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Curvature Perturbations from First-Order Phase Transitions: Implications to Black Holes and Gravitational Waves

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We employ a covariant formalism to study the evolution of cosmological perturbations during a first-order phase transition, addressing in particular their gauge dependence that have been overlooked so far. Our results reveal that non-covariant treatments employed in previous studies can substantially overestimate the production of primordial black holes and scalar-induced gravitational waves. Once gauge dependencies are properly accounted for, we find that both effects occur at significantly lower levels than previously estimated.

Introduction. The vacuum right after reheating is expected to possess a higher degree of symmetry compared to the one of today [1]. The breaking of these symmetries as the universe cools down can arise via first-order phase transitions (FOPTs). Bubbles of the true vacuum nucleate and expand within the false vacuum background, converting the vacuum energy difference into heat and kinetic motion of the fluid (see e.g. Ref. [2] for a review). If bubbles carry a large energy fraction of the universe, FOPTs are expected to leave important imprints on the space-time metric, including gravitational waves [3, 4], primordial black holes (PBHs) [5–30], and curvature perturbations [20, 31–37]. The latter are known to generate scalar-induced gravitational waves (SIGWs) [38–49]. Cosmological scalar perturbations exhibit gauge dependence in general [50], and therefore the validity of PBH, curvature, and SIGW predictions relies on employing a proper covariant treatment of perturbations and their evolution equations.

In this *letter*, we derive the statistical properties of cosmological perturbations produced during a strong first-order phase transition, with a particular emphasis on the gauge choice and the calculation of gauge-invariant quantities. We evaluate the PBH abundance and the spectrum of SIGWs using the most appropriate quantities for each: the density contrast in the comoving gauge $\delta^{(C)}$ for the former, to match the convention used to report the threshold for collapse in numerical simulations, while the gauge-invariant curvature perturbation $\mathcal{R}$ for the latter.

Covariant linear perturbations from FOPTs. We model the nucleation rate by

$$\Gamma = H_n^4\, e^{\beta(t-t_n)}, \tag{1}$$

where H_n is the Hubble factor ($H \equiv \dot{a}(t)/a$, with a being the scale factor) at $t = t_n$, defined as the nucleation time. During an FOPT, the energy and pressure density decomposes as Adopting a relativistic bag equation of state, the energy and pressure density decompose as

$$\rho = \rho_R + \rho_V, \qquad p = \rho_R/3 - \rho_V, \tag{2}$$

where R and V stand for radiation and vacuum components, respectively. We consider the supercooled limit for which the latent heat ΔV is larger than the radiation energy density just before nucleation $\alpha \equiv \Delta V/\rho_R \gg 1$. At the background level, the radiation and vacuum energy density obeys the continuity equation

$$\overline{\rho_{\rm R}}' + 4\mathcal{H}\overline{\rho_{\rm R}} = -\overline{\rho_{\rm V}}', \qquad \text{with} \quad \overline{\rho_{\rm V}} = \overline{F}\Delta V, \tag{3}$$

and the Friedman equation $3M_{\rm pl}^2 H^2 = \overline{\rho_{\rm R}} + \overline{\rho_{\rm V}}$. The prime denotes the derivative with respect to the conformal time η related to the cosmic time t by $\eta(t_2) = \eta(t_1) + \int_{t_1}^{t_2} d\tilde{t}/a(\tilde{t})$, and we defined and $\mathcal{H} = aH$. The false vacuum fraction $\overline{F}(t)$ is given by

$$\overline{F}(t) = \exp\left[-\frac{4\pi}{3}\int_{-\infty}^{t} \mathrm{d}t_n\, \Gamma(t_n)a(t_n)^3 \left(\int_{t_n}^{t}\frac{d\tilde{t}}{a(\tilde{t})}\right)^3\right], \tag{4}$$

where we assumed bubble walls to expand at the speed of light $v_w \simeq 1$.

Due to the stochastic nature of bubble nucleation, the stress-energy tensor $T^{\mu}{}_{\nu}$ acquires fluctuations – assumed to be statistically isotropic – $T^0{}_0 = \overline{\rho} + \delta\rho$, $T^i{}_j = -(\overline{p} + \delta p)\delta^i_j$, and $T^i{}_0 = (\overline{\rho} + \overline{p})v^i$ where $v^i = dx^i/dt$ is the coordinate velocity, and we use $\delta \equiv \delta\rho/\overline{\rho}$ in the following. Cosmological linear perturbations are gauge-dependent [50]. Fixing the gauge to be spatially-flat (F), linearised Einstein equations give in Fourier space

$$\tilde{\delta}_{\mathbf{k}}^{(F)\prime} + 3\mathcal{H}(c_s^2 - \omega)\tilde{\delta}_{\mathbf{k}}^{(F)} = (1+\omega)\tilde{\mathcal{V}}_{\mathbf{k}} - 3\mathcal{H}\tilde{\delta}_{p,\mathrm{nad},\mathbf{k}}, \tag{5}$$

where $\tilde{X}_{\mathbf{k}}(t)$ denotes the Fourier-transform of $X(\mathbf{x}, t)$. The quantities $\omega \equiv \overline{p}/\overline{\rho}$ and $c_s^2 \equiv \dot{\overline{p}}/\dot{\overline{\rho}}$ are the equation of state and speed of sound, respectively. The quantity $\delta_{p,\mathrm{nad}}$ is the non-adiabatic pressure perturbation

$$\delta_{p,\mathrm{nad}} \equiv \frac{\delta p_{\mathrm{nad}}}{\overline{\rho}}, \quad \text{with} \quad \delta p_{\mathrm{nad}} \equiv \delta p^{(F)} - c_s^2\delta\rho^{(F)}, \tag{6}$$

which is gauge-invariant in spite of the superscript (F) on the right hand-side. Using Eq. (2), Eq. (6) becomes

$$\delta p_{\mathrm{nad}} = \frac{1-3c_s^2}{3}\overline{\rho}\,\delta^{(F)} + \frac{4}{3}\Delta V\,\delta F^{(F)}, \tag{7}$$

where $\delta F^{(F)} = F^{(F)} - \overline{F}$ with $F^{(F)} \equiv \rho_V^{(F)}/\Delta V$. The quantity $\mathcal{V}$ in Eq. (5) is the gauge-invariant (GI) scalar velocity $\mathcal{V} = v + E'$ with $\partial_i(v) = \delta_{ij}v^j$ and E being the scalar component of the 3-dimensional spatial metric h_{ij}. Einstein equation along $0i$ gives

$$\mathcal{V} = -\frac{2}{3(1+\omega)}\frac{\mathcal{H}\Phi + \Phi'}{\mathcal{H}^2}, \tag{8}$$

where Φ is the GI Newtonian potential [50]. A combination of Einstein equations along 00 and ij gives

$$\tilde{\Phi}''_{\mathbf{k}}+3(1+c_s^2)\mathcal{H}\tilde{\Phi}'_{\mathbf{k}}+[3(c_s^2-\omega)\mathcal{H}^2+c_s^2k^2]\tilde{\Phi}_{\mathbf{k}} = \frac{3}{2}\mathcal{H}^2\tilde{\delta}_{p,\mathrm{nad},\mathbf{k}}. \tag{9}$$

The initial conditions for the system Eq. (5) and (9) are

$$\tilde{\delta}_{\mathbf{k}}^{(F)}(0) = 0, \qquad \tilde{\Phi}_{\mathbf{k}}(0) = 0, \qquad \tilde{\Phi}'_{\mathbf{k}}(0) = 0, \tag{10}$$

where we have neglected the primordial curvature perturbations (i.e. generated during inflation), which we assume to be small at these scales. For more details on cosmological perturbation theory, we refer the reader to the companion paper [51] or Refs. [52–56].

In the spatially-flat gauge, constant-time hypersurfaces are flat $ds^2\big|_{\eta=\mathrm{const.}} = -a^2dx^2$. This is particularly convenient since by plugging Eq. (7) into Eq. (5), we get a perturbation equation

$$\delta\tilde{\rho}_{R,\mathbf{k}}^{(F)\prime} + 4\mathcal{H}\delta\tilde{\rho}_{R,\mathbf{k}}^{(F)} = -\delta\tilde{\rho}_{V,\mathbf{k}}^{(F)\prime} + \frac{4}{3}\overline{\rho_R}\,k^2\tilde{\mathcal{V}}_{\mathbf{k}}, \tag{11}$$

which coincides with the background equation in Eq. (3) in the super-Hubble limit $k \ll \mathcal{H}$. Hence, in the spatially-flat gauge and in the super-Hubble limit, different patches evolve as distinct FLRW universes [57], and we can identify

$$\tilde{F}_{\mathbf{k}}^{(F)} \underset{k\ll\mathcal{H}}{\simeq} \tilde{F}_{\mathbf{k}}^{(\mathrm{FLRW})}, \tag{12}$$

where $F^{(\mathrm{FLRW})}$ is the false vacuum fraction in a flat FLRW universe. Notice that such equality does not hold in other gauges. The code `deltaPT` developed in Ref. [20] calculates the false vacuum fraction $F_{\mathrm{avg}}^{(\mathrm{FLRW})}(R,t)$ in flat FLRW universe averaged over a ball of radius R. In this work, we rely on the approximation[1]

$$\tilde{F}_{\mathbf{k}}^{(\mathrm{FLRW})}(t) \simeq V F_{\mathrm{avg}}^{(\mathrm{FLRW})}(R=k^{-1},t), \tag{13}$$

with $V = 4\pi R^3/3$. The factor V ensures that Fourier transforms carry the usual volume dimension. Due to Eq. (13), it follows that $\tilde{X}_{\mathbf{k}}(t) \simeq VX_{\mathrm{avg}}(R = k^{-1}, t)$ for any perturbations X derived in this work. This approximation should become exact in the super-Hubble limit $k \ll \mathcal{H}$ when gradient terms are negligible. We now denote perturbations by $X \equiv \tilde{X}_{\mathbf{k}}/V \simeq X_{\mathrm{avg}}(R = k^{-1})$.

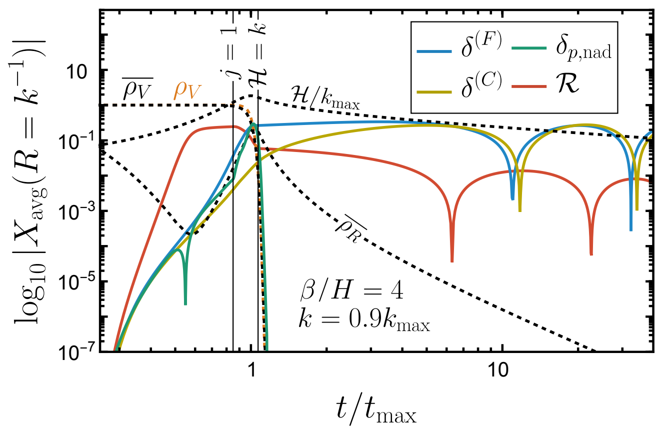

Figure 1. Time evolution of the GI curvature perturbation $\mathcal{R}$, and the density contrast in the spatially-flat $\delta^{(F)}$ and comoving $\delta^{(C)}$ gauge during an FOPT. We show perturbations X averaged over a volume $4\pi k^{-3}/3$. We also report the non-adiabatic pressure δp_{nad} peaking close to Hubble crossing epoch $\mathcal{H} = k$. For presentation purposes we assume $\beta/H = 4$ and show the wavenumber $k = 0.9k_{\mathrm{max}}$, where we identify k_{max} as the maximal $\mathcal{H}$. $j = 1$ indicates the time of first bubble nucleation, which is slightly delayed with respect to average (orange vs black dashed line).

In practice, the evolution of perturbations can be derived as follows: from Eq. (12), (13) and `deltaPT`, we calculate δp_{nad} in Eq. (7), then we solve the closed system of equations in (5) and (9) to determine Φ and $\delta^{(F)}$, see Ref. [51] for more details. The density contrast in the comoving gauge can be derived using the transformation rules (cf. Ref. [51])

$$\delta^{(C)} = \delta^{(F)} + (5+3\omega)\Phi + \frac{2\Phi'}{\mathcal{H}}. \tag{14}$$

Knowing Φ, we can deduce the comoving curvature perturbation $\mathcal{R}$ given by the gauge-invariant expression [50]

$$\mathcal{R} \;\equiv\; \Phi - \mathcal{H}\mathcal{V} \;=\; \frac{5+3\omega}{3+3\omega}\Phi + \frac{2\Phi'}{3(1+\omega)\mathcal{H}}. \tag{15}$$

We show the time evolution of $\delta^{(F)}$, $\delta^{(C)}$, $\mathcal{R}$ and δp_{nap} in Fig. 1, for a single realization of the local nucleation dynamics. One can derive the statistics for these quantities by making multiple realizations of the random bubble nucleation histories using `deltaPT` which relies on the semi-analytical formula introduced in App. A of Ref. [20]. We show the probability distribution function (PDF) of the density contrast in Fig. 2. We show the variance of the density contrast in the spatially-flat and comoving gauges, as well as the GI curvature perturbation in Fig. 3.

Primordial Black Hole formation. Once the density contrast in a Hubble patch exceeds a threshold, it collapses into a PBH [58–62]. Numerical-relativity simulations [63] spec-

[1] We leave a proper de-convolution from the top-hat window function $W(r,R) = \Theta(R-r)/V$ followed by inverse-Fourier transformation for future work.

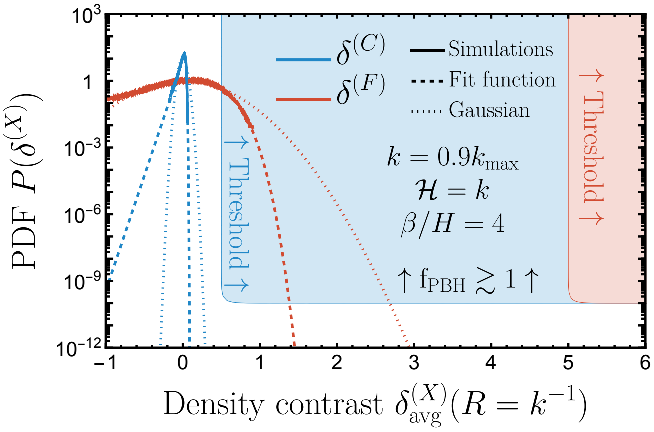

Figure 2. Probability distribution of the density contrast in the comoving gauge averaged over a volume $4\pi k^{-3}/3$, from $N_{\rm sim} = 10^6$ patches caused by the random nucleations time and position of $j_c = 50$ bubbles per patch. The dashed lines show the best fits using the distribution in Eq. (22). The dotted lines shows the best fits using a Gaussian distribution. We choose $k = 0.9k_{\max}$ where $k_{\max}$ is the scale entering at percolation. The colored boxes indicate the collapse condition in the comoving (blue) and spatially-flat (red) gauges. We cut the box at the indicative amplitude for a stellar mass PBH.

ify this threshold using the compaction function at the Hubble crossing epoch t_H [64, 65], related to the density contrast averaged over a Hubble patch $\delta_{\rm avg}(R = k^{-1}, t_H)$ [66, 67], which in our notation is $\delta(k \simeq \mathcal{H})$. As we have seen, the latter is gauge-dependent, see also Ref. [68]. Recent simulations have relied on the comoving gauge, see e.g. [66]. This gauge is advantageous since the 00 component of Einstein's equations reduces to Poisson's equation $\Delta\Phi = 3\mathcal{H}^2\delta^{(C)}/2$. In this gauge, a patch collapses into a PBH if [63, 66, 67]

$$\delta^{(C)}(k \simeq \mathcal{H}) \gtrsim \delta_c^{(C)} \in [0.40,\, 0.67], \tag{16}$$

with the exact value depending on the curvature profile [69]. These simulations assume adiabatic evolution from super-Hubble scales [63, 66, 67]. In our scenario, non-adiabatic effects quickly vanish after percolation (see Fig. 1), so we adopt the usual collapse condition at $k = \mathcal{H}$.

We proceed to estimate the abundance of primordial black holes. Denoting $P(\delta^{(C)})$ as the probability distribution for $\delta^{(C)}$ at $k = \mathcal{H}$, the PBH mass fraction at formation reads

$$\beta_k(M) = \int_{\delta_c^{(C)}}^{\infty} {\rm d}\delta^{(C)} \left(\frac{M(\delta^{(C)})}{M_k}\right) P(\delta^{(C)}). \tag{17}$$

In the first line of Eq. (17), the mass M of the PBH is related to the density contrast through the critical scaling law [70–73]

$$M(\delta^{(C)}) = \mathcal{K} M_k (\delta^{(C)} - \delta_c^{(C)})^{\gamma}, \tag{18}$$

with $\gamma = 0.36$, $\mathcal{K} \sim 3$, and $M_k = 4\pi M_{\rm pl}^2/H_k$ represents the mass within the Hubble sphere when scale k re-enters and $H_k = k/a$. Using that PBHs redshift like matter, we obtain the DM abundance composed of PBH $f_{\rm PBH}$

$$\frac{df_{\rm PBH}}{d\ln(M)} \simeq \int d\ln k \left(\frac{\beta_k(M)}{3\times 10^{-10}}\right)\left(\frac{T_k}{\rm GeV}\right). \tag{19}$$

where T_k is the temperature at Hubble crossing. Approximating spectrum to a nearly monochromatic distribution close to $k \sim k_{\max}$, one parametrically finds

$$f_{\rm PBH} \sim \left(\frac{P(\delta_c^{(C)})}{10^{-10}}\right)\left(\frac{T_k}{\rm GeV}\right), \tag{20}$$

where the Hubble crossing temperature is related to the Hubble mass as

$$M_k(T_k) = 4.8\times 10^{-2} M_\odot \left(\frac{106.75}{g_*}\right)^{1/2}\left(\frac{\rm GeV}{T_k}\right)^2, \tag{21}$$

and g_* is the number of relativistic degrees of freedom.

In Fig. 2 we plot $P(\delta^{(C)})$ at $k = 0.9k_{\max}$. The distribution is negatively skewed—with an extended tail in the negative range and an exponential suppression of large overdensities—reflecting the exponentially low probability of late-blooming patches. As in Ref. [20], the fit

$$P(\delta) \propto \exp\left[\frac{\epsilon}{2}(\delta-\mu) - \frac{2}{\epsilon^2\sigma^2}\left(1 - e^{\frac{\epsilon}{2}(\delta-\mu)}\right)^2\right] \tag{22}$$

(with $\epsilon, \sigma > 0$) accurately describes the data (see companion paper [51]).

Due to the negative non-Gaussianity (NG), even for a completion rate as low as $\beta/H = 4$, $P(\delta^{(C)})$ drops sharply around $\delta^{(C)} \sim 0.1$ and is strongly suppressed above the optimistic PBH threshold $\delta_c^{(C)} \gtrsim 0.4$ in Eq. (16). We found similar results for $\beta/H = 2$, just above the no-completion boundary. We conclude that within the framework considered here, slow FOPTs do not appear to yield PBHs in observable amounts. The PDF for $\delta^{(F)}$ in the flat gauge is broader but still heavily skewed, and with the collapse threshold shifted to $\delta_c^{(F)} \simeq 10\,\delta_c^{(C)}$ – following from Eq. (14) – the conclusion remains unchanged. Note that our analysis neglects non-linear corrections to the density–curvature relation, which would slightly reduce the variance of $\delta^{(C)}$ further [74, 75].

Induced Gravitational Waves. We now focus on the GW spectrum from FOPT dynamics, seizing the relevance of SIGWs from curvature perturbations. The GW abundance today can be written as

$$\Omega_{\rm GW}(k) = \frac{1.7\times 10^{-5}}{g_*^{1/3}(T_k)}\Big[\Omega_{\rm PGW}(k, T_k) + \Omega_{\rm SIGW}(k, T_k)\Big], \tag{23}$$

where $\Omega_{\rm PGW}$ and $\Omega_{\rm SIGW}$ are the spectrum of primary GWs (PGWs) and SIGWs. Primary GW are produced from bubble collision and relativistic shells. Assuming that the latter remain thin and conserve their energy after collision, their GW spectrum is approximated by the broken power-law bulk flow formula [76–78]

$$\Omega_{\rm PGW} \simeq 0.06\left(\frac{H}{\beta}\right)^2 \frac{(a+b) f^a f_\star^b}{\left(a f^{a+b} + b f_\star^{a+b}\right)} S_H(f), \tag{24}$$

with $f_\star = 0.8(\beta/H)(\mathcal{H}/2\pi)$, $a = 0.9$, $b = 2.1$, and $S_H(f)$ imposing a f^3 behavior for $f < \mathcal{H}/2\pi$. SIGWs are given

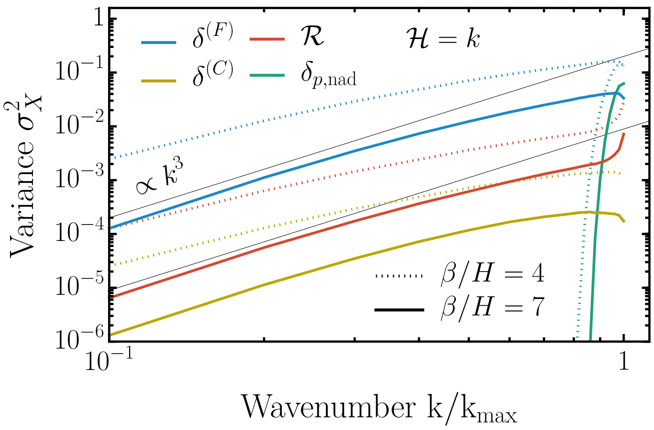

Figure 3. Variance of the perturbations X averaged over a ball of radius k^{-1}, $\sigma_X^2 \equiv \langle X_{\text{avg}}^2(R = k^{-1}, t_H)\rangle$, evaluated at Hubble crossing epoch t_H. We show the comoving curvature perturbation $\mathcal{R}$ (red), the density contrast in the spatially-flat $\delta^{(F)}$ (blue) and comoving gauge $\delta^{(C)}$ (yellow), and the non-adiabatic pressure perturbation $\delta_{p,\text{nad}}$ (green).

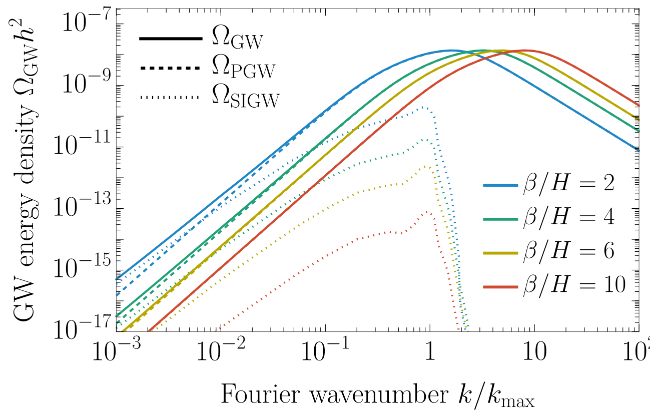

Figure 4. SIGW contribution to the GWs from FOPT. The solid line reports the total Ω_{GW}, while the dotted line the subdominant SIGWs, which includes the NG corrections.

by [38–49]

$$\Omega_{\text{SIGW}}(k,\eta) = \frac{1}{24}\left(\frac{k}{\mathcal{H}(\eta)}\right)^2 \overline{\mathcal{P}_h(k,\eta)}, \tag{25}$$

where the power spectrum induced at second-order from curvature perturbations is

$$\overline{\mathcal{P}_h(k,\eta)}(2\pi)^3\delta_D^{(3)}(\mathbf{k}+\mathbf{k}') = \frac{16k^3}{2\pi^2}\sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \times \overline{J_s(\mathbf{k},\mathbf{p},\eta)J_s(\mathbf{k}',\mathbf{q},\eta)}\langle\tilde{\mathcal{R}}_{\mathbf{p}}\tilde{\mathcal{R}}_{\mathbf{k}-\mathbf{p}}\tilde{\mathcal{R}}_{\mathbf{q}}\tilde{\mathcal{R}}_{\mathbf{k}'-\mathbf{q}}\rangle\,, \tag{26}$$

having defined

$$J_s(\mathbf{k},\mathbf{p},\eta) \equiv Q_s(\mathbf{k},\mathbf{p})I(\mathbf{k},\mathbf{p},\eta) \tag{27}$$

as the product of the polarization tensor contracted with the loop momenta $Q_s(\mathbf{k},\mathbf{p}) = e_s^{ij}(\hat{\mathbf{k}})p_ip_j$, and the kernel function $I(\mathbf{k},\mathbf{p},\eta)$ being the convolution of source and Green function, see e.g. Ref. [79] and references therein for the explicit expressions. The overline denotes a time average over oscillations [80] and s labels the two GW polarizations. In the Gaussian limit, only disconnected contractions contribute to the four-point function in (26), while NGs introduce additional terms proportional to the curvature trispectrum [81–95]. We extract the curvature power spectrum, $\langle\tilde{\mathcal{R}}(\mathbf{k})\tilde{\mathcal{R}}(\mathbf{k}')\rangle = (2\pi)^3\delta_D^{(3)}(\mathbf{k}+\mathbf{k}')(2\pi^2/k^3)\mathcal{P}_{\mathcal{R}}(k)$, from the variance of the volume-averaged curvature perturbation $\sigma_{\mathcal{R}}^2$ shown in Fig. 3 by inverting the formula

$$\sigma_{\mathcal{R}}^2 \equiv \langle\mathcal{R}_{\text{avg}}^2\rangle = \int \frac{\mathrm{d}k}{k}\,|\tilde{W}(k,R)|^2\mathcal{P}_{\mathcal{R}}(k), \tag{28}$$

where $\tilde{W}(k,R)$ is the Fourier-transform of the top-hat window function. Using $\mathcal{P}_{\mathcal{R}}(k) \propto k^3$ we obtain $\mathcal{P}_{\mathcal{R}}(k = R^{-1}) \simeq 2\sigma_{\mathcal{R}}^2/3\pi$. Omission of this factor $2/3\pi$ would lead to an overestimation of the SIGW amplitude by a factor of 20 when employing the variance of the spatially-averaged $\mathcal{R}$ instead of its power spectrum. The NG parameter is defined via the skewness as [96, 97]

$$F_{\text{NL}} = \frac{\langle\mathcal{R}^3\rangle}{6\langle\mathcal{R}^2\rangle^2}. \tag{29}$$

The coefficient F_{NL} depends on k. We leave the study of the scale-dependence for future work and fix $k \simeq 0.9k_{\text{max}}$, close enough to the maximum of $\mathcal{P}_{\mathcal{R}}(k)$ which dominates the GW emission. We find that, for a representative value of $\beta/H = 5$, $F_{\text{NL}} \simeq -2.5$. Even though F_{NL} is negative, the correction to the SIGW spectrum are proportional to F_{NL}^2 from the curvature four-point function, and thus can enhance the SIGW abundance. Fig. 4 shows the resulting GW spectra for various β/H. We conclude that SIGWs are never strong enough to form a secondary peak in the total spectrum and at most affect the far-IR tail due to the logarithmic scaling $\Omega_{\text{SIGW}}(k \ll k_\star) \propto k^3(1 + \tilde{A}\ln^2(k/\tilde{k}))$ [98].

Conclusions. In this work we have employed a covariant formalism to study cosmological perturbations originating from an FOPT, with implications for PBH formation and SIGWs. Upon modeling bubble dynamics on a FLRW background, previous works have implicitly chosen the spatially-flat gauge (F). However, the values of the density contrast at the PBH formation threshold $\delta_c^{(C)} \in [0.4, 0.67]$ and its relation to the comoving curvature perturbation at Hubble crossing $\mathcal{R} \simeq -9\delta^{(C)}/4$ – widely quoted in the PBH literature – are valid in the comoving gauge (C). At Hubble crossing, the density contrast in the spatially flat and comoving gauges are related by $\delta^{(F)} \simeq 10\delta^{(C)}$. Misidentifying these two gauges leads to an underestimation of the PBH collapse threshold by a factor of 10, leading to a dramatic overestimation of the PBH abundance. The same gauge confusion over-estimates the comoving curvature perturbation $\mathcal{R}$ by a factor 10 and therefore overestimates SIGWs by a factor of 10^4. Additionally, we identified a factor of $2/3\pi$ between the variance of the spatially averaged $\mathcal{R}$ and its power spectrum, further suppressing SIGWs by a factor of 20.

Hence, PBH and SIGW production from FOPTs are far smaller than previously thought, potentially placing them beyond experimental reach. This finding has significant implications for ongoing and future gravitational-wave experiments, where the loudest predicted signals from FOPTs are not necessarily ruled out by PBH overproduction constraints [14, 99].

We have neglected gravitational effects on nucleation rates [100] and on the expansion of bubbles [33, 37], as well as the gradient energy of bubble walls [11, 19, 26]. The actual distribution of radiation converted from the vacuum energy is also important in understanding perturbations on small scales $k \gtrsim \mathcal{H}$ [101, 102]. Future work should incorporate these effects and perform full numerical simulations tracking non-linear curvature perturbations and non-adiabatic pressure effects to refine PBH formation criteria.

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