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FOCUSING WITH QUADRUPOLES, DOUBLETS AND TRIPLETS

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INTRODUCTION

This paper partially reproduces a series of lectures given at the University of Rennes (Faculté des Sciences) during the year 1963-64.

Although some results are new either in fact or in presentation, no claim to originality is being made; on the contrary, free use has been made of the existing literature.

It was thought worthwhile to write down these notes during a visit by the author to the NPA Division, CERN. It is a great pleasure to acknowledge the facilities made available during this stay by Dr. C.A. Ramm, Leader of the NPA Division.

SUMMARY

1. GENERAL PROPERTIES

1.1 Matrix of a drift space

Consider a field-free region limited by two planes normal to the optical axis (Fig. 1). There are no forces in this region and, therefore, the equation of a ray is

$$x = Az + B \quad . \tag{1}$$

Taking into account the initial conditions, this may be put in the form

$$\begin{aligned} x &= x_0 + L\, x'_0 \\ x' &= \qquad\; x'_0 \end{aligned} \tag{2}$$

where $L = z - z_0$ represents the length of the drift space. In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{Vmatrix} 1 & L \\ 0 & 1 \end{Vmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} . \tag{3}$$

Consequently, the matrix of a drift space is

$$m_d = \begin{Vmatrix} 1 & L \\ 0 & 1 \end{Vmatrix} . \tag{4}$$

1.2 Matrix of a lens system

In linear theory, any lens system may be described by a matrix

$$m_\ell = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \tag{5}$$

where the elements a,b,c, and d depend on the geometry of the system and its physical parameters (focusing strengths), but not on the incoming or outgoing ray.

If the lens system is preceded by a drift length p and followed by another drift length q, the complete transfer from the "object point" to the "image point" writes

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{Vmatrix} 1 & q \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \times \begin{Vmatrix} 1 & p \\ 0 & 1 \end{Vmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} . \qquad (6)$$

Here, p stands for the object distance which will be counted positive towards the left starting from the entrance plane, and q stands for the image distance q which will be counted positive towards the right starting from the exit plane (Fig. 2).

1.3 Optical characteristics

Multiplying out Eq. (6) we find

$$\begin{aligned} x &= (a + cq)x_0 + [ap + b + q(cp + d)]x'_0 \\ x' &= cx_0 + (cp + d)x'_0 . \end{aligned} \qquad (7)$$

These relations reveal all first-order characteristics of the optical system:

i) Object-image conjugation

We must have $x = 0$ for $x_0 = 0$ and arbitrary x'_0 ; consequently, the conjugation relation writes

$$cpq + ap + dq + b = 0 . \qquad (8)$$

ii) Image position

Eq. (8) immediately gives

$$q = - \frac{ap + b}{cp + d} . \qquad (9)$$

iii) Magnification

The conjugation relation being satisfied, one has

$$g = \frac{x}{x_0} = a + cq = a - c \frac{ap + b}{cp + d}$$

i.e.

$$g = \frac{\Delta}{cp + d}$$

where

$$\Delta = ac - bd \tag{10}$$

is the determinant of the transfer matrix. In what follows, we shall assume $\Delta = 1$, this condition being always satisfied when the space on both sides of the lens combination is at the same potential. The magnification then becomes

$$y = \frac{1}{cp + d} \, . \tag{11}$$

iv) Focal planes

An incoming ray parallel to the axis will, after going through the lens system intersect the axis at a point whose distance from the exit plane is given by

$$F_i = - \frac{a}{c} \, . \tag{12}$$

Similarly, an outgoing ray which is parallel to the axis originates from a point whose distance from the entry plane is given by

$$F_o = - \frac{d}{c} \, . \tag{13}$$

v) Principal planes

These planes are conjugate to each other and defined by unit magnification.
From

$$a + cq = 1$$

we infer

$$H_i = \frac{1 - a}{c} \tag{14}$$

whereas

$$cp + d = 1$$

leads to

$$H_o = \frac{1-d}{c} . \qquad (15)$$

vi) Focal distances

The focal distance which is the reciprocal value of the lens strength is defined as the distance between the focus and the corresponding principal plane. Therefore, we have for the image side

$$f_i = -\frac{a}{c} - \frac{1-a}{c} = -\frac{1}{c} , \qquad (16)$$

and for the object side

$$f_o = -\frac{d}{c} - \frac{1-d}{c} = -\frac{1}{c} . \qquad (17)$$

The equality

$$f_o = f_i = f$$

is essentially due to the fact that $\Delta = 1$.

vii) Antiprincipal planes

These planes are again conjugate to each other and defined by negative unit magnification.
From

$$a + cq = -1$$

we have

$$\bar{H}_i = -\frac{1+a}{c} \qquad (18)$$

whereas

$$cp + d = -1$$

gives

$$\bar{H}_o = -\frac{1+d}{c} . \qquad (19)$$

In the case of a symmetric lens system, a = d, and these relations become

conjugation	$cpq + a(p+q) + b = 0$	(20)
image distance	$q = -\frac{ap+b}{cp+a}$	(21)
magnification	$g = \frac{1}{cp+a}$	(22)
focal planes	$F_i = F_o = -\frac{a}{c}$	(23)
principal planes	$H_i = H_o = \frac{1-a}{c}$	(24)
focal distance	$f_i = f_o = -\frac{1}{c}$	(25)
antiprincipal planes	$\bar{H}_i = \bar{H}_o = -\frac{1+a}{c}$	(26)

At the risk of repetition, it might be worthwhile to point out again that all image characteristics, i, are counted positive towards the right starting from the exit plane of the lens combination, whereas all object parameters, o, are counted positive towards the left starting from the entrance plane.

1.4 Ideal and practical focusing

An efficient focusing system would require, in the electric case, a purely radial field and, in the magnetic case, a purely tangential field (Fig. 3). In both cases, the restoring force (directed towards the axis) should be proportional to the distance from the axis for optimum focusing conditions.

In the standard axially-symmetric electric lens, $\vec{E}$ has only a small radial component and similarly, in solenoidal focusing devices, $\vec{B}$ has only a small tangential component. In both cases, the focusing forces are weak.

The situation displayed in Fig. 3, with restoring forces proportional to the distance from the axis, has not yet been achieved. However, a way of focusing far superior to the old devices based on axial symmetry, was proposed by Corant, Livingston and Snyder in 1952: they were the first to show that, associating a convergent lens and a divergent lens, strong overall focusing might be achieved in electron and corpuscular optics. The lenses are either electric or magnetic quadrupoles and, in both cases, the field increases almost linearly with the distance from the axis. The price one has to pay for this improvement lies in the appearance of a certain amount of astigmatism or, more generally, the deterioration of the optical quality of the image. However, in many cases and especially in beam transport systems associated with accelerating machines, strong focusing is far more important than image quality.

2. THE QUADRUPOLE LENS

2.1 Field configuration

In both the electric and magnetic lenses, the focusing field is obtained by means of four poles. We shall confine ourselves to the magnetic case, breakdown difficulties preventing the use of electrostatic lenses at energies much higher than 10 MeV. Figure 4 shows the field configuration in a 4-pole geometry and Fig. 5 gives the direction of the B_x and B_y components of the field in the four quadrants. The beam is supposed to travel in the z direction, perpendicular to the plane of the paper. It is readily seen that B_x changes sign upon traversal of the x0z plane, whereas B_y changes sign when passing through the y0z plane.

To evaluate the field distribution quantitatively, one must solve Laplace's equation $\nabla^2 V = 0$ with the given boundary conditions. This is easily done if the 4 poles are equilateral hyperbolas. In this case, one finds for the potential distribution

$$V = -Gxy \qquad (27)$$

with G constant. The field components are, therefore, proportional to the distance from the axis

$$B_x = -\frac{\partial V}{\partial x} = Gy \qquad B_y = -\frac{\partial V}{\partial y} = Gx \, . \qquad (28)$$

On the other hand, the magnetic scalar potential is

$$V = \mu_0 n I \qquad (29)$$

where nI is the number of exciting-ampere turns per pole; if R_0 denotes the aperture of the quadrupole (Fig. 6), one has

$$G = \mu_0 \frac{2nI}{R_0^2} . \qquad (30)$$

Practically, the hyperbolas will never go to infinity and, therefore, higher-order terms will appear in the expression of the field components; however, their contribution will be quite small in general and the field gradient

$$G = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} \qquad (31)$$

can be considered as a constant to a fairly high degree of accuracy.

2.2 Forces acting on the beam

The magnetic force on a particle of charge q is given by

$$\vec{F} = q\vec{v} \times \vec{B} . \qquad (32)$$

The transverse velocity components being very small compared to the longitudinal velocity v, the components of the force acting on a particle may be written

$$\begin{aligned} F_x &= - q v B_y \\ F_y &= q v B_x . \end{aligned} \qquad (33)$$

Figure 7 shows the direction of the force components in the four quadrants. It is seen that wherever the particle happens to be, the x component will tend to pull it towards the y axis and the y component will tend to push it away from the x axis. Focusing will therefore be achieved in the x plane, whereas defocusing will take place in the y plane.

If one reverses the polarities in Fig. 7, the situation will be reversed, i.e. the quadrupole will be focusing in the y plane and defocusing in the x plane.

2.3 Orbits in the quadrupole

If one excepts the small energy transfer from longitudinal to transverse motion, the longitudinal velocity v stays constant during the traversal of the quadrupole, for there is no accelerating force to act on the particle. The equations of motion may therefore be written

$$m \frac{d^2x}{dt^2} = F_x \qquad m \frac{d^2y}{dt^2} = F_y \tag{34}$$

where m is the relativistic mass of the particle

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{35}$$

(m_0 = rest mass; c = velocity of light) which is constant here as v is constant.

Replacing the force components in Eq. (34) we get

$$m \frac{d^2x}{dt^2} = - q\, v\, B_y = - q\, v\, Gx \qquad m \frac{d^2y}{dt^2} = q\, v\, B_x = q\, v\, Gy\ . \tag{36}$$

It is usually more interesting to know the equation of the trajectories x = x(z), y = y(z) than the position of the particle as a function of time. Therefore, using (for v constant) the transformation relations

$$\frac{d^2x}{dt^2} = v^2 \frac{d^2x}{dz^2} \qquad \frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dz^2} \tag{37}$$

we can write for the differential equations of the orbit

$$\frac{d^2x}{dz^2} + \frac{qG}{mv}x = 0$$
$$\frac{d^2y}{dz^2} - \frac{qG}{mv}y = 0 \ . \tag{38}$$

Putting

$$k^2 = \frac{qG}{mv} = \frac{qG}{p} = \frac{G}{Br}$$

where p is the (relativistic) momentum of the particle and Br is its magnetic rigidity (momentum per unit charge), we finally have for the equations of motion

$$\frac{d^2x}{dz^2} + k^2x = 0$$
$$\frac{d^2y}{dz^2} - k^2y = 0 \ . \tag{39}$$

The solutions can be written immediately in the form

$$x = a \cos kz + b \sin kz$$
$$y = c \operatorname{ch} kz + d \operatorname{sh} kz \ . \tag{40}$$

2.4 Transfer matrices of a quadrupole

Associating to the equation of the orbit in the focusing plane

$$x = a \cos kz + b \sin kz \tag{41}$$

the relation one obtains by differentiation

$$x' = \frac{dx}{dz} = - ak \sin kz + bk \cos kz \ , \tag{42}$$

one can easily express the constants a and b in terms of the initial conditions x_0, x'_0 which specify the position and the slope of the ray entering the quadrupole. If we denote the length of the quadrupole by L, we then obtain

$$x = x_0 \cos kL + \frac{x'_0}{k} \sin kL$$
$$x' = -x_0 k \sin kL + x'_0 \cos kL \; . \qquad (43)$$

A similar calculation for the defocusing plane gives

$$y = y_0 \operatorname{ch} kL + \frac{y'_0}{k} \operatorname{sh} kL$$
$$y' = y_0 k \operatorname{sh} kL + y'_0 \operatorname{ch} kL \; . \qquad (44)$$

Replacing the dimensionless quantity kL by ϑ

$$kL = \vartheta \; , \qquad (45)$$

the transfer matrices can be written in the form

$$T_c = \left\| \begin{matrix} \cos \vartheta & \frac{1}{k} \sin \vartheta \\ -k \sin \vartheta & \cos \vartheta \end{matrix} \right\| \qquad (46)$$

for the convergent or focusing plane and

$$T_d = \left\| \begin{matrix} \operatorname{ch} \vartheta & \frac{1}{k} \operatorname{sh} \vartheta \\ k \operatorname{sh} \vartheta & \operatorname{ch} \vartheta \end{matrix} \right\| \qquad (47)$$

for the divergent or defocusing plane.

2.5 Optical properties of the quadrupole

All first-order optical properties of the quadrupole can be derived from these matrices. It is sufficient to apply our general relations (20) to (26) noting that, for both matrices, $a = d$.

2.5.1 Focal planes

From Eq. (23) we have for the position of the foci

$$F_{ic} = F_{oc} = F_c = \frac{1}{k} \cot \vartheta$$
$$F_{id} = F_{od} = F_d = -\frac{1}{k} \coth \vartheta \ . \qquad (48)$$

2.5.2 Focal distances

From Eq. (25) we get

$$f_c = \frac{1}{k \sin \vartheta} \qquad f_d = -\frac{1}{k \,\mathrm{sh}\, \vartheta} \qquad (49)$$

which can be written in non-dimensional form

$$f_c/L = \frac{1}{\vartheta \sin \vartheta} \qquad f_d/L = -\frac{1}{\vartheta \,\mathrm{sh}\, \vartheta} \ . \qquad (50)$$

The relations (49) and (50) display the strong astigmatic properties of the quadrupole and suggest the use of quadrupole doublets, triplets or multiplets to achieve some amount of stigmatism. Moreover, it is readily seen that a quadrupole can be defocusing in its focusing plane but can never be focusing in its defocusing plane. In the vast majority of cases, however, $\vartheta < \pi/2$ and the intersection of the ray with the axis occurs outside the quadrupole.

2.5.3 Principal planes

These are given by Eq. (24)

$$H_{ic} = H_{oc} = H_c = -\frac{1 - \cos \vartheta}{k \sin \vartheta}$$
$$H_{id} = H_{od} = H_d = \frac{1 - \mathrm{ch}\, \vartheta}{k \,\mathrm{sh}\, \vartheta} \ . \qquad (51)$$

2.5.4 Image distances

Equation (21) shows that

$$q_c = \frac{p \cos \vartheta + (1/k) \sin \vartheta}{pk \sin \vartheta - \cos \vartheta}$$

$$q_d = - \frac{p \operatorname{ch} \vartheta + (1/k) \operatorname{sh} \vartheta}{pk \operatorname{sh} \vartheta + \operatorname{ch} \vartheta} . \tag{52}$$

It is again possible to rewrite these relations in non-dimensional form which makes them more suitable for graphical representation.

Putting

$$P = \frac{p}{L} \qquad Q = \frac{q}{L} \tag{53}$$

we have

$$Q_c = \frac{1}{\vartheta} \frac{P\vartheta \cot \vartheta + 1}{P\vartheta - \cot \vartheta}$$

$$Q_d = - \frac{1}{\vartheta} \frac{P\vartheta \coth \vartheta + 1}{P\vartheta + \coth \vartheta} . \tag{54}$$

2.5.5 Magnifications

Using the same notations, one finds from Eq. (11)

$$g_c = \frac{1}{\cos \vartheta - P\vartheta \sin \vartheta}$$

$$g_d = \frac{1}{P\vartheta \operatorname{sh} \vartheta + \operatorname{ch} \vartheta} . \tag{55}$$

2.6. Thin-lens approximation

2.6.1 First definition

A quadrupole is considered to be thin if its longitudinal spacial extension is small compared to its focal length, i.e. if

$$L \leq \frac{1}{k \operatorname{sh} \vartheta} < \frac{1}{k \sin \vartheta} . \tag{56}$$

This leads to the inequality

$$kL \ll 1 . \tag{57}$$

Under these conditions, the transfer matrices T_c and T_d become

$$T_c = \begin{Vmatrix} 1 & 0 \\ -\delta & 1 \end{Vmatrix} \tag{58}$$

$$T_d = \begin{Vmatrix} 1 & 0 \\ \delta & 1 \end{Vmatrix} \tag{59}$$

where we have put

$$\delta = k^2 L = \frac{G}{Br} L . \tag{60}$$

The focal distances are then

$$f_o = f_c = -f_d = \frac{1}{\delta} = \frac{Br}{GL} . \tag{61}$$

2.6.2 Second definition

A quadrupole is considered to be thin if it does not modify the position of the trajectory but only its slope (Fig. 8); therefore, it acts as a prism (impulse approximation).

Quantitatively, this means that in the equations of motion

$$\begin{aligned} \frac{d(dx/dz)}{dz} + \frac{Gx}{Br} &= 0 \\ \frac{d(dy/dz)}{dz} - \frac{Gy}{Br} &= 0 , \end{aligned} \tag{62}$$

x and y can be considered as constants during the traversal of the quadrupole. Upon integration, one finds for the change in slope

$$\Delta(dx/dz) = -\frac{GL}{Br}\,x$$

$$\Delta(dy/dz) = +\frac{GL}{Br}\,y \tag{63}$$

and the focal distances (61) follow immodiately.

2.6.3 Third definition

A lens can be considered as being thin to the extent that the distance between its principal planes can be considered small compared to it focal length. From the formulae given above, it follows that this definition is equivalent to the other two.

2.6.4 Second-order approximation

In many cases, the first-order approximation is not sufficient. One can then use a series expansion of cos ϑ and ch ϑ and replace them in the formulae giving the optical characteristics.

Carrying out the calculations, one finds to the second order

$$f_c = f_o + \frac{L}{6}$$

$$f_d = -f_o + \frac{L}{6} \tag{64}$$

$$H_{ic} = H_{oc} = H_c = -\frac{L}{2}\left(1 + \frac{L}{12\,f_o}\right)$$

$$H_{id} = H_{od} = H_d = -\frac{L}{2}\left(1 - \frac{L}{12\,f_o}\right). \tag{65}$$

Figure 9 shows the position of the image principal planes in the two basic directions.

Obviously if, in the formulae given above, L can be neglected with respect to f_o, one again finds the simple expressions of the thin-lens theory.

The second-order expressions (64), as well as the thick-lens formulae (49), show that in its focusing plane, a quadrupole is less focusing than the ideal thin lens, whereas in its defocusing plane, the defocusing is stronger than that displayed by the thin-lens approximation.

2.7 Equivalent representations

Any quadrupole can be replaced by a thin lens having a drift space on each side (Fig. 10). Writing out the equivalence of the transfer matrices in the focusing plane

$$\left\| \begin{matrix} \cos\vartheta & \frac{1}{k}\sin\vartheta \\ -k\sin\vartheta & \cos\vartheta \end{matrix} \right\| = \left\| \begin{matrix} 1 & s_c \\ 0 & 1 \end{matrix} \right\| \times \left\| \begin{matrix} 1 & 0 \\ -\frac{1}{f_c} & 1 \end{matrix} \right\| \times \left\| \begin{matrix} 1 & s_c \\ 0 & 1 \end{matrix} \right\| \qquad (66)$$

one finds

$$\frac{1}{f_c} = k \sin\vartheta$$

$$s_c = \frac{1-\cos\vartheta}{k\sin\vartheta} \;. \qquad (67)$$

Whereas the focusing strength of the equivalent lens is the same as that of the original lens, the associated drift length is larger than L/2.

By similar considerations, one finds in the defocusing plane

$$\frac{1}{f_d} = -k \,\mathrm{sh}\,\vartheta$$

$$s_d = \frac{\mathrm{ch}\,\vartheta - 1}{k\,\mathrm{sh}\,\vartheta} \qquad (68)$$

and, therefore, the virtual drift length is smaller than L/2.

In some applications it is useful to replace an actual quadrupole by two thin lenses separated by a drift space of length s (Fig. 11). Writing out again the equality of the transfer matrices, one finds

$$\frac{1}{f_c} = k\,\frac{1 - \cos\vartheta}{\sin\vartheta}$$

$$s_c = \frac{1}{k}\sin\vartheta \qquad (69)$$

for the focusing plane, and

$$\frac{1}{f_d} = -k\,\frac{\text{ch}\,\vartheta - 1}{\text{sh}\,\vartheta}$$

$$s_d = \frac{1}{k}\,\text{sh}\,\vartheta \qquad (70)$$

for the defocusing plane.

3. THE DOUBLET

3.1 General properties

Starting from a real object, a simple quadrupole gives a real image point in its converging plane and a virtual image in its diverging plane. However, the practical use of a focusing system requires, in general, a real image in both planes. The simplest device achieving this requirement is a doublet (Fig. 12), i.e. a set of two quadrupoles of opposite polarity separated by a drift space.

Figure 13 shows the envelope of the beam in the plane where the first quadrupole is diverging and the second is converging, whereas Fig. 14 gives the shape of the same envelope in the plane where the first quadrupole is converging and the second is diverging. The over-all effect is that of an AG lens whose focusing strength is generally different in the two basic planes. Although Q_{dc} is different from Q_{cd}, bidirectional focusing is possible. It is indeed seen that in both cases the excursion of the particle is greater in the quadrupole which is focusing; the restoring force being proportional to the excursion of the particle, the focusing effect will win over the defocusing effect.

It has been shown that the focusing action of a quadrupole is weaker than the thin-lens approximation would indicate, whereas the defocusing action of a quadrupole is stronger than that indicated by the same approximation. The over-all focusing effect of a doublet being due to the difference of these partial effects, it follows that the converging power of a doublet might be substantially smaller than the value given by the thin-lens approximation. The thin-lens theory is, however, quite useful in determining general properties of doublets, especially in high-energy work where the focusing power is small, and also in calculating some secondary effects like aberrations.

3.2 The antisymmetric doublet

We shall call a doublet antisymmetric if its component quadrupoles are of equal length and gradient. If d denotes the distance between the effective end planes of the quadrupoles (Fig. 15) the transfer matrices in the two basic planes are

$$T_{cd} = \begin{Vmatrix} \mathrm{ch}\,\vartheta & \frac{1}{k}\,\mathrm{sh}\,\vartheta \\ k\,\mathrm{sh}\,\vartheta & \mathrm{ch}\,\vartheta \end{Vmatrix} \times \begin{Vmatrix} 1 & d \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \cos\vartheta & \frac{1}{k}\sin\vartheta \\ -k\sin\vartheta & \cos\vartheta \end{Vmatrix} \qquad (71)$$

$$T_{dc} = \begin{Vmatrix} \cos\vartheta & \frac{1}{k}\sin\vartheta \\ -k\sin\vartheta & \cos\vartheta \end{Vmatrix} \times \begin{Vmatrix} 1 & d \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \mathrm{ch}\,\vartheta & \frac{1}{k}\,\mathrm{sh}\,\vartheta \\ k\,\mathrm{sh}\,\vartheta & \mathrm{ch}\,\vartheta \end{Vmatrix} \qquad (72)$$

Carrying out the calculations, we find the two expressions

$$T_{cd} = \begin{Vmatrix} \cos\vartheta\,\mathrm{ch}\,\vartheta - \sin\vartheta\,\mathrm{sh}\,\vartheta - dk\sin\vartheta\,\mathrm{ch}\,\vartheta & \frac{1}{k}(\cos\vartheta\,\mathrm{sh}\,\vartheta + \sin\vartheta\,\mathrm{ch}\,\vartheta) + d\cos\vartheta\,\mathrm{ch}\,\vartheta \\ k(\cos\vartheta\,\mathrm{sh}\,\vartheta - \sin\vartheta\,\mathrm{ch}\,\vartheta) - dk^2\sin\vartheta\,\mathrm{sh}\,\vartheta & \cos\vartheta\,\mathrm{ch}\,\vartheta + \sin\vartheta\,\mathrm{sh}\,\vartheta + dk\cos\vartheta\,\mathrm{sh}\,\vartheta \end{Vmatrix} \qquad (73)$$

and

$$T_{dc} = \begin{Vmatrix} \cos\vartheta\,\mathrm{ch}\,\vartheta + \sin\vartheta\,\mathrm{sh}\,\vartheta + dk\cos\vartheta\,\mathrm{sh}\,\vartheta & \frac{1}{k}(\cos\vartheta\,\mathrm{sh}\,\vartheta + \sin\vartheta\,\mathrm{ch}\,\vartheta) + d\cos\vartheta\,\mathrm{ch}\,\vartheta \\ k(\cos\vartheta\,\mathrm{sh}\,\vartheta - \sin\vartheta\,\mathrm{ch}\,\vartheta) - dk^2\sin\vartheta\,\mathrm{sh}\,\vartheta & \cos\vartheta\,\mathrm{ch}\,\vartheta - \sin\vartheta\,\mathrm{sh}\,\vartheta - dk\sin\vartheta\,\mathrm{ch}\,\vartheta \end{Vmatrix} \qquad (74)$$

Comparing the last two transfer matrices and using the general matrix symbolism (5), we infer the following important relations:

$$\begin{aligned} c_{cd} &= c_{dc} = c \\ b_{cd} &= b_{dc} = b \\ a_{cd} &= d_{dc} \\ d_{cd} &= a_{dc} \,. \end{aligned} \tag{75}$$

One matrix is therefore sufficient to describe the antisymmetric doublet.

3.3 Optical properties of the antisymmetric doublet

Applying the general relations (8) to (19) and using relations (75), the optical characteristics of the antisymmetric doublet can readily be derived.

3.3.1 Focal distances

From Eq. (16) we have

$$f_{cd} = f_{dc} = \frac{1}{k(\sin\vartheta \ \mathrm{ch}\ \vartheta - \cos\vartheta \ \mathrm{sh}\ \vartheta + dk \sin\vartheta \ \mathrm{sh}\ \vartheta)} \,. \tag{76}$$

Putting

$$D = \frac{d}{L} \tag{77}$$

we can write this in the non-dimensional form

$$\frac{f_{cd}}{L} = \frac{f_{dc}}{L} = \frac{1}{\vartheta(\sin\vartheta \ \mathrm{ch}\ \vartheta - \cos\vartheta \ \mathrm{sh}\ \vartheta + D\vartheta \sin\vartheta \ \mathrm{sh}\ \vartheta)} \,. \tag{78}$$

3.3.2 Focal planes

Equation (23) gives for the image foci

$$F_{icd} = \frac{\cos\vartheta \ \mathrm{ch}\ \vartheta - \sin\vartheta \ \mathrm{sh}\ \vartheta - dk \sin\vartheta \ \mathrm{ch}\ \vartheta}{k(\sin\vartheta \ \mathrm{ch}\ \vartheta - \cos\vartheta \ \mathrm{sh}\ \vartheta + dk \sin\vartheta \ \mathrm{sh}\ \vartheta)} \tag{79}$$

$$F_{idc} = \frac{\cos\vartheta\ ch\ \vartheta + \sin\vartheta\ sh\ \vartheta + dk\ \cos\vartheta\ sh\ \vartheta}{k(\sin\vartheta\ ch\ \vartheta - \cos\vartheta\ sh\ \vartheta + dk\ \sin\vartheta\ sh\ \vartheta)} . \tag{79}$$

These relations can also be written in the non-dimensional form

$$F_{icd}/L = \frac{1}{\vartheta} \cdot \frac{1 - \cot\vartheta\ \coth\vartheta + D\vartheta\ \coth\vartheta}{\cot\vartheta - \coth\vartheta - D\vartheta}$$

$$F_{idc}/L = \frac{1}{\vartheta} \cdot \frac{1 + \cot\vartheta\ \coth\vartheta + D\vartheta\ \cot\vartheta}{\coth\vartheta - \text{cotg}\ \vartheta + D\vartheta} \tag{80}$$

or by using Eq. (48)

$$F_{icd} = \frac{F_d(F_c - d) + \frac{1}{k^2}}{F_c + F_d - d}$$

$$F_{idc} = \frac{F_c(F_d - d) - \frac{1}{k^2}}{F_c + F_d - d} . \tag{81}$$

The last two relations show that it is not possible to achieve coincidence of the two foci. $F_{icd} = F_{idc}$ would indeed imply

$$d = \frac{2}{k^2(F_d - F_c)} \tag{82}$$

which is not possible, F_d being positive and F_c being generally positive. In exceptional cases, F_c could be negative so that Eq. (82) would be satisfied; however, these cases do not seem to be of interest in beam transport systems for they would imply very strong focusing and intersection of the ray with the axis occurring inside the lens.

3.3.3 Principal planes

From Eq. (14) we have

$$H_{icd}/L = \frac{1 - \cos\vartheta\ ch\ \vartheta + \sin\vartheta\ sh\ \vartheta + D\vartheta\ \sin\vartheta\ ch\ \vartheta}{\vartheta(\cos\vartheta\ sh\ \vartheta - \sin\vartheta\ ch\ \vartheta - D\vartheta\ \sin\vartheta\ sh\ \vartheta)}$$

$$H_{idc}/L = \frac{1 - \cos\vartheta\ ch\ \vartheta - \sin\vartheta\ sh\ \vartheta - D\vartheta\ \cos\vartheta\ sh\ \vartheta}{\vartheta(\cos\vartheta\ sh\ \vartheta - \sin\vartheta\ ch\ \vartheta - D\vartheta\ \sin\vartheta\ sh\ \vartheta)} . \tag{83}$$

Again, it is not possible to bring into coincidence the principal planes in the two basic directions, for $H_{icd} = H_{idc}$ would demand

$$2 \sin \vartheta \ \mathrm{sh}\ \vartheta + D\vartheta \left(\cos \vartheta \ \mathrm{sh}\ \vartheta + \sin \vartheta \ \mathrm{ch}\ \vartheta\right) = 0 \tag{84}$$

which, expressed in other terms, is exactly the same relation as (82).

3.3.4 Magnifications

From Eq. (11) we have

$$\begin{aligned} \frac{1}{g_{cd}} &= P\vartheta \left(\cos \vartheta \ \mathrm{sh}\ \vartheta - \sin \vartheta \ \mathrm{ch}\ \vartheta - D\vartheta \sin \vartheta \ \mathrm{sh}\ \vartheta\right) \\ &\quad + \cos \vartheta \ \mathrm{ch}\ \vartheta + \sin \vartheta \ \mathrm{sh}\ \vartheta + D\vartheta \cos \vartheta \ \mathrm{sh}\ \vartheta \\ \frac{1}{g_{dc}} &= P\vartheta \left(\cos \vartheta \ \mathrm{sh}\ \vartheta - \sin \vartheta \ \mathrm{ch}\ \vartheta - D\vartheta \sin \vartheta \ \mathrm{sh}\ \vartheta\right) \\ &\quad + \cos \vartheta \ \mathrm{ch}\ \vartheta - \sin \vartheta \ \mathrm{sh}\ \vartheta - D\vartheta \sin \vartheta \ \mathrm{ch}\ \vartheta \ . \end{aligned} \tag{85}$$

It is seen that in an antisymmetric doublet

$$g_{cd} < g_{dc} \ . \tag{86}$$

The equality of the two magnifications would again lead to Eq. (82) or (84).

3.3.5 Image position

Using Eq. (9) the image positions may be written

$$\begin{aligned} q_{cd} &= F_{icd} + \frac{F_{icd} F_{idc} + t}{p - F_{idc}} \\ q_{dc} &= F_{idc} + \frac{F_{icd} F_{idc} + t}{p - F_{icd}} \end{aligned} \tag{87}$$

with

$$t = -\frac{b}{c} = \frac{F_d - F_c + dk^2 F_c F_d}{k^2 (F_c + F_d - d)} \ , \tag{88}$$

noting that the elements b and c are the same for both transfer matrices (73) and (74).

The image positions can also be written in non-dimensional form

$$Q_{cd} = \frac{P\,F_{icd}/L + T}{P - F_{idc}/L}$$

$$Q_{dc} = \frac{P\,F_{idc}/L + T}{P - F_{idc}/L} \tag{89}$$

where

$$T = \frac{t}{L^2} = \frac{1}{\vartheta^2}\,\frac{\cot\vartheta + \coth\vartheta + D\vartheta\cot\vartheta\coth\vartheta}{D\vartheta + \coth\vartheta - \cot\vartheta}\;. \tag{90}$$

Finally, replacing in Eq. (89), F_{icd}/L, F_{idc}/L and T by their explicit values, the image distances can be written in a form suitable for graphical plotting of universal curves

$$Q_{cd} = \frac{1}{\vartheta}\,\frac{P\vartheta\,(1 - \cot\vartheta\coth\vartheta + D\vartheta\coth\vartheta) - (\cot\vartheta + \coth\vartheta + D\vartheta\cot\vartheta\coth\vartheta)}{P\vartheta\,(\cot\vartheta - \coth\vartheta - D\vartheta) + 1 + \cot\vartheta\coth\vartheta + D\vartheta\cot\vartheta}$$

$$Q_{dc} = \frac{1}{\vartheta}\,\frac{P\vartheta\,(1 + \cot\vartheta\coth\vartheta + D\vartheta\cot\vartheta) + (\cot\vartheta + \coth\vartheta + D\vartheta\cot\vartheta\coth\vartheta)}{P\vartheta\,(\coth\vartheta - \cot\vartheta + D\vartheta) + 1 - \cot\vartheta\coth\vartheta + D\vartheta\coth\vartheta} \tag{91}$$

3.3.5 Characteristics of the object space

The object focal planes and the object principal planes can easily be derived from the characteristics of the image space.

One has, indeed, by virtue of Eq. (75)

$$F_{ocd} = -\frac{d_{cd}}{c} = -\frac{a_{dc}}{c} = F_{idc}$$

$$F_{odc} = -\frac{d_{dc}}{c} = -\frac{a_{cd}}{c} = F_{icd}\;. \tag{92}$$

On the other hand

$$H_{ocd} = \frac{1 - d_{cd}}{c} = \frac{1 - a_{dc}}{c} = H_{idc}$$
$$H_{odc} = \frac{1 - d_{dc}}{c} = \frac{1 - a_{cd}}{c} = H_{icd} \; . \qquad (93)$$

3.3.7 Stigmatic operation of an antisymmetric doublet

In general, $q_{cd} \neq q_{dc}$ and the same object point yields two different image points according to whether one considers the cd plane or the dc plane. However, for a given position of the object point, it is possible to choose the parameters of the lens system so as to obtain a one-to-one correspondence.

The condition of stigmatism writes

$$q_{cd} = q_{dc} = q_o \qquad (94)$$

and writing this out in the two basic planes in terms of the object position

$$q_o = - \frac{a_{cd}\, p_o + b}{c\, p_o + d_{cd}} = - \frac{a_{dc}\, p_o + b}{c\, p_o + d_{dc}} \; . \qquad (95)$$

Taking into account Eq. (75), this becomes

$$- q_o = \frac{a_{cd}\, p_o + b}{c\, p_o + a_{dc}} = \frac{a_{dc}\, p_o + b}{c\, p_o + a_{cd}} = \frac{a_{cd} - a_{dc}}{a_{dc} - a_{cd}}\, p_o \qquad (96)$$

and as $a_{cd} \neq a_{dc}$ we have

$$q_o = p_o \; , \qquad (97)$$

which is also evident for symmetry reasons. Next, for the equation in p_o, we obtain

$$- p_o = \frac{a_{cd}\, p_o + b}{c\, p_o + a_{dc}} \; , \qquad (98)$$

so that

$$p_o^2 + \frac{a_{cd} + a_{dc}}{c} p_o + \frac{b}{c} = 0 \tag{99}$$

or

$$p_o^2 - (F_{cd} + F_{dc}) p_o - t = 0 . \tag{100}$$

The solution is therefore

$$p_o = \frac{F_{cd} + F_{dc}}{2} \pm \sqrt{\frac{(F_{cd} + F_{dc})^2}{4} + t} . \tag{101}$$

For real objects and images, only the upper sign is to be conserved.

In non-dimensional form, Eq. (99) can be written

$$P_o = Q_o = \frac{D\vartheta\,(\coth\vartheta - \cot\vartheta) - 2\cot\vartheta\coth\vartheta \pm \sqrt{[D\vartheta\,(\cot\vartheta + \coth\vartheta) + 2]^2 + \frac{4}{\sin^2\vartheta\,\mathrm{sh}^2\vartheta}}}{2\vartheta(\cot\vartheta - \coth\vartheta - D\vartheta)} . \tag{102}$$

If p_o is given, Eq. (100) gives the inter-lens spacing corresponding to stigmatic operation

$$d = \frac{p_o^2\,(F_c + F_d) - 2p_o\,F_cF_d + (F_c - F_o)/k^2}{p_o^2 - p_o(F_c + F_d) + F_cF_d} \tag{103}$$

or, in non-dimensional form,

$$D\vartheta = \frac{(P_o\vartheta)^2(\cot\vartheta - \coth\vartheta) + 2P_o\vartheta\,\cot\vartheta\coth\vartheta + \cot\vartheta + \coth\vartheta}{(P_o\vartheta)^2 + P_o\vartheta(\coth\vartheta - \cot\vartheta) - \cot\vartheta\coth\vartheta} . \tag{104}$$

3.3.8 Stigmatism and magnifications

For stigmatic operation, there is an important relation between the magnifications in the two basic planes.

One has, indeed,

$$g_{cd} \times g_{dc} = \frac{1}{cp_o + d_{cd}} \cdot \frac{1}{cp_o + d_{dc}}$$

$$= \frac{1}{cp_o + a_{dc}} \cdot \frac{1}{cp_o + a_{cd}}$$

$$= \frac{1}{c^2 p_o^2 + c\,(a_{cd} + a_{dc})p_o + a_{cd}\,a_{dc}}$$

$$= \frac{1}{c\left[cp_o^2 + (a_{cd} + a_{dc})p_o + \frac{a_{cd}\,a_{dc}}{c}\right]} \,.$$

Using Eqs. (75) and (99), this becomes

$$g_{cd} \times g_{dc} = \frac{1}{c\left(-b + \frac{a_{cd}\,a_{dc}}{c}\right)} = \frac{1}{-bc + a_{cd}\,d_{cd}} ,$$

that is

$$g_{cd} \times g_{dc} = 1 \,. \tag{105}$$

The product of the magnifications being equal to unity, there cannot be real stigmatism: a small circle will be deformed into an ellipse for it gets larger in one plane and smaller in the other. This situation of pseudo-stigmatism is not acceptable in electron microscopy, for example, where the requirements on the image quality are high, but it can be quite satisfactory and interesting in beam-transport systems where the source may present some inherent astigmatism.

3.4 The general doublet

The main interest of the general doublet, as compared to the anti-symmetric doublet, lies in the possibility of adjusting independently the excitations of the individual quadrupoles and, therefore, to have one more degree of freedom at hand. In most cases, the lengths of the two elements will be equal, but this does not imply any simplification in the mathematical aspects of the problem.

Figure 16 shows the notations used.

3.4.1 Transfer matrices

One has, for the two basic planes,

$$T_{cd} = \begin{Vmatrix} \mathrm{ch}\,\vartheta_2 & \frac{1}{k_2}\,\mathrm{sh}\,\vartheta_2 \\ k_2\,\mathrm{sh}\,\vartheta_2 & \mathrm{ch}\,\vartheta_2 \end{Vmatrix} \times \begin{Vmatrix} 1 & d \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \cos\vartheta_1 & \frac{1}{k_1}\,\sin\vartheta_1 \\ -k_1\,\sin\vartheta_1 & \cos\vartheta_1 \end{Vmatrix} \tag{106}$$

$$T_{dc} = \begin{Vmatrix} \cos\vartheta_2 & \frac{1}{k_2}\,\sin\vartheta_2 \\ -k_2\,\sin\vartheta_2 & \cos\vartheta_2 \end{Vmatrix} \times \begin{Vmatrix} 1 & d \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \mathrm{ch}\,\vartheta_1 & \frac{1}{k_1}\,\mathrm{sh}\,\vartheta_1 \\ k_1\,\mathrm{sh}\,\vartheta_1 & \mathrm{ch}\,\vartheta_1 \end{Vmatrix}. \tag{107}$$

Carrying out the calculations, one finds for the various matrix elements

$$\begin{aligned}
a_{cd} &= \cos\vartheta_1\,\mathrm{ch}\,\vartheta_2 - d\,k_1\,\sin\vartheta_1\,\mathrm{ch}\,\vartheta_2 - \frac{k_1}{k_2}\,\sin\vartheta_1\,\mathrm{sh}\,\vartheta_2 \\
b_{cd} &= d\,\cos\vartheta_1\,\mathrm{ch}\,\vartheta_2 + \frac{1}{k_1}\,\sin\vartheta_1\,\mathrm{ch}\,\vartheta_2 + \frac{1}{k_2}\,\cos\vartheta_1\,\mathrm{sh}\,\vartheta_2 \\
c_{cd} &= k_2\,\cos\vartheta_1\,\mathrm{sh}\,\vartheta_2 - k_1\,\sin\vartheta_1\,\mathrm{ch}\,\vartheta_2 - d\,k_1\,k_2\,\sin\vartheta_1\,\mathrm{sh}\,\vartheta_2 \\
d_{cd} &= \cos\vartheta_1\,\mathrm{ch}\,\vartheta_2 + d\,k_2\,\cos\vartheta_1\,\mathrm{sh}\,\vartheta_2 + \frac{k_2}{k_1}\,\sin\vartheta_1\,\mathrm{sh}\,\vartheta_2
\end{aligned} \tag{108}$$

$$\begin{aligned}
a_{dc} &= \mathrm{ch}\,\vartheta_1\,\cos\vartheta_2 + d\,k_1\,\mathrm{sh}\,\vartheta_1\,\cos\vartheta_2 + \frac{k_1}{k_2}\,\mathrm{sh}\,\vartheta_1\,\sin\vartheta_2 \\
b_{dc} &= d\,\mathrm{ch}\,\vartheta_1\,\cos\vartheta_2 + \frac{1}{k_1}\,\mathrm{sh}\,\vartheta_1\,\cos\vartheta_2 + \frac{1}{k_2}\,\mathrm{ch}\,\vartheta_1\,\sin\vartheta_2 \\
c_{dc} &= -k_2\,\mathrm{ch}\,\vartheta_1\,\sin\vartheta_2 + k_1\,\mathrm{sh}\,\vartheta_1\,\cos\vartheta_2 - d\,k_1\,k_2\,\mathrm{sh}\,\vartheta_1\,\sin\vartheta_2 \\
d_{dc} &= \mathrm{ch}\,\vartheta_1\,\cos\vartheta_2 - d\,k_2\,\mathrm{ch}\,\vartheta_1\,\sin\vartheta_2 - \frac{k_2}{k_1}\,\mathrm{sh}\,\vartheta_1\,\sin\vartheta_2\;.
\end{aligned} \tag{109}$$

3.4.2 Optical characteristics

All optical characteristics are contained in the matrix elements. One finds for the focal distances

$$f_{cd} = \frac{1}{d k_1 k_2 \sin\vartheta_1 \operatorname{sh}\vartheta_2 + k_1 \sin\vartheta_1 \operatorname{ch}\vartheta_2 - k_2 \cos\vartheta_1 \operatorname{sh}\vartheta_2}$$

$$f_{dc} = \frac{1}{d k_1 k_2 \operatorname{sh}\vartheta_1 \sin\vartheta_2 - k_1 \operatorname{sh}\vartheta_1 \cos\vartheta_2 + k_2 \operatorname{ch}\vartheta_1 \sin\vartheta_2} \qquad (110)$$

and, in general, the focal distance will be different.

For the position of the focal planes one finds

$$F_{icd} = \frac{1}{k_2} \frac{k_1 - k_2 \cot\vartheta_1 \coth\vartheta_2 + d k_1 k_2 \coth\vartheta_2}{k_2 \cot\vartheta_1 - k_1 \coth\vartheta_2 - d k_1 k_2}$$

$$F_{idc} = \frac{1}{k_2} \frac{k_1 + k_2 \coth\vartheta_1 \cot\vartheta_2 + d k_1 k_2 \cot\vartheta_2}{k_2 \coth\vartheta_1 - k_1 \cot\vartheta_2 + d k_1 k_2} . \qquad (111)$$

The abscissae of these foci are different in general but it is possible to bring them into coincidence except in the case $k_1 = k_2$, $\vartheta_1 = \vartheta_2$, which corresponds to the antisymmetric doublet.

By reversing the direction of the incident ray, it is easily seen that the optical characteristics of the object space in the cd(dc) plane can be found simply by using the formulae of the optical characteristics of the image space in the dc(cd) plane and interchanging the subscripts 1 and 2. Therefore, for example,

$$F_{ocd} = \frac{1}{k_1} \frac{k_2 + k_1 \coth\vartheta_2 \cot\vartheta_1 + d k_1 k_2 \cot\vartheta_1}{k_1 \coth\vartheta_2 - k_2 \cot\vartheta_1 + d k_1 k_2}$$

$$F_{odc} = \frac{1}{k_1} \frac{k_2 - k_1 \cot\vartheta_2 \coth\vartheta_1 + d k_1 k_2 \coth\vartheta_1}{k_1 \cot\vartheta_2 - k_2 \coth\vartheta_1 - d k_1 k_2} . \qquad (112)$$

3.4.3 Stigmatic operation

The general condition of stigmatism can be written here

$$-q_o = \frac{a_{cd}\, p_o + b_{cd}}{c_{cd}\, p_o + d_{cd}} = \frac{a_{dc}\, p_o + b_{dc}}{c_{dc}\, p_o + d_{dc}} \tag{113}$$

Putting

$$t_{cd} = -\frac{b_{cd}}{c_{cd}} \qquad t_{dc} = -\frac{b_{dc}}{c_{dc}} \tag{114}$$

Eq. (113) can be written

$$q_o = \frac{F_{icd}\, p_o + t_{cd}}{p_o - F_{ocd}} = \frac{F_{idc}\, p_o + t_{dc}}{p_o - F_{odc}} \quad . \tag{115}$$

The last equality gives the value of p_o for stigmatic operation; the corresponding value of q_o can then be found from

$$(F_{ocd} - F_{odc})\, q_o = (F_{idc} - F_{icd})\, p_o + t_{dc} - t_{cd} \tag{116}$$

which results from Eq. (115).

The product of the two magnifications becomes, under stigmatic conditions,

$$g_{cd} \cdot g_{dc} = \frac{1}{c_{cd}\, p_o + d_{cd}} \cdot \frac{1}{c_{dc}\, p_o + d_{dc}}$$

$$= \frac{1}{c_{cd}\left(p_o + \frac{d_{cd}}{c_{cd}}\right)} \cdot \frac{1}{c_{dc}\left(p_o + \frac{d_{dc}}{c_{dc}}\right)}$$

and this can be written

$$g_{cd} \cdot g_{dc} = \frac{f_{cd}\, f_{dc}}{(p_o - F_{ocd})\,(p_o - F_{odc})} \quad . \tag{117}$$

3.4.4 The doublet in the thin-lens approximation

The thin-lens approximation can be of considerable usefulness as a first step in determining the properties of a doublet. Of course, its validity is restricted to weak lenses, i.e. to high-energy work, but its correctness can be improved by considering each of the quadrupoles composing the doublet as a thick lens and using the thin-lens approximation only in calculating the properties of the combination.

Let s be the separation of the mid-planes of the quadrupoles whose characteristics are L_1, k_1 and L_2, k_2 (Fig. 17). We then have

$$s = d + \frac{L_1 + L_2}{2} \; . \tag{118}$$

On the other hand, it turns out to be convenient to introduce the absolute values of the focusing strengths of the individual quadrupoles and take care of their signs in writing the transfer matrices. Therefore, we shall put

$$\begin{aligned} \frac{1}{f_{1c}} &= k_1 \sin \vartheta_1 \qquad & \frac{1}{f_{1d}} &= k_1 \,\mathrm{sh}\, \vartheta_1 \\ \frac{1}{f_{2c}} &= k_2 \sin \vartheta_2 \qquad & \frac{1}{f_{2d}} &= k_2 \,\mathrm{sh}\, \vartheta_2 \; . \end{aligned} \tag{119}$$

With these sign conventions, the transfer matrices are

$$T_{cd} = \begin{Vmatrix} 1 & 0 \\ \frac{1}{f_{2d}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ -\frac{1}{f_{1c}} & 1 \end{Vmatrix} \tag{120}$$

$$T_{dc} = \begin{Vmatrix} 1 & 0 \\ -\frac{1}{f_{2c}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ \frac{1}{f_{1d}} & 1 \end{Vmatrix} \; . \tag{121}$$

Multiplying out one finds

$$T_{cd} = \begin{Vmatrix} 1 - \frac{s}{f_{1c}} & s \\ -\left(\frac{1}{f_{1c}} - \frac{1}{f_{2d}} + \frac{1}{f_{1c}\, f_{2d}}\right) & 1 + \frac{s}{f_{2d}} \end{Vmatrix} \tag{122}$$

$$T_{dc} = \begin{Vmatrix} 1 + \frac{s}{f_{1d}} & s \\ -\left(-\frac{1}{f_{1d}} + \frac{1}{f_{2c}} + \frac{s}{f_{1d}\, f_{2c}}\right) & 1 - \frac{s}{f_{2c}} \end{Vmatrix} \quad . \tag{123}$$

3.4.5 Optical properties derived from the thin-lens approximation

From the two matrices we first derive the focal distances of the doublet combination

$$\begin{aligned} \frac{1}{f_{cd}} &= \frac{1}{f_{1c}} - \frac{1}{f_{2d}} + \frac{s}{f_{1c}\, f_{2d}} \\ \frac{1}{f_{dc}} &= -\frac{1}{f_{1d}} + \frac{1}{f_{2c}} + \frac{s}{f_{1d}\, f_{2c}} \quad . \end{aligned} \tag{124}$$

Under the conditions where one may apply the thin-lens approximation to the individual quadrupoles, one can drop the subscripts c and d for these elements and write

$$\begin{aligned} \frac{1}{f_{cd}} &= \frac{1}{f_1} - \frac{1}{f_2} + \frac{s}{f_1 f_2} \\ \frac{1}{f_{dc}} &= -\frac{1}{f_1} + \frac{1}{f_2} + \frac{s}{f_1 f_2} \end{aligned} \tag{125}$$

with

$$\frac{1}{f_1} = k_1^2 L_1 \qquad \frac{1}{f_2} = k_2^2 L_2 \quad . \tag{126}$$

It is readily seen that the doublet provides bidirectional focusing if

$$|f_1 - f_2| < s \, . \tag{127}$$

For an antisymmetric doublet

$$f_1 = f_2 = f_0 = \frac{1}{k^2 L} \tag{128}$$

and

$$f_{cd} = f_{dc} = \frac{f_0^2}{s} \, . \tag{129}$$

Considering next the positions of the principal planes, we find from the transfer matrices

$$H_{icd} = -\frac{s}{f_{1c}} f_{cd} \qquad H_{idc} = \frac{s}{f_{1d}} f_{dc} \tag{130}$$

for the image elements and

$$H_{ocd} = \frac{s}{f_{2d}} f_{cd} \qquad H_{odc} = \frac{s}{f_{2c}} f_{dc} \tag{131}$$

for the object elements.

The distance between corresponding principal planes is therefore

$$\Delta z\,(H)_{cd} = \frac{s^2}{f_{1c}\, f_{2d}} f_{cd} \tag{132}$$

in one plane and

$$\Delta z\,(H)_{dc} = \frac{s^2}{f_{1d}\, f_{2c}} f_{dc} \tag{133}$$

in the other. The doublet itself can be considered as a thin lens if

$$\frac{\Delta z\,(H)_{cd}}{f_{cd}} \ll 1 \qquad \frac{\Delta z\,(H)_{dc}}{f_{dc}} \ll 1 \, . \tag{134}$$

These conditions can be written

$$s^2 \ll f_{1c}\, f_{2d} \qquad s^2 \ll f_{1d}\, f_{2c} \ . \tag{135}$$

In the case where one may drop the subscripts c and d of the individual quadrupoles, the conditions reduce to

$$s^2 \ll f_1 f_2 \tag{136}$$

or

$$s\sqrt{L_1\, L_2} \ll \frac{1}{k_1\, k_2} \ . \tag{137}$$

For an antisymmetric doublet, this becomes

$$s\, L \ll \frac{1}{k^2} \tag{138}$$

or in terms of the beam rigidity

$$Br \gg s\, L\, G \ . \tag{139}$$

3.5 Practical use of a doublet

A doublet is a major device in a beam transport system as it can be used for a variety of purposes. Typical examples are focusing of a parallel beam, rendering parallel a divergent beam, focusing of a divergent beam, focusing of an astigmatic beam. These cases will be considered in more detail in what follows.

3.5.1 P-F problem

This is a common problem encountered in beam transport. A beam, initially parallel, is brought to a focus (Fig. 18) in both the cd and dc planes.

In the thin-lens approximation, the position of the common focus is found from Eqs. (12), (122) and (123) to be

$$F = f_{cd}\left(1 - \frac{s}{f_{1c}}\right) = f_{dc}\left(1 + \frac{s}{f_{1d}}\right) \ . \tag{140}$$

Putting

$$\sigma_1 = \frac{1}{f_{1c}} + \frac{1}{f_{1d}} \qquad \sigma_2 = \frac{1}{f_{2c}} + \frac{1}{f_{2d}}$$
$$\vartheta_1 = \frac{1}{f_{1c}} - \frac{1}{f_{1d}} \qquad \vartheta_2 = \frac{1}{f_{2c}} - \frac{1}{f_{2d}} \tag{141}$$

the solution of Eq. (140) can be written in the form

$$s = 2 \frac{\sigma_2 - \sigma_1}{\sigma_2 \vartheta_1 + \sqrt{\sigma_1 \sigma_2 (\vartheta_1^2 - \sigma_1^2 + \sigma_1 \sigma_2)}}$$
$$F = 2 \frac{\sigma_1}{\sigma_1 \vartheta_2 + \sqrt{\sigma_1 \sigma_2 (\vartheta_1^2 - \sigma_1^2 + \sigma_1 \sigma_2)}} \quad . \tag{142}$$

If one may drop the subscripts c and d of the individual quadrupoles, these relations become

$$s = f_1 \sqrt{\frac{f_1 - f_2}{f_1}}$$
$$F = f_2 \sqrt{\frac{f_1}{f_1 - f_2}} \quad . \tag{143}$$

Therefore, it is necessary that f_1 be larger than f_2. More often, s and F will be given and f_1 and f_2 wanted. From Eq. (143) we find

$$f_1 = \sqrt{s^2 + sF}$$
$$f_2 = \frac{sF}{\sqrt{s^2 + sF}} \quad . \tag{144}$$

The practical design of the two quadrupoles (excitations, geometry, etc.) can proceed from these data.

3.5.2 F-P problem

This is the inverse problem: one wants to render parallel a beam diverging from a point source (Fig. 19). To solve this problem it suffices to interchange the subscripts 1 and 2 in the preceding one. Thus,

$$s = 2\,\frac{\sigma_1 - \sigma_2}{\sigma_1 \vartheta_2 + \sqrt{\sigma_1 \sigma_2}\,(\vartheta_2^2 - \sigma_2^2 + \sigma_1 \sigma_2)}$$

$$F = 2\,\frac{\sigma_2}{\sigma_2 \vartheta_1 + \sqrt{\sigma_1 \sigma_2}\,(\vartheta_2^2 - \sigma_2^2 + \sigma_1 \sigma_2)} \qquad (145)$$

and, if one may drop the subscripts of the individual quadrupoles,

$$s = f_2 \sqrt{\frac{f_2 - f_1}{f_2}}$$

$$F = f_1 \sqrt{\frac{f_2}{f_2 - f_1}} \; . \qquad (146)$$

Here, f_2 should be larger than f_1.

If s and F are given (which is the usual case), the focal distances follow from

$$f_1 = \frac{sF}{\sqrt{s^2 + sF}}$$

$$f_2 = \sqrt{s^2 + sF} \; . \qquad (147)$$

It is sometimes useful to know the width of the emerging parallel beam as a function of the initial divergence. From the transfer matrix, this is found to be

$$R = \frac{x}{x_0'} = aF + b \; . \qquad (148)$$

Particularizing to the two basic planes, one has

$$\begin{aligned} R_{cd} &= \sqrt{s+F}\ (\sqrt{s+F} - \sqrt{s}) \\ R_{dc} &= \sqrt{s+F}\ (\sqrt{s+F} + \sqrt{s})\ . \end{aligned} \tag{149}$$

The beam is therefore wider in the dc plane.

In many cases, it is necessary to go beyond the first approximation and use the thick-lens approach. With our previous notations we find from the conjugation relation for the case under consideration and considering the two planes (Fig. 20)

$$\begin{aligned} c_{cd}\ p + d_{cd} &= 0 \\ c_{dc}\ p + d_{dc} &= 0 \end{aligned} \tag{150}$$

$$\begin{aligned} R_{cd} &= a_{cd}\ p + b_{cd} \\ R_{dc} &= a_{dc}\ p + b_{dc}\ . \end{aligned} \tag{151}$$

Putting*)

$$\begin{aligned} \alpha_1 &= \cos\vartheta_1 - pk_1 \sin\vartheta_1 \\ \beta_1 &= \mathrm{ch}\ \vartheta_1 + pk_1\ \mathrm{sh}\ \vartheta_1 \\ A_1 &= d\alpha_1 + p\cos\vartheta_1 + \frac{1}{k_1}\sin\vartheta_1 \\ B_1 &= d\beta_1 + p\ \mathrm{ch}\ \vartheta_1 + \frac{1}{k_1}\ \mathrm{sh}\ \vartheta_1\ , \end{aligned} \tag{152}$$

using Eqs. (108) and (109), and carrying out the calculations, one finds from Eqs. (150)

$$\begin{aligned} k_2\ \mathrm{th}\ \vartheta_2 &= -\frac{\alpha_1}{A_1} \\ k_2\ \mathrm{tg}\ \vartheta_2 &= \frac{\beta_1}{B_1}\ . \end{aligned} \tag{153}$$

*) There is no possible confusion between the matrix element d and the distance between the end faces of the quadrupoles, which is also denoted by d.

The thin-lens approximation proves its usefulness here as it yields approximate values, the knowledge of which considerably facilitates the solution of the system (153). It should be noted that in this set all characteristic parameters of the first quadrupole are localized on the r.h.s whereas the l.h.s. contains only the parameters of the second quadrupole.

The problem is simplified if one of the quadrupoles is given both in geometry and focusing strength, the other either in geometry or focusing strength, and the distance d is sought to obtain a parallel beam from an initially divergent one. Combining the last two equations, one has

$$\frac{1}{k_2}(\cot\vartheta_2 + \coth\vartheta_2) = \frac{B_1}{\beta_1} - \frac{A_1}{\alpha_1} \tag{154}$$

and upon substitution of

$$\begin{aligned} \frac{B_1}{\beta_1} &= d + \frac{p\ \mathrm{ch}\ \vartheta_1 + \frac{1}{k_1}\ \mathrm{sh}\ \vartheta_1}{\mathrm{ch}\ \vartheta_1 + pk_1\ \mathrm{sh}\ \vartheta_1} \\ \frac{A_1}{\alpha_1} &= d + \frac{p\ \cos\vartheta_1 + \frac{1}{k_1}\ \sin\vartheta_1}{\cos\vartheta_1 - pk_1\ \sin\vartheta_1} \end{aligned} \tag{155}$$

one finds

$$\frac{1}{k_2}(\cot\vartheta_2 + \coth\vartheta_2) = \frac{1}{k_1}\left(\frac{pk_1\ \mathrm{ch}\ \vartheta_1 + \mathrm{sh}\ \vartheta_1}{pk_1\ \mathrm{sh}\ \vartheta_1 + \mathrm{ch}\ \vartheta_1} + \frac{pk_1\ \cos\vartheta_1 + \sin\vartheta_1}{pk_1\ \sin\vartheta_1 - \cos\vartheta_1}\right). \tag{156}$$

Here, one has only one transcendental equation to solve and again the parameters of the two quadrupoles are concentrated respectively in the two sides of the equation. After solving this equation, d can be found by one or the other of Eq. (155).

The width of the beam can be found from Eqs. (151). Carrying out the calculations, one finds

$$\begin{aligned} R_{cd} &= \frac{A_1}{\mathrm{ch}\ \vartheta_2} \\ R_{dc} &= \frac{B_1}{\cos\vartheta_2}\ . \end{aligned} \tag{157}$$

Knowledge of these values will permit determination of the lens aperture.

3.5.3 F-F problem

In trying to focus an initially divergent beam, it is again convenient to first use the thin-lens approximation (Fig. 21) to obtain some parameters with which to start in more accurate calculations. In fact, the problem under consideration is one of stigmatism. From the two transfer matrices (122) and (123) we can write the condition of stigmatism in the form

$$F_2 = -\frac{s + F_1\left(1 - \frac{s}{f_{1c}}\right)}{1 + \frac{s}{f_{2d}} - \frac{F_1}{f_{cd}}} = -\frac{s + F_1\left(1 + \frac{s}{f_{1d}}\right)}{1 - \frac{s}{f_{2c}} - \frac{F_1}{f_{dc}}} \quad . \tag{158}$$

Using the same notations as above and carrying out the calculations, one finds

$$\begin{aligned} F_1 &= \frac{s}{\sqrt{\left(\frac{s\sigma_1}{2}\right)^2 + \frac{\sigma_1}{\sigma_2} + \frac{s\vartheta_1}{2} - 1}} \\ F_2 &= \frac{s}{\sqrt{\left(\frac{s\sigma_2}{2}\right)^2 + \frac{\sigma_2}{\sigma_1} + \frac{s\vartheta_2}{2} - 1}} \quad . \end{aligned} \tag{159}$$

In the case where the subscripts can be dropped, these relations reduce to

$$\begin{aligned} F_1 &= \frac{sf_1}{\sqrt{s^2 + f_1 f_2} - f_1} \\ F_2 &= \frac{sf_2}{\sqrt{s^2 + f_1 f_2} - f_2} \quad . \end{aligned} \tag{160}$$

Usually F_1 and F_2 will be known; the required focusing strengths are then obtained from

$$\begin{aligned} f_1 &= F_1\sqrt{\frac{s(s + F_2)}{(s + F_1)(F_1 + F_2 + s)}} \\ f_2 &= F_2\sqrt{\frac{s(s + F_1)}{(s + F_2)(F_1 + F_2 + s)}} \quad . \end{aligned} \tag{161}$$

It may be of some use to know the magnifications under stigmatic operating conditions. From Eqs. (11) and (160) one finds

$$g_{cd} = -\frac{\sqrt{s^2+f_1f_2}+s}{f_1}\,\frac{\sqrt{s^2+f_1f_2}-f_1}{\sqrt{s^2+f_1f_2}-f_2}$$

$$g_{dc} = -\frac{\sqrt{s^2+f_1f_2}-s}{f_1}\,\frac{\sqrt{s^2+f_1f_2}-f_1}{\sqrt{s^2+f_1f_2}-f_2} \qquad (162)$$

and consequently

$$\frac{g_{cd}}{g_{dc}} = \frac{\sqrt{s^2+f_1f_2}+s}{\sqrt{s^2+f_1f_2}-s} \quad . \qquad (163)$$

The magnification is therefore larger in the cd plane.

If $F_1 = F_2 = F_0$, Eqs. (159) become

$$F_0 = \frac{s}{\sqrt{1+\left(\frac{s\sigma}{2}\right)^2+\frac{s\vartheta}{2}}-1} \quad . \qquad (164)$$

In the case where one may drop the individual subscripts, one finds for the focusing strengths

$$f_1 = f_2 = f_0 = F_0\sqrt{\frac{s}{s+2F_0}} \quad . \qquad (165)$$

In this case

$$g_{cd} = -\frac{\sqrt{s^2+f_0^2}+s}{f_0}$$

$$g_{dc} = -\frac{\sqrt{s^2+f_0^2}-s}{f_0} \qquad (166)$$

and therefore $g_{cd} \cdot g_{dc} = 1$, as given by the thick-lens theory of the anti-symmetric doublet [Eq. (105)].

The values one obtains by means of the thin-lens approximation are usually good starting parameters for the more exact procedure of the thick-lens approach.

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In the latter case (Fig. 22), using the conditions of stigmatism and our previous notations, one finds

$$k_2 \ \text{th} \ \vartheta_2 = - \frac{\alpha_1 q + A_1}{A_1 q + \frac{\alpha_1}{k_2^2}}$$

$$k_2 \ \text{tg} \ \vartheta_2 = \frac{\beta_1 q + B_1}{B_1 q + \frac{\beta_1}{k_2^2}} \quad . \tag{167}$$

The magnifications are then found to be

$$\frac{1}{g_{cd}} = \alpha_1 \ \text{ch} \ \vartheta_2 + A_1 k_2 \ \text{sh} \ \vartheta_2$$

$$\frac{1}{g_{dc}} = \beta_1 \ \cos \ \vartheta_2 - B_1 k_2 \ \text{sh} \ \vartheta_2 \quad . \tag{168}$$

If d is unknown, it can be eliminated by means of Eq. (155) as in the PF case and the condition of stigmatism becomes

$$\frac{1}{k_1}\left(\frac{pk_1 + \text{th} \ \vartheta_1}{pk_1 \ \text{th} \ \vartheta_1 + 1} + \frac{pk_1 + \text{th} \ \vartheta_1}{pk_1 \ \text{tg} \ \vartheta_1 - 1}\right) = \frac{1}{k_2}\left(\frac{qk_2 + \text{th} \ \vartheta_2}{qk_2 \ \text{th} \ \vartheta_2 + 1} + \frac{qk_2 + \text{tg} \ \vartheta_2}{qk_2 \ \text{tg} \ \vartheta_2 - 1}\right) \quad .$$

The distance can then be calculated by one or the other of Eqs. (155).

3.5.4 Focusing of an astigmatic beam

This problem is encountered in a cyclotron for example where the outcoming beam may have an apparent source which is different in the horizontal and in the vertical plane (Fig. 23). The problem is then to focus the two sources in one point.

Considering first the thin-lens approximation, dropping the subscripts of the individual quadrupoles and writing out the conjugation relation (8), one finds for the two planes

$$\left(1+\frac{s}{f_2}-\frac{1}{f_{cd}}F_{cd}\right)F+s+\left(1-\frac{s}{f_1}\right)F_{cd} = 0$$
$$\left(1-\frac{s}{f_2}-\frac{1}{f_{dc}}F_{dc}\right)F+s+\left(1+\frac{s}{f_1}\right)F_{dc} = 0 \qquad (169)$$

Using Eqs. (124) these relations can be written

$$(F+F_{cd}+s)f_1f_2+F(F_{cd}+s)f_1-F_{cd}(F+s)f_2-sFF_{cd} = 0$$
$$(F+F_{dc}+s)f_1f_2-F(F_{dc}+s)f_1+F_{dc}(F+s)f_2-sFF_{dc} = 0\ . \qquad (170)$$

Putting

$$F_{cd}+F_{dc} = S$$
$$F_{cd}-F_{dc} = D \qquad (171)$$
$$F_{cd}\cdot F_{dc} = P$$

the solution of Eq. (170) can be written

$$f_2 = F\sqrt{\frac{s(sS+2P)}{(F+s)[(F+s)S+2P]}}$$
$$f_1 = \frac{F+s}{(F+s)(S+2s)+sS+2P}\left\{sD+\frac{f_2}{F}[(F+s)S+2P]\right\}\ . \qquad (172)$$

Knowledge of these solutions will greatly facilitate the calculations of the more rigourous thick-lens method.

Putting, in the latter case,

$$\alpha_1 = \cos\vartheta_1 - p_{cd}\,k_1\,\sin\vartheta_1$$
$$\beta_1 = \mathrm{ch}\,\vartheta_1 + p_{dc}\,k_2\,\mathrm{sh}\,\vartheta_1$$
$$A_1 = d\alpha_1 + p_{cd}\cos\vartheta_1 + \frac{1}{k_1}\sin\vartheta_1 \qquad (173)$$
$$B_1 = d\beta_1 + p_{dc}\,\mathrm{ch}\,\vartheta_1 + \frac{1}{k_1}\,\mathrm{sh}\,\vartheta_1\ ,$$

one is once again led to the set of transcendental equations

$$k_2 \ \mathrm{th} \ \vartheta_2 = - \frac{\alpha_1 q + A_1}{A_1 q + \frac{\alpha_1}{k_2^2}} \qquad (174)$$

$$k_2 \ \mathrm{tg} \ \vartheta_2 = \frac{\beta_1 q + B_1}{B_1 q - \frac{\beta_1}{k_2^2}}$$

which will have to be solved by numerical procedures.

4. THE TRIPLET

4.1 General properties

Although the doublet achieves a significant improvement with respect to a single quadrupole, and although its construction is inherently simpler than that of a three-lens system, the latter system will be preferred in many cases. The chief disadvantage of the doublet lies in the fact that the variation of a parameter in one of the basic planes may entail an important variation of the parameters in the other plane. If one tries, for example, to adjust the position of the principal plane in the cd direction, the position of the principal plane in the dc direction will equally vary. Supposing that the doublet can be assimilated to a thin lens, the position of this lens may then be quite different in the two perpendicular planes as the variation of the excitation will modify not only the focusing strength of the lens but also its position in space.

In a symmetric triplet, the principal planes are symmetric with respect to the median plane of the lens and this applies equally well to the cdc as to the dcd direction and is independent of the excitation level. If a symmetric triplet can be considered as a thin lens, the position of this equivalent lens is fixed in space and independent of excitation.

4.2 The general triplet

Figure 24 shows, qualitatively, the behaviour of the beam envelope in a general triplet. In most cases, the calculations are carried out in the frame of the thin-lens approximation, i.e. each quadrupole is considered to be concentrated in its median plane (Fig. 25). If G_1, G_2, and G_3 are the gradients of the magnetic field of the three quadrupoles, one has for the characteristic parameters k and ϑ

$$k^2_{1,2,3} = \frac{\vartheta_{1,2,3}}{Br} \qquad\qquad (175)$$

$$\vartheta_{1,2,3} = k_{1,2,3}\ L_{1,2,3}\ .$$

As already mentioned in the case of the doublet, the procedure of the lens association can be improved if one takes, for the focal distances, the values one finds from the thick lens approach applied to each individual quadrupole. Under these conditions, one has

$$\frac{1}{f_{1c}} = k_1 \sin\vartheta_1 \qquad \frac{1}{f_{1d}} = k_1 \operatorname{sh}\vartheta_1$$

$$\frac{1}{f_{2c}} = k_2 \sin\vartheta_2 \qquad \frac{1}{f_{2d}} = k_2 \operatorname{sh}\vartheta_2 \qquad\qquad (176)$$

$$\frac{1}{f_{3c}} = k_3 \sin\vartheta_3 \qquad \frac{1}{f_{3d}} = k_3 \operatorname{sh}\vartheta_3 \ .$$

All quantities are counted positive in Eq. (176), the proper sign being taken care of in the expressions of the transfer matrices.

4.3 Transfer matrices

The triplet being made up of three quadrupoles and two drift spaces, the basic transfer matrices are

$$T_{cdc} = \begin{Vmatrix} 1 & 0 \\ -\frac{1}{f_{3c}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s_2 \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ \frac{1}{f_{2d}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s_1 \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ -\frac{1}{f_{1c}} & 1 \end{Vmatrix} \qquad (177)$$

$$T_{dcd} = \begin{Vmatrix} 1 & 0 \\ \frac{1}{f_{3d}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s_2 \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ -\frac{1}{f_{2c}} & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & s_1 \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} 1 & 0 \\ \frac{1}{f_{1d}} & 1 \end{Vmatrix} \quad . \tag{178}$$

Multiplying out and using the abbreviations

$$\begin{aligned} \frac{s_i}{f_{jc}} &= x_{ij} \qquad i = 1,2 \qquad j = 1,2,3 \\ \frac{s_i}{f_{jd}} &= y_{ij} \qquad i = 1,2 \qquad j = 1,2,3 \quad , \end{aligned} \tag{179}$$

one finds

$$T_{cdc} = \begin{Vmatrix} (1-x_{11})(1+y_{22}) - x_{21} & s_1 + s_2(1+y_{12}) \\ (1-x_{23})\left[\frac{1}{f_{2d}} - \frac{1}{f_{1c}}(1+y_{12})\right] - \frac{1}{f_{3c}}(1-x_{11}) & (1+y_{12})(1-x_{23}) - x_{13} \end{Vmatrix} \tag{180}$$

$$T_{dcd} = \begin{Vmatrix} (1+y_{11})(1-x_{22}) + y_{21} & s_1 + s_2(1-x_{12}) \\ (1+y_{23})\left[\frac{1}{f_{1d}}(1-x_{12}) - \frac{1}{f_{2c}}\right] + \frac{1}{f_{3d}}(1+y_{11}) & (1-x_{12})(1+y_{23}) + y_{13} \end{Vmatrix} \tag{181}$$

4.4 Optical properties

a) Focal distances

From the transfer matrices, the focal distances can readily be derived

$$\frac{1}{f_{cdc}} = (1 - x_{23})\left[\frac{1}{f_{1c}}(1 + y_{12}) - \frac{1}{f_{2d}}\right] + \frac{1}{f_{3c}}(1 - x_{11})$$

$$\frac{1}{f_{dcd}} = (1 + y_{23})\left[\frac{1}{f_{2c}} - \frac{1}{f_{1d}}(1 - x_{12})\right] - \frac{1}{f_{3d}}(1 + y_{11}) \; . \qquad (182)$$

b) Position of the focal planes

$$F_{icdc} = f_{cdc}\left[(1 - x_{11})(1 + y_{22}) - x_{21}\right]$$

$$F_{idcd} = f_{dcd}\left[(1 + y_{11})(1 - x_{22}) + y_{21}\right] \; . \qquad (183)$$

Similar expressions can be derived for the object focal planes.

c) Principal planes

Again, from the transfer matrices, one has

$$H_{icdc} = f_{cdc}\left[(1 - x_{11})(1 + y_{22}) - (1 + x_{21})\right]$$

$$H_{idcd} = f_{dcd}\left[(1 + y_{11})(1 - x_{22}) - (1 - y_{21})\right] \; . \qquad (184)$$

In many cases, it will be simpler to calculate the optical elements by numerical procedures directly from the transfer matrices. Algebraic calculations are more useful in the case of the symmetric triplet where some specific properties can be derived.

4.5 The symmetric triplet

In this case we have (Fig. 26)

$$\begin{array}{ll} s_1 = s_2 = s & \\ L_1 = L_3 = L_e & L_2 = L_i \\ G_1 = G_3 = G_e & G_2 = G_i \\ k_1 = k_3 = k_e & k_2 = k_i \\ \vartheta_1 = \vartheta_3 = \vartheta_e & \vartheta_2 = \vartheta_i \end{array} \qquad (185)$$

so that the focal lengths of the individual quadrupoles are

$$\frac{1}{f_{ec}} = k_e \sin \vartheta_e \qquad \frac{1}{f_{ed}} = k_e \ \mathrm{sh}\ \vartheta_2$$

$$\frac{1}{f_{ic}} = k_i \sin \vartheta_i \qquad \frac{1}{f_{id}} = k_i \ \mathrm{sh}\ \vartheta_i \ . \qquad (186)$$

It is useful to introduce the dimensionless quantities

$$\frac{s}{f_{ec}} = x_e \qquad \frac{s}{f_{ed}} = y_e$$

$$\frac{s}{f_{ic}} = x_i \qquad \frac{s}{f_{id}} = y_i \ . \qquad (187)$$

4.5.1 Transfer matrices

With the above notations, the transfer matrices can be written

$$T_{cdc} = \begin{Vmatrix} 1 - 2x_e + y_i(1 - x_e) & s(2 + y_i) \\ \frac{1 - x_e}{s}[-2x_e + y_i(1 - x_e)] & 1 - 2x_e + y_i(1 - x_e) \end{Vmatrix} \qquad (188)$$

$$T_{dcd} = \begin{Vmatrix} 1 + 2y_e - x_i(1 + y_e) & s(2 - x_i) \\ \frac{1 + y_e}{s}[2y_e - x_i(1 + y_e)] & 1 + 2y_e - x_i(1 + y_e) \end{Vmatrix} \ . \qquad (189)$$

4.5.2 Focal distances

From the matrices we have, using non-dimensional notation,

$$\frac{s}{f_{cdc}} = (1 - x_e)[2x_e - y_i(1 - x_e)]$$

$$\frac{s}{f_{dcd}} = (1 + y_e)[-2y_e + x_i(1 + y_e)] \ . \qquad (190)$$

4.5.3 Position of focal planes

$$\frac{F_{icdc}}{s} = \frac{1 - 2x_e + y_i(1 - x_e)}{(1 - x_e)[2x_e - y_i(1 - x_e)]}$$

$$\frac{F_{idcd}}{s} = \frac{1 + 2y_e - x_i(1 + y_e)}{(1 + y_e)[-2y_e + x_i(1 + y_e)]} \quad . \qquad (191)$$

Due to the symmetry properties of the lens system, the same expressions give the position of the object focal planes, the abscissae, however, being taken here towards the left starting from the entrance plane.

4.5.4 Position of principal planes

Very simple expressions follow for these quantities from the matrix elements.

$$\frac{H_{icdc}}{s} = \frac{H_{ocdc}}{s} = -\frac{1}{1 - x_e}$$

$$\frac{H_{idcd}}{s} = \frac{H_{odcd}}{s} = -\frac{1}{1 + y_e} \quad . \qquad (192)$$

Figure 27 shows the position of the two principal planes in the two basic directions for a symmetric triplet.

The relations (192) display two properties:

a) Provided $x_e, y_e << 1$, i.e. provided the excitation of the outer quadrupole is small, all principal planes coincide with the geometric centre plane of the lens system. Under these conditions, the symmetric triplet can be considered as a thin lens of fixed position.

b) To adjust the optical properties of the triplet, one can vary x_i, y_i, i.e. the excitation of the inner quadrupole, without modifying the position of the principal planes.

These properties have no analogy in doublet behaviour.

In writing Eqs. (190) and (191), we have tacitly assumed that

$$\begin{aligned} 2x_e &\neq y_i(1-x_e) \\ 2y_e &\neq x_i(1+y_e) \; ; \end{aligned} \tag{193}$$

the equalities corresponding indeed to infinite focal lengths or afocal systems. Moreover, all higher-order terms must be kept in the brackets of expressions (190) giving the focal lengths, otherwise the condition of bidirectional focusing

$$\frac{2y_e}{1+y_e} < x_i < y_i < \frac{2x_e}{1-x_e} \tag{194}$$

cannot be satisfied.

4.5.5 Stigmatic operation of a symmetric triplet

Writing out the condition of stigmatism, we have here

$$-q_o = \frac{a_{cdc}\, p_o + b_{cdc}}{c_{cdc}\, p_o + a_{cdc}} = \frac{a_{dcd}\, p_o + b_{dcd}}{c_{dcd}\, p_o + a_{dcd}} \; . \tag{195}$$

Using the same notations as before and putting, moreover,

$$\begin{aligned} x_i + y_i &= 4S_i \\ x_e + y_e &= S_e \\ x_i - y_i &= 4(1-A_i) \\ x_e - y_e &= -2A_e \\ (2-x_i)(2+y_i) &= 4\lambda \\ \lambda S_e - 2S_i &= S \end{aligned} \tag{196}$$

one finds, after carrying out the calculations,

$$p_o = q_o = 2s \frac{S + S_i}{A_i\ S_e - A_o(S + S_i) + \sqrt{S_e^2\ S^2 - S_e\ S}} \quad . \tag{197}$$

For the same reasons as before, all higher-order terms must be kept in this expression.

Under operating conditions where the subscipts c and d of the individual quadrupoles can be dropped, one has

$$x_i = y_i$$
$$x_e = y_e \quad . \tag{198}$$

The focal lengths can then be written

$$\frac{s}{f_{cdc}} = (1 - x_e)[2x_e - x_i(1 - x_e)]$$
$$\frac{s}{f_{dcd}} = (1 + x_o)[-2x_e + x_i(1 + x_e)] \quad . \tag{199}$$

Here again, all terms must be kept in the brackets if the condition of bidirectional focusing

$$\frac{2x_e}{1 + x_e} < x_i < \frac{2x_e}{1 - x_e} \tag{200}$$

is to be satisfied.

Taking into account Eq. (198), Eq. (197) giving the couple of stigmatic points, becomes

$$p_o = q_o = s \frac{4x_e - x_i(1 + x_i x_e)}{-2x_e + x_i(1 + x_i x_e) + \sqrt{[x_i x_e(2 + x_i x_e) - 4x_e^2]^2 + x_i x_e(2 + x_i x_e) - 4x_e^2}} \quad . \tag{201}$$

As in the general case, all higher-order terms must be kept in this expression in order to comply with the bidirectional focusing condition.

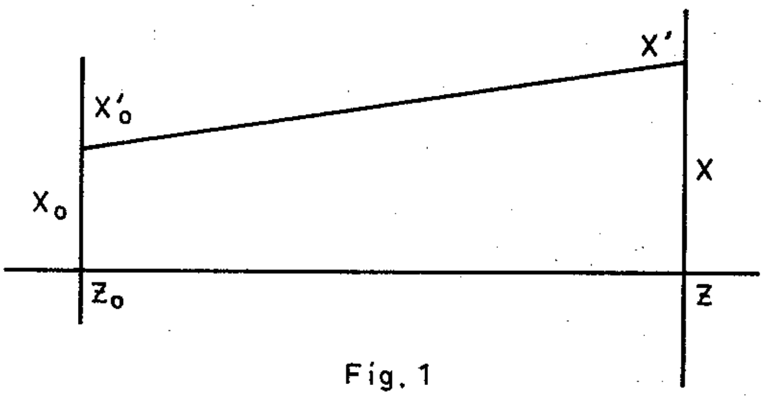

Fig. 1

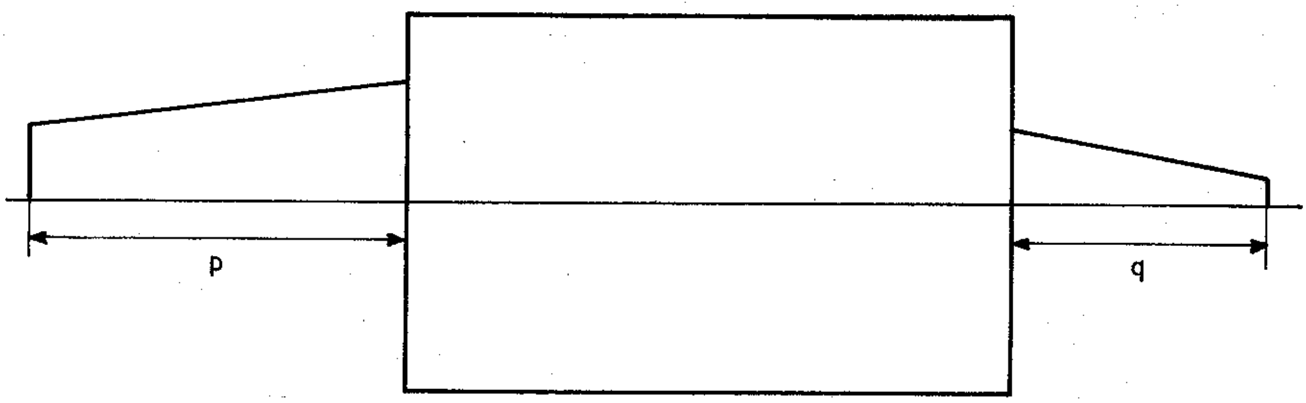

Fig. 2

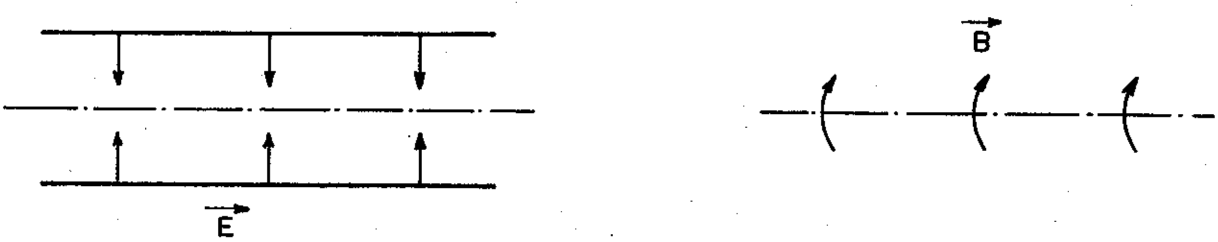

Fig. 3

A. G.

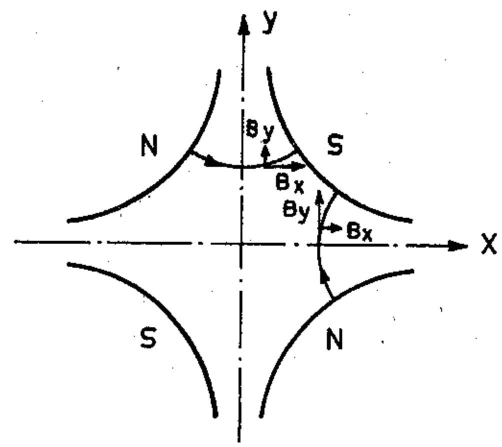

Fig. 4

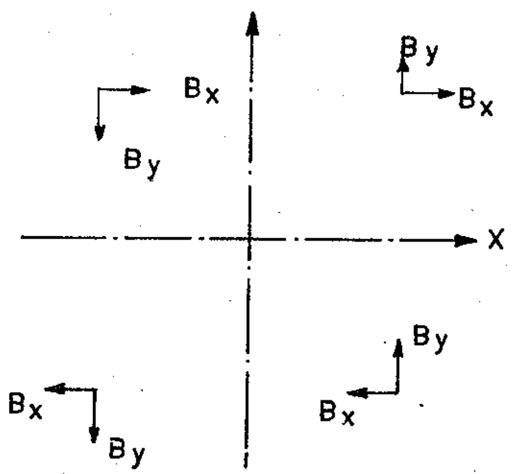

Fig. 5

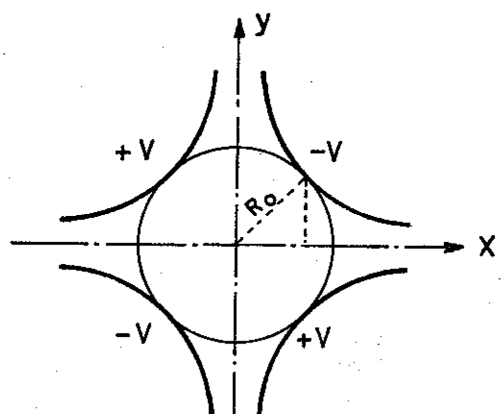

Fig. 6

A. G.

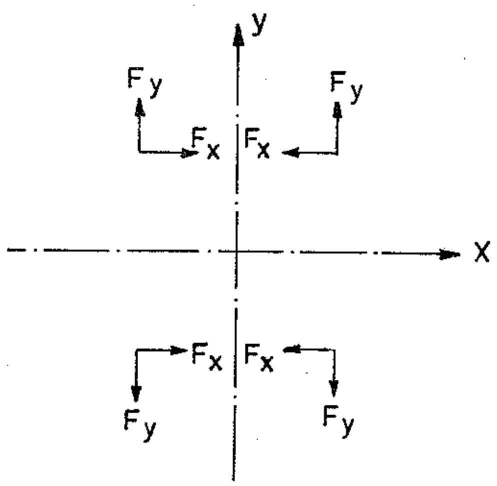

Fig. 7

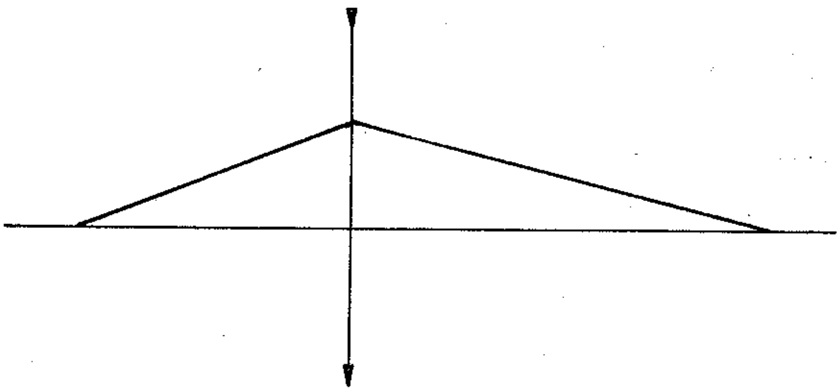

Fig 8

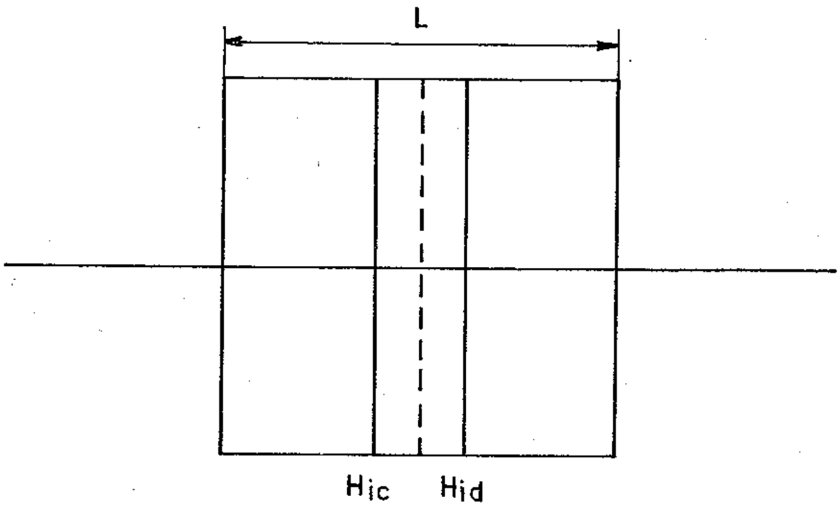

Fig. 9

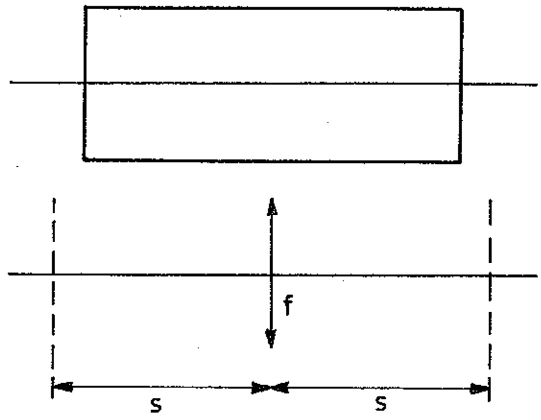

Fig. 10

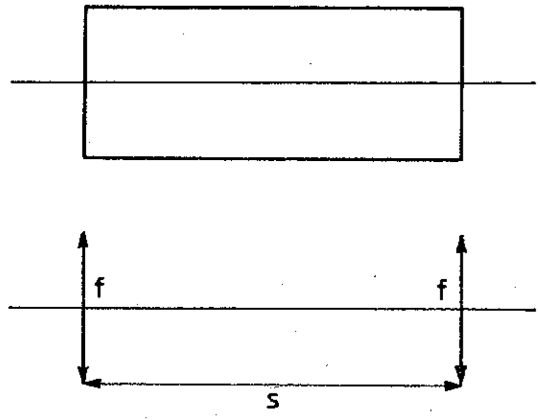

Fig. 11

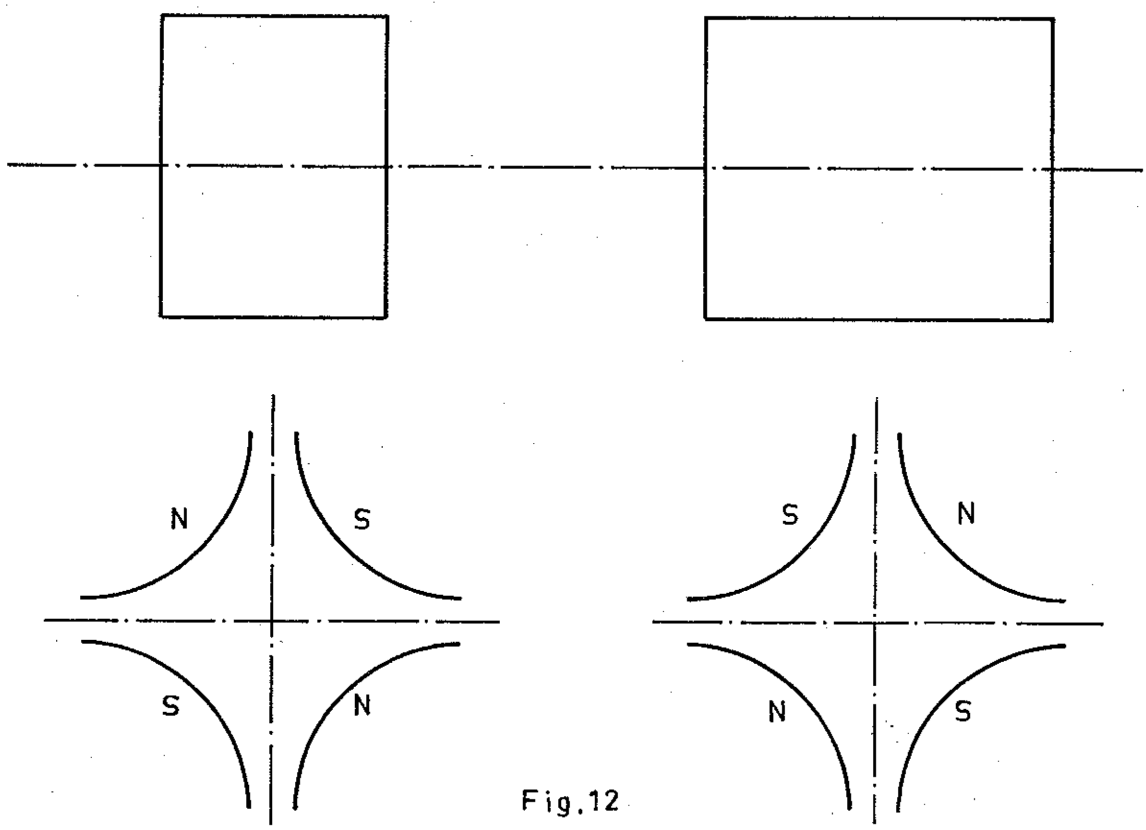

Fig. 12

A.G.

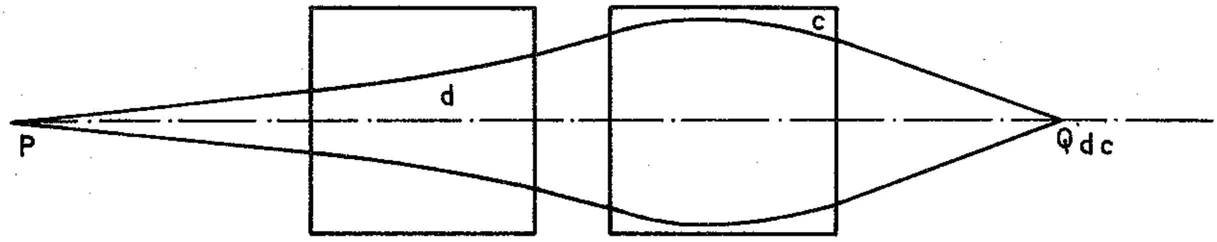

Fig. 13

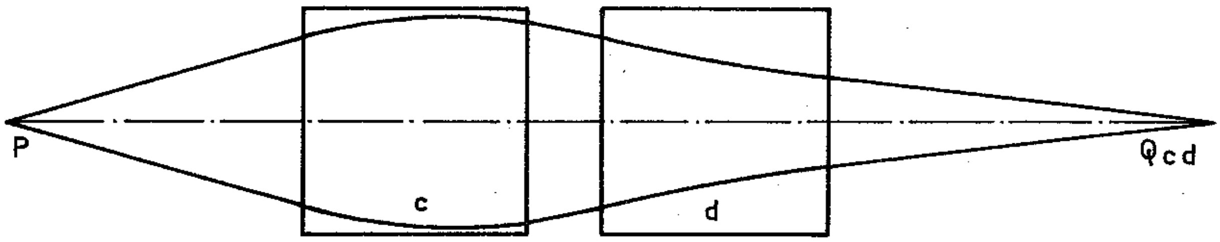

Fig. 14

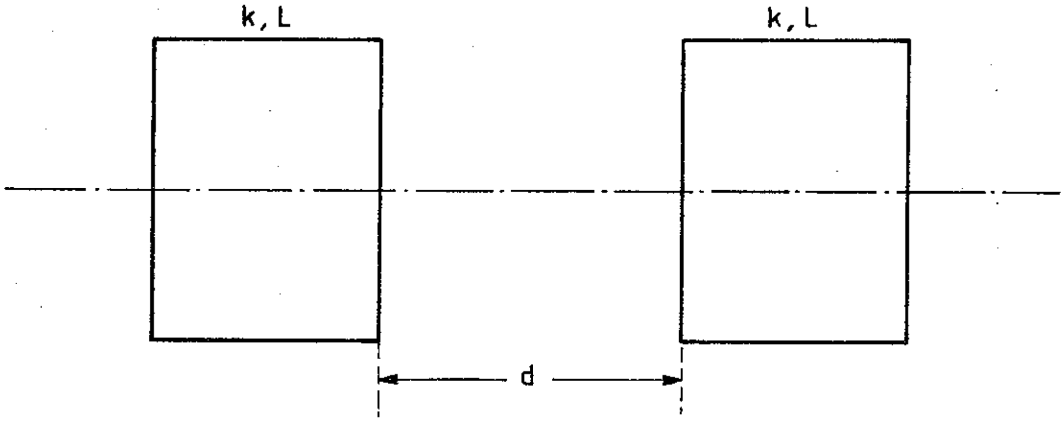

Fig. 15

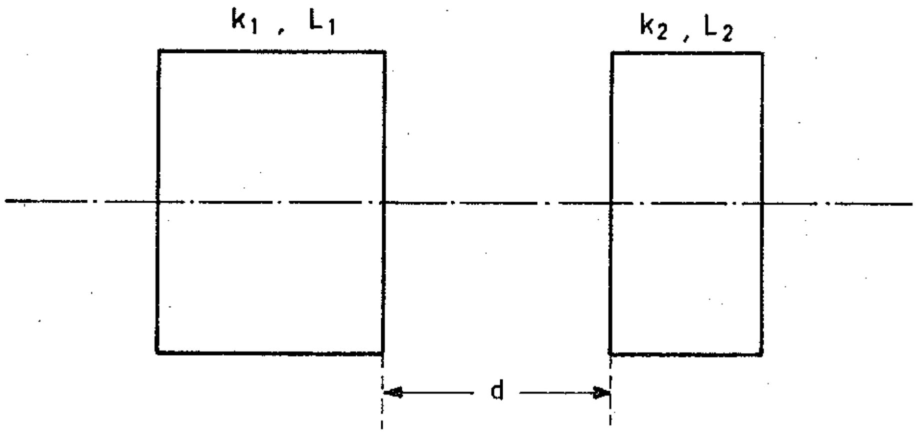

Fig. 16

A.G.

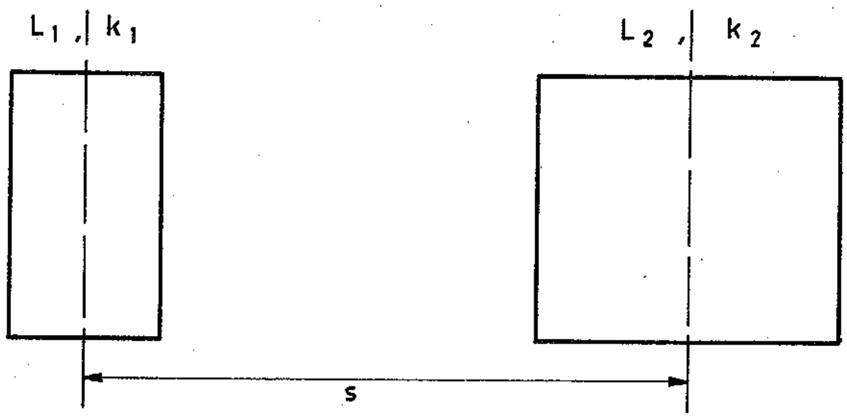

Fig. 17

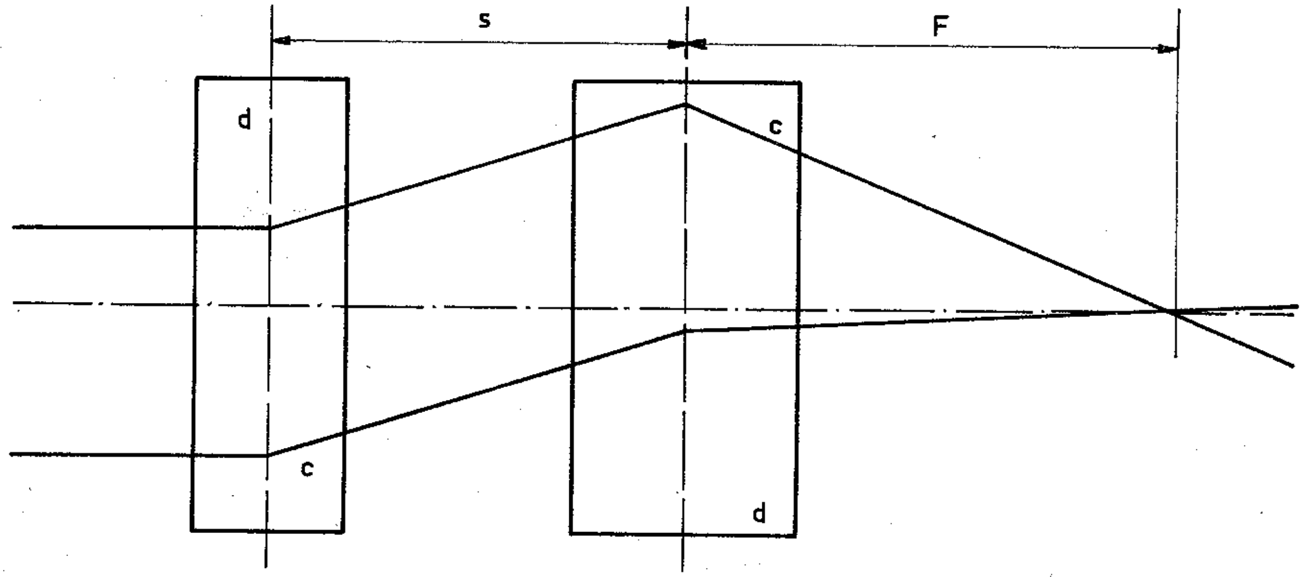

Fig. 18

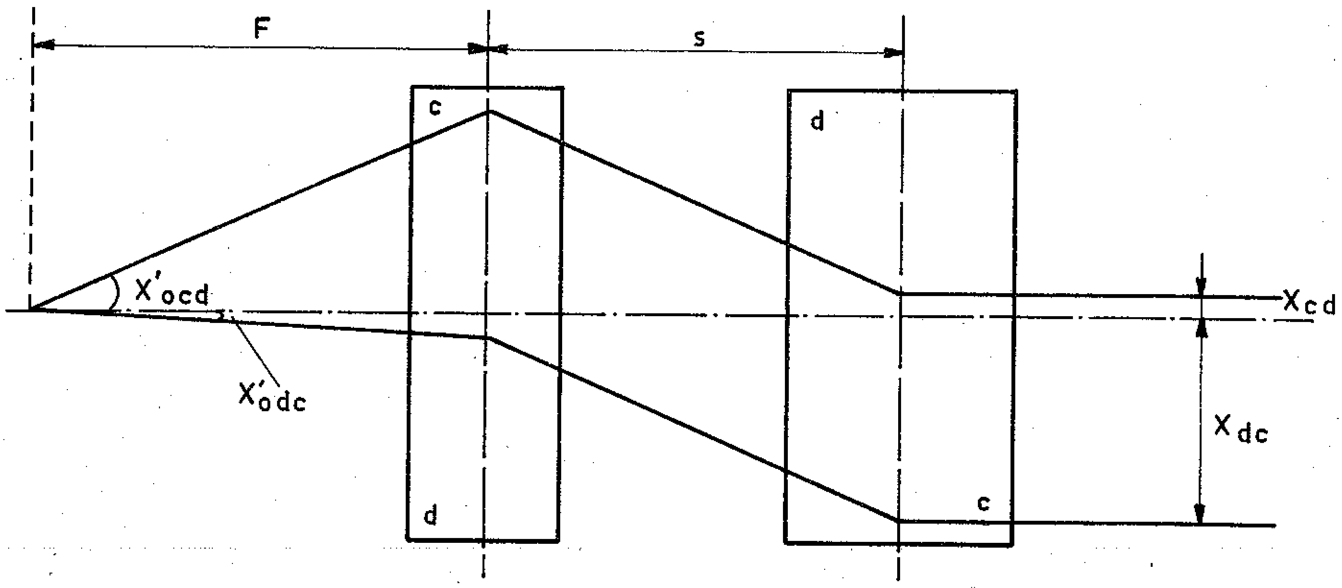

Fig. 19

A. G.

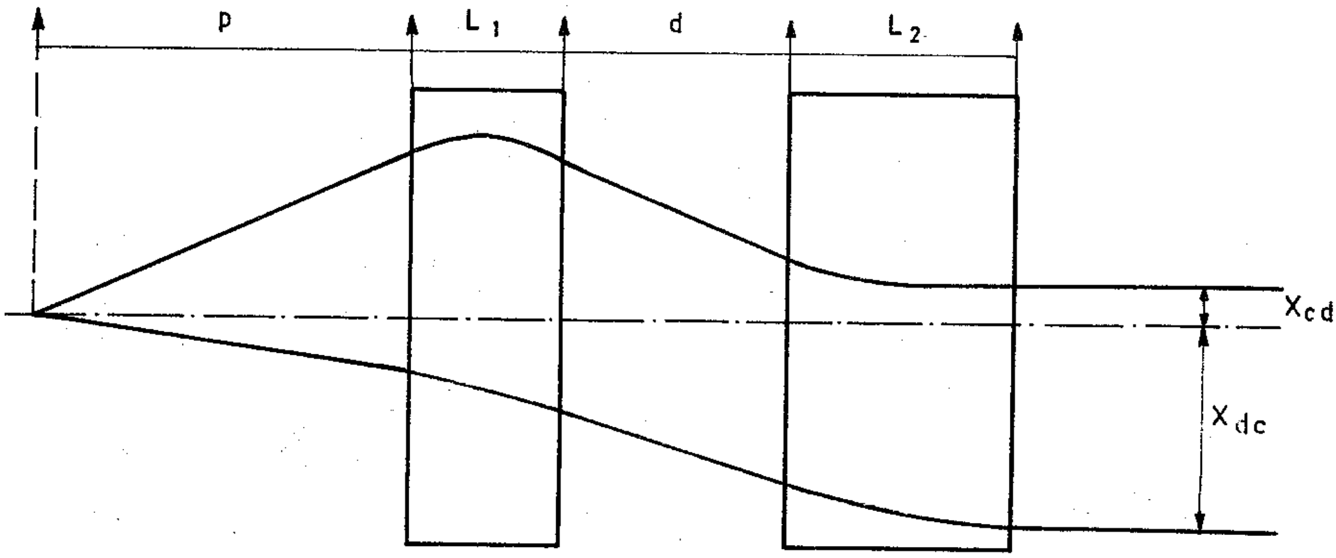

Fig. 20

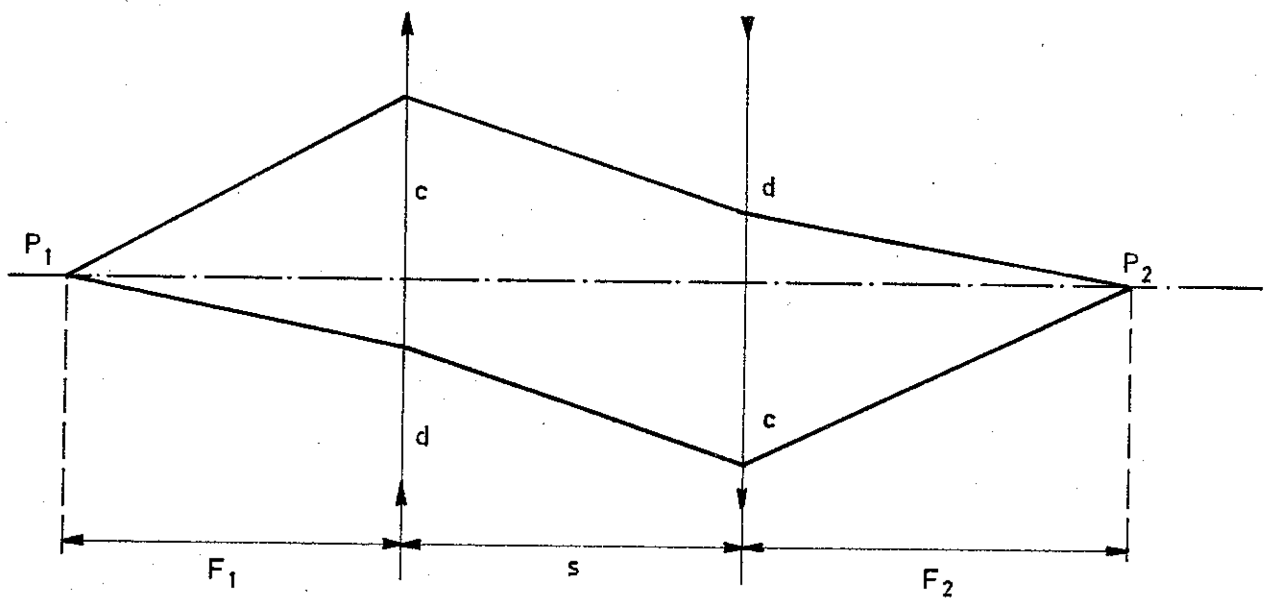

Fig. 21

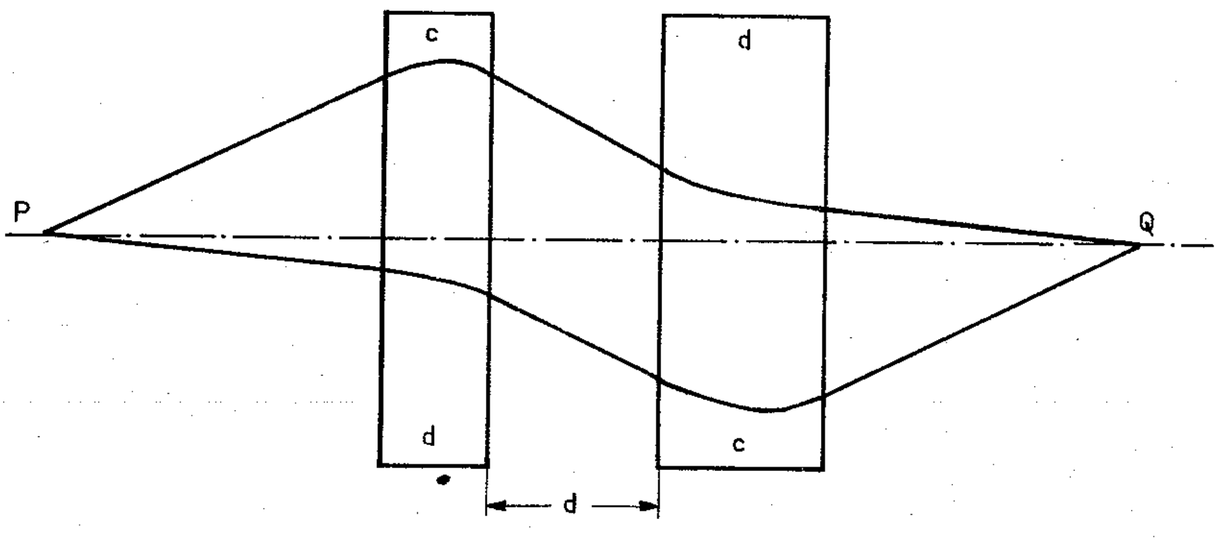

Fig. 22

A. G.

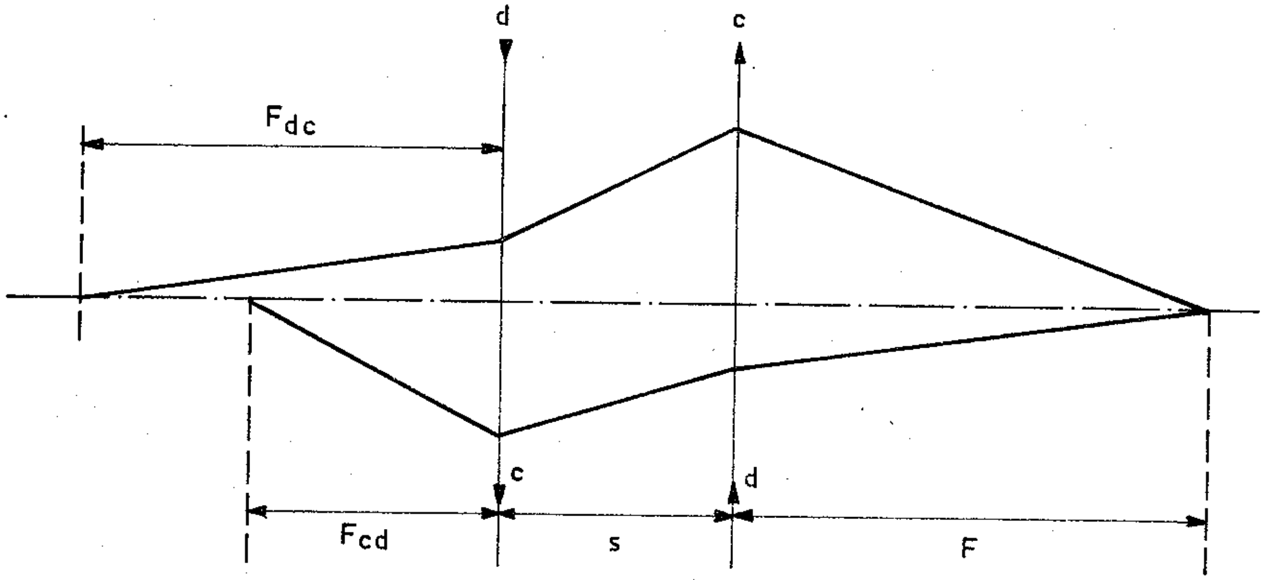

Fig. 23

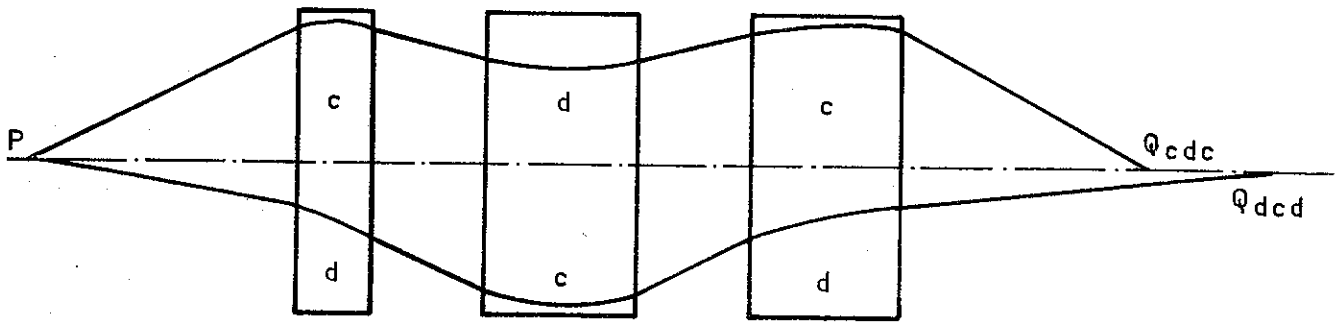

Fig. 24

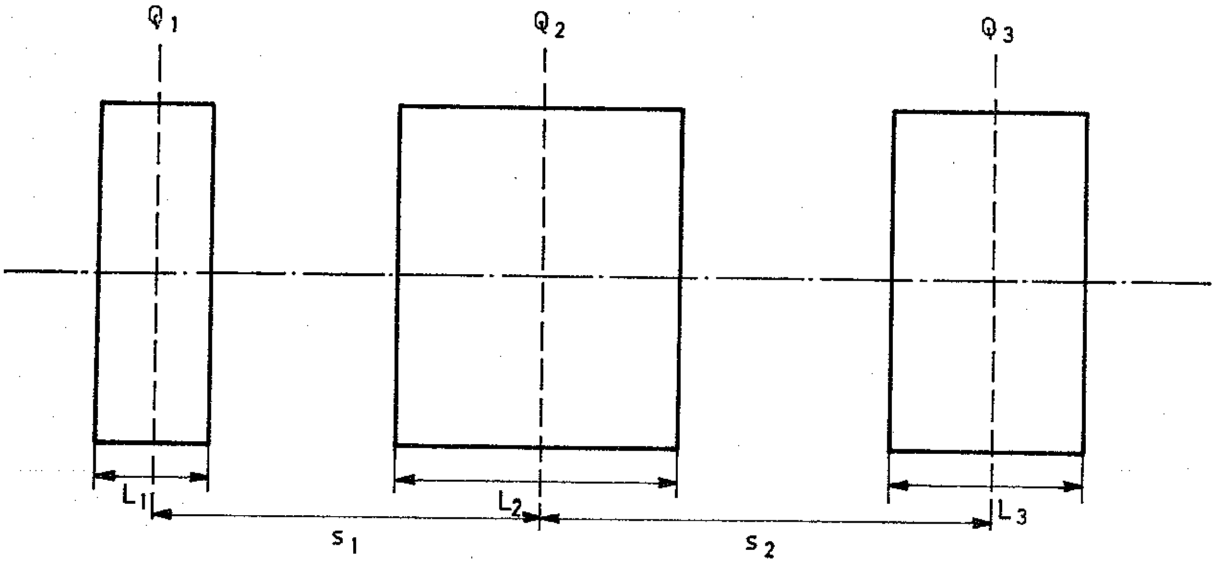

Fig. 25

A.G.

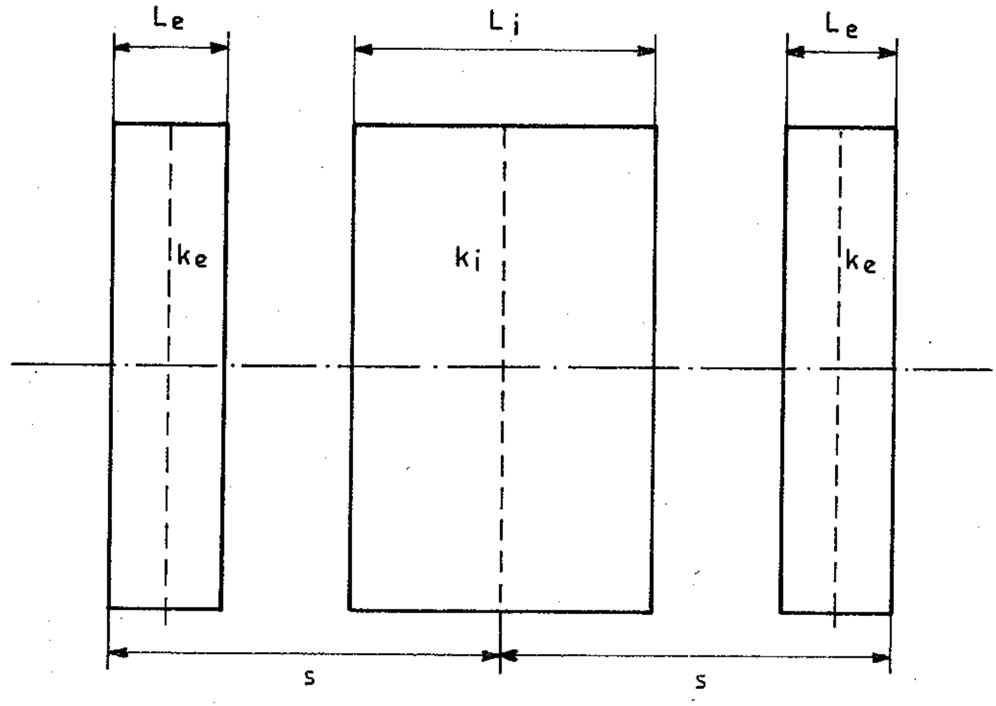

Fig. 26

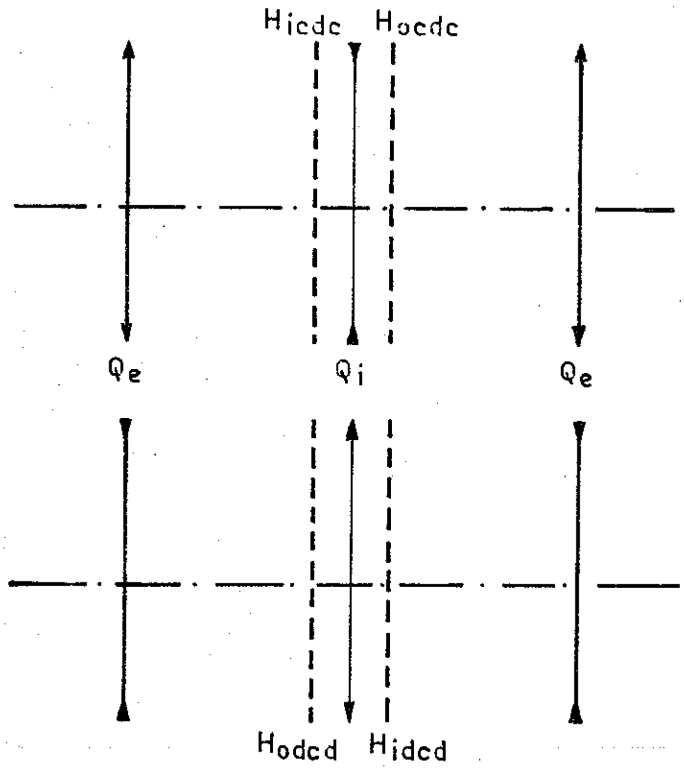

Fig. 27