CERN-TH/96-213

A NEW APPROACH TO THE COULOMB GAUGE: SPLIT DIMENSIONAL REGULARIZATION [1]

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ABSTRACT

Feymann integrals in the noncovariant Coulomb gauge, $\vec{\nabla} \cdot \vec{A}^a(x) = 0$, are regulated with a novel procedure, called *split dimensional regularization*, which employs *two* complex- dimensional parameters. The new technique leads to well-defined integrals , and yields a local Yang-Mills self energy, ${\Pi_{\mu\nu}}^{ab}$, that respects the corresponding BRS identity. Ghosts play an essential role. The method of *split dimensional regularization* may be applied to both Abelian and non-Abelian gauge theories.

CERN-TH/96-213
August 1996

[1] Talk presented at the XXVIII International Conference on High Energy Physics, Warsaw, Poland, July 25-31, 1996

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1 Introduction

Although the Coulomb gauge, $\vec{\nabla} \cdot \vec{A}^a(x) = 0$, and the axial gauges, $n \cdot A^a(x) = 0$, are both *bona fide* members of the set of noncovariant gauges, their mathematical properties and computational requirements seem to be almost orthogonal to each other. (Here, $A_\mu{}^a(x)$ denotes a massless gauge field, with $a = 1, 2, ..., N^2 - 1$, for SU(N), while $n_\mu \equiv (n_0, \vec{n})$ is a fixed four-vector.) In the case of the axial-type gauges, the spurious singularities of $(q \cdot n)^{-1}$ can be handled by the $n_\mu{}^*$-prescription[1,2], or its generalization[3,4], where $n_\mu{}^* = (n_0, -\vec{n}), \mu = 0, 1, 2, 3$.

For the Coulomb gauge, on the other hand, progress has been much slower, especially in the context of non-Abelian theories.[5,6,7,8] To see the source of the difficulties, we just need to look at the Coulomb-gauge propagator $G_{\mu\nu}{}^{ab}(q)$,

$$\begin{aligned} G_{\mu\nu}{}^{ab}(q) = & \frac{-i\delta^{ab}}{(2\pi)^4(q^2 + i\epsilon)} \\ & \times \Big[g_{\mu\nu} - \frac{n^2}{\vec{q}^2} q_\mu q_\nu + \\ & \frac{q \cdot n}{q^2}(q_\mu n_\nu + q_\nu n_\mu) \Big], \\ n_\mu = (1, 0, 0, 0), q^2 = & {q_0}^2 - \vec{q}^{\,2}, \\ & \epsilon > 0, \end{aligned} \tag{1}$$

which is seen to contain both $(q^2)^{-1}$ and $(\vec{q}^2)^{-1}$.It is the *absence* of ${q_0}^2$ in $\vec{q}^{\,2}$, $(0 \cdot {q_0}^2 - \vec{q}^{\,2})^{-1}$, that is responsible for a loss in damping power and, therefore, causes the ambiguities in the dq_0-integrations. The problem becomes particularly acute for certain integrals which appear for the first time at two and three loops.[9,10] As shown by Doust and Taylor[9,10], the divergences of those integrals cannot be regulated consistently by standard dimensional regularization.

In this talk, I shall present a new method for regulating Coulomb-gauge integrals, confining the discussion strictly to one-loop examples, such as

$$\begin{aligned} & \int d^4q \{ \left[(q-p)^2 + i\epsilon \right] \vec{q}^{\,2} \}^{-1}, \\ & \int d^4q \{ (q^2 + i\epsilon)\vec{q}^{\,2}(\vec{q} + \vec{p}\,)^2 \}^{-1}, etc. \end{aligned} \tag{2}$$

2 Coulomb-gauge integrals: a new approach

2.1 General Procedure

Consider the Minkowski-space integral

$$\begin{aligned} I_M = & \int^{Mink} dq_0 d^3\vec{q} f(q_0, \vec{q}; p), \\ & q_\mu = (q_0, \vec{q}\,), \mu = 0, 1, 2, 3, \end{aligned} \tag{3}$$

where p_μ is an external momentum. The highlights of the new technique may then be stated as follows:

(1) We perform a Wick rotation on $I_M, q_0 = iq_4, \vec{q} = \vec{q}$, to obtain the Euclidean-space integral

$$\begin{aligned} I_E \equiv & \int_{-\infty}^{+\infty} dq_4 \int d^3\vec{q} f(q_4, \vec{q}; p), \\ & {q_E}^2 = {q_4}^2 + \vec{q}^{\,2}. \end{aligned} \tag{4}$$

(2) In order to help us regulate the troublesome q_4-integral, we employ two complex-dimensional regularization parameters, σ and ω.[11] Splitting d^4q into two factors $d^4q = dq_4 d^3\vec{q}$, we write each of these factors as

$$dq_4 = d^{2\sigma} S|^{\sigma = \frac{1}{2}^+}, d^3\vec{q} = d^{2\omega}\vec{Q}|^{\omega = \frac{3}{2}^+}. \tag{5}$$

In Minkowski four-space, σ and ω satisfy $(\sigma + \omega) = 2$. We call this procedure *split dimensional regularization.*[11]

2.2 Example

To illustrate the new approach, let us evaluate the Euclidean-space integral I,

$$I = \int \frac{d^4q \quad {q_4}^2}{q^2(\vec{q} + \vec{p}\,)^2}, \tag{6}$$

by employing the exponential representation with Schwinger parameters α, x. Thus

$$\begin{aligned} I = & \int (\alpha, x) \int_{-\infty}^{+\infty} dq_4 \quad {q_4}^2 exp(-\alpha x {q_4}^2) \\ & \times \int d^3\vec{q} \quad exp[-\alpha(\vec{q}^{\,2} + 2\vec{q} \cdot \vec{p}(1-x))]. \end{aligned} \tag{7}$$

Applying the method of split dimensional regularization, as outlined in Section 2.1, we readily obtain

$$I = \lim_{\sigma\to\frac{1}{2}^+} \lim_{\omega\to\frac{3}{2}^+} \int (\alpha, x) \times \int d^{2\sigma} S S^2 exp(-\alpha x S^2) \times \int d^{2\omega}\vec{Q} \quad exp[-\alpha(\vec{Q}^{\,2} + 2\vec{Q}\cdot\vec{P}(1-x))], \quad (8)$$

where $\vec{P}$ is now the 2ω-dimensional version of $\vec{p}$. Before proceeding with the momentum integrations, we note that the coefficients of S^2 and $\vec{Q}^2$ in the two exponentials are actually *different* (in contrast to covariant gauges, where the coefficients are the *same*!). Therefore, rescaling the S-vector according to $xS^2 = R^2$, with $d^{2\sigma}S = x^{-\sigma} d^{2\sigma}R$, we finally get, in the limit as $\sigma \to \frac{1}{2}^+, \omega \to \frac{3}{2}^+$,

$$div \int \frac{d^4q \quad {q_4}^2}{q^2(\vec{q}+\vec{p})^2} = \frac{2}{3}\vec{p}^{\,2}\pi^2 \; \Gamma(1-\omega-\sigma)\Big|_{\omega=\frac{3}{2}^+}^{\sigma=\frac{1}{2}^+}. \quad (9)$$

Before leaving this example, we should stress that the parameter integrations in eq.(8) impose several constraints on σ and ω, namely $Re(\omega+\sigma) < 1$ and $\{Re(\omega+\sigma) > 0, Re\omega > 1\}$. Accordingly, there exists a region D in the complex ω-plane, where the α- and x-integrals are indeed well defined.
The technique of split dimensional regularization permits us to compute all one-loop Coulomb-gauge integrals in a consistent manner. It turns out that some of these integrals are actually *nonlocal*.[11]

3 Application

In order to test the method of split dimensional regularization on a nontrivial example, we decided to calculate the Yang-Mills self-energy ${\Pi_{\mu\nu}}^{ab}(p)$ to one loop[11], and to check the result against the BRS identity

$$p^\mu {\Pi_{\mu\nu}}^{ab}(p) + \left(g_{\mu\nu}p^2 - p_\mu p_\nu\right) H^{\mu ab}(p) = 0, \quad (10)$$

where $H^{\mu ab}(p)$ represents a ghost diagram. For ${\Pi_{\mu\nu}}^{ab}$ we obtained

$$\begin{aligned}{\Pi_{\mu\nu}}^{ab}(p) = \quad & c^{ab}\tfrac{11}{3}[(g_{\mu\nu}p^2 - p_\mu p_\nu) \\ & -\tfrac{8}{3}(g_{\mu\nu}p^2 - p_\mu p_\nu) \\ & -\tfrac{4}{3}\tfrac{p\cdot n}{n^2}(p_\mu n_\nu + p_\nu n_\mu) \\ & +\tfrac{8p^2}{3n^2}n_\mu n_\nu]I^*, \end{aligned} \quad (11)$$

which is seen to be strictly *local*, despite the appearance of *nonlocal* integrals at intermediate stages of the calculation, while the ghost-diagram yielded the non-vanishing expression

$$H^{\mu ab}(p) = \frac{4}{3}c^{ab}\left(p^\mu - \frac{p\cdot n}{n^2}n^\mu\right)I^*. \quad (12)$$

Here, $n_\mu = (1,0,0,0)$,

$$c^{ab} = g^2 c_{YM}\frac{\delta^{ab}}{4\pi^2},$$

$c_{YM}\delta^{ab} = f^{acd}f^{bcd}$, and $I^* \equiv \textit{pole term} \sim \Gamma(2-\omega-\sigma)$. It is easily checked that ${\Pi_{\mu\nu}}^{ab}$ and $H^{\mu ab}$ together satisfy the theoretical BRS identity.

4 Concluding remarks

In this paper we have described a new approach of quantizing non-Abelian theories in the physical Coulomb gauge. As with axial-type gauges, the primary goal in the Coulomb gauge is to find a sensible prescription for the spurious singularities, i.e. to succeed in computing all Feynman integrals in a consistent, unambiguous manner.
What has become clear from earlier work on the Coulomb gauge is that such a prescription is unlikely to exist in the framework of conventional dimensional regularization.[12] Split dimensional regularization, on the other hand, appears powerful enough, at least to one-loop order, to cope with the various divergences,specifically those arising from the dq_0-integrations. The fact that split dimensional regularization respects the BRS identity for the Yang-Mills self-energy is another positive signal, of course, but it still is no guarantee that the new method will work to all orders in perturbation theory. What we really need to do next is compute some typical two- and three-loop integrals in Yang-Mills theory or QCD.

Acknowledgments

I am indebted to Professor J.C. Taylor for his constructive criticisms concerning the Coulomb gauge, and for numerous letters on the subject dating back to the winter of 1990. It also gives me great pleasure to thank Professor G. Veneziano, and the staff at CERN for their kind hospitality during the summers of 1995 and 1996. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grant No. A8063.

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