

Generalized Wilson lines and the gravitational scattering of spinning bodies

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Abstract

A generalization of Wilson line operators at subleading power in the soft expansion has been recently introduced as an efficient building block of gravitational scattering amplitudes for non-spinning objects. The classical limit in this picture corresponds to the strict Regge limit, where the Post-Minkowskian (PM) expansion corresponds to the soft expansion, interpreted as a sum over correlations of soft emissions. Building on the well-studied worldline model with $\mathcal{N} = 1$ supersymmetry, in this work we extend the generalized Wilson line (GWL) approach to the case of spinning gravitating bodies. Specifically, at the quantum level we derive from first-principles a representation for the spin 1/2 GWL that is relevant for the all-order factorization of next-to-soft gravitons with fermionic matter, thus generalizing the exponentiation of single-emission next-to-soft theorems. At the classical level, we identify the suitable generalization of Wilson line operators that enables the generation of classical spin observables at linear order in spin. Thanks to the crucial role played by the soft expansion, the map from Grassmann variables to classical spin is manifest. We also comment on the relation between the GWL approach and the Worldline Quantum Field Theory as well as the Heavy Mass Effective Theory formalism. We validate the approach by rederiving known results in the conservative sector at 2PM order.

1 Introduction

Wilson lines are ubiquitous objects in quantum field theories. In particular, when evaluated on paths composed of straight (infinite) segments, they exhibit universal renormalization properties [1] that are extremely powerful in a variety of perturbative calculations. For example, they provide an invaluable tool for a gauge invariant formulation of factorization theorems in QCD [2] which are vital for an all-order resummation of many collider observables. They are crucial building blocks for the construction of effective field theories both in gauge theories [3] and gravity [4] and the systematic treatment of power corrections [5–7]. They are also of primary importance for supersymmetric theories, where maximally helicity violating (MHV) n -legs scattering amplitudes in the planar limit are dual to n -cusped Wilson loops [8]. However, a naive application of Wilson lines is sometimes not enough to cope with more general problems. In these situations, in order to minimally modify the formalism, it is often desirable to generalize these Wilson line operators. This is the case for the aforementioned duality, where supersymmetric generalizations have shown how to extend the relation beyond the MHV case [9]. It is also the case for the soft expansion, where Generalized Wilson Lines (GWLs) have been defined in gauge theories [10, 11] and gravity [12, 13] in order to include power corrections in factorization theorems, thus reorganizing the effect of next-to-soft theorems to an arbitrary number of emissions.

In the light of this wide range of applications, it is natural to ask whether (generalized) Wilson lines could provide a useful representation for the development of analytic tools in the gravitational wave program. In fact, a wealth of new methods has been proposed over the recent years, some of them driven by the possibility to compute observables in classical General Relativity (GR) with Quantum Field Theory (QFT) techniques [14–40]. In this regard, the possibility to map the bound state problem to scattering data [41–47] makes the study of the classical limit of gravitational scattering amplitudes particularly interesting. For weak gravitational fields and large impact parameters, in particular, gravitating bodies can be represented by highly energetic particles glancing off each other and interacting via low energetic gravitons. Therefore, one could naively guess that the soft expansion and the Regge limit, both of which have a natural representation in terms of Wilson lines [48–50], might be key for an efficient identification of classical contributions. Motivated by this picture and building on more recent developments [51, 52], in [13] we argued that suitably defined *classical* Generalized Wilson lines provide indeed a natural language for the computation of observables in the Post Minkowskian (PM) expansion.

More specifically, focusing on the non-spinning case in the conservative sector up to 2PM, in [13] we showed how the classical limit of scattering amplitudes can be extracted by taking their strict Regge limit. This is in contrast to the traditional eikonal approach [53–57] where the PM expansion corresponds to an expansion in¹ t/s . To succeed in this task, we constructed the scalar GWL starting from the worldline quantization of the relativistic particle. This strategy, which is the starting point also for the recent

¹Throughout this work we denote with s, t, u the standard Mandelstam variables.

Worldline Quantum Field Theory (WQFT) approach [21,22,58,59], consists of generating the perturbative series from a path integration over both the graviton field $h^{\mu\nu}$ and the trajectory x of the gravitating body (neglecting of course quantum terms). This aspect differentiates both GWL and WQFT from worldline EFT approaches [60–69], where the only propagating field is the graviton and trajectory deflections are obtained by solving the equations of motion iteratively. However, unlike the WQFT approach, the GWL representation is derived by integrating over x each external line of the scattering amplitude before any graviton integration, thus generating observables by a VEV of suitably defined operators. This provides a connection with the computation of generalized soft functions that have been a focus of recent studies in QCD phenomenology [70–74]. As such, the GWL representation establishes a clear relation between the PM expansion and the soft expansion, interpreted as a sum over correlations among multiple soft emissions rather than a sum over the momentum of the graviton.²

In this paper we generalize the work of [13] by considering the scattering of spinning objects, focusing separately on the quantum case with spin $1/2$ particles and contributions linear in spin at the classical level. Specifically, starting from the well-known $\mathcal{N} = 1$ supersymmetric model for the relativistic particle in curved space, we first derive the exponentiation of soft gravitons dressing the external Dirac states of a quantum scattering amplitude in terms of *soft* GWLs. We then move to the classical problem and derive a suitable representation of 2PM observables in terms of a VEV of *classical* GWLs. There are numerous reasons for carrying out this program.

At the pure classical level, the motivation is fairly obvious, given the high demand of accurate theoretical predictions in gravitational wave astronomy and the pivotal role of spin. In this regard, state-of-the-art PM calculations include the 3PM scattering angle and impulse to quadratic order in spin [78–80], while at 2PM the observables are known to even higher orders in spin [81–87]. In particular, in worldline supersymmetric models, higher-spin degrees of freedom are notoriously difficult to implement, because of well-known no-go theorems [88], though a worldline formulation that includes spin effects at fourth-order in spin at 2PM has been recently proposed [89]. We consider our work as a first step in the development of a new method which might facilitate the computation of higher spin corrections on the worldline [90]. Specifically, although the starting point is based on worldline supersymmetric models as in the WQFT [21], by integrating out all worldline degrees of freedom (i.e. trajectory x and spin variables ψ) we obtain a suitable definition for classical GWLs such that its VEV generates classical observables, hence opening up the possibility to use renormalization techniques developed for Wilson line correlators.

On top of this, at the classical level, we clarify an aspect which has not been addressed in previous supersymmetric worldline calculations and that becomes particularly transparent in our approach. Specifically, when defining classical spin on the worldline one has to map the worldline Grassmann variables ψ representing quantum spin (which are a

²The possibility to control the classical scattering of gravitating body via the soft expansion has been also described in [55,75–77].

necessary ingredient of the supersymmetric theory) to the classical spin tensor $S^{\mu\nu}$ in GR, defined as the canonical momentum conjugated to the angular velocity of an extended body. The procedure discussed in WQFT consists of mapping these supersymmetric degrees of freedom to the worldline EFT, where spin is not represented by supersymmetric Grassmann variables, but rather by a tower of operators containing powers of $S^{\mu\nu}$ multiplied by the corresponding matching coefficients. In our approach, however, the soft expansion in the construction of the GWL removes the need to map to the EFT. Specifically, as we will discuss, the presence of the Lorenz spin generator $\sigma_{\mu\nu}$ in the soft GWL yields a map between the supersymmetric Grassmann variables and the classical spin tensor, such that the classical GWL can be derived in terms of a classical spin tensor.

However, the motivation for this work is not restricted to the pure classical regime. Indeed, at the quantum level, an all-order treatment of subleading soft theorems for spinning particles [91] has continued to attract attention in the recent years [75, 92]. It is therefore natural to ask whether the soft GWL representation of [12] and [13] can be extended to the case of fermionic matter. This generalization is not immediate. Indeed, while it is clear that for a fixed number of emissions one should recover the result of soft theorems, thanks to which spin effects are described by a coupling to the Lorenz spin generator, the role of this generator to all orders in perturbation theory is not immediately evident in a diagrammatic analysis. In particular, it is not clear how correlated subleading soft emissions depend on the spin of the hard emitter. Secondly, in the gauge theory case it has been shown in [11] that worldline supersymmetry is key to prove that the background field in the numerator of dressed propagators does not contribute in the asymptotic limit, making the derivation similar to the more studied quantization on a closed worldline which yields the one-loop effective action. Does a similar mechanism occur in gravity? We shall see that by deriving the soft GWL from first-principles in the supersymmetric worldline model we will generalize single-emission on-shell soft theorems to the case of an arbitrary number of off-shell graviton emissions, including pairwise correlations, and we will thus address all these questions.

The parallel between the soft and the classical GWL is not only useful for the identification of the map to classical spin but also to highlight the different exponentiation of classical and quantum spin variables at a given order in the soft expansion. Specifically, as we shall discuss, the enhancement of spin in the classical limit yields a dependence over classical spin both in the leading eikonal (E) and in the subleading (i.e. next-to-eikonal (NE)) terms, in sharp contrast with the soft GWL which depends on the quantum spin only through single emissions at NE level.

Finally, we shall note that constructing the spinning GWL from first principles in the supersymmetric worldline model enables a relation between the WQFT and other amplitude-based methods. In particular, the relation with the Heavy Mass EFT (HEFT) of [93–95] emerges neatly, although classical observables in terms of on-shell amplitudes in a heavy mass expansion seems quite different from a supersymmetric worldline model where the classical limit is achieved at the Lagrangian level. The GWL approach is somewhat intermediate and can thus be used to provide a map between the two approaches,

and more generally between worldline models and on-shell methods.

The structure of the paper is as follows. In Section 2 we derive the factorization properties of subleading soft gravitons dressing the external states of a quantum scattering amplitude in terms of soft GWLs. In doing so, we devote special care to the quantization of the supersymmetric model since, in order to derive the GWL, we adopt conventions which are somewhat less common in the worldline literature, such as px -ordering and a worldline that extends from a localized hard interaction to infinity. In Section 3 we consider the classical limit, and derive a representation for the classical GWL. We cross-check the formalism with known results at 2PM order. We then comment on the relation between the WQFT and the HEFT approaches. We finally conclude in Section 4. Some technical calculations are presented in separated appendices.

2 Soft exponentiation on the spinning worldline

Building on the well-known $\mathcal{N} = 1$ supersymmetric model for the relativistic particle in curved space [88], in this section we discuss the exponentiation of an arbitrary number of (subleading) soft graviton emissions dressing a fermionic external state of a quantum scattering amplitude. Specifically, we provide a first-principles derivation of the fermionic gravitational generalized Wilson line, thus generalizing the scalar case discussed in [12] and [13].

A number of features make the spinning case distinct from the scalar case and related work discussed in the literature [96–99]. First of all, we consider dressed propagators extending from a localized hard interaction to infinity. This choice, which was also made in previous work on the GWL [10–13], is particular to the case of soft exponentiation and will be revisited in the Regge limit discussed in the second part of this paper. It is a somewhat less explored case in the literature about the worldline formalism, where the length of the line is usually assumed to be finite or, as in the case of the WQFT, from $-\infty$ to $+\infty$. On a more technical side, in contrast to the existing literature on the quantization of the $\mathcal{N} = 1$ model that assumes Weyl ordering, we choose a rather uncommon px -ordering prescription³. This choice is however purely conventional, motivated by the fact that a path integral representation for the GWL in time-slicing regularization is more easily derived from a dressed propagator between an initial state of definite position and a final state of definite momentum.

We begin our discussion in Section 2.1 with a brief review of how a dressed propagator for a spin 1/2 particle in curved space can be expressed in terms of the supersymmetric charges constructed from the corresponding worldline degrees of freedom. We then work out a path integral representation for this dressed propagator in Section 2.2. In doing so, we pay special attention to some technical details concerning the construction of the fermionic path integral. In particular, following a well-established procedure that goes

³In this work px -ordering means that we put all \hat{p} 's to the left of the \hat{x} 's (e.g. $[\hat{x}\hat{p}]_{\text{px}} = \hat{p}\hat{x} + [\hat{x}, \hat{p}]$). For a comparison with the more standard Weyl-ordering see [13].

by the name of fermion doubling [97], we add spurious fermionic degrees of freedom (that we eventually integrate out) to cope with the real nature of the Grassmann fields. In this construction, we also show the elegant role that worldline supersymmetry plays in demonstrating that the background soft graviton field does not contribute to the numerator of the dressed propagator in the asymptotic limit. This feature, which is relevant for spinning emitters, was already discussed in the gauge theory case in [11]. Equipped with the path integral representation, in Section 2.3 we finally solve the integral in a weak field expansion, obtaining the sought representation for the soft Generalized Wilson line.

2.1 From supersymmetric charges to dressed propagators

We begin by considering a free scalar point particle. Apart from convention purposes, this is useful since the scalar contribution will be isolated when discussing the spinor case later on. In flat spacetime, the classical worldline action in phase space with canonical variables $x^\mu(t)$ and $p^\mu(t)$, which fulfill the Poisson bracket $\{x^\mu, p_\nu\}_{\text{PB}} = -\delta_\nu^\mu$. Such action is obtained by enforcing the constraint⁴

$$-p_\mu p_\nu g^{\mu\nu} + m^2 \equiv 2H = 0, \quad (2.1)$$

with Lagrange multiplier $e(t)$, to ensure the on-shell condition. This constraint, which in Dirac terminology is first class and thus generates gauge transformations (time reparameterization), is conventionally defined to be (twice) the Hamiltonian H . The Klein-Gordon equation in flat spacetime then trivially follows after canonical quantization, by identifying $\hat{p} = i\partial_\mu$ and requiring the physical states $|\phi_{\text{phys}}\rangle$ to fulfill $\hat{H}|\phi_{\text{phys}}\rangle = 0$.

The transition from flat to curved spacetime introduces some technical difficulties, mainly due to the metric $g_{\mu\nu}$ being a function of worldline operator $\hat{x}^\mu(t)$. In particular, in analogy with the gauge theory case discussed in [11], the definitions of the Hamiltonian and the corresponding covariant momentum operator $\hat{\Pi}_\mu$ are sensitive to the choice of the operator ordering. For gravity in particular, as discussed in [13], one can define the following Hamiltonian

$$\begin{aligned} \hat{H} &= -\frac{1}{2}\hat{\Pi}_\mu g^{\mu\nu}(-g)^{1/2}\hat{\Pi}_\nu(-g)^{-1/2} + \frac{1}{2}m^2 \\ &= \frac{-g^{\mu\nu}\hat{\Pi}_\mu\hat{\Pi}_\nu + m^2}{2} + \frac{i}{2}g^{\mu\nu}\Gamma_{\mu\nu}^e\hat{\Pi}_e. \end{aligned}$$

Although such Hamiltonian is hermitian, this definition requires the following (non-hermitian) momentum

$$\hat{\Pi}_\mu = (-g)^{-1/2}\hat{p}_\mu(-g)^{1/2}. \quad (2.2)$$

In px -ordering this yields the following Hamiltonian

$$H_{px} = \frac{1}{2} \left(-p_\mu p_\nu g^{\mu\nu} + m^2 + ip_\mu (\partial_\nu g^{\mu\nu} + g^{\mu\nu}\Gamma_{\nu\alpha}^\alpha) \right). \quad (2.3)$$

⁴We adopt the $+, -, -, -$ signature throughout.

The inclusion of additional degrees of freedom such as color or spin can be achieved by enlarging the phase space by additional Grassmann fields. A particle of spin $\frac{N}{2}$ in particular requires \mathcal{N} real fermionic variables ψ_i fulfilling $\{\psi_i^\mu, \psi_j^\nu\}_{\text{PB}} = -\eta^{\mu\nu}\delta_{ij}$. The corresponding action is constructed by adding \mathcal{N} further constraints $Q_i \equiv p_\mu \psi_i^\mu = 0$ and $J_{ij} \equiv i\psi_i^\mu \psi_{j\mu}$, with the corresponding Lagrange multipliers. These constraints generate the \mathcal{N} -extended supersymmetry algebra, which closes under Poisson brackets. In particular, one has

$$i\{Q_i, Q_j\}_{\text{PB}} = 2H\delta_{ij} . \quad (2.4)$$

After quantization, the constraints lead to the massless Bargmann-Wigner equations for the multispinor wave function labeled by \mathcal{N} spin indices. The massive case is more subtle, as it can be obtained by dimensional reduction of a massless model in $d + 1$ dimensions. For practical calculations in four dimensions, it boils down to the introduction of an auxiliary variable ψ^5 .

In this paper we focus on the massive spin $\frac{1}{2}$ case. In flat space, the corresponding worldline action takes the form

$$S = \int d\sigma \left(-p_\mu \dot{x}^\mu + \frac{i}{2} \psi^\mu \dot{\psi}^\nu \eta_{\mu\nu} - \frac{i}{2} \psi^5 \dot{\psi}^5 - eH - \chi Q \right) . \quad (2.5)$$

Here $\chi(t)$ and $e(t)$ denote the gravitino and the einbein fields, which together form a supergravity multiplet of this one-dimensional theory. The constraint Q is then given by

$$Q = i(p_\mu \psi^\mu + m\psi^5) . \quad (2.6)$$

The Grassmann variables ψ^μ and ψ^5 fulfill the following Poisson bracket relations

$$\{\psi^\mu, \psi^\nu\}_{\text{PB}} = -i\eta^{\mu\nu}, \quad \{\psi^5, \psi^5\}_{\text{PB}} = i, \quad (2.7)$$

which immediately results in

$$\{Q, Q\}_{\text{PB}} = i(p^2 - m^2) = -2iH, \quad (2.8)$$

in agreement with the general relation in eq. (2.4). Canonical quantization transforms eq. (2.7) to the following (anti-)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = -i\delta_\nu^\mu, \quad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu}, \quad \{\hat{\psi}^5, \hat{\psi}^5\} = -1 . \quad (2.9)$$

which are realized with

$$\hat{p}_\mu = i\partial_\mu, \quad \hat{\psi}^\mu = \frac{1}{\sqrt{2}}\gamma^\mu, \quad \hat{\psi}^5 = \frac{i}{\sqrt{2}}\gamma^5. \quad (2.10)$$

Correspondingly, \hat{Q} then takes the form

$$\hat{Q} = \frac{\gamma^5}{\sqrt{2}}(i\tilde{\gamma}^\mu \partial_\mu - m), \quad (2.11)$$

where $i\gamma^5\gamma^\mu = \tilde{\gamma}^\mu$ is also a representation of the γ -matrices. As expected, upon implementing the first class constraint for \hat{Q} in Fock space via $\hat{Q}|\psi_{\text{phys.}}\rangle = 0$, we recover the standard Dirac equation in flat space. This observation is key in order to identify the constraints \hat{Q} and \hat{H} in curved space, as we are going to discuss.

Classically, the covariant Dirac equation in curved space reads

$$\left[i\gamma^a e_a^\mu \left(\partial_\mu - \frac{i}{2} \omega_\mu^{ab} \sigma_{ab} \right) - m \right] \Psi = 0, \quad (2.12)$$

where $\omega_\mu^{ab} = e_\nu^a \partial_\mu e^{b\nu} + e_\nu^b \Gamma_{\mu\sigma}^\nu e^{a\sigma}$ is the spin connection expressed via the vierbein e_μ^a and $\sigma_{ab} = \frac{i}{4} [\gamma_a, \gamma_b]$ is the spin 1/2 Lorentz generator. As usual in GR the γ -matrices are defined in the locally flat reference frame defined by the vierbein, which we denote with Roman indices. Correspondingly, after making use of the momentum defined in eq. (2.2) and recalling the flat space expression in eq. (2.6), it is not difficult to see that the quantum mechanical operator \hat{Q} leading to the Dirac equation in curved space should read

$$\hat{Q} = i\hat{\psi}^a \hat{e}_a^\mu \hat{\Pi}_\mu + im\hat{\psi}^5. \quad (2.13)$$

Here, in analogy with eq. (2.2), we defined the covariant momentum as

$$\hat{\Pi}_\mu = \frac{1}{\sqrt{-g}} \hat{p}_\mu \sqrt{-g} + \hat{\omega}_\mu^{ab} \hat{\sigma}_{ab}, \quad (2.14)$$

where

$$\hat{\sigma}_{ab} = \frac{i}{4} [\hat{\psi}_a, \hat{\psi}_b]. \quad (2.15)$$

The definition in eq. (2.14) corresponds to a minimal coupling of the spinning particle to a gravitational background.

Subsequently, to find \hat{H} , we exploit the fact that the supersymmetry algebra is closed, i.e.

$$\{\hat{Q}, \hat{Q}\} = 2\hat{H}. \quad (2.16)$$

After some tedious manipulations, one obtains

$$\hat{H} = \frac{-\hat{g}^{\mu\nu} \hat{\Pi}_\mu \hat{\Pi}_\nu + m^2}{2} + \frac{i}{2} \hat{g}^{\mu\nu} \hat{\Gamma}_{\mu\nu}^e \hat{\Pi}_e - \hat{R}_{ab}{}^{cd} \hat{\sigma}^{ab} \hat{\sigma}_{cd}. \quad (2.17)$$

This Hamiltonian bears some similarity to the scalar case of eq. (2.3). The first and the second term in particular are of precisely the same form, with the covariant momentum of eq. (2.2) modified as in eq. (2.14) to include the spin dependent term. The last term instead contains the coupling of the spinning degrees of freedom to the curvature $R^{\mu\nu\rho\sigma}$, in analogy with the gauge theory case where one has the coupling between $\sigma_{\mu\nu}$ and the field strength tensor $F^{\mu\nu}$ [11].

After having identified the correct quantum operators \hat{Q} and \hat{H} that correspond to the classical first class constraints of the worldline model, we can finally consider the dressed propagator, which is defined as the matrix element of the ratio of these operators.

Specifically, we define the dressed propagator $\mathcal{P}_1(p_f, x_i, \eta)$ for a spin 1/2 particle in the background of a gravitational field as

$$\mathcal{P}_1(p_f, x_i, \eta, \eta^5) = \frac{1}{\langle \bar{\chi}_f | \chi_i \rangle \langle \bar{\chi}_f^5 | \chi_i^5 \rangle \langle p_f | x_i \rangle} \langle f | \frac{\hat{Q}}{-2\hat{H} + i\epsilon} | i \rangle , \quad (2.18)$$

where the initial and final states are

$$\langle f | = \langle p_f | \otimes \langle \bar{\chi}_f, \bar{\chi}_f^5 | \quad \text{and} \quad | i \rangle = | x_i \rangle \otimes | \chi_i, \chi_i^5 \rangle . \quad (2.19)$$

Here, fermions fulfill the twisted boundary conditions

$$\bar{\chi}_f + \chi_i = \sqrt{2}\eta , \quad \bar{\chi}_f^5 + \chi_i^5 = \sqrt{2}\eta^5 , \quad (2.20)$$

where η^μ and η^5 are a set of constant Grassmann variables that eventually generate the spinor structure of the propagator. More specifically, as in the flat spacetime case where $g_{\mu\nu} = \eta_{\mu\nu}$, we need to map⁵

$$\eta^5 \mapsto \frac{i\gamma^5}{\sqrt{2}} , \quad (2.21)$$

$$\eta^{\mu_1} \dots \eta^{\mu_n} \mapsto \frac{1}{n! 2^{\frac{n}{2}}} \epsilon_{j_1, \dots, j_n} \gamma^{\mu_{j_1}} \dots \gamma^{\mu_{j_n}} , \quad (2.22)$$

leaving us with standard gamma matrices. In the next section we work out a path integral representation for the dressed propagator.

2.2 Setting up the path integral

A path integral representation for the dressed propagator in eq. (2.18) can be obtained by using the curved space action corresponding to eq. (2.5) with the appropriate Hamiltonian eq. (2.17) and integrating over all dynamical fields $x^\mu(t), p^\mu(t), \psi^\mu(t), e(t), \chi(t)$. In order to see the equivalence to the operator definition, it is customary to gauge fix the multiplet via $(e(t), \chi(t)) = (T, \theta)$, so that the path integrations over $e(t)$ and $\chi(t)$ reduce to integrations over the proper time T and the fermionic ‘‘supertime’’ θ , respectively. In this way we get

$$\mathcal{P}_1(p_f, x_i, \eta, \eta^5) = e^{-ip_f x_i - \bar{\chi}_f \chi_i + \bar{\chi}_f^5 \chi_i^5} \frac{1}{2} \int d\theta \int_0^\infty dT \mathcal{M}(T, \theta) , \quad (2.23)$$

where the integrations over the the remaining fields give the following matrix element

$$\mathcal{M}(T, \theta) = \langle f | \exp \left\{ -i\theta \hat{Q} - i(\hat{H} - i\epsilon)T \right\} | i \rangle . \quad (2.24)$$

By comparing eq. (2.24) with eq. (2.18), we see that, in analogy with the gauge theory case of [11], the role of the T is to exponentiate the numerator \hat{H} while the Grassmann nature of θ allows the exponentiation of the denominator \hat{Q} .

⁵In the literature this is called symbol map [100].

We aim at a rigorous path integral representation for this matrix element. Thanks to eq. (2.19), the representation for the bosonic part of eq. (2.24) is the same as in the scalar case discussed in [13]. In px -ordering and time slicing it reads

$$\langle p_f | \exp \left\{ i \hat{A}(\hat{x}, \hat{p}) T \right\} | x_i \rangle = e^{ip_f \cdot x_i} \int \mathcal{D}x \mathcal{D}p \exp \left\{ ip_f x(T) + i \int_0^T dt (-p \cdot \dot{x} + A_{\text{px}}(x, p)) \right\} . \quad (2.25)$$

For most applications, $A_{\text{px}}(x, p)$ yields a trivial integration over p , and thus for perturbative calculations the only relevant propagator is the one for the x field, which reads

$$\overline{x^\mu(t) x^\nu(t')} = -\eta^{\mu\nu} \min(t, t') . \quad (2.26)$$

However, it is sometimes necessary to consider derivatives of eq. (2.26), which at equal time contain $\delta(0)$ and are thus divergent. These divergences are canceled by auxiliary ghosts fields, which are introduced by exponentiating the factors $\sqrt{-g(x)}$ in the integration measure, which in turn emerge after integrating over p . The role of ghost fields in the construction of the GWL have been discussed in great detail in [13] and will not be repeated here. We can safely ignore ghosts in the following.

The construction of fermionic path integrals, on the other hand, relies on coherent states, which in turn require the notion of creation and annihilation operators. Since the supersymmetric $\mathcal{N} = 1$ worldline model only contains a single real (i.e. Majorana) Grassmann field $\hat{\psi}_a$, there is no natural way to construct creation and annihilation operators. A well-known possibility consists of doubling the fermionic degrees of freedom by introducing unphysical fields that eventually must be integrated out. Although this procedure has been thoroughly discussed in the literature [97], one has to carefully verify that the method works with the ordering prescription and the somewhat unusual boundary conditions we are adopting. We discuss this in the next section.

2.2.1 Fermionic path integral

We first rename $\hat{\psi}^a \rightarrow \hat{\psi}_1^a$ and $\hat{\psi}^5 \rightarrow \hat{\psi}_1^5$, and introduce free, unphysical fermionic operators $\hat{\psi}_2^a$ and $\hat{\psi}_2^5$. The linear combinations

$$\hat{\psi}_a = \frac{1}{\sqrt{2}} \left(\hat{\psi}_1^a + i \hat{\psi}_2^a \right), \quad \hat{\psi}^5 = \frac{1}{\sqrt{2}} \left(\hat{\psi}_1^5 + i \hat{\psi}_2^5 \right), \quad (2.27)$$

provide then the desired algebra, since they fulfill

$$\left\{ \hat{\psi}^a, \hat{\psi}^{b\dagger} \right\} = \eta^{ab} \mathbf{1}, \quad \left\{ \hat{\psi}^5, \hat{\psi}^{5\dagger} \right\} = -\mathbf{1}. \quad (2.28)$$

This enables the construction of a Fock space, where the vacuum state $|\Omega\rangle$ is annihilated by $\hat{\psi}^a$ and $\hat{\psi}^5$, while the excited states fulfill

$$|a\rangle \equiv \hat{\psi}_a^\dagger |\Omega\rangle, \quad \langle a| \equiv \langle \Omega| \hat{\psi}^a, \quad \langle a|b\rangle = \delta_b^a, \quad (2.29)$$

with analogous construction for ψ^5 . A coherent state is then given by

$$|\chi\rangle = \exp\left\{\hat{\psi}_a^\dagger \chi^a\right\} |\Omega\rangle, \quad (2.30)$$

with normalization

$$\langle \bar{\chi} | \chi \rangle = e^{\bar{\chi}^a \chi^a}. \quad (2.31)$$

Equipped with coherent states, we now aim at writing a path integral representation for matrix elements of the form

$$\langle \bar{\chi}_f | \exp\left\{\hat{A}(\hat{\psi}, \hat{\psi}^\dagger)T\right\} |\chi_i\rangle. \quad (2.32)$$

We first write the identity operator in terms of coherent states as

$$\mathbb{1} = \int d^4 \bar{\chi} d^4 \chi |\chi\rangle e^{-\bar{\chi} \chi} \langle \bar{\chi} |, \quad (2.33)$$

with the product measure defined as

$$d^n \bar{\chi} d^n \chi = \prod_{a=1}^n d\bar{\chi}_a d\chi^a. \quad (2.34)$$

As usual in time slicing, we split the length of the time interval T into N pieces τ and insert $N - 1$ coherent state completeness relations $\mathbb{1}_i$, defined as

$$\mathbb{1}_i = \int d^4 \bar{\chi}_i d^4 \chi_i |\chi_i\rangle e^{-\bar{\chi}_i \chi_i} \langle \bar{\chi}_i |. \quad (2.35)$$

Setting $\bar{\chi}_f = \bar{\chi}_N$ and $\chi_i = \chi_0$ we get

$$\begin{aligned} \langle \bar{\chi}_f | \exp\left\{\hat{A}(\hat{\psi}, \hat{\psi}^\dagger)T\right\} |\chi_i\rangle &= \langle \bar{\chi}_N | e^{\hat{A}\tau} \mathbb{1}_{N-1} \dots \mathbb{1}_1 e^{\hat{A}\tau} |\chi_0\rangle \\ &= \prod_{j=1}^{N-1} (d^4 \bar{\chi}_j d^4 \chi_j e^{-\bar{\chi}_j \chi_j}) \prod_{n=1}^N \langle \bar{\chi}_n | e^{\tau \hat{A}} |\chi_{n-1}\rangle. \end{aligned} \quad (2.36)$$

We then expand in τ the matrix elements between states at n and $n - 1$

$$\langle \bar{\chi}_n | e^{\hat{A}\tau} |\chi_{n-1}\rangle = \langle \bar{\chi}_n | \mathbb{1} + \hat{A}_{\text{px}} \tau + \dots | \chi_{n-1}\rangle = \exp\left\{A_{\text{px}}(\chi_{n-1}, \bar{\chi}_n) \tau + \mathcal{O}(\tau^2)\right\} \langle \bar{\chi}_n | \chi_{n-1}\rangle. \quad (2.37)$$

Here we defined px -ordering for the Grassmann fields via

$$\left(\hat{\psi}^a \hat{\psi}_b^\dagger\right)_{\text{px}} = \delta_b^a - \hat{\psi}_b^\dagger \hat{\psi}^a, \quad (2.38)$$

in analogy to the bosonic operators and the function $A_{\text{px}}(\chi_{n-1}, \bar{\chi}_n)$ is obtained by replacing in the ordered operator all operators with their eigenvalues on the coherent states. With the shorthand $A_n = A_{\text{px}}(\chi_{n-1}, \bar{\chi}_n)$, we get

$$\begin{aligned} &\langle \bar{\chi}_f | \exp\left\{\hat{A}(\hat{\psi}, \hat{\psi}^\dagger)T\right\} |\chi_i\rangle \\ &= \prod_{j=1}^{N-1} (d^4 \bar{\chi}_j d^4 \chi_j) \exp\left\{\bar{\chi}_N \chi_{N-1} - \sum_{n=1}^{N-1} \bar{\chi}_n (\chi_n - \chi_{n-1}) + \sum_{n=1}^N \tau A_n\right\}, \end{aligned} \quad (2.39)$$

which in the continuum limit becomes

$$\langle \bar{\chi}_f | \exp \left\{ \hat{A}(\hat{\psi}, \hat{\psi}^\dagger) T \right\} | \chi_i \rangle = \int_{\chi^{(0)=\chi_i}}^{\bar{\chi}^{(T)=\bar{\chi}_f}} \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left\{ \bar{\chi}_f \chi(T) + \int_0^T dt (-\bar{\chi} \dot{\chi} + A) \right\}. \quad (2.40)$$

At this point it seems we achieved our goal, with eq. (2.40) being the fermionic equivalent of eq. (2.25). However, the path integral in eq. (2.40) depends on the nonphysical imaginary part ψ_2 . In principle this is not a problem, since by construction the final result must depend only on the physical operator $\hat{\psi}_1$. The representation in eq. (2.40) is in fact what is typically used for computations in the literature (see e.g. [97]). However, in order to make manifest that these spurious degrees of freedom do not alter physical predictions, it would be desirable to have a path integral representation that explicitly depends only on the physical fields. Since to the best of our knowledge this has not been discussed in the literature, and because we are adopting a rather unconventional ordering prescription, we present a first-principle construction for such representation in Appendix A. The final result is that one can integrate out ψ_2 such that eq. (2.40) becomes

$$\langle \bar{\chi}_f | \exp \left\{ \hat{A}(\hat{\psi}, \hat{\psi}^\dagger) T \right\} | \chi_i \rangle = \int_{\psi_1(0)+\psi_1(T)=2\eta} \mathcal{D}\psi_1 \exp \left\{ \bar{\chi}_f \chi_i + \int_0^T dt \left(-\frac{1}{2} \psi_1 \dot{\psi}_1 + A(\psi_1) \right) \right\}. \quad (2.41)$$

with the corresponding propagator

$$\overline{\psi_1^a(t) \psi_1^b(t')} = \frac{1}{2} \eta^{ab} (\theta(t-t') - \theta(t'-t)). \quad (2.42)$$

Few comments are in order here. First, the boundary condition of the path integral follows from eq. (2.20) and the fact that $\psi_1(0) = \sqrt{2}\Re(\chi_i)$ and $\psi_1(T) = \sqrt{2}\Re(\bar{\chi}_f)$. Secondly, the continuum propagator for ψ_1^a is in agreement with the intuitive picture of inverting the differential operator in the kinetic term. Finally, as mentioned, the operator $\hat{A}(\psi, \psi^\dagger)$ needs to be related to the function $A(\psi_1)$ via the chosen ordering prescription. We discuss this aspect in the next section.

2.2.2 An ordering prescription for \hat{Q} and \hat{H}

The results of eq. (2.25) and eq. (2.41) yields a path integral representation for eq. (2.24). However, there is still a crucial ingredient missing, namely that we need to map the operators \hat{Q} of eq. (2.13) and \hat{H} of eq. (2.17) to classical expressions, in analogy to the scalar case of eq. (2.3). In particular, on top of ordering the operators \hat{p} and \hat{x} , one has to assign a classical phase space function to products of the operator $\hat{\psi}^a$, contained both in \hat{Q} and \hat{H} through the operator $\hat{\sigma}^{ab}$ defined in eq. (2.15). Once again, fermion doubling turns out to be a powerful tool.

First, using eq. (2.9) we px -order the bosonic operators in eq. (2.17) thereby isolating the scalar contribution

$$\hat{H}_{\text{px}} = \hat{H}_{\text{px}}^{\text{scalar}} - \hat{p}_\mu \hat{g}^{\mu\nu} \hat{\omega}_\nu + \frac{1}{2} [i \nabla_\nu \hat{\omega}^\nu - \hat{\omega}_\mu \hat{\omega}_\nu \hat{g}^{\mu\nu}] - \hat{R}_{ab}{}^{cd} \hat{\sigma}^{ab} \hat{\sigma}_{cd}, \quad (2.43)$$

where $\hat{\omega}_\mu = \hat{\omega}_\mu^{ab} \hat{\sigma}_{ab}$ and the scalar Hamiltonian reads [13]

$$\hat{H}_{\text{px}}^{\text{scalar}} = \frac{1}{2} \left(-\hat{p}_\mu \hat{p}_\nu \hat{g}^{\mu\nu} + m^2 + i\hat{p}_\mu \left(\partial_\nu \hat{g}^{\mu\nu} + \hat{g}^{\mu\nu} \hat{\Gamma}_{\nu\alpha}^\alpha \right) \right). \quad (2.44)$$

For the Grassmann operators, we replace $\hat{\psi}_1$ with $\hat{\psi}$ and $\hat{\psi}^\dagger$ by applying eq. (2.27) and move all $\hat{\psi}^\dagger$'s to the left. After ordering, the operators can be evaluated on the coherent states at their time slice to obtain the phase space function $A_{\text{px}}(\bar{\chi}, \chi)$ depending on $\bar{\chi}$ and χ . Integrating out the spurious imaginary part leaves just a dependence $A(\psi_1)$ on the original real Grassmann field ψ_1 (see eq. (2.41)). This lengthy procedure can be conveniently bypassed by means of the following Wick-like theorem for ordered products of Majorana operators, which reads

$$\left\langle \prod_{i=1}^n \hat{\psi}_1^{a_i} \right\rangle = \sum_{\substack{\text{possible} \\ \text{contractions}}} \left[\text{contractions} \times \prod_{i \text{ not contracted}} \langle \hat{\psi}_1^{a_i} \rangle \right]. \quad (2.45)$$

Here, we defined the expectation value via

$$\psi_1^a \equiv \langle \hat{\psi}_1^a \rangle = \left\langle \frac{\hat{\psi}^a + \hat{\psi}^{a\dagger}}{\sqrt{2}} \right\rangle \equiv \frac{1}{\langle \bar{\chi} | \chi \rangle} \langle \bar{\chi} | \frac{\hat{\psi}^a + \hat{\psi}^{a\dagger}}{\sqrt{2}} | \chi \rangle = \frac{\bar{\chi}^a + \chi^a}{\sqrt{2}}, \quad (2.46)$$

and the contraction of two Majorana operators as

$$\overline{\hat{\psi}_1^a \hat{\psi}_1^b} = \frac{1}{2} \left\{ \hat{\psi}_1^a, \hat{\psi}_1^b \right\} = \frac{\eta^{ab}}{2}. \quad (2.47)$$

We also adopted the convention

$$\overline{\hat{\psi}_1^a \hat{\psi}_1^b \hat{\psi}_1^c} = -\overline{\hat{\psi}_1^a \hat{\psi}_1^c \hat{\psi}_1^b}. \quad (2.48)$$

The outcome of eq. (2.45) is a correspondence between products of operators $\hat{\psi}_1$ and the function of ψ_1 to be used in the path integral. In particular, the operators appearing in the Hamiltonian and supersymmetry get replaced as follows

$$\hat{\sigma}_{ab} \rightarrow \frac{i}{2} \left(\langle \psi_a \rangle \langle \psi_b \rangle + \overline{\psi_a \psi_b} \right) = \frac{i}{2} \left(\psi_a \psi_b + \frac{\eta_{ab}}{2} \right), \quad (2.49)$$

$$\begin{aligned} \hat{\sigma}_{ab} \hat{\sigma}_{cd} \rightarrow & -\frac{1}{4} \left(\psi_a \psi_b \psi_c \psi_d + \frac{1}{4} (\eta_{ab} \eta_{cd} + \eta_{ad} \eta_{cb} - \eta_{ac} \eta_{bd}) \right. \\ & \left. + \frac{1}{2} (\eta_{ab} \psi_c \psi_d + \eta_{cd} \psi_a \psi_b - \eta_{ac} \psi_b \psi_d - \eta_{bd} \psi_a \psi_c + \eta_{bc} \psi_a \psi_d + \eta_{ad} \psi_b \psi_c) \right). \end{aligned} \quad (2.50)$$

We further notice that when inserting these terms in the Hamiltonian their contributions drastically simplify due to the following contractions

$$\hat{\omega}_\mu^{ab} \hat{\sigma}_{ab} = \frac{i}{2} \omega_\mu^{ab} \psi_a \psi_b, \quad (2.51)$$

$$\hat{R}^{abcd} \hat{\sigma}_{ab} \hat{\sigma}_{cd} = -\frac{1}{4} R^{abcd} \psi_a \psi_b \psi_c \psi_d + \frac{1}{8} R = \frac{1}{8} R, \quad (2.52)$$

$$\hat{g}^{\mu\nu} \hat{\omega}_\mu^{ab} \hat{\omega}_\nu^{cd} \hat{\sigma}_{ab} \hat{\sigma}_{cd} = -\frac{1}{4} g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{cd} \psi_a \psi_b \psi_c \psi_d + \frac{1}{8} g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{ab}, \quad (2.53)$$

where in eq. (2.51) we made use of the antisymmetry of the spin connection and in eq. (2.52) we used $R^{a[bcd]} = 0$.

In conclusion, by making use of eq. (2.51), eq. (2.52) and eq. (2.53) in eq. (2.43) and eq. (2.13), the classical expressions for H and Q that we use in the path integral take the following form

$$Q_{\text{px}} = i \left(\psi^a e_a^\mu (p_\mu + \frac{i}{2} \omega_\mu^{cd} \psi_c \psi_d) + m \psi_5 - \psi^a i \partial_\mu (e_a^\mu \sqrt{-g}) \frac{1}{\sqrt{-g}} + \frac{i}{2} e_c^\mu \omega_\mu^{cd} \psi^d \right) \quad (2.54)$$

and

$$H_{\text{px}} = H_{\text{px}}^{\text{scalar}} - \frac{i}{2} p_\mu g^{\mu\nu} \omega_\nu^{ab} \psi_a \psi_b - \frac{1}{4} \nabla_\nu g^{\nu\sigma} \omega_\sigma^{ab} \psi_a \psi_b + \frac{1}{8} g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{cd} \psi_a \psi_b \psi_c \psi_d - \frac{1}{16} g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{ab} - \frac{1}{8} R. \quad (2.55)$$

With eq. (2.25), eq. (2.41), eq. (2.54) and eq. (2.55) we have all ingredients to write a path integral representation for the matrix element of eq. (2.24), which reads

$$\begin{aligned} \mathcal{M}(T, \theta) = & \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}x \mathcal{D}p \int_{\psi_1(0)+\psi_1(T)=2\eta}^{\psi_1^5(0)+\psi_1^5(T)=2\eta^5} \mathcal{D}\psi_1 \int \mathcal{D}\psi_1^5 \exp \left\{ ip_f x(T) + \bar{\chi}_f \chi_i - \bar{\chi}_f^5 \chi_i^5 \right. \\ & \left. + i \int_0^T dt \left(-p\dot{x} + \frac{i}{2} \psi_1 \dot{\psi}_1 - \frac{i}{2} \psi_1^5 \dot{\psi}_1^5 - H_{\text{px}} - \frac{\theta}{T} Q_{\text{px}} \right) \right\}, \quad (2.56) \end{aligned}$$

where the constraints H_{px} and Q_{px} refer to the px-ordered operators \hat{H}_{px} and \hat{Q}_{px} evaluated on the eigenstates of the phase space variables at time t .

2.2.3 A trick with conserved charges and the asymptotic limit

At this point, to perform the integration over ψ^5 and further simplify the expression in eq. (2.56), we perform a trick that has been already exploited in the gauge theory case [11]⁶ and that becomes particularly useful when taking the asymptotic limit $T \rightarrow \infty$. We consider the integration over θ in eq. (2.23) and eq. (2.24) and we first simplify

$$\int d\theta \exp \left\{ -i\theta \frac{1}{T} \int_0^T Q_{\text{px}}(t) dt \right\} = -i \frac{1}{T} \int_0^T Q_{\text{px}}(t) dt. \quad (2.57)$$

The integration over the phase space variables in eq. (2.56) thus corresponds to the expectation value $\langle Q_{\text{px}} \rangle$. However, \hat{Q} is a conserved charge due to the supersymmetry of the worldline model and it is therefore constant. This can also be seen by observing that $[\hat{H}, \hat{Q}] = 0$. The above expression can therefore be simplified further into

$$-i \frac{1}{T} \int_0^T Q_{\text{px}}(t) dt \rightarrow -i \frac{1}{T} \int_0^T \langle \hat{Q}_{\text{px}}(t) \rangle dt = -i \langle \hat{Q}_{\text{px}}(t) \rangle. \quad (2.58)$$

⁶See also [101].

Being a constant, $Q_{\text{px}}(t)$ can be evaluated at an arbitrary time. We choose $t = T$, to get

$$\begin{aligned} \mathcal{M}(T, \theta) = \langle -i\theta \hat{Q}_{\text{px}}(T) \rangle & \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}x \mathcal{D}p \int_{\psi(0)+\psi(T)=2\eta}^{\psi^5(0)+\psi^5(T)=2\eta^5} \mathcal{D}\psi \int \mathcal{D}\psi^5 \exp \left\{ ip_f x(T) \right. \\ & \left. + \bar{\chi}_f \chi_i - \bar{\chi}_f^5 \chi_i^5 + i \int_0^T dt \left(-p\dot{x} + \frac{i}{2}\psi\dot{\psi} - \frac{i}{2}\psi^5\dot{\psi}^5 - H_{\text{px}} \right) \right\}. \end{aligned} \quad (2.59)$$

where we dropped the subscript of ψ_1 for notational convenience.

There are a number of advantages that follow from having factorized the expectation value of $\hat{Q}(T)$. The most obvious one is that the integration over ψ^5 can be done straightforwardly, since the dependence on ψ^5 is entirely in the free kinetic term, giving unity. Similarly, eq. (2.55) depends quadratically on the momentum and thus the integral over p^μ is Gaussian and yields

$$\mathcal{M}(T, \theta) = \langle -i\theta Q_{\text{px}}(T) \rangle e^{ip_f x(T) + \bar{\chi}_f \chi_i - \bar{\chi}_f^5 \chi_i^5 - \frac{i}{2}m^2 T} f(x^\mu, \psi, g_{\mu\nu}, T) .$$

where

$$\begin{aligned} f(x^\mu, \psi, g_{\mu\nu}, T) = \int_{x(0)=x_i}^{\psi(0)+\psi(T)=2\eta} \mathcal{D}x \int \mathcal{D}\psi \exp \left\{ i \int_0^T dt \left(\frac{i}{2}\psi\dot{\psi} - \frac{1}{2}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} + \frac{i}{2}\dot{x}^\mu g_{\mu\nu} g^{\tau\lambda} \Gamma_{\tau\lambda}^\nu + \right. \right. \\ \left. \left. + \frac{i}{2}\dot{x}^\mu \omega_\mu^{ab} \psi_a \psi_b + \frac{1}{4}g^{\mu\nu} \partial_\nu \omega_\mu^{ab} \psi_a \psi_b + \frac{1}{8} \left(R + \frac{1}{2}g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{ab} + g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \Gamma_{\alpha\beta}^\mu \Gamma_{\gamma\delta}^\nu \right) \right) \right\}. \end{aligned} \quad (2.60)$$

However, to fully appreciate the factorization of $Q(T)$ above, we have to proceed as in the previous work on the GWL [10, 11, 13] and LSZ truncate the dressed propagator. In doing so, we parameterize the path integral over x via

$$x(t) = x_i + p_f t + \tilde{x}(t) , \quad (2.61)$$

and eventually consider the on-shell limit $p_f^2 \rightarrow m^2$. Specifically, we consider the following chain of equalities

$$\begin{aligned} \bar{u}(p_f) \frac{p_f^2 - m^2 + i\epsilon}{i(\not{p}_f + m)} \mathcal{P}_1(p_f, x_i, \eta, \eta^5) \\ = \bar{u}(p_f) \int_0^\infty dT \frac{d}{dT} \left(e^{\frac{i}{2}(p_f^2 - m^2)T} \right) \int d\theta \theta \frac{\langle -iQ_{\text{px}}(T) \rangle}{i(\not{p}_f + m)} e^{-\frac{1}{2}\epsilon T} f(x_i, p_f, \eta, T) \\ = \bar{u}(p_f) \lim_{T \rightarrow \infty} \frac{\langle -iQ_{\text{px}}(T) \rangle}{i(\not{p}_f + m)} f(x_i, p_f, \eta, T) , \end{aligned} \quad (2.62)$$

where we first integrated by parts, then we set the Feynman ϵ to zero and only at the very end we took the on-shell limit. In this way, we are left with the asymptotic boundary term i.e. the dressed propagator evaluated at $T \rightarrow \infty$.

The value of having the charge $\hat{Q}(t)$ evaluated at $t = T$ is now evident. In fact, what seemed at first glance a very technical trick that allowed us to trivially perform the integration over ψ^5 , has in fact a clear physical interpretation. By evaluating the charge \hat{Q} asymptotically for $T \rightarrow \infty$, we are in fact replacing the charge Q in the presence of a gravitational background defined in eq. (2.13) with the free charge in flat spacetime, i.e.

$$\lim_{T \rightarrow \infty} \langle \hat{Q}_{\text{px}}(T) \rangle = \lim_{T \rightarrow \infty} (i\eta^a e_a^\mu(x(T)) p_\mu + im\eta^5) = i\eta^\mu p_\mu + im\eta^5, \quad (2.63)$$

Stated differently, the background field does not contribute to the numerator Q of the asymptotic propagator (which determines the structure of its spin indices) but only to the denominator H , which contains the coupling between spin and curvature. We have thus extended the gauge-theory result of [11], by showing that also in gravitational theories there is a clear connection between the worldline representation of an asymptotic dressed propagator and the more studied worldline quantization on the circle (i.e. the one-loop effective action), which by definition is constructed with denominators only (see e.g. [102]).

In conclusion, we are left with the following path-integral representation for the asymptotic propagator (i.e. the LSZ-truncated dressed propagator)

$$\frac{p^2 - m^2 + i\epsilon}{i(p+m)} \mathcal{P}_1(x_i, p_f, \eta) = \int_{x(0)=0}^{\psi(0)+\psi(\infty)=2\eta} \mathcal{D}x \int \mathcal{D}\psi \exp \left\{ i \int_0^\infty dt e^{-ct} L_{\text{spin}}(t) \right\}, \quad (2.64)$$

where

$$L_{\text{spin}}(t) = L_{\text{scalar}} + \frac{i}{2} \psi \dot{\psi} + \frac{i}{2} (\dot{x} + p)^\mu \omega_\mu^{ab} \psi_a \psi_b + \frac{1}{4} g^{\mu\nu} \partial_\nu \omega_\mu^{ab} \psi_a \psi_b + \frac{1}{8} R + \frac{1}{16} g^{\mu\nu} \omega_\mu^{ab} \omega_\nu^{ab}. \quad (2.65)$$

Here, L_{scalar} is just the x kinetic term together with the same counter terms in the second line as in [13] (with eq. (4.13) inserted into eq. (4.19) in that paper) and reads

$$L_{\text{scalar}}(t) = -\frac{1}{2} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \frac{1}{2} (p^\mu p^\nu + 2p^\mu \dot{x}^\nu + \dot{x}^\mu \dot{x}^\nu) (g_{\mu\nu} - \eta_{\mu\nu}) + \frac{i}{2} (p + \dot{x})^\mu g_{\mu\nu} \Gamma_{\rho\sigma}^\nu g^{\rho\sigma} + \frac{1}{8} g_{\mu\nu} \Gamma_{\rho\sigma}^\nu g^{\rho\sigma} \Gamma_{\alpha\beta}^\mu g^{\alpha\beta}. \quad (2.66)$$

Note that although the last two terms in eq. (2.65) are ψ -independent they are not included in L_{scalar} , since they are counterterms that originate from the ordering prescription for the Grassmann fields, and hence are absent in the scalar case. We also dropped the notation from eq. (2.61) and eq. (2.27) and renamed the variables $\tilde{x} \rightarrow x$, $p_f \rightarrow p$ and $\psi_1 \rightarrow \psi$. Note that the dependence on x_i is now hidden in the argument of the graviton field, which needs to be evaluated at $x_i + pt + x(t)$. The remaining integrations over x and ψ cannot be carried out exactly. In the next section we will perform this task perturbatively in the soft expansion. The final result will be an exponential form for the LSZ-truncated dressed propagator that we call GWL.

2.3 Generalized Wilson line

In the previous section we derived an exact result for the path integral representation of the LSZ-truncated Dirac propagator in the presence of a background gravitational field. No assumption on this background has been made. We now perform the first approximation and expand the metric around a Minkowski background by defining the graviton $h_{\mu\nu}$ via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} , \quad (2.67)$$

where $\kappa^2 = 32\pi G$, with G the Newton constant. Correspondingly, the Lagrangian of eq. (2.65) expanded to second order in κ reads

$$\begin{aligned} L_{\text{spin}}(t) = & L_{\text{scalar}}(t) + \frac{i}{2}\psi\dot{\psi} + \frac{i\kappa}{2}(\dot{x}^\mu + p^\mu)\partial^b h_\mu^a \psi_a \psi_b \\ & + \frac{i\kappa^2}{2}(\dot{x}^\mu + p^\mu) \left(\frac{1}{2} (h^{a\lambda}\partial_\lambda h_\mu^b + h^{b\lambda}\partial^a h_{\mu\lambda}) \psi_a \psi_b + \frac{1}{4} h^b_\lambda \partial_\mu h^{a\lambda} \psi_a \psi_b \right) \\ & + \frac{\kappa^2}{4} \left(\partial^b \partial_\mu h^{\mu a} - h^{\mu\nu} \partial_\nu \partial^b h_\mu^a + \frac{1}{4} h^b_\mu \square h^{\mu a} \right. \\ & \left. + \frac{1}{2} (h^{a\lambda} \partial_\mu \partial_\lambda h_\mu^b + \partial_\mu h^{b\varrho} \partial^a h_{\mu\varrho} + h^{b\varrho} \partial_\mu \partial^a h_{\mu\varrho}) \right) \psi_a \psi_b + \\ & + \frac{\kappa}{8} (\partial_\mu \partial_\nu h^{\mu\nu} - \square h) + \frac{\kappa^2}{8} \left(h^{\mu\nu} \square h_{\mu\nu} + h^{\mu\nu} \partial_\mu \partial_\nu h - \left(\partial_\mu h^{\mu\alpha} - \frac{1}{2} \partial^\alpha h \right)^2 \right. \\ & \left. + \frac{3}{4} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} - h^{\mu\nu} \partial_\mu \partial_\alpha h_\nu^\alpha - \frac{1}{2} \partial_\mu h_{\mu\alpha} \partial^\alpha h_{\mu\nu} + \frac{1}{2} \partial^{[\alpha} h^{\beta]\mu} \partial^{[\alpha} h_\mu^{\beta]} \right) , \quad (2.68) \end{aligned}$$

where the square bracket in the last term denote index antisymmetrization and

$$\begin{aligned} L_{\text{scalar}}(t) = & -\frac{1}{2} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \frac{\kappa}{2} h_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + p^\mu p^\nu + 2\dot{x}^\mu p^\nu) \\ & + \frac{i\kappa}{2} (\dot{x}^\mu + p^\mu) \left(\partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h \right) + \frac{i\kappa^2}{2} (\dot{x}^\mu + p^\mu) \left(\frac{1}{4} \partial_\mu (h^{\varrho\sigma} h_{\varrho\sigma}) - h^{\nu\sigma} \partial_\sigma h_{\mu\nu} \right) \\ & + \frac{\kappa^2}{8} \left(\partial_\alpha h^{\alpha\mu} - \frac{1}{2} \partial^\mu h \right)^2 \quad (2.69) \end{aligned}$$

Once again, we explicitly isolated the scalar contribution, previously computed in [13].

The second approximation, which is key in deriving the exponential form of the GWL, is to perform the soft expansion, defined by the limit $p \gg k$, with k being the momentum of a soft emission dressing the fermion propagator. In the gauge theory case discussed in [10, 11] this procedure is straightforward, since one can solve the path integral order by order in the soft expansion, but to all-orders in the coupling constant. This must be clearly revisited in gravity since, unlike the gauge theory case, the background gravitational field is intrinsically defined perturbatively via eq. (2.67). Apart from this aspect, the gauge-gravity parallel holds. We thus rescale $p \rightarrow \lambda p$, $t \rightarrow t/\lambda$, and $\kappa \rightarrow \kappa/\lambda$, where λ is

a bookkeeping parameter (eventually set to one) that controls the soft expansion in the limit $\lambda \rightarrow \infty$. Borrowing a terminology used in [10], we denote the leading term as eikonal (E), the subleading as next-to-eikonal (NE), and so forth. At NE level, the Lagrangian then becomes⁷

$$\begin{aligned} L_{\text{spin}}(t) &= \frac{i}{2}\psi\dot{\psi} - \frac{\lambda}{2}\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} - \frac{\kappa}{2}h_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu + p^\mu p^\nu + 2\dot{x}^\mu p^\nu) + \\ &\quad + \frac{i\kappa}{2\lambda}p^\mu \left(\partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h + \partial^b h_\mu^a \psi_a \psi_b \right) + \mathcal{O}(\lambda^{-2}) \\ &= L_{\text{scalar}}(t) + \frac{i}{2}\psi\dot{\psi} + \frac{i\kappa}{2\lambda}p^\mu \partial^b h_\mu^a \psi_a \psi_b. \end{aligned} \quad (2.70)$$

Note that only the first line of eq. (2.68) contributes to eq. (2.70), since the other terms are $\mathcal{O}(\lambda^{-2})$ hence sub-subleading. Recalling that we substituted $x(t) \rightarrow x_i + pt + x(t)$ in the path integral representation of the asymptotic propagator, Feynman rules for the worldline fields x and ψ can then be generated by expanding $h^{\mu\nu}(x_i + pt + x(t))$ in powers of $x(t)$ around $x = 0$. Crucially, as evident in the kinetic terms of eq. (2.70), the two-point correlator of x and its time derivatives scale like $1/\lambda$ while the fermionic correlator scales like λ^0 . Since the fermionic vertex is of order $1/\lambda$, only a finite number of diagrams are necessary at any given order in $1/\lambda$. We do not repeat the calculation here for the purely bosonic diagrams, which is identical to the one presented in [13].

Spin-dependent diagrams instead follow from one term only, namely

$$\frac{i\kappa}{2\lambda}p^\mu \partial^b h_\mu^a \psi_a \psi_b, \quad (2.71)$$

which generates the following tower of vertices

$$\partial^b h_\mu^a(x_i + pt)\psi_a \psi_b, \quad (2.72)$$

$$x^{\alpha_1} \partial_{\alpha_1} \partial^b h_\mu^a(x_i + pt)\psi_a \psi_b, \quad (2.73)$$

$$\frac{1}{2}x^{\alpha_1} x^{\alpha_2} \partial_{\alpha_1} \partial_{\alpha_2} \partial^b h_\mu^a(x_i + pt)\psi_a \psi_b, \quad (2.74)$$

and so forth. At NE order no x -propagator is required, which means that the path integral over ψ is decoupled from the one over x . Therefore, to carry out the ψ integration it is enough to consider the boundary condition $\psi(0) + \psi(\infty) = 2\eta$ and expand around the constants η_a via

$$\psi_a(t) = \eta_a + \tilde{\psi}_a(t), \quad (2.75)$$

where the fluctuations now obey the boundary condition

$$\tilde{\psi}_a(0) + \lim_{T \rightarrow \infty} \tilde{\psi}_a(T) = 0. \quad (2.76)$$

Inserting this expansion into eq. (2.71) yields

$$\frac{i\kappa}{2\lambda}p^\mu \partial^b h_\mu^a \left(\eta_a \eta_b + \eta_a \tilde{\psi}_b + \tilde{\psi}_a \eta_b + \tilde{\psi}_a \tilde{\psi}_b \right). \quad (2.77)$$

⁷We omit $+\frac{i\kappa}{2\lambda}\dot{x}^\mu \left(\partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h + \partial^b h_\mu^a \psi_a \psi_b \right)$ in eq. (2.70) since the appearance of \dot{x} makes it subleading.

Notably, ψ -fields in eq. (2.77) have to be contracted at equal time, resulting in $\langle \psi_a(t)\psi_b(t) \rangle = \eta_a\eta_b$. However, thanks to eq. (2.42), the equal-time propagator for ψ vanishes. Hence, only the first term of eq. (2.77) contributes to the path integral. We finally employ the symbol map of eq. (2.21), which gives

$$\eta_a\eta_b \mapsto \frac{1}{4} [\gamma_a, \gamma_b] = -i\sigma_{ab}. \quad (2.78)$$

In conclusion, the contribution from eq. (2.71) to the GWL is given at NE by a single spin-dependent vertex, which modifies the NE one-graviton emission. It reads

$$\frac{i\kappa}{2\lambda} \int_0^\infty dt p^\mu \partial^b h^a{}_\mu \sigma_{ab}, \quad (2.79)$$

or, in momentum space,

$$\int \frac{d^4k}{(2\pi)^4} \tilde{h}_{\mu\nu} \left[\frac{i\kappa p^\mu k_\rho}{2} \sigma^{\rho\nu} \right]. \quad (2.80)$$

Therefore, combining the vertex of eq. (2.79) with the scalar vertices derived in [13], we obtain the following representation for the spin 1/2 GWL in curved space

$$\begin{aligned} \widetilde{W}_p(0, \infty; x_i) = & \exp \left\{ \frac{i\kappa}{2} \int_0^\infty dt \left[-p_\mu p_\nu + i p_\nu \partial_\mu - \frac{i}{2} \eta_{\mu\nu} p^\alpha \partial_\alpha + \frac{i}{2} t p_\mu p_\nu \partial^2 + \sigma^{\nu\sigma} p^\mu \partial_\sigma \right] h^{\mu\nu}(x_i + pt) \right. \\ & + \frac{i\kappa^2}{2} \int_0^\infty dt \int_0^\infty ds \left[\frac{p^\mu p^\nu p^\rho p^\sigma}{4} \min(t, s) \partial_\alpha h_{\mu\nu}(x_i + pt) \partial^\alpha h_{\rho\sigma}(x_i + ps) + \right. \\ & \left. \left. + p^\mu p^\nu p^\rho \theta(t-s) h_{\rho\sigma}(x_i + ps) \partial_\sigma h_{\mu\nu}(x_i + pt) + p^\nu p^\sigma \delta(t-s) h^\mu{}_\sigma(x_i + ps) h_{\mu\nu}(x_i + pt) \right] \right\}, \end{aligned} \quad (2.81)$$

where the arguments 0 and ∞ refer to the boundaries of the time integrations, which have been made explicit for upcoming generalizations. Eq. (2.81) is one of the main results of this paper. A few comments are in order.

The spin dependent term in eq. (2.79) and eq. (2.80), which appears in the first line of eq. (2.81), is the only modification w.r.t. the scalar GWL. It represents the universal soft factor which arises in the factorization of a single soft emission in a scattering amplitude, in agreement with single-emission next-to-soft theorems [91]. Note, however, that the result of eq. (2.81) is more general, since it involves correlations between two gravitons. It shows in fact that no spin-dependent modification is necessary for two-graviton emissions. This aspect is analogous to the gauge theory case, where the spin dependent contribution in the double-emission is proportional to the commutator $\sigma_{\mu\nu}[A^\mu, A^\nu]$ and thus vanishes in the abelian limit. Therefore, the absence of spin dependent terms in two-graviton emissions is due to the abelian nature of the graviton field. Note however that at sub-subleading (i.e. NNE) level correlations among three gravitons need to be taken into account.

Secondly, we note the presence of an explicit dependence of the GWL on x_i , which was introduced as an integration constant while solving the free equations of motion in flat space. Its effect is to shift the starting point of the particle trajectory away from the origin and is due to finite-size effects of the hard interaction. This NE effect captures (part of) the orbital contribution of the soft theorem for gravitons and in the case of classical scattering becomes the impact parameter. Indeed, as discussed in [12, 13], the presence of $x_i \neq 0$ mixes with derivatives of the hard interaction to generate the full orbital angular momentum.

The soft GWL in eq. (2.81) manifestly demonstrates the exponentiation of subleading soft gravitons dressing the external fermionic state of a scattering amplitude. In the next section we discuss how eq. (2.81) has to be modified in the classical limit. As we shall see, the parallel with the soft exponentiation discussed in this section will turn out to be a powerful tool in this derivation.

3 Classical scattering and GWLs

As shown in [13] for the case of non-spinning objects in the conservative sector, classical observables in the Post Minkowskian (PM) expansion (i.e. the expansion in G) can be computed via vacuum expectation values of classical GWLs. We now extend this method to spin 1/2 particles, thus allowing the inclusion of perturbative corrections which are linear in the classical spin variable S . In doing so, we clarify some aspects concerning the construction of classical GWLs originally proposed in [13].

3.1 Regge VS soft VS classical

The starting point of the GWL approach for classical scattering is to consider the Regge limit of the (quantum) $2 \rightarrow 2$ scattering amplitude, where the Mandelstam variables fulfill⁸ $s \gg t$ and the exchanged gravitons are soft. More specifically, observables at order κ^{2n} (i.e. nPM) require GWLs expanded to order κ^n , which in turn are constructed via the soft expansion at the $N^{n-1}E$ level.

Specifically, for scattering of spin 1/2 objects, we consider the following next-to-eikonal amplitude

$$\begin{aligned} & \mathcal{A}_{E+NE} \\ &= \langle 0 | \bar{u}(p_3) \widetilde{W}_{p_3}(0, \infty; b) \widetilde{W}_{p_1}(-\infty, 0; b) u(p_1) \bar{u}(p_4) \widetilde{W}_{p_4}(0, \infty; 0) \widetilde{W}_{p_2}(-\infty, 0; 0) u(p_4) | 0 \rangle, \end{aligned} \tag{3.1}$$

where the definition for the GWL starting from $-\infty$ to 0 can be trivially obtained from eq. (2.81) by flipping the boundary conditions. The dependence over b in the GWL

⁸Note also that we assume masses to be much larger than t .

with momenta p_1 and p_3 instead represents a constant overall shift which separates the trajectories of the hard particles and therefore plays the role of the impact parameter⁹.

Note that by including subleading soft effects with GWLs, in eq. (3.1) we are in fact including corrections to the strict Regge limit $t/s \rightarrow 0$. However, eq. (3.1) contains quantum effects that one wishes to discard. In order for the method to be efficient for the calculation of classical observables, it would be desirable to isolate the classical contributions at the integrand level, rather than taking $\hbar \rightarrow 0$ of the final result, so that classical observables can be computed from the exponential form of eq. (3.1), i.e.

$$\mathcal{A}_{\text{E+NE}} \xrightarrow{\hbar \rightarrow 0} e^{i(\chi_{\text{E}} + \chi_{\text{NE}})} , \quad (3.2)$$

where $\chi_{(\text{N})\text{E}}$ is called (next-to-)eikonal phase. In fact, the classical limit in this approach corresponds to a strict Regge limit, while subleading Regge corrections contribute to the so-called Regge trajectory of the graviton and are quantum. Accordingly, in order to efficiently compute classical observables through a VEV of GWL operators, one has to replace the GWL defined in eq. (2.81) with a properly-defined classical GWL.

This approach has been carried out in the scalar case [13]. By reinstating \hbar , indeed, one can identify the terms in the worldline Lagrangian (and subsequently in the exponentiated vertices of the GWL) that vanishes in the limit $\hbar \rightarrow 0$. Specifically, NE single emissions in the GWL correspond to a small recoil of the hard particle (hence a subleading Regge effect), which are necessarily quantum phenomena. Double-graviton vertices, on the other hand, survive the classical limit. In this way, the soft expansion can be interpreted in classical sense as a sum over correlations among soft emissions rather than an expansion in the energy of a single emission.

However, in order to properly define a classical GWL, there are still two aspects to be addressed. The first one is that, for spin variables, a naive \hbar power counting is not suitable for the classical limit, since the spin of a point-like particle is proportional to \hbar , hence intrinsically a quantum property. We rather need a map to classical quantities. The second one is that the boundary conditions of the worldline path integral in the GWL must be modified. Although most of these aspects have already been addressed in the recent literature (see [13, 14, 22, 103]) we would like to discuss them in greater detail to elucidate their role in the construction of a classical GWL. We discuss these aspects in turn.

3.2 Classical limit and spin

As already mentioned, to extract the classical limit one has to compute the \hbar scaling of different terms in the worldline Lagrangian. Specifically, momenta of external particles are regarded as classical momenta, but the momenta of massless force carriers are rescaled as $k = \hbar \bar{k}$ [14]. As shown in [13], this power-counting analysis is enough to identify the

⁹It can be shown that due to worldline supersymmetry, one can exploit a gauge symmetry among the variables p_i , b and η , to eliminate two components of b , choosing it for example to be perpendicular to the incoming momenta [22].

classical terms in the scalar GWLs that are necessary for classical observables. In the case of spin, however, we first need to select the relevant classical variable for the spin of an extended body in GR, i.e. the classical spin tensor $S^{\mu\nu}$, defined as the canonical momentum conjugated to the angular velocity $\Omega_{\mu\nu}$. The obvious choice for the spin tensor is the spin Lorentz generator $\sigma^{\mu\nu}$. Recalling that on the worldline path integral this generator naturally emerges from the symbol map $i\eta^a\eta^b \rightarrow \sigma^{ab}$, where η_a represent the boundary condition for the integral over ψ_a , we can make the identification

$$i\eta^a\eta^b \equiv S^{ab}, \quad (3.3)$$

which establishes a map¹⁰ between the worldline spin variable ψ_a and the classical spin tensor S^{ab} .

To determine the scaling of the Grassmann fields ψ_a in the classical limit we then consider the following argument. In the rest frame of the particle the spatial components of the Lorentz generator can be arranged into the spin operator, whose eigenvalue scales as $\hbar s$, which suggests that the quantum spin scales as $\psi^a\psi^b \sim \hbar$. For the spin to survive the classical limit, we therefore need to consider the limit in which $s \rightarrow \infty$ in such a way that the spin remains a macroscopic variable. The standard way to achieve this with amplitude methods is by rescaling $\psi_i^a \rightarrow \frac{1}{\sqrt{\hbar}}\psi_i^a$. As we are going to discuss, in this work we implement it instead by combining the classical limit of the spin degrees of freedom with the soft limit.

We first observe that loops of the worldline fields x and ψ contribute only to the quantum part of the asymptotic propagator. The reason for that is fairly straightforward and was given in [21], namely that the path integral should solve the classical equations of motion for x and ψ , which only requires tree level graphs. This can also be seen by power counting arguments. There is only one term in the Lagrangian that would allow for an x -loop without expanding $h_{\mu\nu}$, but this term diverges and is cancelled by the ghost contributions (see [13] for a detailed discussion). This means for every other x loop it is required to expand $h_{\mu\nu}$ into a power series introducing extra derivatives that add additional powers of k and therefore of \hbar .

The argument for ψ loops works similarly. Upon implementing the boundary conditions for ψ we expand

$$\psi_a\psi_b = (\eta_a + \tilde{\psi}_a)(\eta_b + \tilde{\psi}_b). \quad (3.4)$$

This tells us that the ψ -vertices are divided into three categories: one with all constant Grassmann vectors, which get mapped onto the spin tensor, one with all dynamical fields, and one with every possible mixture of the two. If a worldline diagram contains a $\tilde{\psi}$ -loop the propagators remove one spin tensor more as in the corresponding loop-free diagram. Therefore, the loop diagram is subleading in \hbar and thus non classical. The case of mixed loops is also straightforward. Due to the expansion of the Grassmann vectors into the constant part η plus a dynamical field, every diagram with an x - $\tilde{\psi}$ -loop also exists as an

¹⁰This map facilitates neglecting the spinors in the definition of \mathcal{A}_{NE} , as it is compatible with the usual identification $u(p)\sigma^{\mu\nu}\bar{u}(p') = S^{\mu\nu}u(p)\bar{u}(p') + \mathcal{O}(\hbar)$ used in traditional amplitude approaches.

x -propagator tree-level diagram. The $\tilde{\psi}$ -propagator that would turn this diagram into a loop diagram thus removes a potential spin tensor and the diagram contributes only to the quantum part of the asymptotic propagator. We should thus neglect all loops involving ψ . This can be elegantly achieved by slightly altering the power counting in λ introduced in eq. (2.70) for the soft expansion. For the classical GWL we additionally rescale $\psi \rightarrow \sqrt{\lambda}\psi$. In this way, λ suppresses both x and ψ loops, eliminating all quantum correction from the soft GWL.

Further we note that in the derivation of the soft GWL we rescaled also $\kappa \rightarrow \kappa/\lambda$, which resulted in a correspondence between the soft expansion and the λ -expansion. Such correspondence is independent of the order in the PM expansion. Specifically, the eikonal approximation is order λ^0 , NE approximation is order λ^{-1} , and so forth. However, for the classical limit it is actually more convenient not to rescale the coupling κ . Although this choice spoils the simple relation between the soft and the λ expansion, the classical terms emerge neatly as those of order λ . Therefore, we conclude that the relevant Lagrangian in the classical limit is

$$L_{\text{spin}}^{\text{cl.}} = \frac{\lambda}{2} \left(i\psi_a \dot{\psi}^a - \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \kappa h_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + p^\mu p^\nu + 2\dot{x}^\mu p^\nu) + i\kappa p^\mu (\partial^{[b} h_\mu^{a]} \psi_a \psi_b) \right) + \mathcal{O}(\lambda^0), \quad (3.5)$$

where the graviton field is evaluated at $h_{\mu\nu} = h_{\mu\nu}(x_i + pt + x(t))$.

3.3 The classical GWL

After having identified the terms in the worldline Lagrangian that are relevant in the classical limit, we need to consider the boundary conditions of the path integral that leads to a proper definition of the GWL. In this regard, we first observe that when we derived the soft GWL in Section 2 from a localized hard interaction to an asymptotic state of definite momentum, we considered the following path integral

$$\widetilde{W}_p(0, \infty; x_i) = \int \mathcal{D}x \mathcal{D}\psi \exp \left\{ i \int_0^\infty L[x(t), \psi(t)] dt \right\}. \quad (3.6)$$

We solved the path integration perturbatively by using the boundary conditions $x(0) = 0$ and $\psi(0) + \psi(\infty) = 0$. Then, in order describe factorized amplitudes at next-to-soft level in terms of GWLs and construct a next-to-soft function, we attached each GWL, corresponding to an external state of hard momentum, to the hard function \mathcal{H} at $t = 0$. In classical settings however, we want the incoming and outgoing particles to be causally connected through an uninterrupted matter line, as evident in the strict Regge limit description, which essentially sets $p_1 = p_3$ and $p_2 = p_4$. As we shall see, gluing together two Wilson lines from $-\infty$ to 0 and from 0 to $+\infty$ is not the most efficient way to define a classical GWL.

There is a clear diagrammatic interpretation for this. The boundary conditions at $t = 0$ cause the diagram to pick up a pole that originates from the matter propagator connecting the hard interaction to the first soft graviton emission from that line. Due

to these boundary conditions, naively multiplying two GWLs does not give the correct pole structure in the correlated multi-graviton exchange diagrams (i.e. those involving the vertices in the last two lines of eq. (2.81)). For instance, in the two-graviton emission diagram this can be viewed as the particle not fulfilling its equations of motion at $t = 0$. As this is clearly a non-classical phenomenon, we need to discard it. Discarding these contributions would need to be considered part of the gluing process to obtain a classical result from \mathcal{A}_{NE} .

To restore the classical history of the worldline particle from $-\infty$ to $+\infty$, it is in fact more straightforward not to glue the GWLs but rather the actions for incoming and outgoing particles. In this way the path integral solves for the classical trajectory from the far past to the far future and not for the incoming and outgoing paths separately, in analogy with [21]. This leads to the following definition for the classical GWL

$$\widetilde{W}_p^{\text{cl.}}(x_i) = \int \mathcal{D}x \mathcal{D}\psi \exp \left\{ i \int_{-\infty}^{\infty} L_{\text{spin}}^{\text{cl.}}[x(t), \psi(t)] dt \right\}, \quad (3.7)$$

where $L_{\text{spin}}^{\text{cl.}}$ is given by eq. (3.5) and the dynamical fields have been extended to the entire real line $(-\infty, \infty)$. Specifically, to better appreciate the relation with the boundary condition of the soft GWL, we decompose the action in eq. (3.7) into the sum of two copies of the soft action with flipped boundary conditions, i.e.

$$\int_{-\infty}^{\infty} L[x(t), \psi(t)] = \int_0^{\infty} (L_+[x_+(t), \psi_+(t)] + L_-[x_-(t), \psi_-(t)]) dt, \quad (3.8)$$

where L_- is related to L_+ simply by replacing $p \rightarrow -p$ and $\eta \rightarrow i\eta$ ¹¹. The reader may now check that the Lagrangian is build such that this sign flip is compensated exactly by substituting $t \rightarrow -t$. This allows us to combine the Lagrangians after defining $x(t) = \theta(t)x_+(t) + \theta(-t)x_-(-t)$ (and similarly for ψ) representing one uninterrupted matter line, where the worldline fields are allowed to fluctuate across the previous boundary at $t = 0$.

With this definition, one has to re-derive new worldline Green functions which are compatible with the $(-\infty, +\infty)$ boundary conditions. In fact, for the soft GWL we used

$$\langle x^\mu(t)x^\nu(s) \rangle = i\eta^{\mu\nu} G^{\text{soft}}(t, s) \quad \text{where} \quad G^{\text{soft}}(t, s) = -\min(t, s). \quad (3.9)$$

Here instead we need to use a classical Green function $G^{\text{cl.}}$ defined by

$$\langle x^\mu(t)x^\nu(s) \rangle = i\eta^{\mu\nu} G^{\text{cl.}}(t, s) \quad \text{where} \quad G^{\text{cl.}}(t, s) = \frac{|t-s|}{2}. \quad (3.10)$$

Note that the soft and classical Green functions are related via

$$2G^{\text{cl.}} = G^{\text{soft}}(t, s) + G^{\text{soft}}(-t, -s). \quad (3.11)$$

The Fourier representation might at times be more helpful. It reads

$$\tilde{G}(\omega) = -\frac{1}{2} \left(\frac{1}{(\omega + i\epsilon)^2} + \frac{1}{(\omega - i\epsilon)^2} \right), \quad (3.12)$$

¹¹The sign flip in the spin tensor can be achieved by multiplying the Grassmann fields by a factor of i .

where $G(t-s) = \int \frac{d\omega}{2\pi} \tilde{G} e^{-i\omega(t-s)}$. In Fourier space it is also more natural to find the derivatives of the Green functions. Finally, note that the Green function of the fermionic field $\psi(t)$, which in the soft calculation reads

$$\langle \psi^a(t) \psi^b(s) \rangle = \frac{1}{2} \eta^{ab} (\theta(t-s) - \theta(s-t)) = \eta^{ab} \mathcal{G}^{\text{cl.}}, \quad (3.13)$$

needs no modifications.

We can thus solve the path integral for the classical GWL and from there extract the vertices. We start with the Lagrangian in eq. (3.5) and expand the Grassmann variables over the background via eq. (3.4), to obtain

$$\begin{aligned} L_{\text{spin}}^{\text{cl.}} = & \frac{i\lambda}{2} \psi \dot{\psi} - \frac{\lambda}{2} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \frac{\kappa\lambda}{2} h_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + p^\mu p^\nu + 2\dot{x}^\mu p^\nu) \\ & + \lambda \frac{i}{2} (p^\mu + \dot{x}^\mu) \omega_\mu^{ab} (\eta_a \eta_b + 2\eta_a \psi_b + \psi_a \psi_b), \end{aligned} \quad (3.14)$$

where we have neglected non-classical terms. Up to order κ^2 and setting $\lambda = 1$, we need the following worldline vertices¹²

$$\textcircled{0} = \frac{i}{2} \int_{-\infty}^{\infty} d\tau \left(-\kappa p^\mu p^\nu h_{\mu\nu}(x_i + p\tau) + p^\mu \omega_\mu^{ab}(x_i + p\tau) S_{ab} \right) \quad (3.15)$$

$$\textcircled{1} = \frac{i}{2} \int_{-\infty}^{\infty} d\tau \left(-\kappa p^\mu p^\nu \partial_\alpha h_{\mu\nu}(x_i + p\tau) + p^\mu \partial_\alpha \omega_\mu^{ab}(x_i + p\tau) S_{ab} \right) x^\alpha(\tau) \quad (3.16)$$

$$\textcircled{2} = \frac{i}{2} \int_{-\infty}^{\infty} d\tau \left(-2\kappa p^\nu h_{\mu\nu}(x_i + p\tau) + \omega_\mu^{ab}(x_i + p\tau) S_{ab} \right) \dot{x}^\mu(\tau) \quad (3.17)$$

$$\textcircled{3} = i \int_{-\infty}^{\infty} d\tau \left(i p^\mu \omega_\mu^{ad}(x_i + p\tau) \eta_a \right) \psi_d(\tau). \quad (3.18)$$

They assemble into the classical GWL using the usual exponentiation of connected diagrams in QFT, i.e. schematically

$$\widetilde{W}^{\text{cl.}} = \exp \left\{ \textcircled{0} + \frac{1}{2} \langle \textcircled{1}-\textcircled{1} + \textcircled{2}-\textcircled{2} + \textcircled{3}-\textcircled{3} + 2\textcircled{1}-\textcircled{2} \rangle \right\}. \quad (3.19)$$

¹²The $\mathcal{O}(\kappa)$ expansion of ω_μ^{ab} is left implicit to avoid clutter.

Explicitly, we obtain in position space

$$\begin{aligned}
\widetilde{W}_p^{\text{cl.}}(x_i) = \exp \left\{ -\frac{i\kappa}{2} \int_{-\infty}^{\infty} d\tau_1 [p^\mu p^\nu - p^\mu S^{\nu\varrho} \partial_\varrho] h_{\mu\nu}^1 - \frac{i\kappa^2}{8} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \times \right. \\
\times \left[p^\mu p^\nu p^\varrho p^\sigma h_{\mu\nu,\alpha}^1 h_{\varrho\sigma}^{2,\alpha} G^{\text{cl.}} + 4p^\nu p^\sigma \eta^{\mu\varrho} h_{\mu\nu}^1 h_{\varrho\sigma}^2 \dot{G}^{\text{cl.}} + 4p^\mu p^\nu p^\sigma h_{\mu\nu,\varrho}^1 h_{\sigma}^{2,\varrho} \dot{G}^{\text{cl.}} \right. \\
+ S_{ab} \left(-4p^\mu p^\varrho h_{\mu[a,d]}^1 h_{\varrho[b,d]}^2 \mathcal{G}^{\text{cl.}} - 2p^\mu p^\nu p^\varrho h_{\mu\nu,\alpha}^1 h_{\varrho[a,b]}^{2,\alpha} G^{\text{cl.}} \right. \\
+ 2(p^\mu p^\nu h_{\mu\nu,\varrho}^1 h^{2\varrho[a,b]} + 2p^\mu p^\varrho h_{\mu[a,b],\sigma}^1 h^{2\varrho\sigma}) \dot{G}^{\text{cl.}} - 4p_\nu h_1^{\mu\nu} h_{\mu[a,b]}^2 \ddot{G}^{\text{cl.}} \\
\left. \left. + 2p^\mu \left(h_1^{\varrho a} h_{\mu,\varrho}^{2,b} + h_1^{b\varrho} h_{\mu\varrho}^{2,a} + \frac{1}{2} h_1^{b\varrho} h_{\varrho}^{2,a} \right) \ddot{G}^{\text{cl.}} \right) \right] \Bigg\}, \quad (3.20)
\end{aligned}$$

where we are using the following shorthand notation $h_{\mu\nu}^1 = h_{\mu\nu}(p\tau_1 + x_i)$, $h_{\mu\nu}^2 = h_{\mu\nu}(p\tau_2 + x_i)$, $\dot{G}^{\text{cl.}} = \frac{\partial}{\partial t} G^{\text{cl.}}$. We also used $\ddot{G}^{\text{cl.}} = \frac{\partial^2}{\partial t \partial s} G^{\text{cl.}}$. Alternatively, decomposing the graviton into its momentum modes allows us to solve the time integrals, yielding

$$\begin{aligned}
\widetilde{W}_p^{\text{cl.}}(x_i) = \exp \left\{ -\frac{i\kappa}{2} \int_k \tilde{h}_{\mu\nu}(k) e^{-ikx_i} \hat{\delta}(pk) [p^\mu p^\nu + ip^\mu (S \cdot k)^\nu] \right. \\
- \frac{i\kappa^2}{8} \int_{k,l} \frac{\tilde{h}_{\mu\nu}(k) \tilde{h}_{\varrho\sigma}(l)}{2} e^{-i(k+l)x_i} \left[-2I_1(kl) p^\mu p^\nu p^\sigma (p^\varrho + 2i(S \cdot l)^\varrho) \right. \\
- 4I_2 p^\nu k^\varrho (-2ip^\mu p^\sigma + p^\mu (S \cdot l)^\sigma + 2p^\sigma (S \cdot k)^\mu) + 8I_4 p^\mu p^\varrho S_{ab} k^{[c} \eta^{a]\nu} l^{[c} \eta^{b]\sigma} \\
\left. \left. + 8I_3 \eta^{\mu\varrho} (p^\nu p^\sigma + ip^\nu (S \cdot l)^\sigma) \right] \right\}, \quad (3.21)
\end{aligned}$$

where $(S \cdot k)^\sigma = S^{\sigma\nu} k_\nu$, we used the shorthand notation $\int \frac{d^D k}{(2\pi)^D} = \int_k$ and we defined

$$I_1 = \int_{-\infty}^{\infty} dt ds G^{\text{cl.}}(t, s) e^{-ipkt - ips} = -2\hat{\delta}(p(k+l)) \left(\frac{1}{(p(k-l) + i\epsilon)^2} + \frac{1}{(p(k-l) - i\epsilon)^2} \right) \quad (3.22)$$

$$I_2 = \int_{-\infty}^{\infty} dt ds \dot{G}^{\text{cl.}}(t, s) e^{-ipkt - ips} = -i\hat{\delta}(p(k+l)) \left(\frac{1}{(p(k-l) + i\epsilon)} + \frac{1}{(p(k-l) - i\epsilon)} \right) \quad (3.23)$$

$$I_3 = \int_{-\infty}^{\infty} dt ds \ddot{G}^{\text{cl.}}(t, s) e^{-ipkt - ips} = -\hat{\delta}(p(k+l)) \quad (3.24)$$

$$I_4 = \int_{-\infty}^{\infty} dt ds \mathcal{G}^{\text{cl.}}(t, s) e^{-ipkt - ips} = I_2 \quad (3.25)$$

Equations (3.20) and (3.21) form one the main result of this paper.

At this point it is instructive to compare eq. (3.20) and eq. (3.21) with the soft GWL of eq. (2.81). The single-graviton terms have an easy correspondence. Specifically, both the scalar and the spin terms in the first line of eq. (3.21) correspond to the sum of two soft emissions from $-\infty$ to 0 and from 0 to $+\infty$, respectively. These can be identified in eq. (3.21) by neglecting purely quantum recoil terms and (in momentum space) using

$$2\pi i \delta(p \cdot k) = \frac{1}{p \cdot k + i\epsilon} - \frac{1}{p \cdot k - i\epsilon} . \quad (3.26)$$

Note, however, that while the spin term is a subleading effect in the soft GWL, it becomes a leading effect in the classical limit. The reason is that while in the purely soft limit $k \cdot S$ is subleading w.r.t. terms with no powers of the soft momentum k , the scaling of the spin variable S is enhanced in the classical limit, such that $k \cdot S = \mathcal{O}(1)$. Similarly, the second and third line of eq. (3.20) can be mapped to eq. (2.81) via eq. (3.11), which again corresponds to a sum of soft emissions from $-\infty$ to 0 and from 0 to $+\infty$, respectively. On top of that, we note an explicit dependence over spin for two-graviton emissions, which has no equivalent in eq. (2.81). The reason is again the enhancement in the scaling of the spin variable. We conclude that the soft and the classical GWLs have the same structure, with the only differences due to the presence of different Green functions (because of the boundary conditions) and different spin dependence (because of different scaling in the classical limit).

3.4 Observables at 2PM

Equipped with the classical GWL derived in the previous section, we can now discuss how classical observables are computed. Specifically, if we limit the analysis to the conservative sector up to 2PM, the strategy is the same as the one outlined in the scalar case [13]. In particular, classical observables are generated from the (next-to-)eikonal phase, which for large impact parameter b can be computed as the vacuum expectation value (VEV) of two classical GWLs, i.e.

$$e^{i(\chi_E + \chi_{NE})} = \langle 0 | \widetilde{W}_{m_1 u_1}^{\text{cl.}}(0) \widetilde{W}_{m_2 u_2}^{\text{cl.}}(b) | 0 \rangle , \quad (3.27)$$

where we have introduced the velocities u_i^μ via $p_i^\mu = m_i u_i^\mu$. Scalar and spin observables are then obtained after differentiating χ w.r.t. the impact parameter b . The VEV in eq. (3.27) is computed by using the standard (gauge-fixed) Einstein-Hilbert action. Diagrams are thus generated perturbatively by both the vertices contained in each GWL and by the vertices in the bulk. For sake of completeness, we list all relevant Feynman rules in Appendix C.1.

At 1PM we can then assemble the Feynman rules from the GWL into three diagrams, as shown in fig. 1. The computation of the three diagrams is presented in Appendix C.2. The final result reads

$$i\chi_{E,0} = i \frac{\kappa^2 m_1 m_2}{16\pi} \frac{1}{\sqrt{\gamma^2 - 1}} \Gamma(-\epsilon) \left(\gamma^2 - \frac{1}{2} \right) (-b^2)^\epsilon , \quad (3.28)$$

and

$$i\chi_{E,1} = i \frac{\kappa^2 m_1 m_2}{16\pi} \frac{1}{\sqrt{\gamma^2 - 1}} (-b^2)^{\epsilon-1} \gamma b_\mu (S_1^{\mu\nu} u_{2\nu} - S_2^{\mu\nu} u_{1\nu}) , \quad (3.29)$$

where $\gamma = u_1 \cdot u_2$ and we denoted with $\chi_{E,i}$ terms of order S^i . The result is in agreement with the literature (see e.g. [22, 104–107]).

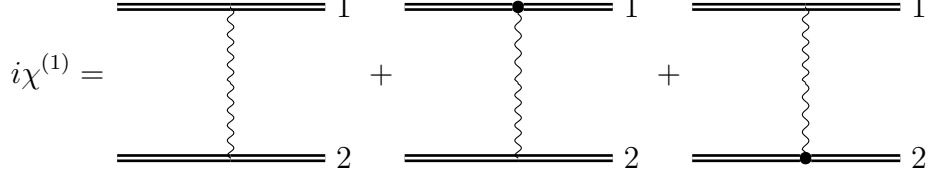


Figure 1: Diagrams contributing to the eikonal phase at 1PM. Dots represent spin vertices.

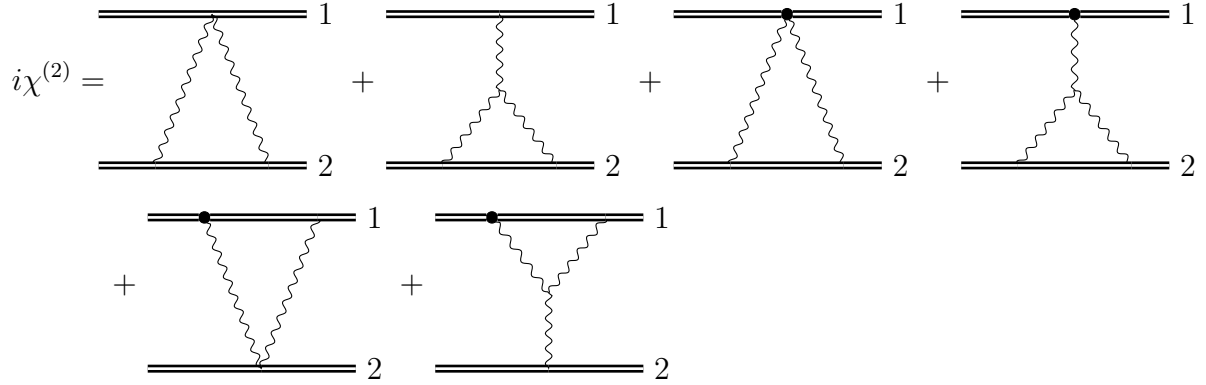


Figure 2: Diagrams contributing to the 2PM eikonal phase. Dots represent spin vertices. Mirrored diagrams are omitted.

At 2PM there are more diagrams involved, as shown in fig. 2. The final result for the scalar contribution reads

$$i\chi_{NE,0} = \frac{i\kappa^4 m_1 m_2 (m_1 + m_2) (15\gamma^2 - 3)}{8 \cdot 32^2 \pi (-b^2) \sqrt{\gamma^2 - 1}} , \quad (3.30)$$

while the term linear in spin is

$$i\chi_{NE,1} = \frac{i\kappa^4 m_1 m_2 \gamma (5\gamma^2 - 3)}{4 \cdot 32^2 \pi (-b^2)^{\frac{3}{2}} (\gamma^2 - 1)^{\frac{3}{2}}} ((4m_1 + 3m_2) (b \cdot S_1 \cdot u_2) - (4m_2 + 3m_1) (b \cdot S_2 \cdot u_1)) . \quad (3.31)$$

We find again complete agreement with the literature (see e.g. [22, 104–107]).

3.5 A brief comparison to WQFT and HEFT approaches

We now make a comparison of the GWL approach with similar methods that have been recently proposed in the literature. In particular, the analogy between this work and the WQFT method of [22] is evident. They both start from the same supersymmetric worldline model and the classical limit is performed at the Lagrangian level. The main difference is that the WQFT approach treats worldline fields and the bulk fields (i.e. the graviton) on the same footing, whereas in our approach all worldline integrals are solved first, leading to the representation of the eikonal phase as the VEV of GWLs. What seems at first a mere prescription for the order of integrations (leading to a re-organization of diagrams and Feynman rules), has in fact deeper consequences. By solving all worldline integral first, we are able to identify how Wilson-line operators must be modified in order to capture the classical dynamics of spinning objects interacting gravitationally at higher orders in the PM expansion. This opens up the possibility to exploit techniques that have been developed over the years for the renormalization of Wilson loops and their generalizations [8, 9, 48, 49, 70–72, 74].

Moreover, there is another interesting feature of the GWL approach, which follows from the crucial role played by the soft expansion. By taking the route over the soft GWL we have been able to identify the correct spin tensor in terms of the initial data of the Grassmann fields ψ_a without ever comparing to the classical EFT. Therefore, the GWL approach brings the advantage that one does not need knowledge of the correct EFT of gravity in order to identify the classical degrees of freedom.

We can also comment on the connection with a conceptually related approach [93–95], which uses a Heavy Mass Effective Theory (HEFT) to construct the classical gravitational scattering amplitude. This approach builds on the fact that in the classical limit the masses of the matter particles are much larger than the momentum transfer. In this limit one can define the so called HEFT amplitudes $\mathcal{A}_n(q, \bar{p})$, which are the classical $\mathcal{O}(\bar{m}^2)$ pieces of the two-massive $n - 2$ -graviton amplitude. Here $\bar{p} = \frac{1}{2}(p_1 + p'_1)$ is the average¹³ of incoming and outgoing momentum and $\bar{m}^2 = \bar{p}^2$. These on-shell amplitudes are then used to construct loop integrands via unitarity methods.

In spite of this computational difference in constructing the amplitudes there is a clear connection between the two approaches, which originates in the way the limit is implemented to extract the classical pieces of the amplitudes. The clear advantage of the HEFT approach lies in the fact that by expanding in the masses and choosing the appropriate dynamical variables one can show that only two massive particle irreducible (2mPI) cut diagrams contribute to the classical amplitude, while all other diagrams are either quantum or iteration pieces. The use of dressed propagators in the GWL approach exponentiates the amplitude, hence taking care of all iteration pieces automatically. Furthermore, the use of time symmetric worldline correlators in the GWL approach leads

¹³Using the more conventional $\hbar \rightarrow 0$ limit for scattering amplitudes one would run into the problem of having feed down terms since $q^2 = \pm 2pq$ does not have a homogeneous scaling in \hbar , which is avoided with the barred variables.

to the identification of the background parameters p_1 and p_2 of the particle trajectories with the average of the particles momenta in the far past and far future. We therefore expect the on-shell HEFT amplitudes to directly correspond to the on-shell version of the classical GWL.

To better appreciate the close relation between the two methods, we consider the following simple calculation. Contracting the classical one-graviton vertex (we do not distinguish between spinning and non-spinning vertices here) with the polarization tensor $\varepsilon_{\mu\nu} = \varepsilon_\mu\varepsilon_\nu$, we obtain

$$\varepsilon_\mu\varepsilon_\nu \left(\begin{array}{c} k, \mu\nu \\ \text{wavy line} \\ \text{double line} \end{array} p, S \right) = \frac{1}{2} \int_k \hat{\delta}(pk) \mathcal{A}_3^{\text{HEFT}}(p), \quad (3.32)$$

where the diagram on the l.h.s. represents the corresponding vertex in the GWL. A similar analogy holds for the HEFT Compton amplitude, which reads

$$\varepsilon_\mu\varepsilon_\nu\varepsilon_\rho\varepsilon_\sigma \left(\begin{array}{c} k_1, \mu\nu \quad k_2, \rho\sigma \\ \text{wavy line} \quad \text{wavy line} \\ \text{double line} \end{array} p, S \quad + \quad \begin{array}{c} k_1, \mu\nu \quad k_2, \rho\sigma \\ \text{wavy line} \\ \text{double line} \end{array} p, S \right) = \frac{1}{2} \int_{k_1, k_2} \hat{\delta}(p(k_1 + k_2)) \mathcal{A}_4^{\text{HEFT}}(p). \quad (3.33)$$

Note that the relations above always include an additional factor of $1/2$ and the orthogonality delta function. This is simply due to the fact that when the GWL is defined in position space it directly gives the impact parameter space amplitude. A vertex from the upper and lower line then together yields the measure $\frac{1}{4} \int_q \hat{\delta}(qp_1) \hat{\delta}(qp_2) e^{-iqb} = \frac{1}{4m_1m_2\sqrt{\gamma^2-1}} \int_q^{D-2} e^{iq\vec{b}}$.

Finally, we observe that this connection between the HEFT and the GWL methods sheds light on the relation between the HEFT and the WQFT approach. Indeed, by mapping the on-shell amplitudes of HEFT onto the vertices of the GWL, we are in fact mapping the worldline vertices of the WQFT to the operators of the HEFT. In this regard, the GWL method can be considered as somewhat intermediate, since the connection between the different approaches becomes only apparent by solving for the worldline fields and bulk field separately. It is a supersymmetric worldline approach like the WQFT, where the classical limit is performed at the level of an intrinsically off-shell object such as the Lagrangian, unlike the on-shell amplitude framework. However, it is also formulated in terms of operators in a mode-expansion (specifically, the soft expansion) in analogy with the heavy-mass expansion of HEFT.

4 Conclusions

In this work we have provided two representations for Generalized Wilson lines operators that are relevant for gravitational scattering amplitudes at the classical and at the quantum level, respectively. Specifically, we have derived a representation for a soft GWL for spin 1/2 particles (eq. (2.81)) and a classical GWL (eq. (3.20) and eq. (3.21)) which generates observables with linear corrections in spin. In this way we have extended results for the scalar case presented in [11].

The soft GWL is a compact tool to describe the factorization and the exponentiation of subleading soft gravitons dressing the external states of a quantum scattering amplitude. As in the scalar case, up to the order in perturbation theory implemented in this work (i.e. next-to-eikonal), soft GWLs exponentiate not only the single-graviton factor of soft theorems but also pairwise correlations of soft emissions. The final result shows a remarkable simplicity, where the only modification w.r.t. the scalar case is given by the single-graviton soft factor depending on the Lorenz spin generator. No modification is necessary for the correlated emissions, which thus remain spin-independent.

The classical GWL is constructed in analogy with the soft one, with two main differences: a classical limit that is performed at the Lagrangian level and different boundary conditions. The boundary conditions in particular have been chosen according to an interrupted straight line from $-\infty$ to $+\infty$. Classical spin variables are mapped to quantum spin numbers which tend to infinity in the classical limit and are thus enhanced compared to the soft limit. This different power counting implies an important difference between the soft and the classical GWL: both single and double graviton emissions depend on spin in the classical case, in agreement with other methods explored in the literature. We have cross-checked our representation by rederiving the eikonal phase at 2PM.

In spite of these differences, it is actually the analogy between the soft and the classical GWL that turns out to be particularly useful. Both of them are derived in the supersymmetric worldline formalism by assuming a soft expansion for the background field. In this way, the correspondence between the soft expansion and the PM expansion provides a map between quantum spin (represented by the worldline Grassmann variables) and the classical spin, thus not requiring any reference to the worldline EFT.

We have also discussed the relation with similar methods in the literature, and specifically with the WQFT and the HEFT. The relation with former is straightforward, since GWLs are constructed by integrating out the same worldline degrees of freedom that appear in the WQFT. We have also shown a clear connection between the GWL and the HEFT, by matching the single and double graviton emissions of the GWL with the on-shell HEFT amplitudes. We therefore conclude that the classical GWL provides a bridge between the WQFT and the HEFT, making their equivalence manifest.

On a more technical side, some new features have been analyzed in the derivation of the GWL from first principles in the supersymmetric worldline formalism. Specifically, we have thoroughly discussed the role of fermion doubling in the construction of the GWL and how to consistently integrate out the auxiliary Grassmann fields. Additionally, in

analogy with the gauge theory case discussed in [11], we have showed that the open line topology of the dressed propagator with twisted boundary conditions enables to evaluate the supersymmetry charge Q at asymptotic time $t_f \rightarrow \infty$, such that it reduces to the numerator contribution of the free Dirac propagator.

The results of this work can be extended in different directions. For instance, at the quantum level a systematic treatment of collinear interactions, which become relevant at subleading powers as shown in [7], would be desirable. At the classical level, a very interesting direction is the inclusion of higher spin particles in the GWL framework [90].

Acknowledgments

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A Fermionic path integral

In order to integrate out ψ_2 from the fermionic part of the path integral, we have to deal with the dependence on ψ_2 in the boundary term $\bar{\chi}_N \chi_{N-1}$ and in the kinetic term to show that indeed ψ_2 can be integrated out. In doing so we derive the propagator for the field ψ_1 .

We start by taking the discretised expression for the boundary and kinetic term

$$\prod_{j=1}^{N-1} (d^4 \bar{\chi}_j d^4 \chi_j) \exp \left\{ \bar{\chi}_N \chi_{N-1} - \sum_{n=1}^{N-1} \bar{\chi}_n (\chi_n - \chi_{n-1}) \right\}, \quad (\text{A.1})$$

where we then expand χ into real and imaginary parts. Here we need to keep in mind that the sum contains $\chi_0 = \chi_i$, which is our initial condition for χ . We will for this calculation keep the boundaries $\bar{\chi}_f$ and χ_i fixed and explicit. Additionally, with the expansion into real and imaginary parts we need to change the integration measure. Luckily, due to the $\sqrt{2}$ factors, the Jacobian determinant of this transformation has modulus one, thus

$$\prod_{j=1}^{N-1} d^4 \bar{\chi}_j d^4 \chi_j = \prod_{j=1}^{N-1} d^4 \psi_{1,j} d^4 \psi_{2,j}. \quad (\text{A.2})$$

All together this yields

$$\int \prod_{j=1}^{N-1} (d^A \psi_{1,j} d^A \psi_{2,j}) \exp \left\{ -\frac{1}{2} \sum_{n=2}^{N-1} (\psi_{1,n} - i\psi_{2,n})(\psi_{1,n} - \psi_{1,n-1} + i\psi_{2,n} - i\psi_{2,n-1}) + \frac{1}{\sqrt{2}} \bar{\chi}_f (\psi_{1,N-1} + i\psi_{2,N-1}) - \frac{1}{\sqrt{2}} \chi_i (\psi_{1,1} - i\psi_{1,1}) - \frac{1}{2} (\psi_{1,1} - i\psi_{1,1})(\psi_{1,1} + i\psi_{1,1}) \right\}. \quad (\text{A.3})$$

Now let us rewrite the exponential in matrix notation to make the structure more apparent. With the following definitions

$$M = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & \dots \\ 0 & -1 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A.4})$$

$$\vec{\psi}_{2,B} = \begin{pmatrix} \chi_i \\ 0 \\ \vdots \\ 0 \\ \bar{\chi}_f \end{pmatrix} \quad (\text{A.5})$$

$$\vec{\psi}_{\psi_1} = \begin{pmatrix} \psi_{1,2} - 2\psi_{1,1} \\ \psi_{1,3} - 2\psi_{1,2} + \psi_{1,1} \\ \vdots \\ \psi_{1,N-1} - 2\psi_{1,N-2} + \psi_{1,N-3} \\ -2\psi_{1,N-1} + \psi_{1,N-2} \end{pmatrix} \quad (\text{A.6})$$

$$\vec{\psi}_{1,B} = \begin{pmatrix} -\chi_i \\ 0 \\ \vdots \\ 0 \\ \bar{\chi}_f \end{pmatrix} \quad (\text{A.7})$$

$$\vec{\psi}_1 = \begin{pmatrix} \psi_{1,1} \\ \psi_{1,2} \\ \vdots \\ \psi_{1,N-1} \end{pmatrix} \quad \text{and} \quad \vec{\psi}_2 = \begin{pmatrix} \psi_{2,1} \\ \psi_{2,2} \\ \vdots \\ \psi_{2,N-1} \end{pmatrix} \quad (\text{A.8})$$

the exponential takes the form

$$\exp \left\{ -\frac{1}{2} \vec{\psi}_1^T M \vec{\psi}_1 + \frac{1}{\sqrt{2}} \vec{\psi}_{1,B}^T \vec{\psi}_1 - \frac{1}{2} \vec{\psi}_2^T M \vec{\psi}_2 + i \left(\frac{1}{\sqrt{2}} \vec{\psi}_{2,B} + \frac{1}{2} \vec{\psi}_{\psi_1} \right)^T \vec{\psi}_2 \right\}. \quad (\text{A.9})$$

We can integrate out the field ψ_2 using the standard Gaussian integral for Grassmann numbers yielding

$$\exp \left\{ -\frac{1}{2} \vec{\psi}_1^T M \vec{\psi}_1 + \frac{1}{\sqrt{2}} \vec{\psi}_{1,B}^T \vec{\psi}_1 + \frac{1}{2} \left(\frac{1}{\sqrt{2}} \vec{\psi}_{2,B} + \frac{1}{2} \vec{\psi}_{\psi_1} \right)^T M^{-1} \left(\frac{1}{\sqrt{2}} \vec{\psi}_{2,B} + \frac{1}{2} \vec{\psi}_{\psi_1} \right) \right\}. \quad (\text{A.10})$$

The inverse of M is given by

$$M^{-1} = 2 \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & -1 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -1 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (\text{A.11})$$

We have thus successfully¹⁴ integrated out ψ_2 , but apparently we did not simplify things. On the contrary, at first glance we seem to have made things a lot more complicated. Luckily, there is still a lot of simplification that can be done. The vectors ψ_{ψ_1} still depend on ψ_1 , and they do so through this matrix equation

$$\vec{\psi}_{\psi_1} = A \vec{\psi}_1 \quad \text{with} \quad A = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ 0 & 0 & 1 & -2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (\text{A.12})$$

We can also connect the boundary vectors by

$$\vec{\psi}_{1,B} = C \vec{\psi}_{2,B} \quad \text{with} \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{A.13})$$

and a quick calculation gives

$$\frac{1}{4} \vec{\psi}_{2,B}^T M^{-1} \vec{\psi}_{2,B} = \bar{\chi}_f \chi_i. \quad (\text{A.14})$$

Thus, the exponential can be rewritten as

$$\exp \left\{ \bar{\chi}_f \chi_i - \frac{1}{2} \vec{\psi}_1 \underbrace{\left(M - \frac{1}{4} A M^{-1} A \right)}_G \vec{\psi}_1 + \frac{1}{\sqrt{2}} \left(\left(C - \frac{1}{2} A M^{-1} \right) \vec{\psi}_{2,B} \right)^T \vec{\psi}_1 \right\}, \quad (\text{A.15})$$

¹⁴Actually the integration also creates the prefactor $\text{Pf}(M) = \frac{1}{2^{N-1}}$, which is cancelled by the integration over ψ_1 as this gives $\text{Pf}(G) = 2^{N-1}$.

Now we see the desired structure emerge. The matrix G defines the propagator for ψ_1 through its inverse and the boundary term $\bar{\chi}_f \chi_i$ is precisely the one we expect to appear since $\langle \bar{\chi}_f | \chi_i \rangle = e^{\bar{\chi}_f \chi_i}$. All that is left to do is to expand the field ψ_1 into a background field and a quantum fluctuation, add a source term and perform the integration over ψ_1 . The background-quantum split is given by

$$\psi_{1,n} = \frac{\eta}{2} + \tilde{\psi}_{1,n}, \quad (\text{A.16})$$

where the quantum field $\tilde{\psi}_{1,n}$ vanishes at the boundary for $n = 0$ and $n = N$. The source term takes the form $\vec{J}^T \vec{\psi}_1$. Integrating over the remaining Grassmann fields yields

$$\exp \left\{ \bar{\chi}_f \chi_i - \frac{1}{2} \vec{J}^T G^{-1} \vec{J} \right\} = \exp \left\{ \bar{\chi}_f \chi_i - \frac{1}{2} J_i G_{ij}^{-1} J_j \right\}. \quad (\text{A.17})$$

The form of G^{-1} is

$$G_{ij}^{-1} = \begin{cases} -\frac{1}{2} & \text{for } i < j \\ 0 & \text{for } i = j \\ \frac{1}{2} & \text{for } i > j \end{cases}. \quad (\text{A.18})$$

Here we also needed the fact that

$$\left((C - \frac{1}{2} AM^{-1}) \vec{\psi}_{2,B} \right)^T G^{-1} \left((C - \frac{1}{2} AM^{-1}) \vec{\psi}_{2,B} \right) = 0. \quad (\text{A.19})$$

Now G_{ij}^{-1} is precisely the propagator

$$\overline{\psi_{1,i}^a \psi_{1,j}^b} = -\eta^{ab} G_{ij}^{-1}, \quad (\text{A.20})$$

thus the continuum version is

$$\overline{\psi_1^a(t) \psi_1^b(t')} = \frac{1}{2} \eta^{ab} (\theta(t - t') - \theta(t' - t)), \quad (\text{A.21})$$

as quoted in eq. (2.42) in the main text.

Equipped with this propagator, we can elaborate eq. (A.17) to get the final representation for the fermionic path integral. Firstly, we see that the boundary term $\bar{\chi}_f \chi_i$ is entirely generated through the ψ_2 integration and not problematic since it is fixed. Secondly, observe that the continuum propagator for $\psi_1(t)$ is precisely the antisymmetrized Green's function for the differential operator $\frac{d}{dt}$. It is therefore justified to symbolically write the path integral over the real Grassmann field $\psi_1(t)$ in the usual path integral notation

$$\langle \bar{\chi}_f | e^{\hat{A}T} | \chi_i \rangle = \int \mathcal{D}\psi_1 \exp \left\{ \bar{\chi}_f \chi_i + \int_0^T dt \left(-\frac{1}{2} \psi_1 \dot{\psi}_1 + A_{\text{px}}(\psi_1) \right) \right\}, \quad (\text{A.22})$$

in agreement with eq. (2.41).

B A Theorem about fermionic path integrals

We prove a theorem about expectation values of Majorana operators that bears some similarity to Wick's theorem.

Theorem B.1. *For a product of Majorana fermion operators $\hat{\psi}_1^a$ the expectation value is given by*

$$\langle \prod_{i=1}^n \hat{\psi}_1^{a_i} \rangle = \sum_{\substack{\text{possible} \\ \text{contractions}}} \left[\text{contractions} \times \prod_{i \text{ not contracted}} \langle \hat{\psi}_1^{a_i} \rangle \right].$$

Proof. Let us check the $n = 2$ case. Expanding everything out and writing it in $\psi^\dagger\psi$ -ordering gives

$$\langle \hat{\psi}_1^a \hat{\psi}_1^b \rangle = \frac{1}{2} \langle \hat{\psi}^a \hat{\psi}^b + \hat{\psi}^{a\dagger} \hat{\psi}^b + \hat{\psi}^{a\dagger} \hat{\psi}^{b\dagger} - \hat{\psi}^{b\dagger} \hat{\psi}^{a\dagger} + \eta^{ab} \rangle.$$

Given the anticommuting nature of Grassmann numbers we can evaluate these operators directly on the coherent states and rewrite this as

$$\frac{\eta^{ab}}{2} + \langle \hat{\psi}_1^a \rangle \langle \hat{\psi}_1^b \rangle.$$

Now let us assume that the theorem holds for $n - 1$. Then the statement for n follows from examining

$$\left\langle \left(\prod_{i=1}^{n-1} \hat{\psi}_1^{a_i} \right) \frac{\hat{\psi}_1^{a_n} + \hat{\psi}_1^{a_n\dagger}}{\sqrt{2}} \right\rangle.$$

The term with $\hat{\psi}_1^{a_n}$ is already in the correct order, but we need to permute $\hat{\psi}_1^{a_n\dagger}$ through all $\hat{\psi}_1^{a_i}$ to the very left. From the anticommutation relation we can show that

$$\hat{\psi}_1^a \hat{\psi}_1^{b\dagger} = -\hat{\psi}_1^{b\dagger} \hat{\psi}_1^a + \frac{\delta^{ab}}{\sqrt{2}},$$

and we pick up a contraction with all $n - 1$ Majorana fermions and an $(-1)^{n-1}$ in front of the permuted term. Since it is then in the correct order, we can evaluate it on the bra state and permute the Grassmann number back to the right, cancelling the $(-1)^{n-1}$. So all in all we have achieved

$$\left\langle \left(\prod_{i=1}^{n-1} \hat{\psi}_1^{a_i} \right) \hat{\psi}_1^{a_n} \right\rangle = \left\langle \prod_{i=1}^{n-1} \hat{\psi}_1^{a_i} \right\rangle \langle \hat{\psi}_1^{a_n} \rangle + \sum_{j=1}^{n-1} (-1)^{n-j-1} \overbrace{\hat{\psi}_1^{a_j} \hat{\psi}_1^{a_n}} \left\langle \prod_{\substack{i=1 \\ i \neq j}}^n \hat{\psi}_1^{a_i} \right\rangle.$$

On the remaining expectation values we can use the theorem for $(n - 1)$ to get that the desired statement for n . The first term gives all the terms where $\hat{\psi}_1^{a_n}$ is not contracted. The second part gives all possibilities of contractions including the n th operator $\hat{\psi}_1^{a_n}$. \square

C Eikonal phase at 2PM

C.1 Feynman-Rules

The GWL sources the graviton field and we can divide the resulting types of vertices into powers of the gravitational coupling and the spin tensor. Let us first discuss the $\mathcal{O}(\kappa)$ vertices. It will be helpful in the ensuing calculations to divide diagrams into different topologies depending on their scaling with the masses of the different particles. Therefore we will introduce the velocity $p_i^\mu = m_i u_i^\mu$ and also rescale the Grassmann fields as $\psi_i^a \rightarrow \sqrt{m_i} \psi_i$. Additionally, we introduce some helpful shorthand notation

$$\int \frac{d^D k}{(2\pi)^D} = \int_k \quad (\text{C.1})$$

and

$$(2\pi)^D \delta^{(D)}(x) = \hat{\delta}(x). \quad (\text{C.2})$$

The single graviton vertices then become in momentum space

$$\begin{array}{c} k, \mu\nu \\ \text{wavy line} \\ \text{double line} \end{array} u_i = -i \frac{\kappa}{2} m_i u_i^\mu u_i^\nu \int_k \hat{\delta}(u_i k), \quad (\text{C.3})$$

and

$$\begin{array}{c} k, \mu\nu \\ \text{wavy line} \\ \text{double line with dot} \end{array} u_i, S_i = \frac{\kappa}{2} m_i u_i^\mu S_i^{\nu\rho} \int_k \hat{\delta}(u_i k) k_\rho. \quad (\text{C.4})$$

Also at $\mathcal{O}(\kappa)$ we have the three graviton vertex. We adopt the form also used in [15]

$$\begin{array}{c} k_3, ef \\ \text{wavy line} \\ \text{wavy line} \quad \text{wavy line} \\ k_1, ab \quad k_2, cd \end{array} = \frac{i\kappa}{4} \left[4k_1 k_2 (\eta^{af} \eta^{bd} \eta^{ce} + \eta^{ae} \eta^{bd} \eta^{cf}) \right. \\ + 4k_2 k_3 (\eta^{ae} \eta^{bc} \eta^{df} + \eta^{ac} \eta^{be} \eta^{df}) + 4k_1 k_3 (\eta^{ad} \eta^{bf} \eta^{ce} + \eta^{ac} \eta^{bf} \eta^{de}) \\ + (k_1 k_2 + k_2 k_3 + k_1 k_3) (\eta^{ab} \eta^{cd} \eta^{ef} - 2\eta^{ae} \eta^{bf} \eta^{cd} - 2\eta^{ab} \eta^{ce} \eta^{df} - 2\eta^{ac} \eta^{bd} \eta^{ef}) \\ \left. - 4 \left(\eta^{ad} \eta^{ce} k_1^f k_2^b + \eta^{ae} \eta^{bc} k_1^d k_2^f + \eta^{ac} \eta^{de} k_2^f k_3^b + \eta^{ae} \eta^{cf} k_2^b k_3^d + \eta^{af} \eta^{ce} k_1^d k_3^b + \eta^{ac} \eta^{be} k_1^f k_3^d \right) \right. \\ \left. + 2\eta^{ac} \eta^{bd} (k_1^e k_2^f + k_1^f k_2^e) + 2\eta^{ce} \eta^{df} (k_2^a k_3^b + k_2^b k_3^a) + 2\eta^{ae} \eta^{bf} (k_1^c k_3^d + k_1^d k_3^c) \right]. \quad (\text{C.5})$$

This directly tells us that the third diagram evaluates to

$$\frac{i\kappa^2 m_1 m_2 \gamma}{4} \frac{\partial}{\partial b^\mu} S_2^{\mu\nu} u_{1\nu} \int_q \frac{\hat{\delta}(u_1 q) \hat{\delta}(u_2 q)}{q^2} e^{-ibq}. \quad (\text{C.12})$$

The remaining Fourier transform can be performed in a suitably chosen frame. We take the frame where particle 1 is at rest and particle 2 moves along the x -direction. This means $u_1 = (1, \vec{0})$ and $u_2 = (\gamma, \sqrt{\gamma^2 - 1}, \vec{0}_{D-2})$. Integrating over the delta constraints and performing the remaining $D - 2$ -dimensional Fourier transform yields

$$\int_q \frac{\hat{\delta}(u_1 q) \hat{\delta}(u_2 q)}{q^2} e^{-ibq} = -\frac{1}{4\pi^{(D-2)/2}} \frac{\Gamma(D/2 - 2)}{\sqrt{\gamma^2 - 1}} |\vec{b}_\perp|^{4-D} \xrightarrow{D \rightarrow 4-2\epsilon} -\frac{\Gamma(-\epsilon)}{4\pi\sqrt{\gamma^2 - 1}} (-b^2)^\epsilon. \quad (\text{C.13})$$

Here we have written \vec{b}_\perp for the remaining spatial part of b that is orthogonal to both velocities. Also in the last step we have written it in a covariant way, which assumes that we keep in mind that b^μ is taken to be orthogonal to the velocities. Inserting this integral and performing the remaining derivatives with respect to the impact parameter we find eq. (3.29).

C.3 2PM eikonal phase

We consider the diagrams of fig. 2. The first diagram turns out to be (where we have substituted $l \rightarrow l - k$ and then $k \rightarrow l - k$)

$$\begin{aligned} & \frac{\kappa^4 m_2^2 m_1}{32} \int_{k,l} \frac{\hat{\delta}(u_2 k) \hat{\delta}(u_2 l) \hat{\delta}(u_1 l)}{k^2 (l - k)^2} e^{ibl} \left[J_{1,l \rightarrow k} k(l - k) \left(\gamma^2 - \frac{1}{D-2} \right)^2 + 4i \left(\gamma u_2 - \frac{1}{D-2} u_1 \right)^2 \right. \\ & \left. + \frac{4i}{D-2} \left(\gamma^2 - \frac{1}{D-2} \right) \right]. \end{aligned} \quad (\text{C.14})$$

There are a couple of things to point out here. First we can rewrite $k(l - k) = \frac{1}{2}(l^2 - k^2 - (l - k)^2)$. We note that since l appears in the exponential it can be understood diagrammatically as the momentum exchange, whereas k plays the role of the loop momentum. Therefore, from the rewriting only the first term survives because the latter two cancel a propagator pole in the denominator. This has the effect of leading to a $\delta(b)$ contribution and hence a contact interaction. What remains of the first term is the loop integral

$$\int_k \frac{\hat{\delta}(u_2 k)}{k^2 (k - l)^2} \left(\frac{1}{(u_1 k - i\epsilon)^2} + \frac{1}{(u_1 k + i\epsilon)^2} \right). \quad (\text{C.15})$$

We can go into the rest frame of particle 1, which leaves two poles in the k^0 component. For classical physics we need to integrate over the potential region. Although the poles lie inside the potential region, they also only occur in on half of the complex plane. Closing the contour in the other half respectively reveals that the integral does not contribute to the eikonal phase. This will also happen for all other diagrams involving the J_1 function. For the other terms in the first diagram we find

$$\frac{i\kappa^4 m_2^2 m_1}{8} \frac{D-3}{D-2} \gamma^2 \int_{k,l} \frac{\hat{\delta}(u_2 k) \hat{\delta}(u_2 l) \hat{\delta}(u_1 l) e^{ibl}}{k^2 (l - k)^2}. \quad (\text{C.16})$$

We now turn to the second diagram and the result in $D = 4$ is given by

$$\frac{i\kappa^4 m_2^2 m_1}{32} \int_{k,l} \frac{\hat{\delta}(u_2 k) \hat{\delta}(u_2 l) \hat{\delta}(u_1 l) e^{ibl}}{k^2 l^2 (l-k)^2} [kl(2\gamma^2 - 1) - l^2 \gamma^2 - k^2(2\gamma^2 - 1) - (ku_1)^2] \quad (\text{C.17})$$

There are several things to note here. First of all we can express $kl = \frac{1}{2}(k^2 + l^2 - (k-l)^2)$. As before only the l^2 term contributes. The $u_1 k$ term can be evaluated using tensor reduction and as shown in [15] this reduces the k -integral to $\frac{\gamma^2-1}{8} \int_k \frac{\hat{\delta}(u_2 k)}{k^2 (l-k)^2}$. Therefore the three-point graviton diagram gives us

$$\frac{i\kappa^4 m_2^2 m_1}{256} (-\gamma^2 - 3) \int_{k,l} \frac{\hat{\delta}(u_2 k) \hat{\delta}(u_2 l) \hat{\delta}(u_1 l) e^{ibl}}{k^2 (l-k)^2}. \quad (\text{C.18})$$

In combination with the first diagram we find

$$\frac{i\kappa^4 m_2^2 m_1}{256} (15\gamma^2 - 3) \int_{k,l} \frac{\hat{\delta}(u_2 k) \hat{\delta}(u_2 l) \hat{\delta}(u_1 l) e^{ibl}}{k^2 (l-k)^2}, \quad (\text{C.19})$$

and after performing the k and l integration this gives us the 2PM scalar eikonal phase

$$i\chi_{S^0}^{(2)} = \frac{i\kappa^4 m_2^2 m_1 (15\gamma^2 - 3)}{8 \cdot 32^2 \pi \sqrt{\gamma^2 - 1} \sqrt{-b^2}}. \quad (\text{C.20})$$

The four diagrams involving spin can be evaluated along the same lines. Here we need to compute the tensor integral

$$I^{\mu\nu} = \int_k \frac{\hat{\delta}(u_1 k) k^\mu k^\nu}{k^2 (k-l)^2}. \quad (\text{C.21})$$

Expanding the integral on a basis of tensors making heavy use of the delta constraint we find

$$I^{\mu\nu} = \frac{\sqrt{-l^2} P_1^{\mu\nu}}{64}, \quad (\text{C.22})$$

thereby resulting in the spin dipole contribution in eq. (3.31).

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