factorization relations,

$$A_-(l+1)A_+(l) = D(l) + (2l+3), \qquad A_+(l-1)A_-(l) = D(l) + (2l-1), \tag{6}$$

$$B_-(l+1)B_+(l) = D(l) - (2l+3), \qquad B_+(l-1)B_-(l) = D(l) - (2l-1). \tag{7}$$

Using (6), (7) and (3), one may prove

$$\begin{aligned} D(l)[A_+(l-1)\chi_{l-1}] &= (\lambda_{l-1}+2)A_+(l-1)\chi_{l-1}, \qquad (8) \\ D(l)[A_-(l+1)\chi_{l+1}] &= (\lambda_{l+1}-2)A_-(l+1)\chi_{l+1}, \\ D(l)[B_+(l-1)\chi_{l-1}] &= (\lambda_{l-1}-2)B_+(l-1)\chi_{l-1}, \qquad (9) \\ D(l)[B_-(l+1)\chi_{l+1}] &= (\lambda_{l+1}+2)B_-(l+1)\chi_{l+1}. \end{aligned}$$

From (8) we may see that if χ_{l-1} is an eigenstate of $D(l-1)$ with eigenvalue λ_{l-1}, then $A_+(l-1)\chi_{l-1}$ is an eigenstate of $D(l)$ with eigenvalue $\lambda_{l-1}+2$, i.e., the corresponding energy eigenvalue (see (3)) is decreased by 1. Therefore, the effect of A_+ is to increase the angular momentum of an eigenstate by 1 and simultaneously decrease its energy eigenvalue by 1. Similarly, the operation of A_- will decrease the angular momentum by 1 and simultaneously increase the energy eigenvalue by 1. Similarly, from (9) we may see that the effect of B_+ (B_-) is to increase (decrease) the angular momentum by 1 and simultaneously increase (decrease) the energy eigenvalue by 1. To clearly indicate the effects of these operators, we may relabel these operators as follows (see Fig. 1):

$$A_+(l) \to A(l\uparrow, N\downarrow) = \frac{d}{dr} - \frac{l+1}{r} + r, \tag{10}$$

$$\begin{aligned} A_-(l) \to A(l\downarrow, N\uparrow) &= \frac{d}{dr} + \frac{l}{r} - r, \\ B_+(l) \to B(l\uparrow, N\uparrow) &= \frac{d}{dr} - \frac{l+1}{r} - r, \\ B_-(l) \to B(l\downarrow, N\downarrow) &= \frac{d}{dr} + \frac{l}{r} + r, \end{aligned} \tag{11}$$

where N is the quantum number characterizing the energy eigenvalue (see appendix 1), $E = N + 3/2$, $N = l + 2n_r$, and $n_r = 0, 1, 2, \cdots$, is the number of nodes of the radial wave function ($r = 0, \infty$ excluded).

According to the physical meaning of the operators A and B, we may construct the other two kinds of operators C and D (see Figs. 1(b) and (c)). Using (10), (11) and (3), it is easily verified that

$$\begin{aligned} A((l+1)\downarrow, N\uparrow)B(l\uparrow, N\uparrow) &= D(l) - 2r\frac{d}{dr} + 2r^2 - 1, \\ B((l+1)\downarrow, N\downarrow)A(l\uparrow, N\downarrow) &= D(l) + 2r\frac{d}{dr} + 2r^2 - 1. \end{aligned} \tag{12}$$

Eq. (12) operating on the energy eigenstate $|lN\rangle$ and using (3), we get

$$\begin{aligned} A((l+1)\downarrow, N\uparrow)B(l\uparrow, N\uparrow)|lN\rangle &= -2\left(r\frac{d}{dr} - r^2 + N + 2\right)|lN\rangle, \\ B((l+1)\downarrow, N\downarrow)A(l\uparrow, N\downarrow)|lN\rangle &= 2\left(r\frac{d}{dr} + r^2 - N - 1\right)|lN\rangle. \end{aligned} \tag{13}$$

According to the physical meaning of the operators A and B, the effect of the operator $A((l+1)\downarrow, N\uparrow)B(l\uparrow, N\uparrow)$ is to increase the energy (N) by 2, but keep the angular momentum unchanged. Similarly, the effect of the operator $B((l+1)\downarrow, N\downarrow)A(l\uparrow, N\downarrow)$ is to decrease the energy (N) by 2, but keep the angular momentum unchanged. Therefore, we get the following kind of operators

$$C(l, N\uparrow\uparrow) = r\frac{d}{dr} - r^2 + (N+2), \quad C(l, N\downarrow\downarrow) = r\frac{d}{dr} + r^2 - (N+1). \tag{14}$$

Similarly, it can be proved that

$$\begin{aligned} A((l-1)\downarrow, N\uparrow)B(l\downarrow, N\downarrow) &= D(l) + \frac{2l-1}{r}\frac{d}{dr} + \frac{l(2l-1)}{r^2}, \\ A((l+1)\uparrow, N\downarrow)B(l\uparrow, N\uparrow) &= D(l) - \frac{2l+3}{r}\frac{d}{dr} + \frac{(l+1)(2l+3)}{r^2}. \end{aligned} \tag{15}$$

Operating (15) on the energy eigenstate $|lN\rangle$ and using (3), we get

$$A((l-1)\downarrow, N\uparrow)B(l\downarrow, N\downarrow)|lN\rangle = (2l-1)\left[\frac{1}{r}\frac{d}{dr} + \frac{l}{r^2} - \frac{2N+3}{2l-1}\right]|lN\rangle,$$

$$A((l+1)\uparrow,N\downarrow)B(l\uparrow,N\uparrow)|lN\rangle = -(2l+3)\left[\frac{1}{r}\frac{d}{dr}-\frac{l+1}{r^2}+\frac{2N+3}{2l+3}\right]|lN\rangle. \quad (16)$$

It is noted that the effect of the operator $A((l-1)\downarrow,N\uparrow)B(l\downarrow,N\downarrow)$ is to decrease the angular momentum by 2, but keep the energy unchanged, and the operator $A((l+1)\uparrow,N\downarrow)B(l\uparrow,N\uparrow)$ is to increase the angular momentum by 2, but keep the energy unchanged. Therefore, we get another kind of operators

$$D(l\downarrow\downarrow,N)=\left(\frac{1}{r}\frac{d}{dr}+\frac{l}{r^2}-\frac{2N+3}{2l-1}\right),\quad D(l\uparrow\uparrow,N)=\left(\frac{1}{r}\frac{d}{dr}-\frac{l+1}{r^2}+\frac{2N+3}{2l+3}\right). \quad (17)$$

It should be noted that the expressions of the operators C and D given in (14) and (17) also can be derived from the recurrence relations of the confluent hypergeometric function (see Appendix 1).

Thus, we have obtained four kinds of raising and lowering operators A, B, C and D (see Fig. 1). The corresponding section rules and conserved quantum numbers are summarized in Table 1.

Considering the effect of the operator $A(l\uparrow,N\downarrow)$ (see Table 1) and the fact $n_r\geq 0$, we get

$$A(l\uparrow,N\downarrow)\chi_{l\,n_r=0}=\left(\frac{d}{dr}-\frac{l+1}{r}+r\right)\chi_{l0}(r)=0. \quad (18)$$

The solution is

$$\chi_{l0}(r)\sim r^{l+1}e^{-r^2/2},\quad l=0,1,2,\cdots, \quad (19)$$

which is the lowest energy eigenfunction with angular momentum l and without node ($n_r=0$). $A((l+1)\downarrow,N\uparrow)$ operating on (18) and using (6), we get

$$[D(l)+2l+3)]\chi_{l0}=[\lambda_{l0}+(2l+3)]\chi_{l0}=0.$$

Thus, $\lambda_{l0}=-(2l+3)$, and the energy eigenvalue is

$$E_{l0}=(l+3/2),\quad l=0,1,2,\cdots, \quad (20)$$

which is the special case ($n_r=0$, $N=l$) of the general expression of the energy eigenvalue $E=N+3/2$ (see App. 1). Successive operations of the operators $A(l\downarrow,N\uparrow)$, $A((l-1)\downarrow,N\uparrow)$, $\cdots$, on $\chi_{l0}(r)$, one may get all the radial eigenfunction $\chi_{l-1,1}(r)$, $\chi_{l-2,2}(r)$, $\cdots$, $\chi_{0,l}(r)$, ($l=1,2,3,\cdots$).

III. 2D ISOTROPIC HARMONIC OSCILLATOR

For a particle in 2D central potential $V(\rho)$, the energy eigenequation (in polar coordinates) reads

$$H\Psi=\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+V(\rho)\right]\Psi=E\Psi. \quad (21)$$

Ψ may be chosen as the simultaneous eigenstate of the complete set of conserved observables $(H,\ \hat{l}_z=-i\hbar\frac{\partial}{\partial\phi})$, i.e.,

$$\Psi=\frac{\chi_m(\rho)}{\sqrt{\rho}}e^{im\phi},\quad m=0,\pm1,\pm2,\cdots, \quad (22)$$

$\chi_m(\rho)$ satisfies the following radial equation,

$$\begin{aligned}&\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}+\frac{1}{4\rho^2}-\frac{m^2}{\rho^2}\right)+V(\rho)\right]\chi_m(\rho)\\ =&\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}-\frac{(m-1/2)(m+1/2)}{\rho^2}\right)+V(\rho)\right]\chi_m(\rho)=E\chi_m(\rho).\end{aligned} \quad (23)$$

Comparing with the radial equation for a 3D central potential $V(r)$,

$$\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)+V(r)\right]\chi_l(r)=E\chi_l(r), \quad (24)$$

it is seen that, there exist great similarity between 2D and 3D cases, and the correspondence between the parameters is

$$l(l+1)\leftrightarrow(m-1/2)(m+1/2)=(|m|-1/2)(|m|+1/2). \quad (25)$$

For the 2D isotropic harmonic oscillator $V(\rho)=\frac{1}{2}M\omega^2\rho^2$, the radial equation can be expressed as ($\hbar=M=\omega=1$).

$$\begin{aligned}D_m(\rho)\chi_m(\rho)&=\lambda_m\chi_m(\rho),\quad \lambda_m=-2E,\\ \chi_m(0)&=0,\\ D(m)&=\frac{d^2}{d\rho^2}-\frac{(m-1/2)(m+1/2)}{\rho^2}-\rho^2.\end{aligned} \quad (26)$$

At first, let us consider the $m\geq 0$ case (which can be easily extended to $m\leq 0$). In a similar way to 3D harmonic oscillator, we may define two kinds m-dependent operators,

$$A_+(m)=\frac{d}{d\rho}-\frac{m+1/2}{\rho}+\rho,\qquad A_-(m)=\frac{d}{d\rho}+\frac{m-1/2}{\rho}-\rho, \quad (27)$$

$$B_+(m)=\frac{d}{d\rho}-\frac{m+1/2}{\rho}-\rho,\qquad B_-(m)=\frac{d}{d\rho}+\frac{m-1/2}{\rho}+\rho. \quad (28)$$

It is easily verified that these operators satisfy the following factorization relations.

$$A_-(m+1)A_+(m) = D(m) + 2(m+1), \qquad A_+(m-1)A_-(m) = D(m) + 2(m-1), \quad (29)$$

$$B_-(m+1)B_+(m) = D(m) - 2(m+1), \qquad B_+(m-1)B_-(m) = D(m) - 2(m-1). \quad (30)$$

Using (29), (30) and (26), we get

$$\begin{aligned} D(m)[A_+(m-1)\chi_{m-1}] &= (\lambda_{m-1}+2)A_+(m-1)\chi_{m-1}, \qquad (31)\\ D(m)[A_-(m+1)\chi_{m+1}] &= (\lambda_{m+1}-2)A_-(m+1)\chi_{m+1}, \\ D(m)[B_+(m-1)\chi_{m-1}] &= (\lambda_{m-1}-2)B_+(m-1)\chi_{m-1}, \qquad (32)\\ D(m)[B_-(m+1)\chi_{m+1}] &= (\lambda_{m+1}+2)B_-(m+1)\chi_{m+1}. \end{aligned}$$

From (31) it is seen that the effect of the operator A_+ (A_-) is to increase (decrease) the magnetic quantum number m by 1 and simultaneously decrease (increase) the energy eigenvalue by 1. Similarly, the effect of the operators B_+ (B_-) is to increase (decrease) m by 1 and simultaneously increase (decrease) the energy eigenvalue by 1. To clearly indicate the raising and lowering properties, these operators may be relabelled as

$$\begin{aligned} A_+(m) \to A(m\uparrow, N\downarrow) &= \frac{d}{d\rho} - \frac{m+1/2}{\rho} + \rho, \qquad (33)\\ A_-(m) \to A(m\downarrow, N\uparrow) &= \frac{d}{d\rho} + \frac{m-1/2}{\rho} - \rho, \\ B_+(m) \to B(m\uparrow, N\uparrow) &= \frac{d}{d\rho} - \frac{m+1/2}{\rho} - \rho, \qquad (34)\\ B_-(m) \to B(m\downarrow, N\downarrow) &= \frac{d}{d\rho} + \frac{m-1/2}{\rho} + \rho, \end{aligned}$$

where N is the quantum number characterizing the energy eigenvalue (see App. 2, $E = N+1$, $N = |m| + 2n_\rho$, and $n_\rho = 0, 1, 2, \cdots$ is the number of nodes of the radial wave function ($\rho = 0, \infty$ excluded).

Using the argument similar to 3D harmonic oscillator, we may construct the other two kinds of raising and lowering operators C and D (see Fig. 2).

$$\begin{aligned} C(m, N\uparrow\uparrow) &= \rho\frac{d}{d\rho} - \rho^2 + N + 2, \qquad (35)\\ C(m, N\downarrow\downarrow) &= \rho\frac{d}{d\rho} + \rho^2 - N - 1, \end{aligned}$$

$$\begin{aligned} D(m\downarrow\downarrow, N) &= \frac{1}{\rho}\frac{d}{d\rho} + \frac{m-1/2}{\rho^2} - \frac{2N+3}{2m-2}, \qquad (36)\\ D(m\uparrow\uparrow, N) &= \frac{1}{\rho}\frac{d}{d\rho} - \frac{m+1/2}{\rho^2} + \frac{2N+3}{2m+2}. \end{aligned}$$

Considering only the absolute value of m, is involved in the radial equation (23), we have $\chi_{-m,n_\rho} = \chi_{m,n_\rho}$ (see App. 2). Thus, the expressions of the operators A, B, C and D for $m \geq 0$ cases may be easily extended to $m \leq 0$. It is seen that

$$\begin{aligned} C(-m, N\uparrow\uparrow) &= C(m, N\uparrow\uparrow), \quad C(-m, N\downarrow\downarrow) = C(m, N\downarrow\downarrow), \qquad (37)\\ A((-m)\downarrow, N\uparrow) &= B(m\uparrow, N\uparrow), \quad \text{or} \quad B((-m)\uparrow, N\uparrow) = A(m\downarrow, N\uparrow), \qquad (38)\\ A((-m)\uparrow, N\downarrow) &= B(m\downarrow; N\downarrow), \quad \text{or} \quad B((-m)\downarrow, N\downarrow) = A(m\uparrow, N\downarrow). \qquad (39) \end{aligned}$$

As for the operator D, we have

$$D((-m)\uparrow\uparrow, N) = D(m\downarrow\downarrow, N), \quad (m \neq \pm 1), \qquad (40)$$

but for $m = \pm 1$, because $\chi_{-1,n_\rho} = \chi_{1,n_\rho}$, we get

$$D((-1)\uparrow\uparrow, N) = D(1\downarrow\downarrow, N) = I \text{ (identity)}. \qquad (41)$$

Thus, we have obtained the four kinds of raising and lowering operators of 2D isotropic harmonic oscillator. The corresponding selection rules and conserved quantum numbers for these operators are given in Table 2.

Appendix 1

Considering the asymptotic behavior of the solution of (3) as $r \to 0$ and $r \to \infty$, we may put the solution of (3) as [3]

$$\chi_l(r) = r^{l+1}e^{-r^2/2}u(r), \qquad (42)$$

and $u(r)$ satisfies

$$u'' + \frac{2}{r}(l+1-r^2)u' - (\lambda_l + 2l + 3)u = 0. \qquad (43)$$

Let $\xi = r^2$, (42) is reduced to the confluent hypergeometric equation

$$\xi\frac{d^2u}{d\xi^2} + (l+3/2-\xi)\frac{du}{d\xi} - \frac{1}{4}[\lambda_l + (2l+3)]u = 0. \qquad (44)$$

The solution u (satisfying $\chi_l(0) = 0$) may be expressed as the confluent hypergeometric function $F(\alpha, \gamma; r^2)$,

$$\alpha = \frac{1}{4}(\lambda_l + 2l + 3), \quad \gamma = l + 3/2. \tag{45}$$

For bound states the infinite series expression of F must be terminated (i.e., F must be reduced to a polynomial), therefore, it is required that $\alpha = -n_r,\ n_r = 0, 1, 2, \cdots$, then we get $E = (l + 2n_r) + 3/2$, or

$$E = E_N = (N + 3/2), \quad N = l + 2n_r = 0, 1, 2, \cdots. \tag{46}$$

The corresponding radial eigenfunction is

$$\chi_{ln_r}(r) \sim r^{l+1} e^{-r^2/2} F(-n_r, l + 3/2, r^2), \tag{47}$$

and the degeneracy of a given level is $f_N = (N + 1)(N + 2)/2$.

Using the fundamental recurrence formulas of confluent hypergeometric functions [13].

$$\frac{d}{dx} F(\alpha, \gamma, x) = \frac{\alpha}{\gamma} F(\alpha + 1, \gamma + 1, x),$$
$$(\gamma - \alpha) F(\alpha - 1, \gamma, x) - \alpha F(\alpha + 1, \gamma, x) = (\gamma - 2\alpha - x) F(\alpha, \gamma, x),$$
$$\gamma(\gamma - 1) F(\alpha, \gamma - 1, x) + (\gamma - \alpha) F(\alpha, \gamma + 1, x) = \gamma(\gamma - 1 + x) F(\alpha, \gamma, x),$$
$$\gamma F(\alpha - 1, \gamma, x) + x F(\alpha, \gamma + 1, x) = \gamma F(\alpha, \gamma, x),$$
$$(\gamma - 1) F(\alpha, \gamma - 1, x) - \alpha F(\alpha + 1, \gamma, x) = (\gamma - \alpha - 1) F(\alpha, \gamma, x),$$
$$(\gamma - \alpha) x F(\alpha, \gamma + 1, x) + \alpha\gamma F(\alpha + 1, \gamma, x) = \gamma(\alpha + x) F(\alpha, \gamma, x),$$
$$(\gamma - 1) F(\alpha, \gamma - 1, x) - (\gamma - \alpha) F(\alpha - 1, \gamma, x) = (\alpha - 1 + x) F(\alpha, \gamma, x),$$

the following recurrence relations can be derived

$$\gamma F(\alpha + 1, \gamma, x) - x F(\alpha + 1, \gamma + 1, x) = \gamma F(\alpha, \gamma, x),$$
$$\gamma(\gamma - \alpha) F(\alpha - 1, \gamma, x) - \alpha x F(\alpha + 1, \gamma + 1, x) = \gamma(\gamma - \alpha - x) F(\alpha, \gamma, x),$$
$$(\gamma - \alpha) F(\alpha, \gamma + 1, x) + \alpha F(\alpha + 1, \gamma + 1, x) = \gamma F(\alpha, \gamma, x),$$
$$\gamma(\gamma - 1) F(\alpha, \gamma - 1, x) - \alpha x F(\alpha + 1, \gamma + 1, x) = \gamma(\gamma - 1) F(\alpha, \gamma, x).$$

From these relations, the following formulas can be obtained,

$$\left[2\alpha + x\frac{d}{dx}\right] F(\alpha, \gamma, x^2) = 2\alpha F(\alpha + 1, \gamma, x^2), \tag{48}$$

$$\left[2(\gamma - \alpha - x^2) + x\frac{d}{dx}\right] F(\alpha, \gamma, x^2) = 2(\gamma - \alpha) F(\alpha - 1, \gamma, x^2), \tag{49}$$

$$\left[-\frac{2\alpha}{\gamma} + \frac{1}{x}\frac{d}{dx}\right] F(\alpha, \gamma, x^2) = \frac{2\alpha(\gamma - \alpha)}{\gamma^2(\gamma + 1)} x^2 F(\alpha + 1, \gamma + 2, x^2), \tag{50}$$

$$\left[2(\gamma - 1) - \frac{2(\gamma - \alpha - 1)}{\gamma - 2} x^2 + x\frac{d}{dx}\right] F(\alpha, \gamma, x^2) = 2(\gamma - 1) F(\alpha - 1, \gamma - 2, x^2). \tag{51}$$

Using (48–51), one may derive the expressions for the operators C and D given in (14) and (17).

Appendix 2

Considering the asymptotic behavior of the solution of the radial equation (26) as $\rho \to 0$ and $\rho \to \infty$, we may put

$$\chi_m(\rho) = \rho^{|m|-1/2} e^{-\rho^2/2} u(\rho) \tag{52}$$

and $u(\rho)$ satisfies

$$u'' + \left[\frac{2|m| + 1}{\rho} - 2\rho\right] u' - (\lambda_m + 2|m| + 2) u = 0. \tag{53}$$

Let $\xi = \rho^2$, (53) is reduced to the confluent hypergeometric equation

$$\xi \frac{d^2 u}{d\xi^2} (|m| + 1 - \xi) \frac{du}{d\xi} - \frac{1}{4}(2|m| + 2 + \lambda_m) u = 0. \tag{54}$$

The solution u (satisfying $\chi_m(0) = 0$) may be expressed as $F(\alpha, \gamma; \rho^2)$,

$$\alpha = \frac{1}{4}(2|m| + 2 + \lambda_m), \quad \gamma = |m| + 1. \tag{55}$$

For bound states the infinite series expression of F must be terminated, i.e., it is required that $\alpha = -n_\rho,\ n_\rho = 0, 1, 2, \cdots$, then we get

$$E = E_N = (N + 1), \quad N = |m| + 2n_\rho, \quad n_\rho = 0, 1, 2, \cdots. \tag{56}$$

The corresponding radial eigenfunction is

$$\chi_{m, n_\rho}(\rho) \sim \rho^{|m|-1/2} F(-n_\rho, |m| + 1; \rho^2). \tag{57}$$

The degeneracy of a given energy level is $f_N = (N + 1)$, $N = 0,\ 1,\ 2, \cdots$.

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Table Captions

Table 1 Four kinds of raising and lowering operators of the 3D isotropic harmonic oscillator and the corresponding selection rules and conserved quantum numbers.

Table 2 The same as Table 1, but for the 2D isotropic oscillator.

Table 1

raising and lowering operators	l	n_r	$N = l + 2n_r$	conserved quantum number
$A(l\uparrow, N\downarrow) = \frac{d}{dr} - \frac{l+1}{r} + r$	$l \to l+1$	$n_r \to n_r - 1$	$N \to N-1$	$l + n_r,\ N - n_r$
$A(l\downarrow, N\uparrow) = \frac{d}{dr} + \frac{l}{r} - r$	$l \to l-1$	$n_r \to n_r + 1$	$N \to N+1$	
$B(l\uparrow, N\uparrow) = \frac{d}{dr} - \frac{l+1}{r} - r$	$l \to l+1$	$n_r \to n_r$	$N \to N+1$	n_r
$B(l\downarrow, N\downarrow) = \frac{d}{dr} + \frac{l}{r} + r$	$l \to l-1$	$n_r \to n_r$	$N \to N-1$	
$C(l, N\uparrow\uparrow) = r\frac{d}{dr} - r^2 + (N+2)$	$l \to l$	$n_r \to n_r + 1$	$N \to N+2$	l
$C(l, N\downarrow\downarrow) = r\frac{d}{dr} + r^2 - (N+1)$	$l \to l$	$n_r \to n_r - 1$	$N \to N-2$	
$D(l\uparrow\uparrow, N) = \frac{1}{r}\frac{d}{dr} - \frac{l+1}{r^2} + \frac{2N+3}{2l+3}$	$l \to l+2$	$n_r \to n_r - 1$	$N \to N$	N
$D(l\downarrow\downarrow, N) = \frac{1}{r}\frac{d}{dr} + \frac{l}{r^2} - \frac{2N+3}{2l-1}$	$l \to l-2$	$n_r \to n_r + 1$	$N \to N$	

Table 2

raising and lowering operators	m	n_ρ	$N = \lvert m\rvert + 2n_\rho$	conserved quantum number
$A(m\uparrow, N\downarrow) = \frac{d}{d\rho} - \frac{m+1/2}{\rho} + \rho$	$m \to m+1$	$n_\rho \to n_\rho - 1$	$N \to N-1$	$\lvert m\rvert + n_\rho,\ N - n_\rho$
$A(m\downarrow, N\uparrow) = \frac{d}{d\rho} + \frac{m-1/2}{\rho} - \rho$	$m \to m-1$	$n_\rho \to n_\rho + 1$	$N \to N+1$	
$B(m\uparrow, N\uparrow) = \frac{d}{d\rho} - \frac{m+1/2}{\rho} - \rho$	$m \to m+1$	$n_\rho \to n_\rho$	$N \to N+1$	n_ρ
$B(m\downarrow, N\downarrow) = \frac{d}{d\rho} + \frac{m-1/2}{\rho} + \rho$	$m \to m-1$	$n_\rho \to n_\rho$	$N \to N-1$	
$C(m, N\uparrow\uparrow) = \rho\frac{d}{d\rho} - \rho^2 + (N+2)$	$m \to m$	$n_\rho \to n_\rho + 1$	$N \to N+2$	m
$C(m, N\downarrow\downarrow) = \rho\frac{d}{d\rho} + \rho^2 - (N+1)$	$m \to m$	$n_\rho \to n_\rho - 1$	$N \to N-2$	
$D(m\uparrow\uparrow, N) = \frac{1}{\rho}\frac{d}{d\rho} - \frac{m+1/2}{\rho^2} + \frac{2N+3}{2m+2}$	$m \to m+2$	$n_\rho \to n_\rho - 1$	$N \to N$	N
$D(m\downarrow\downarrow, N) = \frac{1}{\rho}\frac{d}{d\rho} + \frac{m+1/2}{\rho^2} - \frac{2N+3}{2m-2}$	$m \to m-2$	$n_\rho \to n_\rho + 1$	$N \to N$	

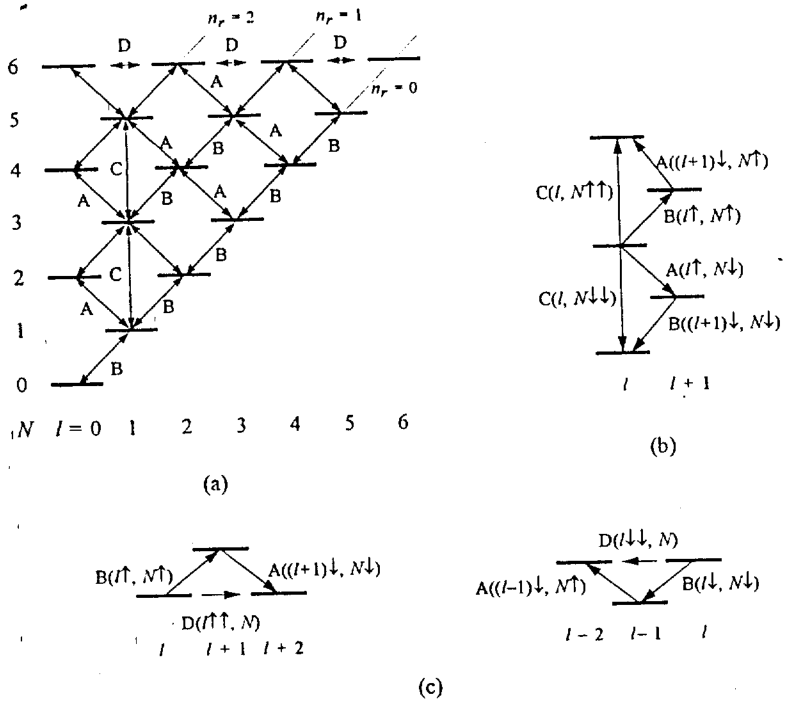
Figure Captions

Fig. 1 (a) The energy levels of a 3D isotropic harmonic oscillator. Operator A connects states with the same $N + n_r$. Operator B connects states with the same radial quantum nubmer n_r. Operator C connects states with the same angular momentum l. and operator D connects states with the same energy N.

(b) The graphical illustration of the operator $C(l, N\uparrow\uparrow) = A((l+1)\downarrow, N\uparrow)B(l\uparrow, N\uparrow)$ and $C(l, N\downarrow\downarrow) = B((l+1)\downarrow, N\downarrow)A(l\uparrow, N\downarrow)$.

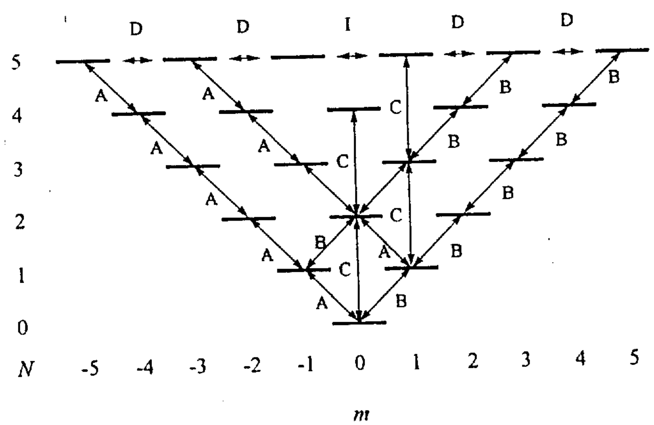
(c) The graphical illustration of the operator $D(l\uparrow\uparrow, N) = A((l+1)\uparrow, N\downarrow)B(l\uparrow, N\uparrow)$ and $D(l\downarrow\downarrow, N) = A((l-1)\downarrow, N\uparrow)B(l\downarrow, N\downarrow)$.

Fig. 2 The same as Fig. 1a, but for the 2D isotropic harmonic oscillator.



$n_r = 2$
$n_r = 1$
$n_r = 0$
D
A
B
C
N
$l = 0$ 1 2 3 4 5 6
(a)
C(l, N↑↑)
A((l+1)↓, N↑)
B(l↑, N↑)
A(l↑, N↓)
C(l, N↓↓)
B((l+1)↓, N↓)
l l + 1
(b)
B(l↑, N↑)
A((l+1)↓, N↓)
D(l↑↑, N)
l l + 1 l + 2
D(l↓↓, N)
A((l−1)↓, N↑)
B(l↓, N↓)
l − 2 l − 1 l
(c)

Fig. 1



D
I
A
B
C
N
-5 -4 -3 -2 -1 0 1 2 3 4 5
m

Fig. 2