EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

A SIMPLE AND PRECISE METHOD

FOR MEASURING THE COUPLING COEFFICIENT OF THE DIFFERENCE RESONANCE

by

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Abstract

The horizontal and vertical coherent oscillations excited on a beam being kicked horizontally or vertically are analysed in detail in the vicinity of the difference coupling resonance. It is pointed out that there are some difficulties with the frequency measurement and that the measurement of the amplitude modulation is preferable. The coupling coefficient of the difference resonance can be obtained quickly and precisely by observing simultaneously the following two quantities of the oscillations excited by a horizontal kick: the period of the amplitude modulation in the vertical plane and the ratio of the minimum to the maximum of the modulated amplitude in the horizontal plane. The proposed method was tested successfully on the ISR. An electronic measurement system based on the proposed method is also suggested, which will be helpful to machine operation.

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1. INTRODUCTION

The ISR are normally operated with the horizontal and vertical betatron oscillation frequencies close to each other ($Q_x - Q_z = \Delta$, with Δ small), in order to avoid the presence of non-linear resonance lines over the wide tune spread required for beam stability. For such choices of the operation line, a difference coupling resonance may have a significant effect on the beam and a lowering of the luminosity may occur from a blow-up of the vertical emittance via the coupling to the horizontal emittance. The extent to which we can approach the diagonal in the tune diagram depends on the strength of the coupling coefficient κ between the horizontal and vertical betatron oscillations. The smaller the coupling coefficient κ, the closer we can approach the diagonal (i.e. the smaller we can make Δ).

Furthermore, there is a future program to install a solenoid inclined to the beam axis for physics experiments and it may be necessary to correct the coupling effects caused by the solenoid by using the correction skew quadrupole magnets. The efficiency of such a correction can be monitored by measuring the coupling coefficient κ. The currents of the correction magnets can be set so that the coupling coefficient is minimized.

Thus, an operational method for measuring the coupling coefficient is required, which tells us the strength of κ quickly and precisely. In addition, if an electronic measurement system for the coupling coefficient is feasible, it will be helpful to machine operation. Fur such an electronic system to be designed, we must first have a method of measurement which is suitable for this application. So far, the coupling effects in the ISR have been investigated several times by different methods[1,2,3,4], one of which consisted in kicking the beam in one plane and observing the resultant coherent coupled oscillations. In this report, the observational potentialities of the kick method are thoroughly analyzed. The characteristics of the stimulated coherent motion are examined in detail and the parameters of the coupled oscillations, which should be measured in order to obtain the most accurate results, are determined.

The proposed method of measurement consists in kicking the beam horizontally by a kicker and observing the following two quantities simultaneously: the period of the amplitude modulation in the vertical plane and the ratio of the minimum to the maximum of the modulated amplitude in the horizontal plane. From these two data taken after a single kick, we can calculate the unperturbed Q-separation ($|\Delta|$) as well as the coupling coefficient ($|\kappa|$). A vertical kick of the beam is equally valid but then the above two quantities have to be measured in the horizontal and vertical planes, respectively.

After a review of theories of the difference coupling resonance in Section 2, some general features of the difference coupling resonance are described in Section 3. In Section 4, we shall derive the expressions for the coherent coupled oscillations excited by a kick and in Section 5 we shall examine the quantities to be measured. Section 6 describes a test experiment on the ISR and Section 7 a possible electronic system based on the proposed method.

2. A REVIEW OF A LINEAR THEORY FOR THE DIFFERENCE RESONANCE

In this section we shall give a review of the theories for the difference resonance in order to provide a theoretical background for the succeeding discussions. A linear theory of coupling resonances (sum resonance and difference resonance) is described in a book by Kolomensky[5], in which the fundamental equation for the coupling resonance is derived by solving the equations of motion. A more elaborate theory has recently been given by Guignard[6], using the Hamiltonian method. Here, we shall start our review according to Ref. 5, since a manipulation of the equations of motion seems more familiar to most of us than the Hamiltonian method, but at the end of the section, we shall find that we must have recourse to the Hamiltonian method in order to have a correct description of the coupling resonance.

The linearized equations of betatron oscillations in the presence of the skew quadrupole fields $\partial b_x/\partial x$, $\partial b_z/\partial z$ and the azimuthal field b_s may be written in the following form:

$$x'' + \frac{1-n}{\rho^2} x = \frac{1}{B_o \rho} \left[- \frac{\partial b_z}{\partial z} z + b_s z' \right] \qquad (2.1)$$

$$z'' + \frac{n}{\rho^2} z = \frac{1}{B_o \rho} \left[\frac{\partial b_x}{\partial x} x - b_s x' \right] \qquad (2.2)$$

where x, z denote the horizontal and vertical displacements of a particle from the equilibrium orbit and the prime ' a differentiation with respect to the distance s along the equilibrium orbit; the coordinate system is taken in such a way that s, x, z form a right-handed system; n denotes the field index of the ideal guiding field, ρ the radius of curvature of the equilibrium orbit, and B_o the z-component of the ideal magnetic field at the equilibrium orbit; $\partial b_x/\partial x$, $\partial b_z/\partial z$, and b_s are also taken at the equilibrium orbit.

The solutions for the homogeneous equations of Eqs. (2.1) and (2.2), which describe the betatron oscillations in the ideal guiding field, are given by the well-known Courant-Snyder theory[7]. Representing by y either x or z, we have:

$$y = a w \cos (Q \phi + \delta) = \tfrac{1}{2} A w e^{iQ\phi} + cc \qquad (2.3)$$

where:

$w = \sqrt{\beta}$, $\phi = \int ds/Q\beta$ and Q denotes the number of betatron oscillations per revolution. The term cc denotes the complex conjugate. The arbitrary constants a, δ or the complex constant $A = a\, e^{i\delta}$ are determined from initial conditions.

Assuming that the skew quadrupole and azimuthal fields are small compared with the ideal guiding field, we shall seek a solution of Eqs. (2.1) and (2.2) in the following form:

$$y = \frac{1}{2} A w e^{iQ\phi} + cc \tag{2.4}$$

$$y' = \frac{1}{2} A (w' + \frac{i}{w}) e^{iQ\phi} + cc \tag{2.5}$$

Here the complex quantity A, which is a constant in the case of the ideal guiding field, has to be considered now as a function of s. The functional form of A will be determined by the usual method of variation of constants. Note that, in writing the expression for y', we have put an additional condition on the function A:

$$A' e^{iQ\phi} + cc = 0 . \tag{2.6}$$

This is possible because the function A is complex and contains two unknown real functions. The expression for y" is given by:

$$y'' = \left(w'' - \frac{1}{w^3}\right) \frac{1}{w} y + A' \frac{i}{w} e^{iQ\phi} \tag{2.7}$$

Substituting x, x', x", z, z', z" having forms given by Eqs. (2.4), (2.5) and (2.7) into Eqs. (2.1) and (2.2), we have:

$$\frac{dA_x}{d\theta} = \frac{R}{2iB_o\rho} w_x e^{-iX_x} e^{-iQ_x\theta} \left\{ \left[-\frac{\partial b_z}{\partial z} w_z + b_s\left(w_z' + \frac{i}{w_z}\right) \right] A_z e^{iX_z} e^{iQ_z\theta} + cc \right\} \tag{2.8}$$

$$\frac{dA_z}{d\theta} = \frac{R}{2iB_o\rho} w_z e^{-iX_z} e^{-iQ_z\theta} \left\{ \left[\frac{\partial b_x}{\partial x} w_x - b_s\left(w_x' + \frac{i}{w_x}\right) \right] A_x e^{iX_x} e^{iQ_x\theta} + cc \right\} \tag{2.9}$$

Here we have introduced an azimuthal variable θ and betatron phase variables X_x, X_z given by:

$$\theta = s/R \tag{2.10}$$

$$Q\phi = Q\theta + \chi \tag{2.11}$$

where $2\pi R$ denotes the length of the equilibrium orbit. In the sinusoidal approximation for betatron oscillations, we have $\chi = 0$ and so χ may be interpreted as representing the difference of the betatron phase advance from the sinusoidal motion.

A difference coupling resonance occurs when:

$$Q_x - Q_z = \Delta + k \; , \quad \Delta \ll 1 \; , \qquad k \; : \; \text{integer} \tag{2.12}$$

Using the principle of the averaging method, we average the right-hand sides of Eqs. (2.8) and (2.9) with respect to θ. When the resonance condition (2.12) is satisfied, the first terms in the $\{....\}$ brackets of Eqs. (2.8) and (2.9) vary slowly with respect to θ. But the cc terms oscillate rapidly with θ and can be considered to give an average of zero in the first approximation of the averaging method. Thus we have:

$$\frac{dA_x}{d\theta} = \kappa_z \, e^{-i\Delta\theta} \, A_z \tag{2.13}$$

$$\frac{dA_z}{d\theta} = \kappa_x \, e^{i\Delta\theta} \, A_x \tag{2.14}$$

with

$$\kappa_z = \frac{R}{2iB_o\rho} \left\langle w_x \left[-\frac{\partial b_z}{\partial z} w_z + b_s \left(w_z' + \frac{i}{w_z} \right) \right] e^{-i(\chi_x - \chi_z) - ik\theta} \right\rangle \tag{2.15}$$

$$\kappa_x = \frac{R}{2iB_o\rho} \left\langle w_z \left[\frac{\partial b_x}{\partial x} w_x - b_s \left(w_x' + \frac{i}{w_x} \right) \right] e^{i(\chi_x - \chi_z) + ik\theta} \right\rangle \tag{2.16}$$

Here $<g(\theta)>$ denotes an average of the periodic function $g(\theta)$ with a period of 2π:

$$<g(\theta)> = \frac{1}{2\pi} \int_{o}^{2\pi} g(\theta) \, d\theta \tag{2.17}$$

The quantities κ_z, κ_x are not independent but are linked by Maxwell's equations. In fact, by using div $\vec{b} = 0$ at the equilibrium orbit it can be shown that $\kappa_z + \kappa_x^*$ reduce to:

$$\kappa_z + \kappa_x^* = \frac{-\Delta}{2B_0\rho} \langle b_s w_x w_z e^{-i(\chi_x - \chi_z) - ik\theta} \rangle \tag{2.18}$$

We notice that this quantity is small in comparison with κ_z, κ_x when Δ is small. To the first approximation with the averaging method, one can simply put:

$$\kappa_z = -\kappa_x^* = \kappa \tag{2.19}$$

Thus, we have the fundamental equations describing the difference coupling resonance.

$$\frac{dA_x}{d\theta} = \kappa e^{-i\Delta\theta} A_z \tag{2.20}$$

$$\frac{dA_z}{d\theta} = -\kappa^* e^{i\Delta\theta} A_x \tag{2.21}$$

To be theoretically rigorous, the above formulation[5] is valid only on the resonance, i.e. $\Delta = 0$. When $\Delta \neq 0$, we have $\kappa_z \neq -\kappa_x^*$ in Eqs. (2.13) and (2.14). This condition admits the possibility of having coupling stronger one way than the other, which is non-symplectic and gives a wrong description of the coupling, as discussed by Guignard[6]. With the use of the Hamiltonian method, Guignard succeeded in deriving the symplectic equations even for $\Delta \neq 0$:

$$\frac{dA_x}{d\theta} = \kappa e^{-i\Delta\theta} A_z + i \frac{R}{8B_0^2\rho^2} \langle b_s^2 \beta_x \rangle A_x \tag{2.22}$$

$$\frac{dA_z}{d\theta} = -\kappa^* e^{i\Delta\theta} A_x + i \frac{R}{8B_0^2\rho^2} \langle b_s^2 \beta_z \rangle A_z \tag{2.23}$$

where the coupling coefficient κ is given by:

$$\kappa = \frac{R}{2iB_o\rho} \left\langle \sqrt{\beta_x \beta_z} \left[\frac{1}{2}\left(\frac{\partial b_x}{\partial x} - \frac{\partial b_z}{\partial z}\right) + \frac{b_s}{2}\left(\frac{\alpha_x}{\beta_x} - \frac{\alpha_z}{\beta_z}\right) + i\frac{b_s}{2}\left(\frac{1}{\beta_x} + \frac{1}{\beta_z}\right) \right] \right.$$

$$\left. \exp i \left[\Delta \frac{s}{R} - \int_o^s \frac{ds}{\beta_x} + \int_o^s \frac{ds}{\beta_z} \right] \right\rangle \qquad (2.24)$$

The reason why we could not get the symplectic equations by solving the equations of motion may be attributed to the wrong choice of the trial function, Eq. (2.5), for solving the perturbation problem. There we have assumed that the perturbed motion in the presence of coupling will be described by x, z, x', and z' having the same functional forms as the unperturbed solutions. Mathematically, we can start with any trial functions. In fact, if we start with trial functions x', z' having the following forms

$$x' = \frac{1}{2} A_x \left(w'_x + \frac{i}{w_x}\right) e^{iQ_x\phi_x} + cc + \frac{b_s}{2B_o\rho} z \qquad (2.25)$$

$$z' = \frac{1}{2} A_z \left(w'_z + \frac{i}{w_z}\right) e^{iQ_z\phi_z} + cc - \frac{b_s}{2B_o\rho} x \qquad (2.26)$$

we are led to the symplectic equations (2.22) and (2.23). The trial functions (2.25) and (2.26) may seem abrupt at a first glance, but they just correspond to the momenta conjugate to x, z in the Hamiltonian method. Thus, we find that there are some ambiguities in the choice of trial functions with the method of solving the equations of motion and that in order to get a correct description of the coupling phenomena, we must have recourse to the Hamiltonian method.

3. GENERAL FEATURES OF THE DIFFERENCE RESONANCE

In Eqs. (2.22) and (2.23), the second term on the right-hand side of the equation has no relevance to coupling and produces only a frequency shift of the second order in the perturbation. In fact, putting:

$$A_x = \widetilde{A}_x \, e^{i\varepsilon_x \theta} \quad , \qquad A_z = \widetilde{A}_z \, e^{i\varepsilon_z \theta}$$

$$\widetilde{Q}_x = Q_x + \varepsilon_x \quad , \qquad \widetilde{Q}_z = Q_z + \varepsilon_z$$

$$\widetilde{\Delta} = \widetilde{Q}_x - \widetilde{Q}_z = \Delta + \varepsilon_x - \varepsilon_z \tag{3.1}$$

where:

$$\varepsilon_x = \frac{R}{8B_o^2 \rho^2} \langle b_s^2 \, \beta_x \rangle, \; \varepsilon_z = \frac{R}{8B_o^2 \rho^2} \langle b_s^2 \, \beta_z \rangle \tag{3.2}$$

we have:

$$\frac{d\widetilde{A}_x}{d\theta} = \kappa \, e^{-i\widetilde{\Delta}\theta} \, \widetilde{A}_z \tag{3.3}$$

$$\frac{d\widetilde{A}_z}{d\theta} = -\kappa^* \, e^{i\widetilde{\Delta}\theta} \, \widetilde{A}_x \tag{3.4}$$

In the following, we shall omit the wavy sign in A_x, A_z, Q_x, Q_z, Δ in order to simplify the formulae, with the understanding that the effects of ε_x, ε_z are to be incorporated into the unperturbed frequencies.

From Eqs. (3.3) and (3.4) we have:

$$\frac{d^2 A_x}{d\theta^2} + i \, \Delta \, \frac{d A_x}{d\theta} + |\kappa|^2 \, A_x = 0 \tag{3.5}$$

This is a second order differential equation for A_x with constant coefficients, the general solution of which is given by:

$$A_x = A \, e^{\frac{i}{2}(\eta-\Delta)\theta} + B \, e^{-\frac{i}{2}(\eta+\Delta)\theta} \tag{3.6}$$

where:

$$\eta = \sqrt{\Delta^2 + 4|\kappa|^2} \qquad (3.7)$$

and A, B are complex constants to be determined from initial conditions.

Substituting Eq. (3.6) into Eq. (3.3), we have the general solution for A_z:

$$A_z = i \frac{\eta-\Delta}{2\kappa} A e^{\frac{i}{2}(\eta+\Delta)\theta} - i \frac{\eta+\Delta}{2\kappa} B e^{-\frac{i}{2}(\eta-\Delta)\theta} \qquad (3.8)$$

In Eq. (3.6) we notice that the horizontal betatron oscillation, which has a monochromatic frequency Q_x in the absence of coupling, is given by a superposition of two oscillations having the frequencies Q_1, Q_2 :

$$Q_1 = Q_x + \frac{1}{2}(\eta-\Delta) = Q_z + \frac{1}{2}(\eta+\Delta) \qquad (3.9)$$

$$Q_2 = Q_x - \frac{1}{2}(\eta+\Delta) = Q_z - \frac{1}{2}(\eta-\Delta) \qquad (3.10)$$

The situation is identical for the vertical betatron oscillation which is also given by a superposition of the two oscillations having the frequencies Q_1, Q_2. The two frequencies Q_1, Q_2 are the so-called normal mode frequencies of the coupled oscillator[8]. In the presence of coupling, the horizontal and vertical oscillations are given by superpositions of the two normal mode oscillations; only differing from each other in the mixing ratio of the two normal modes. Figure 1 shows the Δ-dependence of the normal mode frequencies.

Besides the coupling of frequencies, there also occurs coupling of amplitudes between the horizontal and vertical oscillations. Since the difference between the two frequencies Q_1, Q_2 is small compared with both Q_1 and Q_2 ($Q_1 - Q_2 = \eta$), the horizontal and vertical oscillations are expected to show amplitude beating. The envelopes of the beat oscillations are given by $|A_x|$ and $|A_z|$, apart from the factors of $\sqrt{\beta_x}$ and $\sqrt{\beta_z}$.

$$|A_x|^2 = |A|^2 + |B|^2 + 2 |AB^*| \cos (\eta\theta + \arg AB^*) \tag{3.11}$$

$$|A_z|^2 = \frac{(\eta-\Delta)^2}{4|\kappa|^2} |A|^2 + \frac{(\eta+\Delta)^2}{4|\kappa|^2} |B|^2 - 2 |AB^*| \cos (\eta\theta + \arg AB^*) \tag{3.12}$$

From Eqs. (3.11) and (3.12), the horizontal and vertical oscilla-
tions show an amplitude modulation with the period of $\theta = \Theta$:

$$\Theta = \frac{2\pi}{\sqrt{\Delta^2 + 4|\kappa|^2}}. \tag{3.13}$$

We may regard the $|A_x|^2$ and $|A_z|^2$ as the energies of horizontal and
vertical oscillations, although they do not have the correct dimension
of energy since we are dealing with the amplitudes without the factors
$\sqrt{\beta_x}$, $\sqrt{\beta_z}$. Then the total energy of the coupled oscillation is a con-
stant of motion.

$$|A_x|^2 + |A_z|^2 = \left[1 + \frac{(\eta-\Delta)^2}{4|\kappa|^2}\right] |A|^2 + \left[1 + \frac{(\eta+\Delta)^2}{4|\kappa|^2}\right] |B|^2 \tag{3.14}$$

Thus, in the vicinity of the difference resonance, there occurs a sinus-
oidal transmission of energy from one of the directions of oscillation
to the other and back, but the amplitudes remain limited even exactly on
the resonance.

4. COHERENT COUPLED OSCILLATION EXCITED BY KICK

Let us consider the coherent betatron oscillations of the beam in
the horizontal and vertical planes, when the beam is kicked horizontally
by a kicker. We can also kick the beam vertically and then the oscilla-
tions will be described by simply interchanging the roles of x and z in
the following expressions, together with the transformations: $\Delta \to -\Delta$,
$\eta \to -\eta$, $\kappa \to -\kappa^*$. The induced coherent oscillations will be detected by
the horizontal and vertical beam position monitors.

Before being kicked, the particles in the beam perform their individual betatron oscillations around the equilibrium orbit. In the vicinity of the difference resonance, the individual betatron oscillations are the beat oscillations, as described in Section 3, and the beam envelope will also show beats. The centre of the beam, however, can be assumed to move on the equilibrium orbit. It is just the motion of the beam centre which is relevant to the measurements by kick methods.

Then, the coherent oscillation excited by a horizontal kick will be derived by imposing the following initial conditions:

$$x(o) = 0 , \qquad x'(o) = \theta_x \tag{4.1}$$

$$z(o) = 0 , \qquad z'(o) = 0 \tag{4.2}$$

Here, θ_x denotes the angle of the horizontal kick and the origin of the azimuthal variable s or θ is taken at the kicker position.

With this choice of the origin of s or θ, we have:

$$x(o) = \frac{1}{2} A_x(o) \, w_x(o) + cc \tag{4.3}$$

$$x'(o) = \frac{1}{2} A_x(o) \left[w'_x(o) + \frac{i}{w_x(o)} \right] + cc + \frac{b_s(o)}{2B_o\rho} z(o) \tag{4.4}$$

where:

$$A_x(o) = A + B \tag{4.5}$$

and the similar expressions for z(o), z'(o) with:

$$A_z(o) = \frac{i}{2\kappa} \left[(\eta-\Delta) \, A - (\eta+\Delta) \, B \right] \tag{4.6}$$

Then, the complex constants A, B are determined as follows:

$$A = -i \frac{\eta+\Delta}{2\eta} \theta_x \sqrt{\beta_x(o)} \tag{4.7}$$

$$B = \frac{\eta-\Delta}{\eta+\Delta} A \tag{4.8}$$

Thus, the coherent oscillations excited by the horizontal kick are described by:

$$x = \frac{1}{2} A \left[e^{\frac{i}{2}(\eta-\Delta)\theta} + \frac{\eta-\Delta}{\eta+\Delta} e^{-\frac{i}{2}(\eta+\Delta)\theta} \right] \sqrt{\beta_x} \, e^{iQ_x\phi_x} + cc \qquad (4.9)$$

$$z = \frac{1}{2} \frac{i(\eta-\Delta)}{2\kappa} A \left[e^{\frac{i}{2}(\eta+\Delta)\theta} - e^{-\frac{i}{2}(\eta-\Delta)\theta} \right] \sqrt{\beta_z} \, e^{iQ_z\phi_z} + cc \qquad (4.10)$$

where:

A is given by Eq. (4.7).

Note that the vertical oscillation is given by a superposition of the two normal modes Q_1 and Q_2 with one-to-one mixing ratio. On the other hand, in the horizontal oscillation the normal mode Q_1 is dominant for $\Delta > 0$, and the normal mode Q_2 is dominant for $\Delta < 0$. Exactly on the resonance, $\Delta = 0$, the two normal modes have equal contributions to the horizontal oscillation.

When the above equations are expressed explicitly in terms of real quantities, we have:

$$x = \theta_x \sqrt{\beta_x(o)\,\beta_x} \, \frac{2|\kappa|}{\eta} \left[\frac{\eta^2}{4|\kappa|^2} - \sin^2 \frac{\eta}{2}\theta \right]^{\frac{1}{2}} \sin \left[Q_x\phi_x - \frac{\Delta}{2}\theta + \Phi(\theta) \right]$$
$$(4.11)$$

$$z = -\theta_x \sqrt{\beta_x(o)\beta_z} \, \frac{2|\kappa|}{\eta} \sin \frac{\eta}{2}\theta \sin \left[Q_z\phi_z + \frac{\Delta}{2}\theta - \arg\kappa \right] \qquad (4.12)$$

where:

arg κ denotes the argument of the coupling coefficient κ, and the phase factor $\Phi(\theta)$ is given by:

$$\tan \Phi(\theta) = \frac{\Delta}{\eta} \tan \frac{\eta}{2}\theta \qquad (4.13)$$

Figure 2 shows an illustration of the coherent coupled oscillations excited by a horizontal kick. The envelopes of the amplitudes are given by $|A_x|$ and $|A_z|$, omitting the factor of $\sqrt{\beta}$.

$$|A_x|^2 = \theta_x^2 \, \beta_x(o) \, \frac{4|\kappa|^2}{\eta^2} \left[\frac{\eta^2}{4|\kappa|^2} - \sin^2 \frac{\eta}{2}\theta \right] \qquad (4.14)$$

$$|A_z|^2 = \theta_x^2 \, \beta_x(o) \, \frac{4|\kappa|^2}{\eta^2} \sin^2 \frac{\eta}{2}\theta \qquad (4.15)$$

The total "energy" of the oscillation is:

$$|A_x|^2 + |A_z|^2 = \theta_x^2 \beta_x(o) \tag{4.16}$$

We can define the fraction F of the energy interchanged between the horizontal and vertical oscillations and the total energy as:

$$F = \frac{4|\kappa|^2}{\Delta^2 + 4|\kappa|^2} \tag{4.17}$$

Exactly on resonance, we have F = 1, which means that the horizontal energy given by the horizontal kick is completely transmitted to the vertical oscillation and back in the course of the coupled oscillation.

Another interesting quantity of the amplitude modulation is the ratio R of the minimum to the maximum of the horizontal envelopes.

$$R = \frac{|A_x|_{min}}{|A_x|_{max}} = \frac{|\Delta|}{\sqrt{\Delta^2 + 4|\kappa|^2}} \tag{4.18}$$

Exactly on resonance we have R = 0 and far away from resonance we have R = 1.

The frequency of the vertical oscillation is easily found from Eq. (4.12), since arg κ does not depend on θ. The tune shift ΔQ_z caused by coupling is:

$$\Delta Q_z = \frac{\Delta}{2} \tag{4.19}$$

and the perturbed frequency is:

$$Q_z + \Delta Q_z = \frac{Q_1 + Q_2}{2} = \frac{Q_x + Q_z}{2} \tag{4.20}$$

In the case of the horizontal oscillation frequency, however, an account should be taken of the phase factor $\Phi(\theta)$. Then, the tune shift ΔQ_x caused by coupling is:

$$\Delta Q_x = -\frac{\Delta}{2} + \frac{1}{2\pi} \left[\Phi(\theta + 2\pi) - \Phi(\theta) \right] \tag{4.21}$$

$$\simeq -\frac{\Delta}{2} + \frac{d\Phi}{d\theta} = -\frac{\Delta}{2} + \frac{\Delta}{2} \frac{\eta^2}{\eta^2 - 4|\kappa|^2 \sin^2 \frac{\eta}{2}\theta} \tag{4.22}$$

In obtaining Eq. (4.22) we have assumed that the increment of $\Phi(\theta)$ per turn is small. The horizontal tune shift ΔQ_x is dependent on θ and varies periodically with the period of $\theta = 2\pi/\eta$.

It is an average frequency taken over some region of θ that can be measured. Hence, let us define $\overline{\Delta Q_x}(\tau)$ by the average tune shift taken from $\theta_- = 2n\pi/\eta - \tau$ to $\theta_+ = 2n\pi/\eta + \tau$, where n is an integer (Figure 3). Here, $\theta = 2n\pi/\eta$ corresponds to the maximum of the horizontal amplitude modulation.

$$\overline{\Delta Q_x}(\tau) = -\frac{\Delta}{2} + \frac{1}{2\tau} \left[\text{phase advance of } \Phi \text{ from } \theta_- \text{ to } \theta_+ \right]$$

$$= -\frac{\Delta}{2} + \frac{1}{\tau} \tan^{-1} \left[\frac{\Delta}{\eta} \tan \frac{\eta}{2} \tau \right] \tag{4.23}$$

When the average is taken over the region of multiples of the beat oscillation period, $\tau = m\pi/\eta$ with m being an integer, we have for the average tune $\overline{Q_x} = Q_x + \overline{\Delta Q_x}$:

$$\overline{Q_x}\left(\tau = \frac{m\pi}{\eta}\right) = \begin{vmatrix} Q_x - \frac{\Delta}{2} + \frac{\eta}{2} = Q_1 & \text{for } \Delta > 0 \\ \\ Q_x - \frac{\Delta}{2} - \frac{\eta}{2} = Q_2 & \text{for } \Delta < 0 \end{vmatrix} \tag{4.24}$$

The θ-dependence of the horizontal tune shift ΔQ_x and the average tune shift $\overline{\Delta Q_x}$ is illustrated in Figure 4, for the typical two cases of $\Delta/|\kappa| = 2$ and 1. The smaller the Δ, the larger the variations of ΔQ_x and $\overline{\Delta Q_x}$ with θ become. When the average is taken over a large number of beat oscillation periods, $\overline{\Delta Q_x}$ converges to $\frac{1}{2}(\eta - \Delta)$ for $\Delta > 0$, and to $-\frac{1}{2}(\eta + \Delta)$ for $\Delta < 0$.

$$\overline{Q_x}(\tau \to \infty) = \begin{cases} Q_1 & \text{for } \Delta > 0 \\ \\ Q_2 & \text{for } \Delta < 0 \end{cases} \tag{4.25}$$

In this context, however, it must be noted that the coherent oscillation damps gradually which is mainly due to a finite Q-spread in the beam[9,10] (Landau damping). In the absence of coupling, we can get an idea of the damping effect by writing the oscillation in the following simplified form:

$$x = e^{iQ\theta} + cc \tag{4.26}$$

and assuming that the Q-spread in the beam is given by a Gaussian distribution:

$$f(Q)dQ = \frac{1}{\sqrt{2\pi}\sigma} e^{-(Q-Q_0)^2/2\sigma^2} \tag{4.27}$$

where:

Q_o = mean value of Q

σ = r.m.s. width in the Q-distribution.

Then, the coherent oscillation with a finite Q-spread is given by:

$$\bar{x} = \int dQ\ f(Q)\ y$$

$$= e^{-\frac{\sigma^2}{2}\theta^2} \left[e^{iQ_o\theta} + cc \right] \tag{4.28}$$

The wider the Q-spread, the more rapidly the oscillation damps.

In the presence of coupling, from Eqs. (4.9) and (4.10) we may write the oscillation in the following simplified form:

$$x = \frac{\eta+\Delta}{\eta} e^{\frac{i}{2}(Q_x+Q_z+\eta)\theta} + \frac{\eta-\Delta}{\eta} e^{\frac{i}{2}(Q_x+Q_z-\eta)\theta} + cc \tag{4.29}$$

$$z = \frac{1}{\eta} e^{\frac{i}{2}(Q_x+Q_z+\eta)\theta} - \frac{1}{\eta} e^{\frac{i}{2}(Q_x+Q_z-\eta)\theta} + cc \tag{4.30}$$

Here, $(Q_x+Q_z)/2$ and Δ or $\eta = \sqrt{\Delta^2+4|\kappa|^2}$ have some spreads in the distribution. In this case, not only the frequency factor but also the amplitude factor have contributions to the decay of the coherent oscillation. Thus, its mathematical analysis is inevitably complex and beyond the scope of this report. Anyway, in order to get data of high-quality, one has to reduce the Q-spread in the beam to a certain tolerable amount and make measurements in a time interval before the signal is deteriorated by the damping effects.

5. QUANTITIES TO BE MEASURED

The horizontal and vertical coherent oscillations excited by a kicker are detected by the horizontal and vertical position-sensitive pick-up (PU) electrodes and the signals on the PU electrodes are fed to a direct-reading, automatic Q-meter in the ISR[11]. The signals on the PU electrodes contain a series of frequencies ω_n

$$\omega_n = 2\pi f |n-q| \qquad\qquad n = \pm 0, 1, 2, \ldots\ldots \qquad (5.1)$$

where, q is a non-integer part of the betatron oscillation frequency and f is the revolution frequency of the beam. The amplitude function of the spectrum ω_n depends upon the shape and length of the kicker pulse. When a rectangular kicker pulse with its length equal to the revolution period is applied on a debunched beam, the amplitude function $G(\omega_n)$ of the spectrum ω_n follows the form[9]

$$G(\omega_n) = \left| \frac{\sin \pi(n-q)}{\pi(n-q)} \right| \qquad (5.2)$$

The amplitude function $G(\omega_n)$ decreases with oscillations as $|n-q|$ increases and only the two lowest modes of ω_n are used for Q-measurements in the ISR. The input processors and filters of the automatic Q-meter extract these two modes (q and 1-q). Since the filter output signals are available for general observations besides the Q-measurement, the measurement of the coupling coefficient will be made using either q or 1-q filter output signals.

Then, the time variations of the filter output signals for the horizontal and vertical oscillations excited by a horizontal kick in the presence of coupling may be expressed by:

$$v_x(t) = G_x \theta_x \sqrt{\beta_x(o)\beta_x} \frac{2|\kappa|}{\eta} \left[\frac{\eta^2}{4|\kappa|^2} - \sin^2 \pi f \eta t \right]^{\frac{1}{2}} \sin \left[2\pi f (q_x + \Delta Q_x)t + \delta_x \right] \qquad (5.3)$$

$$v_z(t) = G_z \theta_x \sqrt{\beta_x(o)\beta_z} \frac{2|\kappa|}{\eta} \sin \pi f \eta t \sin \left[2\pi f (q_z + \Delta Q_z)t + \delta_z \right] \qquad (5.4)$$

Here, we have assumed that the damping of the coherent oscillation is negligible. q_x and q_z are non-integer parts of the unperturbed betatron oscillation frequencies and ΔQ_x and ΔQ_z are the tune shifts caused by coupling. The above expressions correspond to the q filter output and in the case of the 1-q filter output, the terms $q + \Delta Q$ have to be replaced by $1 - (q+\Delta Q)$. The factor $\sqrt{\beta_x(o)}$ should be evaluated at the kicker position and the factors $\sqrt{\beta_x}$ and $\sqrt{\beta_z}$ at the PU position. The factors G_x and G_z denote an overall sensitivity of the detector system, including the PU electrodes' sensitivity, the transfer characteristics of the filters, etc. The origin of time t is taken in such a way that $t = 0$ corresponds to the zero of the vertical signal envelope and δ_x, δ_z are the phase factors.

With the above expressions for the filter output signals, we are now in a position to list the possible quantities that can be measured and to examine the suitability and accuracy of each measurement.

5.1 The period T of the amplitude modulation[1,4]

$$T \;=\; \frac{1}{f\sqrt{\Delta^2+4|\kappa|^2}} \;=\; \frac{\Theta}{2\pi f} \tag{5.5}$$

This quantity is easily measured by taking photographs on an oscilloscope of the vertical oscillation signal, since its envelope crosses zero level. The amplitude of the signal has no relevance to the measurement of T. Since the value of f can be known accurately and easily from other sources, we can calculate $\eta = \sqrt{\Delta^2+4|\kappa|^2}$ from this measurement.

5.2 The horizontal and vertical betatron frequencies

The direct-reading, automatic Q-meter for the ISR tells us the non-integer part of the betatron oscillation frequency, by counting the number of waves of the q or 1-q mode. Therefore, one might be tempted to suppose that the Q-meter will give results of high accuracy, without preparing special devices for coupling measurements. As has been pointed out in

Section 4, however, the horizontal betatron frequency varies according to the position where we measure it along the beat oscillation. By choosing appropriately the time interval in which the Q-meter counter gate is open, or by taking a long-term average of the frequency provided the coherent oscillation continues long enough, we can measure $\frac{1}{2}(q_x+q_z) + \frac{1}{2}\eta$ for $\Delta > 0$ or $\frac{1}{2}(q_x+q_z) - \frac{1}{2}\eta$ for $\Delta < 0$. From the vertical signal we can find $\frac{1}{2}(q_x+q_z)$. In this case, however, the Q-meter counter gate should be set in such a way as to avoid the interval when the envelope of the vertical signal crosses zero level, since the Q-meter cannot count the number of the q or 1-q waves correctly for too small amplitudes. From these two measurements we can calculate η by taking the difference between the two data

$$\pm \frac{\eta}{2} = \left[\frac{q_x+q_z}{2} \pm \frac{\eta}{2}\right] - \left[\frac{q_x+q_z}{2}\right] \tag{5.6}$$

where, the upper sign is for $\Delta > 0$ and the lower sign for $\Delta < 0$.

It should be noted that η is measured far more easily by the measurement method described in 5.1. Furthermore, in the frequency measurement, the tune shift caused by coupling accounts for only a small fraction of the measured quantity. Namely, in most cases the values of $|\kappa|$ will lie in a range of $1 \sim 5 \times 10^{-3}$. Then, the unperturbed component predominates in the tune shift by about a factor of 100. In general, it is not good practice to measure a quantity in which the desired information accounts for only a small fraction of the total quantity.

5.3 Fraction of energy interchanged between horizontal and vertical oscillations[4)]

$$F = \frac{4|\kappa|^2}{\Delta^2+4|\kappa|^2} \tag{5.7}$$

This quantity defined by Eq. (4.17) or its square root, $\sqrt{F}$, may be measured by comparing the peak values of the horizontal and vertical signals, with corrections for the factors $\sqrt{\beta}$ and G in Eqs. (5.3) and (5.4).

The correction for the factor $\sqrt{\beta}$ will not present any trouble, but the correction for the factor G may prove to be very tedious and the method does not seem practical. An alternative way of measuring the $\sqrt{F}$ consists of observing the Δ-variation of the peak value of the vertical signals.

$$\hat{v}_z = G_z \, \theta_x \, \sqrt{\beta_x(o)\beta_z} \times \sqrt{F} \qquad (5.8)$$

The factor in front of $\sqrt{F}$ on the right hand side of the above equation can be determined experimentally by measuring $\hat{v}_z$ exactly on resonance $\Delta = 0$.

5.4 Ratio of minimum to maximum of the horizontal envelope

$$R = \frac{|\Delta|}{\sqrt{\Delta^2 + 4|\kappa|^2}} \qquad (5.9)$$

This quantity, which is already introduced by Eq. (4.18), is easily measured by taking photographs on an oscilloscope of the horizontal oscillation signal. Since we take the ratio only in one plane of the oscillations, the corrections as needed in the measurement method 5.3 are no longer necessary and a direct reading of the ratio on an oscilloscope gives the value of R.

Now, for the coupling measurement we need some device to vary Δ and bring the operation line to the vicinity of the resonance. Let us refer to this device as Δ-varying quadrupoles. Far away from the resonance, the coupling effects are negligible and we can know the unperturbed tune Q_x and Q_z from the Q-meter. In these regions, we can calibrate the effects of the Δ-varying quadrupoles, namely have a table of Δ versus the current of the Δ-varying quadrupoles. From these calibrations in large positive Δ and negative Δ, we can interpolate the Δ in the vicinity of the resonances. With this knowledge of the relationship between Δ and the Δ-varying quadrupole current, we can know the coupling coefficient $|\kappa|$ from one of the measurements mentioned above. In the cases of measurements described in 5.1 and 5.2, we can do with-

out any precise calibrations since we can know $|\kappa|$ by seeking the minimum of η as Δ is varied appropriately by the Δ-varying quadrupoles.

A simple and precise measurement of the coupling coefficient is, however, possible if the measurements described in 5.1 and 5.4 are simultaneously made. A knowledge of T and R enables us to get not only $|\kappa|$ but also $|\Delta|$.

$$|\kappa| \;=\; \frac{1}{2fT} \; \sqrt{1 - R^2} \tag{5.10}$$

$$|\Delta| \;=\; \frac{R}{fT} \tag{5.11}$$

Precise calibrations of the Δ-varying quadrupoles and procedures for seeking the minimum of η as Δ is varied are then no longer necessary. Once Δ is set somewhere in the vicinity of the resonance, we can know $|\kappa|$ from a single measurement. The measurement of T and R will not present any trouble, at least when we take photographs on an oscilloscope. Furthermore, an electronic measuring system based on this method seems to be possible, as will be described in Section 7.

Let us examine the accuracy which is to be expected from the proposed method. Since η and R may be considered to be independent quantities that are measured separately, the relative errors of $|\kappa|$ and $|\Delta|$ may be expressed by:

$$\frac{\delta|\kappa|}{|\kappa|} \;=\; \sqrt{\left(\frac{\delta\eta}{\eta}\right)^2 + \left(\frac{\Delta^2}{4|\kappa|^2}\frac{\delta R}{R}\right)^2} \tag{5.12}$$

$$\frac{\delta|\Delta|}{|\Delta|} \;=\; \sqrt{\left(\frac{\delta\eta}{\eta}\right)^2 + \left(\frac{\delta R}{R}\right)^2} \tag{5.13}$$

Thus, it is clear that we should make the measurement at as small a Δ as possible, in order to get data of high accuracy. Exactly on resonance, the error δR has no influence on the accuracy of the coupling coefficient.

$$\frac{\delta|\kappa|}{|\kappa|} \;=\; \frac{\delta\eta}{\eta} \qquad\qquad \text{for } \Delta = 0 \text{ or } R = 0 \tag{5.14}$$

In practice, Δ will be set somewhere in the region of $|\Delta|/|\kappa| \lesssim 2$ or $R \lesssim 0.7$ by observing the variation of R as the Δ-varying quadrupole current is varied. Then, we have:

$$\frac{\delta|\kappa|}{|\kappa|} \lesssim \sqrt{\left(\frac{\delta\eta}{\eta}\right)^2 + \left(\frac{\delta R}{R}\right)^2} \qquad \text{for } \frac{|\Delta|}{|\kappa|} \lesssim 2 \text{ or } R \lesssim 0.7 \qquad (5.15)$$

Even allowing relatively large errors $\delta\eta/\eta \simeq 3$ % and $\delta R/R \simeq 5$ %, we can expect $\delta|\kappa|/|\kappa| \lesssim 6$ % in the region of $R \lesssim 0.7$, which will be a sufficient accuracy for most practical purposes.

So far, we have been concerned solely with the measurement of the magnitude $|\kappa|$ of the complex quantity κ. In some cases, it will be desirable to know the argument of κ in addition to the magnitude $|\kappa|$, or to know the real and imaginary parts of κ separately. Now, the arg κ is contained only in the phase factor δ_z in Eq.(5.4) but it will be hard to detect with the kick method of the beam being kicked horizontally or vertically. For a possible method of measuring the real and imaginary parts of κ by kicking the beam in the inclined plane and observing the oscillations also in the inclined plane, see Ref. 8.

6. A TEST EXPERIMENT ON THE ISR

In order to confirm that the proposed method for coupling measurement works well in practice, a test experiment was made on the ISR Ring 1. A CPS pulse of protons of momentum 26.6 GeV/c was injected into Ring 1, accelerated and deposited on central orbit. Measurements were made by kicking horizontally this debunched beam and taking photographs on an oscilloscope of the horizontal and vertical coherent oscillations which were given by the 1-q filter output signals of the Q-meter. Figure 5 shows an example of photographs taken.

The basic operation line of the Ring was set to the 8C26 line with a slight modification to suppress the Q-spread. The suppression of the Q-spread to about a half of the basic value was necessary, since in the

basic 8C26 line Q' = 1.62 which was too large for the coherent oscilla-
tion to continue long enough for the measurement. The transverse beam
feedback system was switched off. The unperturbed Q-separation Δ was
varied by TD1 (the first series of the Terwilliger quadrupoles) and the
coupling coefficient κ was varied by the skew quadrupoles Q2 (the second
series of the correction skew quadrupoles).

Figures 6.a and 6.b show the data obtained when Δ is varied with the
other conditions fixed. Here, the abscissae denote the percentage of
maximum current in the TD1. The skew quadrupoles Q2 were switched off
and the first series of the skew quadrupoles Q1 were excited at -2.06 %
of their maximum current. Figure 6.a shows the variations of η and R,
where η is obtained from the period measurement of the vertical signal
envelope and R is obtained from the ratio measurement of the horizontal
signal envelope. The obtained data of $|\kappa|$ and $|\Delta|$ are collected in
Table 1. Note that the values of $|\kappa|$ for different readings of the TD1
current coincide with each other quite well. The average value of $|\kappa|$
is found to be $|\kappa|_{av} = 2.95 \times 10^{-3}$ and the spread of the data around
the average value is found to be 0.7 % at r.m.s. The Δ-dependence of the
$|\kappa|$ and $|\Delta|/|\kappa|_{av}$ is plotted in Figure 6.b. From this data it is con-
cluded that we can calculate the coupling coefficient accurately from
a single measurement provided the Δ-varying quadrupoles are set at an
arbitrary value within the range of R $\lesssim$ 0.6.

TABLE 1

Measured values of $|\kappa|$ and $|\Delta|$ for different readings of TD1 current

TD1 reading (%)	1.51	2.02	2.50	2.99	3.51	3.99	4.50	4.99
$|\Delta|$ ($\times 10^{-3}$)	4.24	3.11	1.67	0.52	1.06	2.29	3.47	4.86
$|\kappa|$ ($\times 10^{-3}$)	2.95	2.92	2.97	2.95	2.95	2.93	2.99	2.95

Figures 7.a and 7.b show another set of results which was taken by varying the strength of the skew quadrupoles Q2 with the other conditions fixed. The Δ-varying TD1 currents were set at 3.0 % and the skew quadrupoles Q1 were set at -2.06 %. In Figure 7.a, the coupling coefficients are plotted versus different excitations of the skew quadrupoles Q2. From the figure it is found that the optimum setting of the Q2 is -2.4 % for the Ring conditions investigated, and the corresponding minimum of $|\kappa|$ is around 0.3×10^{-3}. Figure 7.b shows the obtained value of $|\Delta|/|\kappa|$ in the above experiment. There is a conspicuous Δ-jump observed around the minimum of $|\kappa|$. The observed Δ-jump is 0.0015 and may be attributed to some trivial mistakes in the course of measurement, or to small fluctuations in some components of the ISR, or it may be explained by some physical effects. Further investigations on the coupling phenomena in the ISR will clarify this point. For the moment, we can only remark that in spite of the Δ-jump, the observed behaviour of $|\kappa|$ seems quite reasonable when the strength of the correction skew quadrupoles is varied.

7. A POSSIBLE ELECTRONIC SYSTEM FOR COUPLING MEASUREMENT

In the preceding sections it has been shown that the proposed method quickly gives data of high accuracy. We can know the coupling coefficient $|\kappa|$ and the unperturbed Q-separation $|\Delta|$ from a single kick and measurement sequence. However, if an electronic measurement system is possible, which will tell us the magnitude of coupling coefficient quickly in the rigorous sense of the term, it will be helpful to machine operation. In this section, we shall outline one such electronic system for coupling measurement based on the proposed method. Figure 8.a shows a block-diagram of the electronic system and Figure 8.b the expected waveforms of signals at some points in the system.

After passing through the buffers, the horizontal and vertical filter outputs of the q or 1-q filter of the Q-meter are fed to square-law detectors and associated filters, which give signals proportional to the

squares of the amplitude envelopes, $|A_x|^2$ and $|A_z|^2$. The frequency of the amplitude modulation (1/T) will lie in the range of $(1 \sim 10) \times 10^{-3}$ f in most cases of practical interest and the carrier frequency of the q or 1-q filter output will be around 0.25 f or 0.75 f, where f denotes the particle revolution frequency. Since a large separation between the envelope frequency and the carrier frequency is desirable for the square-law detection, we should use the filter output which has the higher carrier frequency. Then the carrier frequency is expected to be higher by a factor of 75 $\sim$ 750 times than the envelope frequency.

In the case of a horizontal kick, the horizontal signal is applied to the R-channel, where the R-filter cuts the high-frequency components to extract a signal proportional to $|A_x|^2$. The vertical signal is applied to the T-channel, where the T-filter cuts the high-frequency and d.c. components to extract a signal proportional to cos $2\pi f \eta t$. The output of the T-filter is phase-shifted by 90° with an integrator and then sent to a zero-crossing detector, where its positive and negative going zero-crossing timings, t_p and t_n, are detected which correspond to the maximum and minimum of the vertical amplitude modulation.

The period T of the amplitude is measured by the counter with a 5 MHz clock, the gate of which is opened during a time interval from t_{p1} to t_{p2}. For the ISR with f $\simeq$ 300 kHz, the period T will lie in the range of 0.3 $\sim$ 3 ms and a 5 MHz clock is sufficient. The counts in the counter are fed to a computer for data processing.

In order to measure the square of the ratio R, the R-filter output is sample-held at the timing of t_p and t_n. The sample-held values a_n and b_n are analog-to-digital converted and fed to the computer. Since a slow decay of the beat oscillation will be present in most cases, the R^2 may be determined by:

$$R^2 = \frac{b_1 + b_2}{2a_1} \tag{7.1}$$

Of course, a higher precision will be attained if we process a larger number of data a_n and b_n. But Eq. (7.1) will be sufficient for most cases in which the Q-spread is suppressed to a moderate level.

From the data of T and R^2, together with the help of the known value of f, the computer calculates the coupling coefficient $|\kappa|$ by Eq. (5.10). The obtained value of $|\kappa|$ is then returned to the system and displayed.

The timings t_p and t_n may have small errors due to imperfections in the integrator and hysteresis effects in the zero-crossing detector. However, since T is obtained from two timings t_{p1} and t_{p2} with the same error, the timing errors do not seem to have an effect on the T-measurement. Also for the R-measurement, the errors caused by the timing errors will be negligible, since the maximum or minimum value of a sinusoidal wave form is only slightly affected by small timing errors.

In the case of the beam being kicked vertically, we have only to interchange the connections of the horizontal and vertical signals of the Q-meter to the system.

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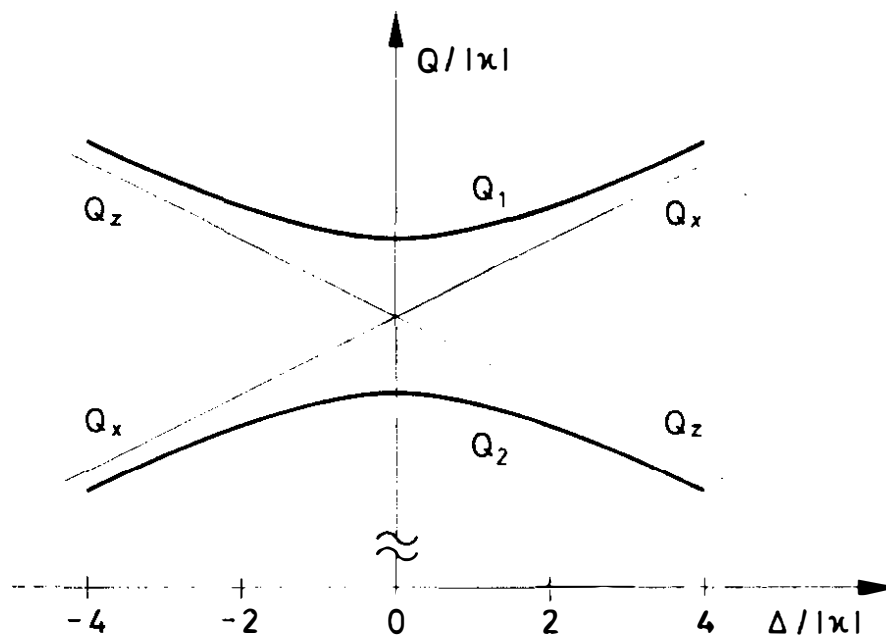
FIGURE 1 - Relationship between the unperturbed frequencies Q_x, Q_z and the normal mode frequencies Q_1, Q_2. Here Q_x, Q_z are assumed to vary perpendicularly to the diagonal of the tune diagram.
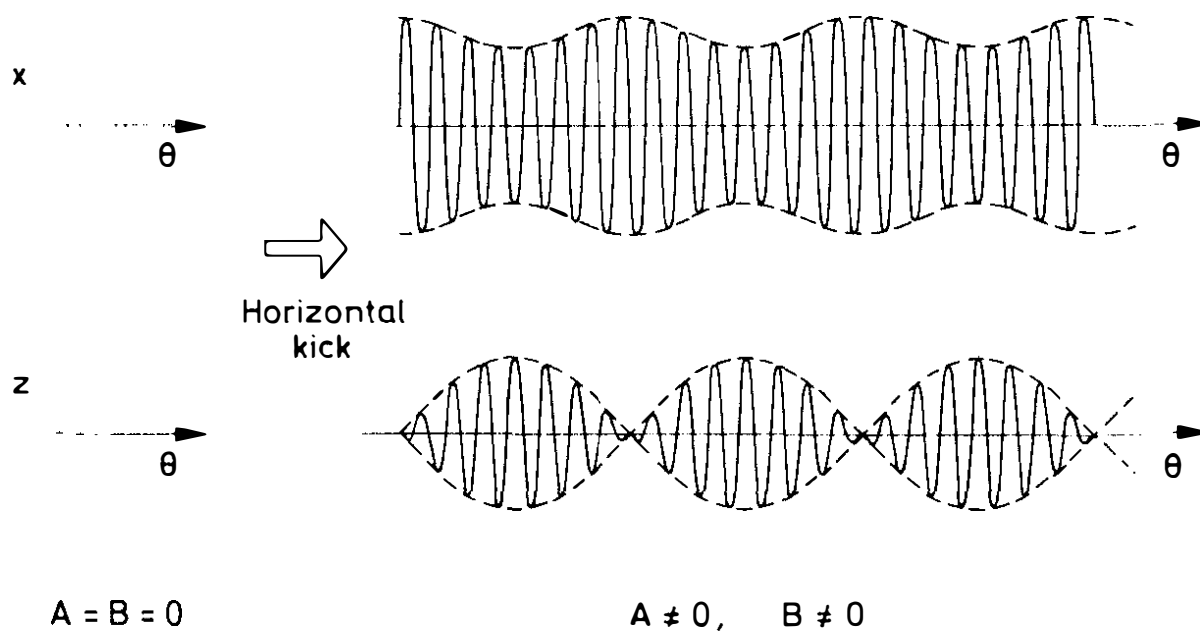
FIGURE 2 - An illustration of the coherent coupled oscillations excited by a horizontal kick. The factor $\sqrt{\beta}$ for the amplitude is omitted.
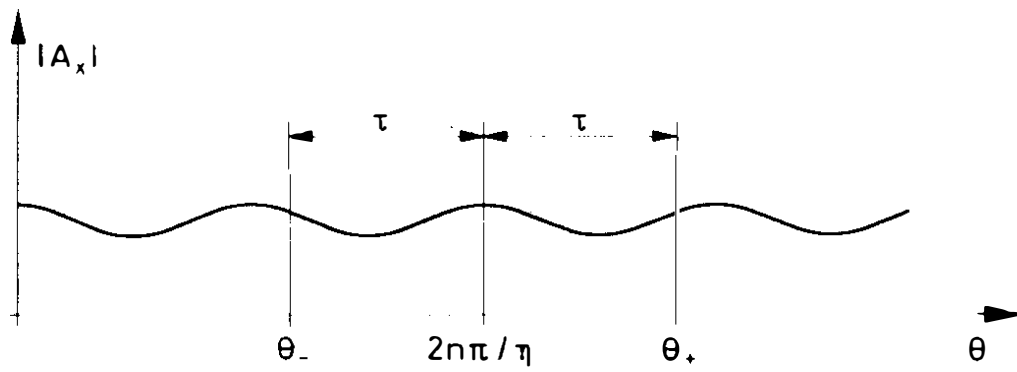
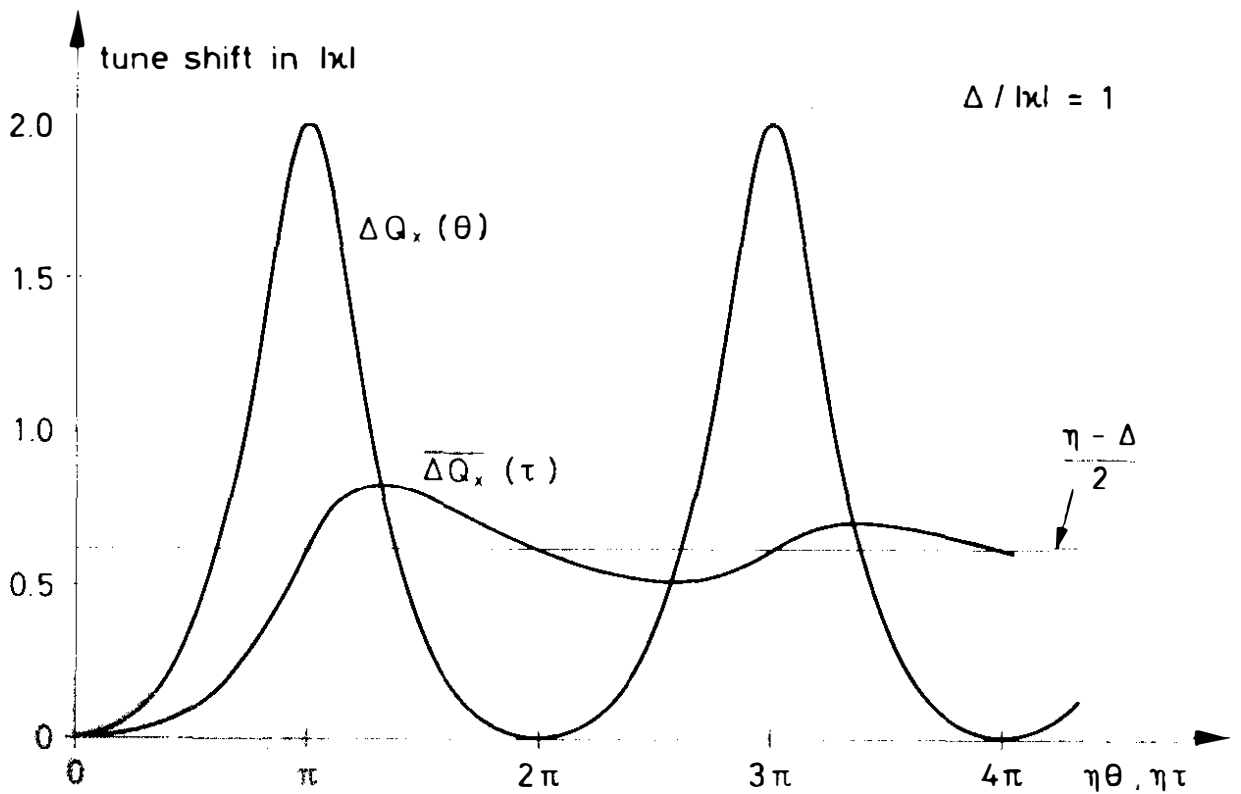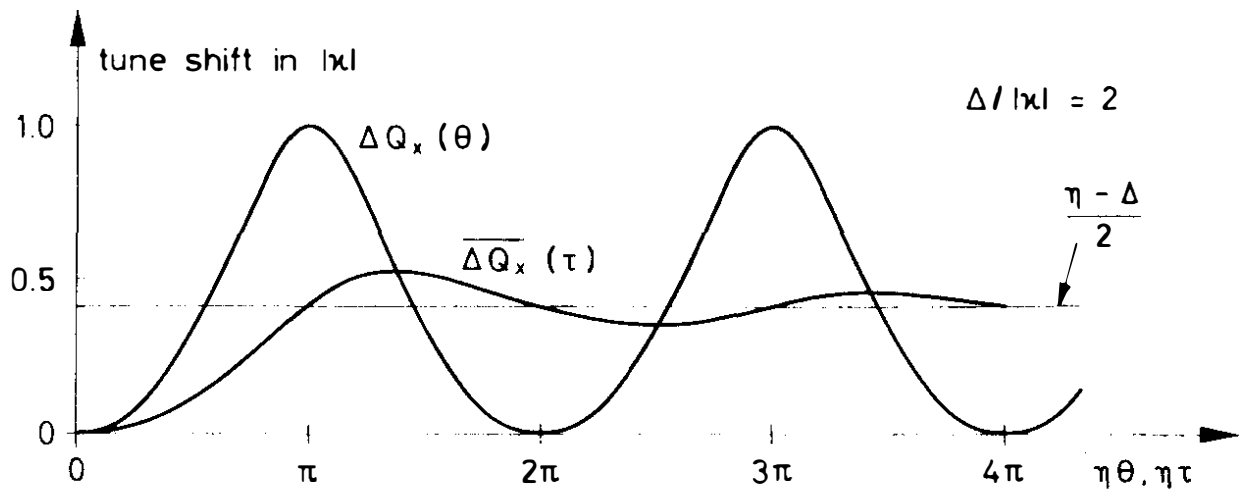FIGURE 3 - A definition of the average horizontal tune shift $\overline{\Delta Q}_x(\tau)$.

FIGURE 4 - The horizontal tune shift ΔQ_x and the average tune shift $\overline{\Delta Q}_x$ for the two cases of $\Delta/|\kappa| = 2$ and 1.
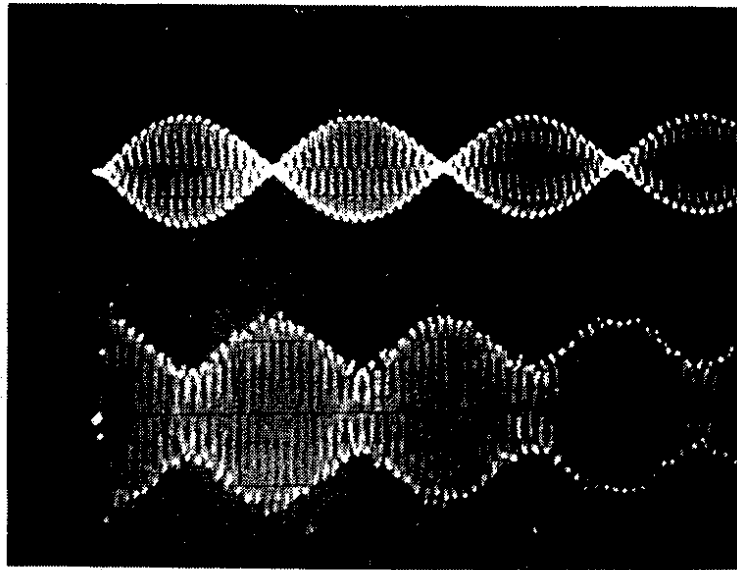
FIGURE 5 - A photograph of the filter output signals of
the Q-meter. The upper trace denotes the ver-
tical signal and the lower trace the horizontal
signal. The time base is 0.2 ms/div. From the
photograph we have $|\Delta|$ = 3.11 × 10^{-3} and
$|\kappa|$ = 2.92 × 10^{-3}, which correspond to the data
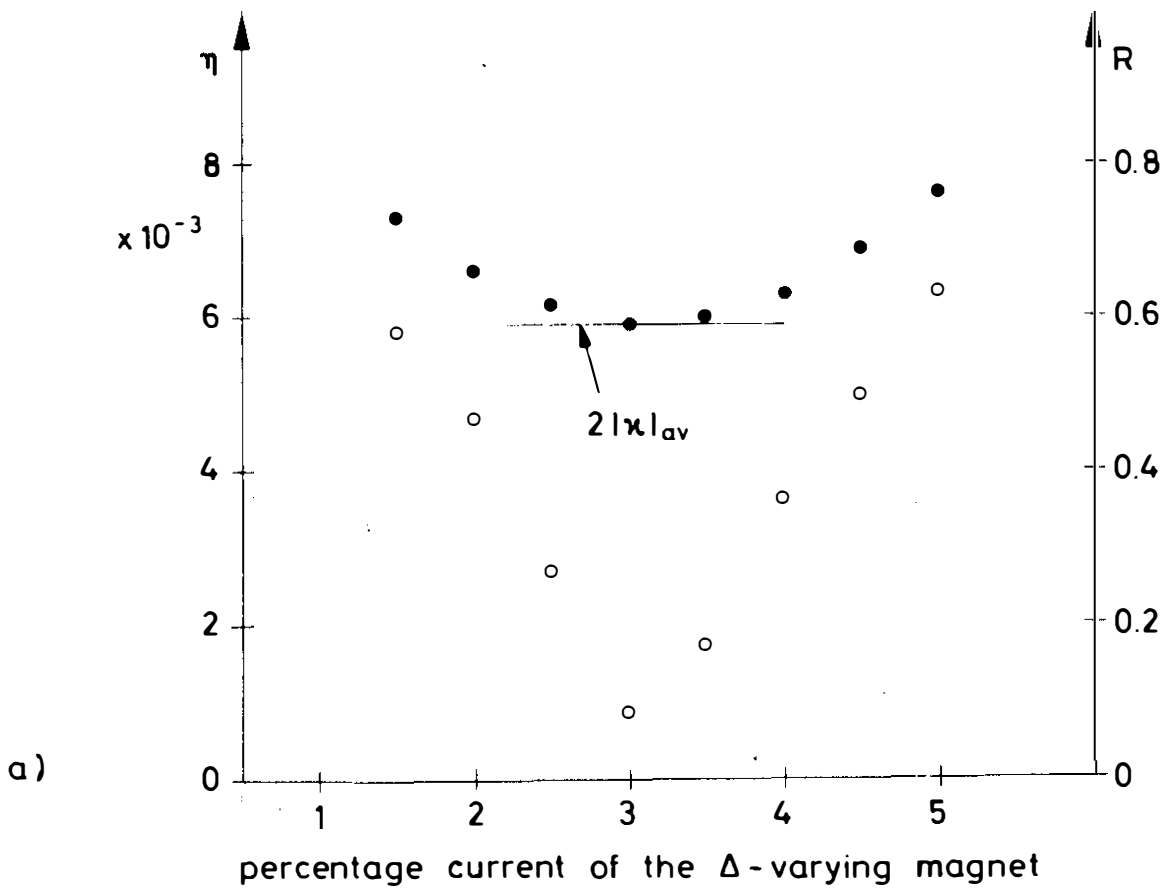shown in the second column of Table 1.
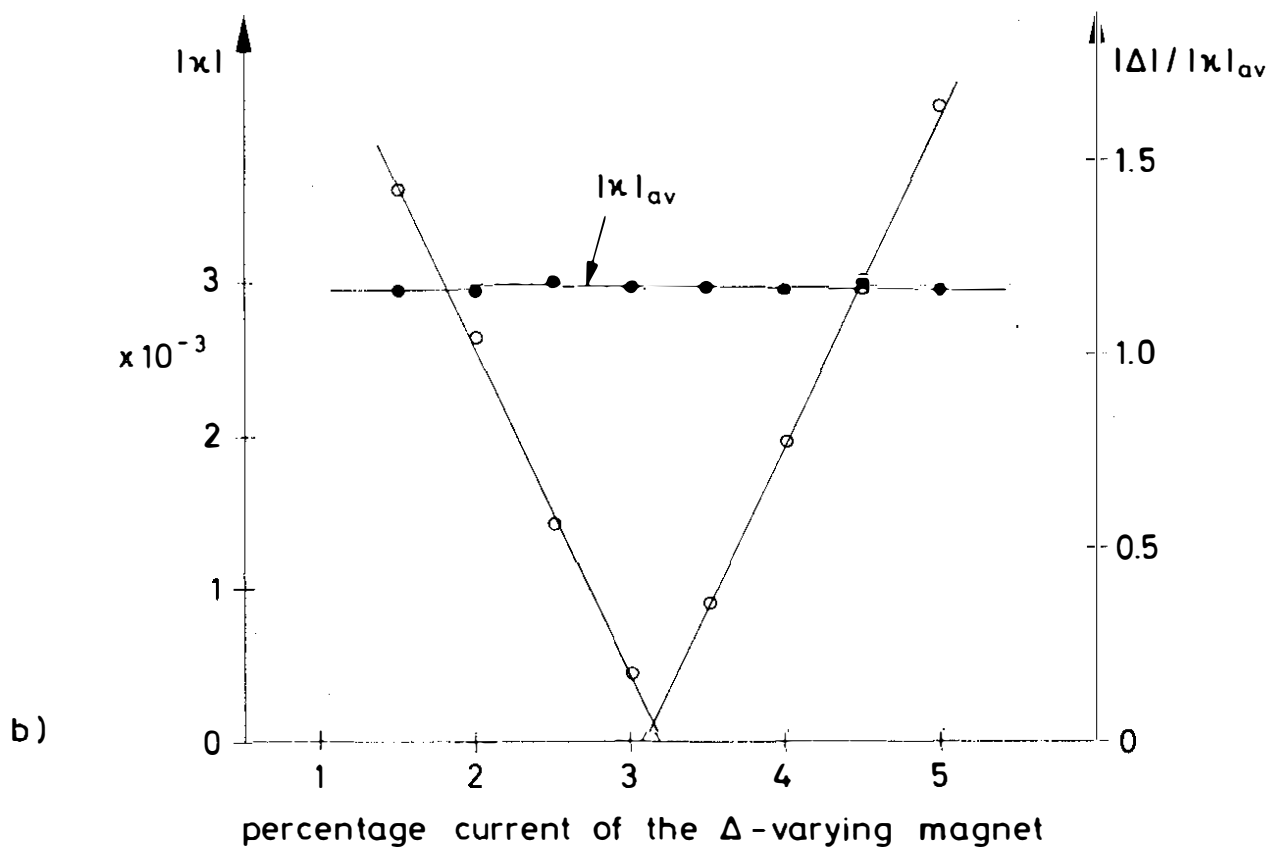
FIGURE 6 - In Figure 6.a), dots denote η and circles R.

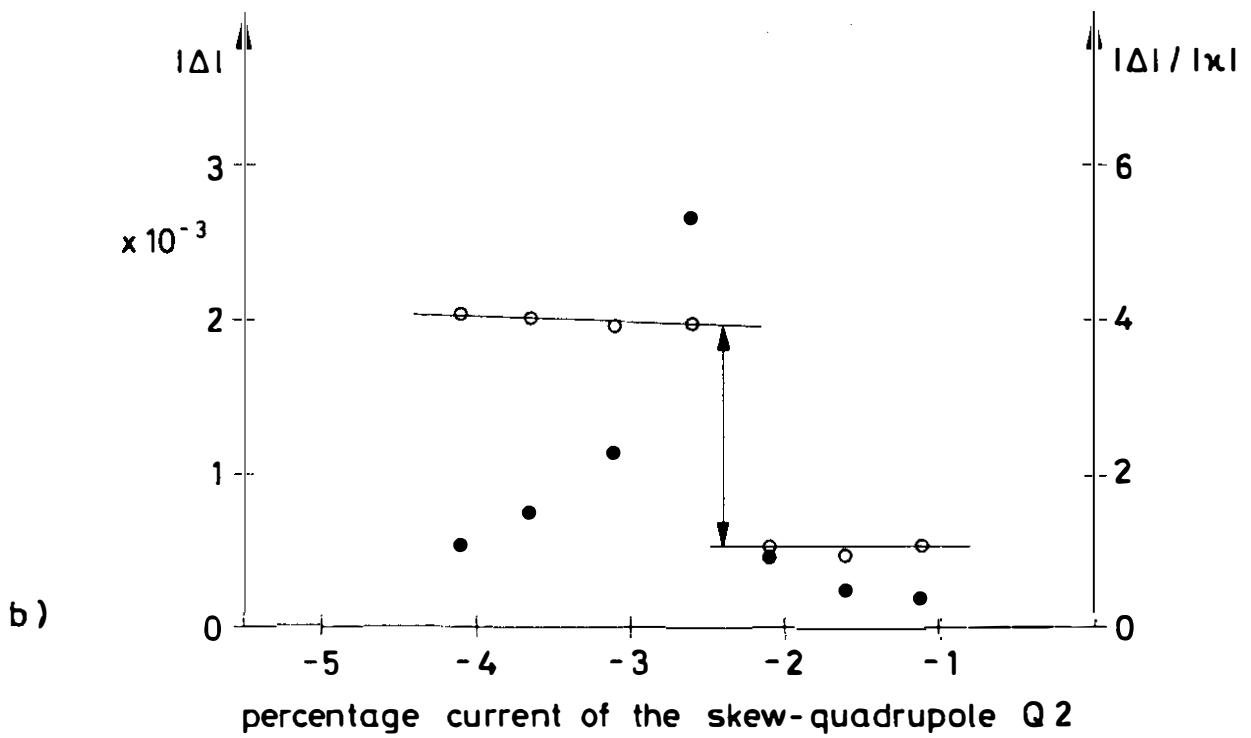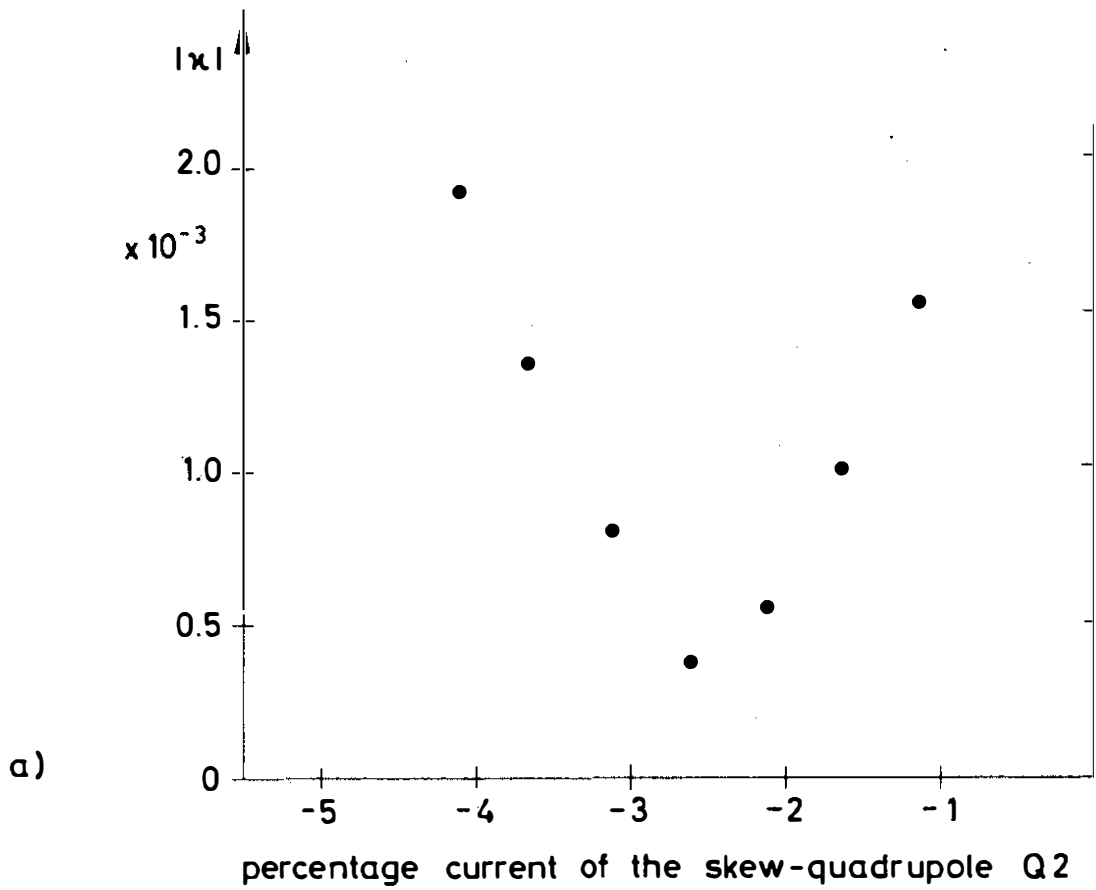In Figure 6.b), dots denote $|\kappa|$ and circles $|\Delta|/|\kappa|_{av}$.

FIGURE 7 — The coupling coefficient $|\kappa|$ and $|\Delta|$ when the strength of
the skew quadrupole Q2 is varied.
In Figure 7.b), dots denote $|\Delta|/|\kappa|$ and circles $|\Delta|$.

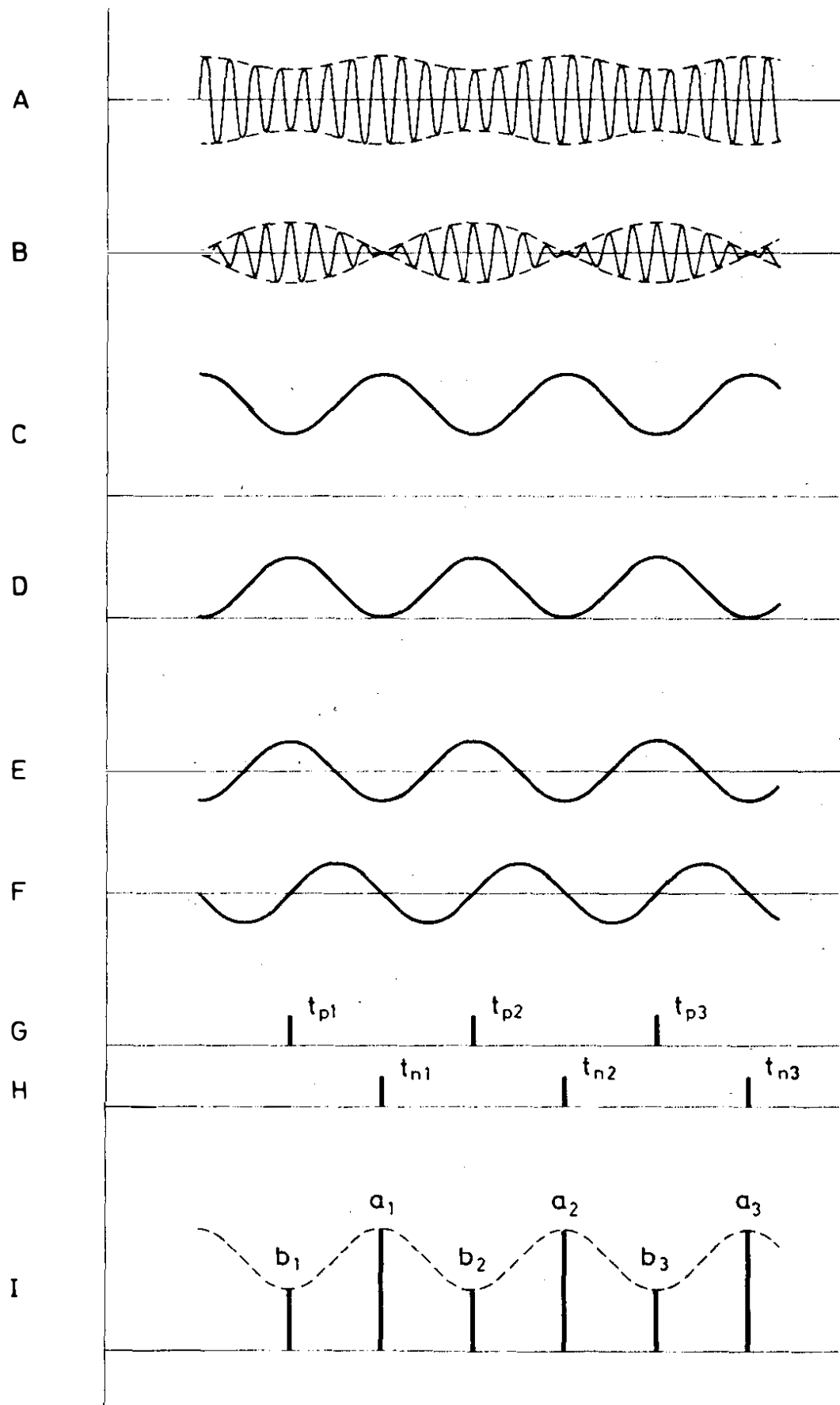FIGURE 8.a) - A block diagram of the electronic system for measuring the coupling coefficient

FIGURE 8.b) - Waveforms of signals at some points in the electronic
system.