Einstein gravity from a matrix integral - Part II

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ABSTRACT: Using supersymmetric localization, we compute the partition function and some protected correlators of the polarized IKKT matrix model. Surprisingly, we find that the original IKKT model is different from polarized IKKT in the limit of vanishing mass deformation. We study different regimes of the localization results and recover the electrostatic problem which defines the gravity dual.

Contents

1 Introduction

Holographic duality [1–3] is one of the most – if not the most – profound ideas in modern theoretical physics as it provides a framework to describe quantum gravity microscopically using quantum field theories in lower dimensions. It builds on 't Hooft's observation [4] that diagrammatic expansions of large N gauge theories share structural similarities with two-dimensional Riemann surfaces, and formalizes this into a concrete correspondence between distinct theories, offering insights both into gauge theories and gravity.

Despite its promise, a precise understanding of how classical Einstein gravity emerges from gauge theories and how it is modified by quantum corrections still remains elusive. The difficulty arises primarily because Einstein gravity is expected to emerge in a strong-coupling regime of gauge theories, a regime for which we currently lack effective analytical tools. A possible way to make progress is to study simpler cases of holography where the gauge theory involved is more tractable. In this paper and its companion work [5], we study one of the simplest examples of gauge theory, a $0+0$ dimensional gauge theory, also known as a matrix integral. The holographic duality between matrix integrals and string theories has been a subject of active research in the past (see e.g. [6–9]). However, the "gravity-side" of these matrix integrals is not conventional Einstein gravity but instead corresponds to non-standard low-dimensional theories with some gravitational features. In contrast the system we will study is, in our view, the simplest model with an emergent description of Einstein gravity.

Matrix integral   More precisely, we study the polarized IKKT matrix model [5, 10–12]. Since this model is in $0+0$ dimension, the partition function is a matrix integral

$$Z = \int [dX^I][d\psi_\alpha] e^{-S}\,, \tag{1.1}$$

and the physical observables are matrix expectation values, expressed as

$$\langle f(X,\psi) \rangle = \frac{1}{Z} \int [dX^I][d\psi_\alpha] f(X,\psi) e^{-S}\,, \tag{1.2}$$

where the action reads

$$S = \mathrm{Tr}\left[-\frac{1}{4}[X_I, X_J]^2 - \frac{i}{2}\bar{\psi}\Gamma^I[X_I, \psi] \right.$$
$$\left. + \frac{3\Omega^2}{4^3}X_i X_i + \frac{\Omega^2}{4^3}X_p X_p + i\frac{\Omega}{3}\epsilon^{ijk}X_i X_j X_k - \frac{1}{8}\Omega\bar{\psi}\Gamma^{123}\psi \right]. \tag{1.3}$$

The first line is the IKKT action, where indices are contracted with Euclidean signature. This ensures that the partition function converges. The second line is a mass-deformation, parameterized by the dimensionless parameter Ω. The matrices X^I are dimensionless $N \times N$ Hermitian matrices. The indices $I, J \in \{1, 2, \ldots, 10\}$ while $i, j, k \in \{1, 2, 3\}$ and $p \in \{4, 5, \ldots 10\}$. The Majorana-Weyl fermions ψ_α (with $\alpha \in \{1, \ldots, 32\}$) are also dimensionless

Hermitian $N \times N$ matrices[1]. The gamma matrices are 32×32 and $\Gamma^{123} = \Gamma^1\Gamma^2\Gamma^3$, while the fermion conjugation is defined as $\bar{\psi} = \psi^\top \mathcal{C}$ where $\mathcal{C}$ is the charge conjugation matrix.

This model has 16 supercharges and global $SO(3) \times SO(7)$ as well as $SU(N)$ gauge symmetry.[2] The matrix integral has classical saddles $X_i = \frac{3\Omega}{8}L_i$, given by representations of $SU(2)$ generators L_i [5, 12]. A generic reducible representation of $SU(2)$ is parameterized by its q different spin j_s irreducible components $s = 1, ..., q$ each of which has n_s copies and dimension $N_s = 2j_s + 1$, such that $N = \sum_{s=1}^{q} n_s N_s$. The number of different saddles is thus given by the number of partitions of N.

Gravity dual   In a recent paper [5] we constructed a family of solutions to Euclidean IIB supergravity dual to the polarized IKKT matrix model. Those solutions are warped products of a 2-sphere and a 6-sphere fibered over the two dimensional (r, z) plane, determined entirely by a single function $V(r, z)$ solving the four dimensional axially symmetric Laplace equation

$$\frac{\partial^2 V}{\partial z^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} = 0, \tag{1.4}$$

where r is a radial variable in $\mathbb{R}^3$ and z is the orthogonal coordinate. We can think of $V(r, z)$ as the solution to an electrostatic problem, that is uniquely determined once we specify proper boundary conditions. As explained in [5], to make the resulting geometry non-singular, we need to consider a background potential

$$V_\infty(r, z) = V_b \left(zr^2 - z^3\right), \tag{1.5}$$

where V_b is a positive constant. In addition we consider a configuration of 3-dimensional conducting balls, each centered on the z-axis, at positions z_s and carrying charges Q_s, with their sizes R_s being determined by the condition that the charge density vanishes at the edge. The electrostatic problem is then entirely specified by the parameters (z_s, Q_s) with $s = 1, ..., q$. This configuration is dual to a saddle of the matrix integral parametrized the degeneracies and irrep dimensions (n_s, N_s), see figure 1.

Results   In this paper, we use localization techniques to derive exact results for the partition function and a family of protected correlation functions. As we show in the following sections, we obtain the $U(N)$ partition function[3]

$$Z = \sum_{\mathcal{R}} Z_{\mathcal{R}},$$

$$Z_{\mathcal{R}} = C_{\mathcal{R}} e^{\frac{9\Omega^4}{2^{15}}\sum_s n_s(N_s^3 - N_s)} \int \left(\prod_s \prod_{i=1}^{n_s} dm_{si}\right) Z_{1-\text{loop}} \exp\left(-\frac{3\Omega^4}{2^7}\sum_s \sum_{i=1}^{n_s} N_s m_{si}^2\right), \tag{1.6}$$

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[1]Note that since they obey Majorana Weyl conditions, they each have 16 real independent components.

[2]We use the word "gauge" in analogy to higher-dimensional gauge theories. However, in 0+0 dimensions, gauge and global symmetries can be treated on the same footing, with the distinction that only gauge-invariant observables are considered.

[3]We use the normalization $[dX] \equiv \prod_{A=1}^{N^2} dX^A$ for bosonic matrices and $[d\psi] \equiv \prod_{A=1}^{N^2} d\psi^A$ for fermionic matrices, where A is an index for the $U(N)$ generators T^A, e.g. $X = X^A T^A$, with normalization $\text{Tr} T^A T^B = \delta^{AB}$. More details on this measure convention are given in appendix C.6.
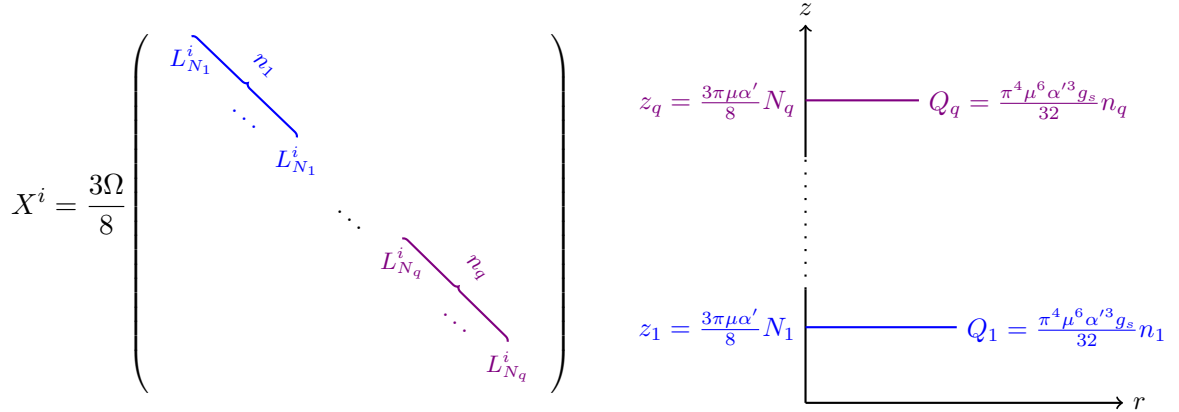
Figure 1: Correspondence between the fuzzy sphere vacua of the matrix model and the dual geometries. For each spin-j_s $SU(2)$ irreducible representation of dimension $N_s = 2j_s + 1$ we put a disk at position $z_s \sim N_s$. The number n_s of copies of this representation appearing determines the charge of that disk $Q_s \sim n_s$.

where

$$Z_{1-\text{loop}} = \prod_{(si,tj)} f_{st}(m_{si} - m_{tj}),$$

$$f_{st}(x) \equiv \prod_{J=\frac{|N_s - N_t|}{2}}^{\frac{N_s + N_t}{2} - 1} \frac{[(2 + 3J)^2 + (8x)^2]^3[(3J)^2 + (8x)^2]}{[(1 + 3J)^2 + (8x)^2]^3[(3 + 3J)^2 + (8x)^2]},$$
(1.7)

$$C_{\mathcal{R}} = \frac{(2\pi)^{5N^2 + N/2}}{\prod_{k=1}^{N-1} k! \prod_s n_s!} \left(\frac{32}{3\pi}\right)^{\sum_s n_s} \prod_s \left(\frac{1}{N_s} \prod_{J=1}^{N_s - 1} \frac{(2 + 3J)^3}{(1 + 3J)^3}\right)^{n_s}.$$
(1.8)

The sum $\sum_{\mathcal{R}}$ is taken over all the inequivalent N-dimensional representations $\mathcal{R}$ of $SU(2)$. The parameters s, t are labels for the different irreducible representations (irreps) appearing in the representation $\mathcal{R}$, and $i = 1, ..., n_s$ is an index going over the multiplicity of the irrep s. The product $\prod_{(si,tj)}$ in $Z_{1-\text{loop}}$ runs over all pairs (si, tj), excluding the identical terms (si, si). The constant $C_{\mathcal{R}}$ is a normalization constant which depends on the saddle. Similarly, correlators of gauge invariant functions of the protected matrix $\phi = X_3 - iX_{10}$ can be computed as shown in (3.63). We provide analyses in various regimes of the localization results (1.6) and (3.63), matching them with direct computations.

As surprising as it may sound, the partition function (1.6) diverges as $\Omega \to 0$, even though the IKKT partition function has been proven to converge [13], namely

$$\lim_{\Omega \to 0} Z(\Omega) = \infty \neq Z_{\text{IKKT}}.$$
(1.9)

This behaviour is due to the fermion mass terms as we illustrate for $N = 2$ in appendix B. We also obtain this divergence for any N from integrating out the off-diagonal matrices

as shown in section 4.2. As we show there, the leading divergence can be interpreted as arising from the trivial vacuum, where each particle is very separated from the others and each contributes $Z_{U(1)} \sim \frac{1}{\Omega^2}$. Nevertheless, we conjecture that the IKKT partition function can be obtained from the $\Omega \to 0$ limit of (1.6) by closing the contour of integration in the complex plane.

Finally, we recover the electrostatic problem describing the backreacted geometries from the density of eigenvalues associated to (1.6). This analysis is very similar to the one relating the BMN model to Lin-Maldacena geometries [14, 15].

Structure of the paper In section 2 we review the basics of the IKKT model and its mass deformation. Section 3 includes the setup and the details of applying supersymmetric localization techniques to compute the polarized IKKT partition function and protected correlation functions. It is intended to be pedagogical. Section 4 presents the analysis of the localization result, including strong and weak coupling limits as well as a conjectured prescription to connect the polarized and the original IKKT partition function. In section 5 we demonstrate the explicit agreement between the polarized IKKT matrix model and the dual geometry constructed in [5]. This is done by showing that in the large N limit the eigenvalue densities from the matrix integral satisfy the same equation as that of the charge densities in the electrostatic system describing the dual geometry (upon an explicit identification of parameters on both sides). We discuss various open problems in section 6. Various technical details are collected in the appendices.

2 Review of the matrix models

2.1 IKKT

The (Euclidean) IKKT integral [10, 16] is

$$Z_{\text{IKKT}} = \int \prod_{A=1}^{N^2-1} \left[\prod_{I=1}^{10} dX_I^A \prod_{\alpha=1}^{16} d\psi_\alpha^A \right] e^{-S_{\text{IKKT}}}, \tag{2.1}$$

where[4]

$$S_{\text{IKKT}} = \text{Tr} \left[-\frac{1}{4}[X_I, X_J]^2 - \frac{i}{2} \sum_{\alpha,\beta=1}^{32} \sum_{I=1}^{10} \psi_\alpha (\mathcal{C}\Gamma^I)_{\alpha\beta}[X_I, \psi_\beta] \right]. \tag{2.2}$$

The bosonic matrices X_I and the fermionic matrices ψ_α are traceless Hermitian matrices, expressed in the basis of $SU(N)$ generators T^A, where we choose $\text{Tr} T^A T^B = \delta^{AB}$ as normalization. The fermions ψ_α^A are 10D Euclidean Majorana-Weyl spinors.[5]

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[4]The action has the same convention as [17–19], which is natural from the perspective of dimensionally reducing 10D $\mathcal{N} = 1$ SYM to a point. This convention is related to that in [12] through a simple redefinition of the charge conjugation matrix: $i\mathcal{C}_{\text{here}} = \mathcal{C}_{\text{there}}$. After Weyl projection to the suitable 16-component spinors, the actions are the same.

[5]Note that after imposing Majorana-Weyl condition on a 32-dimensional complex spinor, it is reduced to 16 real independent components. For an explicit construction, refer to appendix A.

One might expect that the partition function (2.1) is divergent, because the action has infinitely long valleys (flat directions), corresponding to diagonal (commuting) matrices, along which $S_{\text{IKKT}} = 0$. However, it has been shown in [13, 20] that the width of the valleys are in fact highly suppressed so that their volume remains finite. To understand this point, consider the following toy integral

$$\int dxdy e^{-y^2-x^4y^2} = \sqrt{\pi} \int dx \frac{1}{\sqrt{1+x^4}} < \infty .$$

(2.3)

Even though the left-hand-side has flat directions corresponding to $y = 0$, the valley at large x has width $\sim \frac{1}{x^2}$ so that the volume integral over this infinite valley remains finite.

A closed form of the partition function (2.1), first conjectured in [17] and later computed using localization techniques [18] takes the form

$$Z_{\text{IKKT}} = \frac{(2\pi)^{(10N+11)(N-1)/2}}{\sqrt{N} \prod_{k=1}^{N-1}(k!)} \sum_{m|N} \frac{1}{m^2} ,$$

(2.4)

where the sum runs over all divisors of N and the prefactor, determined in [21], has been adapted to our conventions. It can also be expressed in terms of a divisor function $\sigma_2(N) = \sum_{m|N} m^2$ as

$$Z_{\text{IKKT}} = \frac{(2\pi)^{(10N+11)(N-1)/2}}{N^{5/2} \prod_{k=1}^{N-1}(k!)} \sigma_2(N) ,$$

(2.5)

whose asymptotic behavior and variance have been studied in the literature [22].

2.2 Polarized IKKT

The model we are interested in is a mass deformation of IKKT which preserves all sixteen dynamical supersymmetries. Its action, written in (1.3), is invariant under

$$\begin{aligned}
\delta_\epsilon X^I &= \bar{\epsilon}\Gamma^I \psi , \\
\delta_\epsilon \psi &= \frac{i}{2}\Gamma^{IJ}\epsilon[X_I, X_J] + \frac{3\Omega}{8}\Gamma^{123}\Gamma^i\epsilon X_i + \frac{\Omega}{8}\Gamma^{123}\Gamma^p\epsilon X_p ,
\end{aligned}$$

(2.6)

where ϵ is deemed as Grassmann-even and δ_ϵ is Grassman-odd susy generator.

The saddle points of the polarized IKKT integral are minima of the bosonic potential

$$V_B = \text{Tr}\left[-\frac{1}{4}[X_I, X_J]^2 + \frac{3\Omega^2}{4^3}X_iX_i + \frac{\Omega^2}{4^3}X_pX_p + i\frac{\Omega}{3}\epsilon_{ijk}X_iX_jX_k\right] .$$

(2.7)

The minima are given by [5, 12]

$$X_p = 0 , \qquad X_i = \frac{3}{8}\Omega L_i , \qquad [L_i, L_j] = i\epsilon_{ijk}L_k ,$$

(2.8)

with L_i being the $N \times N$ matrix representations of $SU(2)$ Lie algebra (not necessarily irreducible). At the minima the potential takes values

$$V_B = -\frac{9}{2^{13}}\Omega^4 \text{Tr}\, L_iL_i .$$

(2.9)

For a general reducible representation, L_i is block diagonal with q different irreps, each having multiplicity n_s and size $N_s \times N_s$. Thus, we have

$$\operatorname{Tr} L_i L_i = \sum_{s=1}^{q} n_s N_s j_s(j_s + 1) = \sum_{s=1}^{q} n_s N_s \frac{N_s^2 - 1}{4} \,, \qquad N = \sum_{s=1}^{q} n_s N_s \,. \tag{2.10}$$

All these minima preserve supersymmetry [5].

3 Localization in polarized IKKT

We will now use supersymmetric localization to reduce the polarized IKKT partition function, as well as SUSY-protected correlators, to integrals over the moduli space. Given that our model is very similar to the plane-wave matrix model, also called BMN model [23], we will closely follow the formalism developed in [24].

Let us review the main idea [25, 26]. Given a supersymmetric theory, one can choose one supersymmetry δ_s and define a deformed partition function,

$$Z_t \equiv \int [dX^I][d\psi_\alpha] \exp(-(S + t\delta_s V)) \,, \tag{3.1}$$

where V is a Grassmann-odd polynomial of the matrices. If the purely bosonic part of $\delta_s V$ is non-negative, this deformed quantity is converging for any $t \geq 0$. Let us now assume that $\delta_s^2 V = 0$. Then,

$$\frac{d}{dt} Z_t = \int [dX^I][d\psi_\alpha] \delta_s \left(V \exp(-(S + t\delta_s V)) \right) = 0 \,, \tag{3.2}$$

where in the last step we used that δ_s can be written as a sum of derivatives[6]. The assumption that $\delta_s^2 V = 0$ typically restricts V to be invariant under some subgroup of $SU(N) \times SO(3) \times SO(7)$. The fact that Z_t is independent of t has a very useful consequence: The saddle point approximation corresponding to $t \to \infty$ is exact, and is equal to the partition function $Z = Z_{t=0}$. This reduces the integral to the moduli space obeying $\delta_s V = 0$.

Note that this argument can also be applied to correlators of protected operators $\mathcal{O}$ such that $\delta_s \mathcal{O} = 0$, namely

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_t \equiv \frac{1}{Z} \int [dX^I][d\psi_\alpha] \mathcal{O} \exp(-(S + t\delta_s V)) \,, \qquad \forall t \geq 0 \,. \tag{3.3}$$

3.1 Closure of the SUSY algebra

The requirement that δ_s^2 reduces to some combination of gauge and global symmetry generators, under which V should be invariant, requires the SUSY algebra to close on this symmetry group. In the formulation (2.6), this is only true on-shell.

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[6]This follows from the supersymmetries (3.5). We can schematically write

$$\delta_s \sim \psi \frac{\delta}{\delta X} + ([X, X] + \Omega X + K) \frac{\delta}{\delta \psi} + ([X, \psi] + \Omega \psi) \frac{\delta}{\delta K}$$

Note also that all derivatives commute with what's in front of them, so they can be moved to the left.

An off-shell formulation is obtained by including seven auxiliary fields $K_{a=1,...,7}$, following [24, 27]. Namely, we will now consider the action

$$S = \text{Tr}\left[-\frac{1}{4}[X_I, X_J]^2 - \frac{1}{2}\psi^\top \bar{\gamma}^I[X_I, \psi] + \frac{1}{2}K_aK_a \right.$$
$$\left. + \frac{3\Omega^2}{4^3}X_iX_i + \frac{\Omega^2}{4^3}X_pX_p + i\frac{\Omega}{3}\epsilon^{ijk}X_iX_jX_k + \frac{i}{8}\Omega\psi^\top\bar{\gamma}^{123}\psi \right],$$
(3.4)

where we wrote the Majorana-Weyl fermions in 16-component notation, i.e. $\psi_{\alpha=1,...,16}$ are Hermitian matrices of left-handed spinors, and $\bar{\gamma}^I$ are 16×16 Weyl matrices defined in (A.2). Note that the effect of auxiliary fields to the partition function is trivial, which is just a multiplication by $\int[dK_a]e^{-\frac{1}{2}\text{Tr}K_aK_a} = (2\pi)^{7N^2/2}$. The supercharge, leaving the action (3.4) invariant, is now acting as

$$\delta_\epsilon X^I = -i\epsilon^\top\bar{\gamma}^I\psi\,,$$
$$\delta_\epsilon\psi = \frac{i}{2}\bar{\gamma}^{IJ}\epsilon[X_I, X_J] + \frac{3\Omega}{8}\bar{\gamma}^{123}\bar{\gamma}^i\epsilon X_i + \frac{\Omega}{8}\bar{\gamma}^{123}\bar{\gamma}^p\epsilon X_p + i\nu_aK_a\,,$$
(3.5)
$$\delta_\epsilon K_a = i\nu_a^\top\bar{\gamma}^I[X_I, \psi] + \frac{\Omega}{4}\nu_a^\top\bar{\gamma}^{123}\psi\,,$$

where ϵ is deemed as Grassmann-even and δ_ϵ is Grassman-odd. The parameters $\nu_{a=1,2,...,7}$ are 7 arbitrary Grassmann-even Majorana-Weyl spinors at this point. However, we require the SUSY algebra to close, namely,

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}(\text{fields}) = \delta_B(\text{fields})\,,$$
(3.6)

where δ_B is a bosonic symmetry generator (i.e. some combination of $SU(N)$ and $SO(3) \times SO(7)$). This imposes constraints on ν_a, as we show in C.1, which read

$$\epsilon^\top\bar{\gamma}^I\nu_a = 0\,,$$
$$\frac{1}{2}(\epsilon_1^\top\bar{\gamma}^I\epsilon_2)\gamma_{\alpha\beta}^I = (\epsilon_1)_\alpha(\epsilon_2)_\beta + (\nu_a^1)_\alpha(\nu_a^2)_\beta\,,$$
(3.7)
$$(\nu_a^1)^\top\bar{\gamma}^I\nu_b^2 = \delta_{ab}\epsilon_1^\top\bar{\gamma}^I\epsilon_2\,.$$

We solve these constraints for our choice of ϵ, as discussed in the next section and shown explicitly in appendix C.2.

3.2 Localization setup

Throughout this and the following sections, it will prove convenient to use the following notation for the indices

$$I, J, ... = 1, ..., 10\,, \quad i, j, ... = 1, ..., 3\,, \quad p, q, ... = 4, ..., 10\,, \quad a = 1, ..., 7\,,$$
$$I', J', ... = 1, ..., 9\,, \quad i', j', ... = 1, 2\,, \quad p', q', ... = 4, ..., 9\,, \quad a' = 1, ..., 6\,.$$
(3.8)

We now choose a particular supersymmetry transformation δ_s by specifying ϵ. Following [24], we find it natural[7] to choose ϵ such that it preserves a specific linear combination of X^I,

$$\delta_s\phi = 0\,, \qquad \phi \equiv X_3 - iX_{10}\,. \tag{3.9}$$

Choosing a gamma matrix representation such that $\bar{\gamma}^3$ and $\bar{\gamma}^{10}$ are diagonal, taking the same value in the first 8 components, as well as in the last 8 components, as given explicitly in (A.1), (A.2), we obtain that

$$\epsilon = \begin{pmatrix} \eta \\ 0 \end{pmatrix}\,, \tag{3.10}$$

where η is a 8-component spinor. We also choose the normalization $\epsilon^\top\epsilon = \eta^\top\eta = 1$. The constraints on ν_a (3.7) are solved in appendix C.2, where we find that ν_a have to live on the same 8-component space as ϵ, have to be orthogonal to ϵ and have to be orthogonal to each other. This justifies why a has to run from 1 to 7. From now on, we will choose $\eta = (1, 0, ..., 0)$. Correspondingly, we will choose $(\nu_a)_\alpha = \delta_{\alpha,a+1}$.[8]

Let us also package the matrices in a convenient way. As explained in appendix C.2, the spinors $\{\bar{\gamma}^{I'}\epsilon, \nu_a\}$ form an orthonormal basis of 16-component spinors. We thus decompose the fermionic matrices as

$$\psi = v^{I'}\psi_{I'} + \nu^a\chi_a\,, \qquad v^{I'} \equiv \bar{\gamma}^{I'}\epsilon\,. \tag{3.11}$$

This allows us to package matrices into

$$Y \equiv \begin{pmatrix} X_{I'} \\ \chi_a \end{pmatrix}\,, \qquad Y' \equiv \begin{pmatrix} -i\psi_{I'} \\ iH_a \end{pmatrix}\,, \tag{3.12}$$

where

$$H_a \equiv K_a + \frac{\Omega}{8}\nu_a^\top\tilde{\epsilon}X_{10} - is_a\,, \tag{3.13}$$

with

$$\tilde{\epsilon} \equiv \bar{\gamma}^{123}\epsilon\,, \qquad s_a \equiv \frac{3\Omega}{8}\nu_a^\top\gamma^i\tilde{\epsilon}X_i - \frac{\Omega}{8}\nu_a^\top\gamma^{p'}\tilde{\epsilon}X_{p'} + \frac{i}{2}\nu_a^\top\bar{\gamma}^{I'J'}\epsilon[X_{I'}, X_{J'}]\,. \tag{3.14}$$

From now on, we will consider the matrices in Y and Y' as our physical variables. This rewriting is very similar to [24, 26], and it has the advantage of simplifying the SUSY transformations to

$$\delta_s Y = Y'\,, \qquad \delta_s Y' = -i(\delta_\phi + \delta_{U(1)})Y\,, \tag{3.15}$$

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[7]While the final result of the integral (1.1) is independent of the choice of the deformation $\delta_s V$, the partition function written as a reduced (moduli space) integral, e.g. (1.6), can be different. In fact, in [12] a different deformation of the polarized IKKT action was used and it leads to different saddle points and different moduli space integral.

In [15], it was observed that a similar choice of BPS sector (or equivalently ϕ) to ours leads to a direct match with the gravity dual of the BMN model. In section 5, we will see that our choice of ϕ also leads to the matching with the gravity dual to the polarized IKKT model [5].

[8]This is fixed up to $SO(7)$ rotation.

where δ_ϕ is a gauge transformation with parameter ϕ,

$$\delta_\phi = i[\phi, \cdot]\,, \tag{3.16}$$

and $\delta_{U(1)}$ is a $U(1)$ subgroup of $SO(3) \times SO(7)$ which acts as

$$\delta_{U(1)}(X_1 \pm iX_2) = \mp i\frac{3}{8}(X_1 \pm iX_2)\,, \qquad \delta_{U(1)}(\chi_1 \pm i\chi_6) = \pm i\frac{1}{4}(\chi_1 \pm i\chi_6)\,,$$

$$\delta_{U(1)}(X_4 \pm iX_5) = \mp i\frac{1}{8}(X_4 \pm iX_5)\,, \qquad \delta_{U(1)}(\chi_2 \pm i\chi_5) = \mp i\frac{1}{4}(\chi_2 \pm i\chi_5)\,,$$

$$\delta_{U(1)}(X_6 \pm iX_7) = \mp i\frac{1}{8}(X_6 \pm iX_7)\,, \qquad \delta_{U(1)}(\chi_3 \pm i\chi_4) = \pm i\frac{1}{4}(\chi_3 \pm i\chi_4)\,,$$

$$\delta_{U(1)}(X_8 \pm iX_9) = \mp i\frac{1}{8}(X_8 \pm iX_9)\,, \qquad \delta_{U(1)}X_3 = 0\,, \qquad \delta_{U(1)}\chi_7 = 0\,.$$

This notation will prove very useful in the following sections.

Finally we need to choose a Grassmann-odd potential V_0 to deform the action. We take the conventional choice in supersymmetric localization [26, 28], namely

$$V_0 = \text{Tr } \psi_\alpha (\delta_s \psi_\alpha)^\dagger\,. \tag{3.17}$$

This ensures that the purely bosonic part of $\delta_s V_0$ is non-negative, since

$$t\delta_s V_0 = t \text{ Tr } \left(|\delta_s \psi_\alpha|^2 - \psi_\alpha \delta_s \left[(\delta_s \psi_\alpha)^\dagger \right] \right)\,. \tag{3.18}$$

The first term is purely bosonic and positive, whereas the second term is a fermionic bilinear.

3.3 Saddle points

To do the saddle point approximation at $t \to \infty$, we first need to identify the loci of the saddle points in (3.18). This is done by writing explicitly

$$\begin{aligned}
\text{Tr}|\delta_s \psi|^2 = \text{Tr}\bigg\{ &\left(K_7 + \frac{\Omega}{8}X_{10} \right)^2 + K_{a'}^2 + \left(\frac{3\Omega}{8}X_i + \frac{i}{2}\epsilon_{ijk}[X_j, X_k] \right)^2 \\
&+ \left(\frac{\Omega}{8}X_4 - \frac{i}{2}[X_5, X_3] \right)^2 + \left(\frac{\Omega}{8}X_5 - \frac{i}{2}[X_3, X_4] \right)^2 \\
&+ \left(\frac{\Omega}{8}X_6 - \frac{i}{2}[X_7, X_3] \right)^2 + \left(\frac{\Omega}{8}X_7 - \frac{i}{2}[X_3, X_6] \right)^2 \\
&+ \left(\frac{\Omega}{8}X_8 - \frac{i}{2}[X_9, X_3] \right)^2 + \left(\frac{\Omega}{8}X_9 - \frac{i}{2}[X_3, X_8] \right)^2 \\
&- \frac{3}{4}[X_3, X_{p'}]^2 - [X_{i'}, X_{p'}]^2 - \frac{1}{2}[X_{p'}, X_{q'}]^2 - [X_I, X_{10}]^2 \bigg\}\,,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\text{Tr}\left(\psi^\top \delta_s \left[(\delta_s \psi)^* \right] \right) = \text{Tr}\bigg\{ &-\frac{3i\Omega}{4}\psi_1^\top \psi_2 - \frac{i\Omega}{4}\psi_4^\top \psi_5 - \frac{i\Omega}{4}\psi_6^\top \psi_7 - \frac{i\Omega}{4}\psi_8^\top \psi_9 \\
&- \frac{i\Omega}{2}\chi_1^\top \chi_6 + \frac{i\Omega}{2}\chi_2^\top \chi_5 - \frac{i\Omega}{2}\chi_3^\top \chi_4 + \frac{3i\Omega}{4}\chi_7^\top \psi_3 + \dots \bigg\}\,,
\end{aligned} \tag{3.20}$$

where ... denote fermionic bilinears involving bosonic matrices and the notation for the indices follows (3.8). Since the matrices are Hermitian, all terms in the purely bosonic part are positive. Thus, the saddle point loci are determined by imposing that each term in (3.19) vanishes, leading to[9]

$$X_i = \frac{3\Omega}{8}L_i\,, \quad X_{10} = M\,, \quad K_7 = -\frac{\Omega}{8}M\,, \quad X_{p'} = 0\,, \quad K_{a'} = 0\,, \quad \psi_{I'} = 0\,, \quad \chi_a = 0\,,$$
(3.22)

where

$$[L_i, L_j] = i\epsilon_{ijk}L_k\,, \quad [M, L_i] = 0\,.$$
(3.23)

The matrices L_i are $SU(2)$ generators, and the requirement $[M, L_i] = 0$ implies, by Schur's lemma, that M is block-diagonal in the irrep types, and takes constant values in each irrep. For concreteness, L_i and M take the form (recall that $N_s = 2j_s + 1$)

$$L_i = \begin{pmatrix} \mathbb{1}_{n_1} \otimes L_i^{[j_1]} & & & & \\ & \ddots & & & \\ & & \mathbb{1}_{n_s} \otimes L_i^{[j_s]} & & \\ & & & \ddots & \\ & & & & \mathbb{1}_{n_q} \otimes L_i^{[j_q]} \end{pmatrix}\,,$$
(3.24)

$$M = \begin{pmatrix} M_1 \otimes \mathbb{1}_{N_1} & & & & \\ & \ddots & & & \\ & & M_s \otimes \mathbb{1}_{N_s} & & \\ & & & \ddots & \\ & & & & M_q \otimes \mathbb{1}_{N_q} \end{pmatrix}\,.$$
(3.25)

As we will see later, using the $SU(N)$ gauge invariance we can diagonalize M, meaning that each block M_s will be written as

$$M_s = \mathrm{diag}(m_{si})_{i=1,\dots,n_s}\,,$$
(3.26)

which is convenient for localization.

3.4 Gauge fixing

The saddle points $X_i \propto L_i$, for non-trivial L_i have some gauge redundancy $L_i \to UL_iU^\dagger$ (and correspondingly $M \to UMU^\dagger$). It is not easy to gauge fix L_i since this is representation-dependent. For example, the gauge redundancy of the N-dimensional $SU(2)$ irrep is $SU(N)$

---

[9]The reason why fermions are set to zero can be seen through a simple example

$$\lim_{t \to \infty} \int d^n\psi\, d^n\chi\, e^{-t\chi^\top A\psi} f(\chi, \psi) = t^n/n! \int d^n\psi\, d^n\chi\, (\chi^\top A\psi)^n f(0,0) + O(t^{n-1})\,,$$
(3.21)

where χ, ψ are n-component fermions and A is a matrix (operator). Also note that from (3.20) we see there is no fermionic zero modes in $\delta_s V$.

whereas the $SU(2)$ trivial representation $L_i = 0$ has no gauge redundancy. More generally, the non-trivial gauge redundancy of (3.24) is given by the action of the group[10]

$$\mathcal{G}_{\mathcal{R}} = \frac{SU(N)}{\left(\otimes_{s=1}^{q} U(n_s)\right)/U(1)} = \frac{U(N)}{\otimes_{s=1}^{q} U(n_s)},$$ (3.27)

where $\mathcal{R}$ denotes a saddle point in (3.22) or equivalently an N-dimensional representation of $SU(2)$ generators.

The convenient way to gauge fix is to introduce Faddeev-Popov ghosts. This allows us to gauge fix the non-trivial unitary rotations of L_i, in a way that treats all the $SU(2)$ representations on the same footing. Before elaboration, let us first review the main idea. The partition function will be written as a sum over the saddle points $Z = \sum_{\mathcal{R}} Z_{\mathcal{R}}$, and for each $Z_{\mathcal{R}}$ we introduce

$$1 = \int [d\alpha] \det{}' \left(\frac{\delta F}{\delta \alpha}\right) \delta'(F[\Phi]),$$ (3.28)

where $F[\Phi]$ is a gauge-fixing condition on the fields collectively denoted by Φ, α are co-ordinates of a group element (3.27). Due the to slightly unusual gauge group (3.27), the gauge fixing requires extra care for treating the zero modes of the ghosts. The end result is captured by the primes ($\det'$ and δ') above, which indicate the exclusion of zero modes such that the determinant and the Dirac delta function are well-defined (see section 3.4.1). Using gauge invariance, the integral over α factors out, giving the gauge group volume, whereas the determinant and the delta function are written in terms of a ghost action S_{gh}. In our localization setup, this would give[11]

$$Z_{\mathcal{R}} = \int [d\Phi] e^{-(S+t\delta_s V_0)} = \text{Vol}(\mathcal{G}_{\mathcal{R}}) \int [d\Phi][d\Phi_{\text{gh}}] e^{-(S+t\delta_s V_0 + tS_{\text{gh}})},$$ (3.29)

where Φ_{gh} denotes ghost fields collectively.

Written in this form, the exponent in the integrand is no longer of the form $S + t\delta V$ with $\delta^2 V = 0$, which was the form we wanted to localize the integral. However, we can get back to this form by formulating the ghost action in the BRST formalism and combine it with supersymmetry, as was done in [24, 26]. This will allow us to treat $\delta_s V_0$ and S_{gh} on the same footing and write the exponent as $S + tQV$, where the operator $Q = \delta_s + \delta_b$ combines both the supersymmetry δ_s and the BRST δ_b transformations and $Q^2 V = 0$. Let us describe this in more detail.

---

[10]The action of $U(n_s)$ treats each of the n_s $SU(2)$ irrep blocks in (3.24), i.e. $L_j^{[j_s]}$, as a matrix element of an $n_s \times n_s$ matrix. Concretely, elements of $U(n_s)$ have the same matrix form as M in (3.25) and they commute with L_j.

[11]Note that the inclusion of the parameter t in front of the ghost action has no incidence on the analysis, as one can rescale fields in a measure invariant way to make this factor appear.

3.4.1 The ghost action

To begin with, we choose the following gauge fixing condition[12]

$$F[X] \equiv -i[L_j, X_j] \overset{!}{=} 0 \,. \tag{3.30}$$

This is a zero-dimensional version of the background gauge, which was employed e.g. in the analysis of scatterng amplitudes on the Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills [29].

To fix the gauge through the ghosts they should live in the space of the gauge transformation directions, and their fluctuations living in the same space as the physical degrees of freedom should be removed. This will be implemented by introducing additional zero-mode ghosts $\Phi_{\mathrm{gh},0}$ satisfying $[L_j, \Phi_{\mathrm{gh},0}] = 0$, as will be explained in the remainder of this section. Explicitly, we add the ghost fields

$$\{\Phi_{\mathrm{gh}}\} = \{C, C_0, \tilde{C}, \tilde{C}_0, b, b_0, a_0, \tilde{a}_0\} \,, \tag{3.31}$$

$$\{\Phi_{\mathrm{gh},0}\} = \{C_0, \tilde{C}_0, b_0, a_0, \tilde{a}_0\} \,, \tag{3.32}$$

where the ghosts $C, C_0, \tilde{C}, \tilde{C}_0$ are fermionic whereas b, b_0, a_0, $\tilde{a}_0$ are bosonic. The ghosts C, $\tilde{C}$ and b are Hermitian N by N matrices. Since $\Phi_{\mathrm{gh},0}$ commute with L_i, they all take the same form as M, (3.25), parameterized by Hermitian $n_s \times n_s$ matrices. The full ghost action reads

$$\begin{aligned} S_{\mathrm{gh}} = \mathrm{Tr}\Big(& ibF[X] + i\tilde{C}[L_j, [X_j, C]] + ibb_0 + i\tilde{C}C_0 + iC\tilde{C}_0 \\ & + i\left(a_0 + i\frac{3\Omega}{8}L_3 - i\phi + iC^2\right)\tilde{a}_0 + \tilde{C}[L_j, \psi_j]\Big) \,. \end{aligned} \tag{3.33}$$

The first two terms ibF and $i\tilde{C}[...,C]$ are the conventional ones, giving respectively the Dirac δ function for the gauge fixing and the Faddeev-Popov determinant. The terms ibb_0, $i\tilde{C}C_0$ and $iC\tilde{C}_0$ remove the components of $C, \tilde{C}$ and b which are zero-modes under F and $\delta F/\delta\alpha$. The second line, which wouldn't be necessary if our purpose was only to gauge fix, comes from the constraint of BRST symmetry as well as consistency with supersymmetry as shown in section 3.4.2 and in appendix C.3. Let us describe in more details the construction of the zero-modes and why the second line doesn't affect the discussion.

The ghost b has zero modes under $\mathrm{Tr}(F[X]\cdot)$, i.e.,

$$0 \equiv \mathrm{Tr}(F[X]b^{(\mathrm{zero})}) = i\mathrm{Tr}(X_j[L_j, b^{(\mathrm{zero})}]) \,. \tag{3.34}$$

In order for the Dirac delta function in (3.28) to be well-defined, we introduce b_0 and impose

$$[L_j, b_0] = 0 \,, \tag{3.35}$$

---

[12]Schematically we can write $X_j = L_j + \delta X_j^{(\mathrm{phys})} + \delta X_j^{(\mathrm{gauge})}$, and the condition (3.30) indicates that $[L_j, \delta X_j^{(\mathrm{phys})}] = 0$. See [12, Sec.3.2] for an explanation on why fluctuations satisfying this constraint are orthogonal to the gauge orbit. One can also check that (3.30) gives $N^2 - \sum_s n_s^2$ independent constraints on X_j, which equals exactly the number of gauge transformation directions in the gauge group $\mathcal{G}_{\mathcal{R}}$.

such that it indeed lives in the same space as the zero modes of b. Then the integral of $\exp{(ibb_0)}$ gives a Dirac delta on the zero modes of b and eliminates them. Similarly, the ghosts C and $\tilde{C}$ have zero modes under the operator $[L_j, [X_j, \cdot] \equiv \frac{\delta F}{\delta \alpha}$, i.e.,[13]

$$0 \equiv [L_j, [X_j, C^{(\text{zero})}]] = [X_j, [L_j, C^{(\text{zero})}]].\tag{3.37}$$

In the last equality we have used the Jacobi identity as well as the gauge fixing condition (3.30). This allows to identify the zero-mode ghosts as[14]

$$[L_j, C_0] = [L_j, \tilde{C}_0] = 0.\tag{3.38}$$

Then just as for b, the terms $i\tilde{C}C_0$ and $iC\tilde{C}_0$ will remove the zero modes of C, $\tilde{C}$ after integrating out $\tilde{C}_0$, C_0.

Up to this point we have justified the first line of (3.33). The second line comes from the constraint of BRST symmetry as well as consistency with supersymmetry as previously mentioned. Note that the term $i(a_0 + ...)\tilde{a}_0$ does not affect the gauge fixing since the integral over $a_0, \tilde{a}_0$ is a constant. Finally, note the last term $\text{Tr}\tilde{C}[L_j, \psi_j]$ never contributes to physical expectation values without extra ghost insertions.

3.4.2 The BRST+SUSY formalism

To implement the BRST formalism we have introduced a family of *eight* ghosts, denoted by $(C, C_0, \tilde{C}, \tilde{C}_0, b, b_0, a_0, \tilde{a}_0)$, in (3.33). As shown in appendix C.3, it is natural to define the BRST anti-commuting operator as

$$\begin{aligned}
\delta_b Y &= [Y, C], & \delta_b Y' &= [Y', C], \\
\delta_b C &= i\left(a_0 + i\frac{3\Omega}{8}L_3\right) - C^2, & \delta_b \phi &= [\phi, C], \\
\delta_b \tilde{C} &= ib, & \delta_b b &= [\tilde{C}, a_0 + i\frac{3\Omega}{8}L_3], \\
\delta_b \tilde{a}_0 &= -i\tilde{C}_0, & \delta_b \tilde{C}_0 &= -[\tilde{a}_0, a_0 + i\frac{3\Omega}{8}L_3], \\
\delta_b b_0 &= -iC_0, & \\
\delta_b a_0 &= 0, & \delta_b C_0 &= -[b_0, a_0 + i\frac{3\Omega}{8}L_3],
\end{aligned}\tag{3.39}$$

where by the notation $[\cdot, C]$ we mean the commutator for the bosonic components of Y, Y' and minus the anticommutator, $-\{\cdot, C\}$ for the fermionic components of Y, Y'. It obeys the property,

$$\delta_b^2 = i\left[\cdot, a_0 + i\frac{3\Omega}{8}L_3\right], \qquad \delta_b\text{Tr}[\text{physical fields}] = 0 \implies \delta_b V_0 = 0.\tag{3.40}$$

---

[13]One can similarly analyze the zero modes of $\tilde{C}$ using the identity

$$\text{Tr}(\tilde{C}[L_j, [X_j, C]]) = -\text{Tr}(C[X_j, [L_j, \tilde{C}]]).\tag{3.36}$$

[14]Here we only show that (3.38) is sufficient. Using (3.35) and (3.39), one can also show $[L_j, C_0] = 0$.

Since both BRST symmetry and supersymmetry have a cohomology structure, it is convenient to combine them for the supersymmetric localization [26]. Let us define the combined operator

$$Q \equiv \delta_s + \delta_b \,. \tag{3.41}$$

We also define the action of supersymmetry on the ghosts as

$$\delta_s C = \phi \,, \qquad \delta_s(\text{other ghosts}) = 0 \,. \tag{3.42}$$

Then it is straightforward to check that

$$
\begin{aligned}
QY &= Y' + [Y, C] \,, & QY' &= -i(\delta_\phi + \Omega\delta_{U(1)})Y + [Y', C] \,, \\
QC &= \left(\phi - \frac{3\Omega}{8}L_3\right) + ia_0 - C^2 \,, & Q\phi &= [\phi, C] \,, \\
Q\tilde{C} &= ib \,, & Qb &= [\tilde{C}, a_0 + i\frac{3\Omega}{8}L_3] \,, \\
Q\tilde{a}_0 &= -i\tilde{C}_0 \,, & Q\tilde{C}_0 &= -[\tilde{a}_0, a_0 + i\frac{3\Omega}{8}L_3] \,, \\
Qb_0 &= -iC_0 \,, & QC_0 &= -[b_0, a_0 + i\frac{3\Omega}{8}L_3] \,. \\
Qa_0 &= 0 \,,
\end{aligned} \tag{3.43}
$$

The transformation Q satisfies the property[15]

$$Q^2 = R \equiv -i\Omega\delta_{U(1)} + i\left[\cdot, a_0 + i\frac{3\Omega}{8}L_3\right] \,. \tag{3.44}$$

The BRST+SUSY formalism can now be used to write S_{gh} as a Q-exact quantity, namely

$$S_{\text{gh}} = QV_{\text{gh}} \,, \qquad V_{\text{gh}} = \text{Tr}(\tilde{C}(F + b_0) + C\tilde{a}_0) \,. \tag{3.45}$$

Using $\delta_s V_0 = QV_0$, we can define

$$V \equiv V_0 + V_{\text{gh}} \,, \tag{3.46}$$

and write the partition function (3.29) as a Q-exact quantity

$$Z_{\mathcal{R}} = \text{Vol}(\mathcal{G}_{\mathcal{R}}) \int [d\Phi][d\Phi_{\text{gh}}]e^{-(S+tQV)} \,. \tag{3.47}$$

We will use this as our localizing partition function, in view of (3.1).[16]

3.5 The saddle point approximation

We now perform the saddle point approximation of (3.47) around each saddle in the $t \to \infty$ limit[17], in order to compute the partition function. First note that at the saddle $\mathcal{R}$, the action localizes to

$$S_0 = -\frac{9\Omega^4}{2^{13}}\text{Tr}L_i^2 + \frac{3\Omega^2}{2^7}\text{Tr}M^2 = -\frac{9\Omega^4}{2^{15}}\sum_s n_s(N_s^3 - N_s) + \frac{3\Omega^2}{2^7}\sum_s\sum_{i=1}^{n_s} m_{si}^2 \,, \tag{3.48}$$

---

[15]We also choose $\delta_{U(1)}\Phi_{\text{gh}} = 0$ for all ghost fields.

[16]One might worry that QV does not have the same saddle points as $\delta_s V_0$. As explained in e.g. [30, Sec.3.1], the added ghost action does not modify the saddle point loci.

[17]Note that the ghost variables are all set to 0 at the saddle point, except for $a_0 \to M$.

where m_{si} are the eigenvalues of the block s, M_s part of M. The remaining quantity to be computed is the 1-loop determinant. Before computing it, it is useful to do a change of variables of (3.12),

$$
\begin{aligned}
-i\tilde{\psi}_{I'} &\equiv -i\psi_{I'} + [X_{I'}, C], \\
i\tilde{H}_a &\equiv iH_a - \{\chi_a, C\}, \\
\tilde{\phi} &\equiv a_0 + i\frac{3\Omega}{8}L_3 - i\phi + iC^2.
\end{aligned}
\tag{3.49}
$$

The first two are simply translations of $\psi_{I'}$ and H_a, whereas the last can be thought as a translation both on a_0, which commutes with L_i and the components of X_{10} which do not commute with L_i. Note that at the saddle point,

$$
\tilde{H}_a = 0, \quad \text{(all fermions)} = 0, \quad \tilde{\phi} = 0 \implies a_0 = M. \tag{3.50}
$$

These new variables are useful for the following reason. Defining,

$$
\begin{aligned}
Z_b &\equiv (X_{I'}, \tilde{a}_0, b_0), \quad Z_f \equiv (\chi_a, C, \tilde{C}), \\
Z'_f &\equiv (\tilde{\psi}_{I'}, \tilde{C}_0, C_0), \quad Z'_b \equiv (\tilde{H}_a, \tilde{\phi}, b),
\end{aligned}
\tag{3.51}
$$

they transform nicely under Q,

$$
QZ_b = -iZ'_f, \quad QZ_f = iZ'_b, \qquad QZ'_f = iRZ_b, \quad QZ'_b = -iRZ_f, \tag{3.52}
$$

where

$$
R \equiv -i\Omega\delta_{U(1)} + i\left[\cdot, a_0 + i\frac{3\Omega}{8}L_3\right]. \tag{3.53}
$$

To compute the 1-loop determinant, we expand each Z around the saddle point as $Z = \hat{Z} + \tilde{Z}$ where $\hat{Z}$ is the saddle and $\tilde{Z}$ is the fluctuation, obtaining schematically

$$
V_0 + V_{\text{gh}} = \begin{pmatrix} \tilde{Z}'_f & \tilde{Z}_f \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \tilde{Z}_b \\ \tilde{Z}'_b \end{pmatrix} + \mathcal{O}\left(\tilde{Z}^3\right). \tag{3.54}
$$

Indeed, one can check that there is no linear piece in $\tilde{Z}$, as shown in appendix C.4. Then, noting that QZ is always at least linear in the fluctuation,

$$
\begin{aligned}
Q(V_0 + V_{\text{gh}}) =& i\begin{pmatrix} R_0\tilde{Z}_b & \tilde{Z}'_b \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \tilde{Z}_b \\ \tilde{Z}'_b \end{pmatrix} \\
&+ i\begin{pmatrix} \tilde{Z}'_f & \tilde{Z}_f \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \tilde{Z}'_f \\ R_0\tilde{Z}_f \end{pmatrix} + \mathcal{O}\left(\tilde{Z}^3\right),
\end{aligned}
\tag{3.55}
$$

where

$$
R_0 \equiv -i\Omega\delta_{U(1)} - i\left[M + i\frac{3\Omega}{8}L_3, \cdot\right]. \tag{3.56}
$$

Indeed, it is straight-forward to check that $RZ = R_0\tilde{Z} + \dots$ where the additional terms are higher order in $\tilde{Z}$ as shown in (C.60). This is a nice structure because the determinants of the D_{ij} matrices cancel between the fermions and the bosons. We are thus left with

$$
Z_{1-\text{loop}} \propto \left(\frac{\det_{\tilde{Z}_f} R_0}{\det_{\tilde{Z}_b} R_0}\right)^{1/2}. \tag{3.57}
$$

We obtain these determinants by computing the eigenvalues of R_0 on the relevant matrices. To do it, we first define

$$B_1 \equiv X_1 - iX_2 \,, \qquad B_2 \equiv X_4 - iX_5 \,,$$
$$B_3 \equiv X_6 - iX_7 \,, \qquad B_4 \equiv X_8 - iX_9 \,, \qquad (3.58)$$
$$\xi_1 \equiv \chi_1 + i\chi_6 \,, \qquad \xi_2 \equiv \chi_2 - i\chi_5 \,, \qquad \xi_3 \equiv \chi_3 + i\chi_4 \,.$$

These definitions allow to make $-i\Omega\delta_{U(1)}$ diagonal,

$$-i\Omega\delta_{U(1)}B_1 = \frac{3\Omega}{8}B_1 \,, \quad -i\Omega\delta_{U(1)}B_{i=2,3,4} = \frac{\Omega}{8}B_i \,, \quad -i\Omega\delta_{U(1)}\xi_i = \frac{2\Omega}{8}\xi_i \,,$$
$$-i\Omega\delta_{U(1)}(\text{other matrices}) = 0 \,. \qquad (3.59)$$

We also diagonalize M using the remaining $\otimes_s U(n_s)$ gauge symmetry. We denote the diagonal values of M by m_{si}, where s labels the irrep and i goes over its multiplicity. Thus, the operator $[M, \cdot]$ acts diagonally on the matrices. Regarding $[L_3, \cdot]$, the eigenvalues are easily obtained using fuzzy spherical harmonics.

This procedure is detailed in appendix C.5. The eigenvalues of R_0 and the Vandermonde determinant associated to the diagonalization of M combine to the 1-loop determinant[18]

$$Z_{1-\text{loop}} = \prod_{\substack{s,t=1}}^{q} \prod_{\substack{i,j=1 \\ i\neq j}}^{n_s,n_t} \prod_{J=|j_s-j_t|}^{j_s+j_t}$$
$$\left[\frac{\left[\left(\frac{1}{8}\right)^2 (2+3J)^2 + (m_{si}-m_{tj})^2\right]^3 \left[\left(\frac{1}{8}\right)^2 (3J)^2 + (m_{si}-m_{tj})^2\right]}{\left[\left(\frac{1}{8}\right)^2 (1+3J)^2 + (m_{si}-m_{tj})^2\right]^3 \left[\left(\frac{1}{8}\right)^2 (3+3J)^2 + (m_{si}-m_{tj})^2\right]} \right]^{1/2} \,,$$
$$(3.60)$$

which can also be written in the form (1.7).

3.6 The partition function and protected observables

At the saddle point, $S = S_0$ in (3.48), and the 1-loop determinant is (3.60). Since this approximation is exact and independent of t, this is an exact result for the partition function, reduced to a sum over the saddle points, i.e. $SU(2)$-representations, of an integral over the moduli space parameterized by m_{si}. Combining (3.60), (3.48) we obtain, with the rescaling $m_{si} \to \Omega m_{si}$,

$$Z = \sum_{\mathcal{R}} C_{\mathcal{R}} e^{\frac{9\Omega^4}{2^{15}} \sum_s n_s(N_s^3 - N_s)} \int \prod_s \prod_{i=1}^{n_s} dm_{si} Z_{1-\text{loop}} \exp\left(-\frac{3\Omega^4}{2^7} \sum_s \sum_{i=1}^{n_s} N_s m_{si}^2\right) \,, \quad (3.61)$$

$$C_{\mathcal{R}} = \frac{(2\pi)^{5N^2+N/2}}{\prod_{k=1}^{N-1} k! \prod_s n_s!} \left(\frac{32}{3\pi}\right)^{\sum_s n_s} \prod_s \left(\frac{1}{N_s} \prod_{J=1}^{N_s-1} \frac{(2+3J)^3}{(1+3J)^3}\right)^{n_s} \,, \qquad (3.62)$$

---

[18]We also performed a rescaling $M \to \Omega M$ to get rid of the Ω dependence in the 1-loop determinant.

where the normalization factor $C_{\mathcal{R}}$ is computed in appendix C.6.

This analysis also applies to correlators of gauge invariant combinations of the protected operators $\phi = X_3 - iX_{10}$, namely

$$
\langle f(\phi) \rangle = \frac{1}{Z} \sum_{\mathcal{R}} C_{\mathcal{R}} e^{\frac{9\Omega^4}{2^{15}} \sum_s n_s(N_s^3 - N_s)}
$$
$$
\int \prod_s \prod_{i=1}^{n_s} dm_{si} Z_{1-\text{loop}} f\left(\frac{3\Omega}{8} L_3 - i\Omega M\right) \exp\left(-\frac{3\Omega^4}{2^7} \sum_s \sum_{i=1}^{n_s} N_s m_{si}^2\right),
$$

$$(3.63)$$

where f is a gauge invariant function of ϕ. Note that in this equation, L_3 and M take the form (3.24), (3.25), where we diagonalized each block M_s.

So far, we have discussed the partition function of the $U(N)$ theory because we did not impose the matrices to be traceless. The $SU(N)$ partition function is obtained by integrating out the traces of X^I and ψ_α which are purely Gaussian. Writing the matrices in generator components $X_I = X_I^A T^A$, $\psi_\alpha = \psi_\alpha^A T^A$, we define the $SU(N)$ partition function as

$$
Z_{SU(N)} = \int \prod_{A=1}^{N^2-1} \left(\prod_{I=1}^{10} dX_I^A \prod_{\alpha=1}^{16} d\psi_\alpha^A\right) e^{-S},
$$

$$(3.64)$$

where we also use the normalization

$$
\text{Tr} T^A T^B = \delta^{AB}.
$$

$$(3.65)$$

In this case we find that the partition function Z and $Z_{SU(N)}$ are simply related by

$$
Z = Z_{U(1)} Z_{SU(N)}, \qquad Z_{U(1)} = \frac{2^{14} \pi^5}{3\sqrt{3}\Omega^2}.
$$

$$(3.66)$$

Similarly, we can relate $U(N)$ and $SU(N)$ correlators. Expanding ϕ in its traceless part and its trace, we obtain

$$
\langle \text{Tr}\phi^{2n} \rangle = \sum_{k=0}^{n} \binom{2n}{2k} \langle \text{Tr}\phi^{2k} \rangle_{SU(N)} \left\langle \text{Tr}\left(\frac{\phi}{\sqrt{N}}\right)^{2n-2k} \right\rangle_{U(1)}
$$
$$
= \sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{-2^7}{3N\Omega^2}\right)^{n-k} \frac{\Gamma(n-k+\frac{1}{2})}{\sqrt{\pi}} \langle \text{Tr}\phi^{2k} \rangle_{SU(N)}.
$$

$$(3.67)$$

4 Analysis of the localization result

In this section, we perform various analyses of the localization results (3.61) and (3.63).

Firstly, we write what our general results predict for $N = 2$. In this simple case, the integrals can be simplified explicitly in terms of special functions, and we provide plots of some quantities.

Secondly, we discuss the limit of vanishing mass deformation $\Omega \to 0$. Naively, one might expect that the limit $\Omega \to 0$ would allow to obtain the value of the IKKT partition function based on solid grounds. However, (3.61) predicts that it diverges,

$$\lim_{\Omega \to 0} Z(\Omega) = \infty \neq Z_{\text{IKKT}}. \tag{4.1}$$

The reason for this is the presence of massive fermions in the polarized IKKT model. In order to establish the convergence of the IKKT partition function, one crucial factor was that the integral over fermions produces a power-law decay along the commuting modes valley [20, 31]. When giving a tiny mass to the fermions, this power-law decay is modified and the partition function with traceless matrices diverges as $1/\Omega^{2(N-1)}$, as predicted by our localization result. This analysis is carefully carried in the section 4.2. In appendix B, we present a direct argument following the original reasoning for the convergence of the IKKT partition function.

Thirdly, we study the limit $\Omega \to \infty$. In this case, it is interesting to compare our result to the one obtained in [12]. We find precise agreement as shown in section 4.3.

Finally, we show in section 4.4 that we can recover the IKKT partition function for $N = 2$ and $N = 3$ by assuming a contour prescription to avoid the $\Omega \to 0$ singularity, which we conjecture to be valid for arbitrary N.

4.1 Explicit results for $N = 2$

It is enlightening to consider the case $N = 2$ where we can explicitly write the integrals in terms of special functions. There are only two saddles, namely the trivial and the irreducible representations. The trivial saddle involves a double integral, but the integral over the center of mass $m_1 + m_2$ decouples, leaving only one non-trivial integral. The irreducible saddle can be evaluated exactly for any N, as shown in (4.36). Thus the case $N = 2$ trivially follows. Altogether we obtain

$$Z_{SU(2)} = \frac{2^{16}\pi^{29/2}}{3} \int_{-\infty}^{\infty} dx \left(\frac{(4+x^2)^3 x^2}{(1+x^2)^3 (9+x^2)} \right) e^{-\frac{3\Omega^4}{2^{14}}x^2} + 2^8 5^3 \pi^{31/2} e^{\frac{27}{2^{14}}\Omega^4}. \tag{4.2}$$

This can be simplified in terms of the complementary error function $\text{erfc}(x)$, namely

$$Z_{SU(2)} = \frac{2^{23}\pi^{15}}{3\sqrt{3}\Omega^2} - 5 \cdot 3^2 \cdot 2^3 \sqrt{3}\pi^{15}\Omega^2 + \frac{3^3\sqrt{3}\pi^{15}\Omega^6}{2^9}$$
$$+ \pi^{31/2} e^{\frac{3\Omega^4}{2^{14}}} \text{erfc}\left(\frac{\sqrt{3}\Omega^2}{2^7} \right) \left(195 \cdot 2^7 + \frac{3^4}{2^4}\Omega^4 - \frac{3^4}{2^{16}}\Omega^8 \right) \tag{4.3}$$
$$+ 5^3 2^8 \pi^{31/2} e^{\frac{27\Omega^4}{2^{14}}} \left(1 - \frac{1}{2}\text{erfc}\left(\frac{3\sqrt{3}\Omega^2}{2^7} \right) \right).$$

An explicit plot is provided in figure 2. It is also interesting to consider $SU(2)$ correlators. The $U(2)$ localization result precisely takes the form (3.67), namely,

$$\langle \text{Tr}\phi^{2n} \rangle_{U(2)} = \sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{-2^6}{3\Omega^2} \right)^{n-k} \frac{\Gamma(n-k+\frac{1}{2})}{\sqrt{\pi}} \langle \text{Tr}\phi^{2k} \rangle_{SU(2)}, \tag{4.4}$$
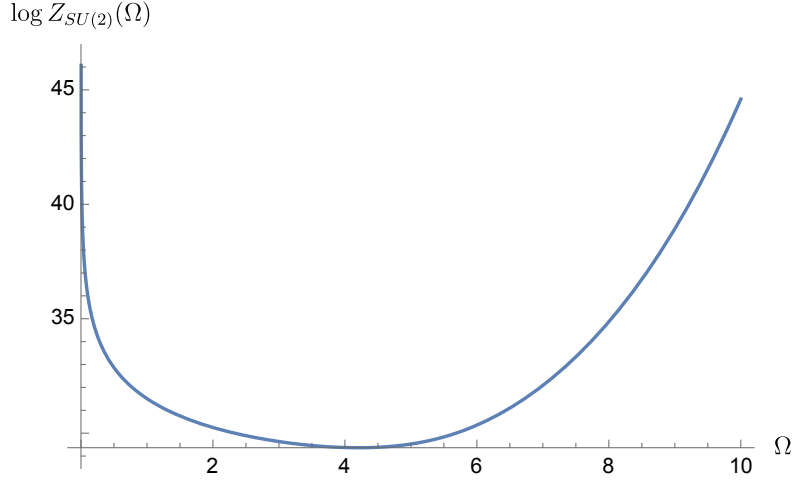
Figure 2: The free energy $\log Z(\Omega)$ of $SU(2)$ polarized IKKT. It has a minimum at $\Omega \approx 4.2046(5)$
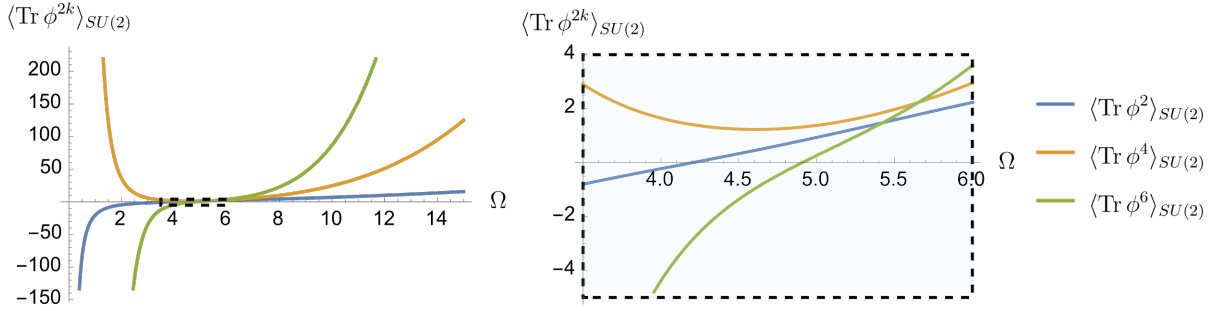

Figure 3: The correlation function $\langle \text{Tr}\phi^{2n} \rangle = \langle \text{Tr}(X_3 - iX_{10})^{2n} \rangle$ of $SU(2)$ polarized IKKT. In the zoomed figure on the right, we can see that $\langle \text{Tr}\phi^4 \rangle$ is always positive while $\langle \text{Tr}\phi^2 \rangle$ and $\langle \text{Tr}\phi^6 \rangle$ change sign at different values of Ω.

where we identify

$$
\langle \text{Tr}\phi^{2k} \rangle_{SU(2)} = \frac{\Omega^{2k}}{Z_{SU(2)}} \left\{ \frac{2^{17}\pi^{29/2}}{3} \left(\frac{-1}{2^8} \right)^k \int_{-\infty}^{\infty} dx \left(\frac{(4+x^2)^3 x^2}{(1+x^2)^3(9+x^2)} \right) x^{2k} e^{-\frac{3\Omega^4}{2^{14}}x^2} \right.
$$
$$
\left. + 5^3 2^9 \pi^{31/2} \left(\frac{3}{16} \right)^{2k} e^{\frac{27}{2^{14}}\Omega^4} \right\}. \tag{4.5}
$$

This can be written in terms of error functions as well. The corresponding results are plotted in figure 3. Notice that in the limit $\Omega \to \infty$, the irreducible saddle dominates and we find

$$
\langle \text{Tr}\phi^{2k} \rangle_{SU(2)} = 2 \left(\frac{3\Omega}{16} \right)^{2k} + O\left(e^{-\frac{27}{2^{14}}\Omega^4} \right), \tag{4.6}
$$

– 20 –

as expected from the leading irreducible saddle contribution (2.8), and discussed in more generality in (4.37).

4.2 The strong coupling limit $\Omega \to 0$

We now study the limit of vanishing mass-deformation $\Omega \to 0$. Written in terms of dimensionful quantities, the polarized IKKT action takes the form

$$
S = \frac{1}{g_{\rm YM}^2} \, {\rm Tr} \left[-\frac{1}{4}[X_I', X_J']^2 - \frac{i}{2}\bar{\psi}'\Gamma^I[X_I', \psi'] \right. \\
\left. + \frac{3\mu^2}{4^3} X_i' X_i' + \frac{\mu^2}{4^3} X_p' X_p' + i\frac{\mu}{3}\epsilon^{ijk} X_i' X_j' X_k' - \frac{1}{8}\mu\bar{\psi}'\Gamma^{123}\psi' \right].
\tag{4.7}
$$

where $g_{\rm YM}$ has dimension two in 0+0 dimensions and μ has dimension one. This is related to (1.3) by rescaling $X_I' = \sqrt{g_{\rm YM}}X_I$, $\psi' = g_{\rm YM}^{3/2}\psi$ and $\Omega = \mu/\sqrt{g_{\rm YM}}$. We thus interpret $\Omega \to 0$ as a strong coupling limit $g_{\rm YM} \to \infty$.

From the localization result: To take the $\Omega \to 0$ limit of the localization result (3.61), it is useful to rescale the moduli space integration by $m_{si} \to m_{si}/\Omega^2$. In the limit $\Omega \to 0$, we obtain

$$
Z = \sum_{\mathcal{R}} C_{\mathcal{R}} \underbrace{\left(\frac{1}{\Omega^2}\right)^{\sum_s n_s} \int \prod_{s=1}^{q}\prod_{i=1}^{n_s} dm_{si}\exp\left(-\frac{3}{2^7}\sum_{s=1}^{q}\sum_{i=1}^{n_s} N_s m_{si}^2\right)}_{\equiv Z_{U(N),\mathcal{R}}^{(\Omega\to 0)}}(1 + \mathcal{O}(\Omega)).
\tag{4.8}
$$

Each saddle contribution in this limit can be written as

$$
\frac{Z_{U(N),\mathcal{R}}^{(\Omega\to 0)}}{A_N} = \prod_s \frac{1}{n_s!}\left(\frac{Z_{U(N_s),{\rm irrep}}^{(\Omega\to 0)}}{A_{N_s}}\right)^{n_s},
\tag{4.9}
$$

where

$$
A_N = \frac{(2\pi)^{5N^2+N/2}}{\prod_{k=1}^{N-1} k!},
\tag{4.10}
$$

and $Z_{U(N_s),{\rm irrep}}$ is the irreducible vacuum contribution (4.35) when $\Omega \to 0$

$$
Z_{U(N),{\rm irrep}}^{(\Omega\to 0)} = \frac{(2\pi)^{5N^2+(N-1)/2}}{\prod_{k=1}^{N-1} k!}\frac{2^9}{3\sqrt{3}N\sqrt{N}\Omega^2}\prod_{J=1}^{N-1}\left(\frac{3J+2}{3J+1}\right)^3.
\tag{4.11}
$$

The notation (4.9) suggests that each irreducible block inside the reducible representation $\mathcal{R}$ decouples from the other blocks in the $\Omega \to 0$ limit. This is very similar to the picture of free particles and their bound states in an harmonic trap that characterizes the strong coupling limit of the BMN model [32]. As we explain in the second part of this section, this can be understood by integrating out the off-diagonal matrix elements that couple the blocks to each other, without using localization.

The result (4.9) predicts that each saddle in the $U(N)$ case diverges as $1/\Omega^{2\sum_s n_s}$. In particular, the leading divergence is governed by the trivial vacuum $n_s = N$, $N_s = 1$

$$Z_{U(N)}^{(\Omega \to 0)} = \frac{(2\pi)^{5N^2}}{\prod_{k=1}^{N} k!} \left(\frac{2^9}{3\sqrt{3}\Omega^2} \right)^N .\tag{4.12}$$

Also note that in the $SU(N)$ case, the trivial saddle $n_s = N$ has the leading divergence $1/\Omega^{2(N-1)}$ whereas the irreducible saddle $n_s = 1$ has a convergent contribution. This convergent behavior of the irreducible saddle was also captured in [12], but the above analysis shows that this is a special feature. All other saddles diverge.

Let us now consider the protected correlators $\langle \mathrm{Tr}\phi^{2n} \rangle$. With the rescaling $M \to \frac{1}{\Omega^2}M$, the dominant contribution also comes from the trivial vacuum and we obtain

$$\langle \mathrm{Tr}\phi^{2n} \rangle_{U(N),\Omega \to 0} = \frac{\int \prod_{i=1}^{N} dm_i \sum_j \left(\frac{-im_j}{\Omega} \right)^{2n} e^{-\frac{3}{2^7}\sum_i m_i^2}}{\int \prod_{i=1}^{N} dm_i e^{-\frac{3}{2^7}\sum_i m_i^2}}$$
$$= N \left(\frac{-2^7}{3\Omega^2} \right)^n \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}} .\tag{4.13}$$

Note that this can be expressed as

$$\langle \mathrm{Tr}\phi^{2n} \rangle_{U(N),\Omega \to 0} = \sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{-2^7}{3N\Omega^2} \right)^{n-k} \frac{\Gamma(n-k+\frac{1}{2})}{\sqrt{\pi}} \left(\frac{-2^7(N-1)}{3N\Omega^2} \right)^k \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}} .\tag{4.14}$$

We thus identify thanks to (3.67),

$$\langle \mathrm{Tr}\phi^{2n} \rangle_{SU(N),\Omega \to 0} = \left(\frac{-2^7(N-1)}{3N\Omega^2} \right)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} .\tag{4.15}$$

From integrating out off-diagonal modes: First note that we can rescale coordinates such that

$$Z(\Omega) = \left(\frac{1}{\Omega^2} \right)^{N^2} \tilde{Z}(\Omega) ,\tag{4.16}$$

where $\tilde{Z} = \int [dX^I][d\psi_\alpha]e^{-\tilde{S}}$ with

$$\tilde{S} = \mathrm{Tr}\left[-\frac{1}{4\Omega^4}[X_I, X_J]^2 - \frac{i}{2\Omega^2}\bar{\psi}\Gamma^I[X_I, \psi] + i\frac{1}{3\Omega^2}\epsilon^{ijk}X_iX_jX_k \right.$$
$$\left. + \frac{3}{4^3}X_iX_i + \frac{1}{4^3}X_pX_p - \frac{1}{8}\bar{\psi}\Gamma^{123}\psi \right] .\tag{4.17}$$

This is better suited for the strong coupling limit $g = \frac{1}{\Omega^2} \gg 1$.[19] Then, we separate the matrices as

$$X^I = r^I + q^I , \qquad \psi_\alpha = \theta_\alpha + \Theta_\alpha ,\tag{4.18}$$

---

[19]Remember that this definition of g is related to the introduction of the dimension-full quantities $\Omega = \mu/\sqrt{g_{\mathrm{YM}}}$. Then, the dimensionless gauge coupling is $g = g_{\mathrm{YM}}/\mu^2 = 1/\Omega^2$.

where r^I, θ_α are diagonal and q^I, Θ_α are off-diagonal. We will use the notation

$$r_{ab}^I \equiv r_a^I - r_b^I \,, \quad \hat{r}_{ab}^I \equiv r_{ab}^I/|r_{ab}| \,, \quad |r_{ab}| \equiv \sqrt{\sum_I r_{ab}^I r_{ab}^I} \,. \tag{4.19}$$

We then use a $SO(10) \supset SO(3) \times SO(7)$ preserving gauge fixing

$$Z(\Omega) = c_N \left(\frac{1}{\Omega^2}\right)^{N^2} \int [dr^I][d\theta_\alpha][dq^I][d\Theta_\alpha]\delta(\hat{r}_{ab} \cdot q_{ab})(\Delta(r) + \mathcal{O}(\Omega))e^{-\tilde{S}} \,, \tag{4.20}$$

where $\Delta(r) = \prod_{a \neq b} |r_{ab}|$ is the Vandermonde determinant, which is the leading term in the change of variables, and $c_N = \frac{1}{N!}\frac{\text{Vol } U(N)}{\text{Vol } U(1)^N} = (2\pi)^{(N^2-N)/2}/\prod_{k=1}^N(k!)$ is a volume factor. Details on this gauge fixing method are given in [32, 33]. Then, we rescale

$$q^I = \Omega^2 y^I \,, \qquad \Theta_\alpha = \Omega\chi_\alpha \,. \tag{4.21}$$

This reduces the partition function to

$$Z(\Omega) = c_N \left(\frac{1}{\Omega^2}\right)^N \int [dr^I][d\theta_\alpha][dy^I][d\chi_\alpha]\delta(\hat{r}_{ab} \cdot y_{ab})\Delta(r)e^{-S_0-S_{y,\chi}} + \text{subleading} \,, \tag{4.22}$$

where

$$S_{y,\chi} = \frac{1}{2}|r_{ab}|^2 y_{ab}^I y_{ba}^I - \frac{1}{2}r_{ab}^I \chi_{ba}^\top \bar{\gamma}^I \chi_{ab} + \mathcal{O}(\Omega) \,, \tag{4.23}$$

$$S_0 = \frac{3}{4^3}r_a^i r_a^i + \frac{1}{4^3}r_a^p r_a^p + \frac{i}{8}\theta_a \gamma^{123}\theta_a \,. \tag{4.24}$$

In the limit $\Omega \to 0$, these are Gaussian integrals,

$$\int [dy^I]\delta(\hat{r}_{ab} \cdot y_{ab})e^{-\frac{1}{2}|r_{ab}|^2 y_{ab}^I y_{ba}^I} = (2\pi)^{9/2}\prod_{a \neq b}|r_{ab}|^{-9} \,, \tag{4.25}$$

$$\int [d\chi_\alpha]e^{\frac{1}{2}r_{ab}^I \chi_{ba}\bar{\gamma}^I \chi_{ab}} = \prod_{a \neq b}|r_{ab}|^8 \,. \tag{4.26}$$

Combined with $\Delta(r)$, all these products cancel. We thus obtain

$$\begin{aligned} Z(\Omega) &= c_N(2\pi)^{\frac{9(N^2-N)}{2}}\left(\frac{1}{\Omega^2}\right)^N \int [dr^I][d\theta_\alpha]e^{-S_0}(1 + \mathcal{O}(\Omega)) \\ &= \frac{(2\pi)^{5N^2}}{\prod_{k=1}^N k!}\left(\frac{2^9}{3\sqrt{3}\Omega^2}\right)^N + \text{subleading} \,. \end{aligned} \tag{4.27}$$

This method can also be applied to compute correlators at small Ω. The original correlator is related to the rescaled coordinates (4.17) by

$$\langle \text{Tr}\phi^{2n}\rangle = \frac{1}{\Omega^{2n}}\langle \text{Tr}\phi^{2n}\rangle_{\tilde{S}} \,, \qquad \langle \mathcal{O}\rangle_{\tilde{S}} \equiv \frac{1}{\tilde{Z}}\int [dX^I][d\psi_\alpha]\mathcal{O}e^{-\tilde{S}} \,. \tag{4.28}$$

To compute $\langle \mathrm{Tr}\phi^{2n}\rangle_{\tilde{S}}$ at small Ω's, we insert $\phi = r^3 - ir^{10} + \mathcal{O}(\Omega)$ in the previous derivation, thus obtaining

$$
\begin{aligned}
\langle \mathrm{Tr}\phi^{2n}\rangle_{\Omega\to 0} &= \left(\frac{1}{\Omega}\right)^{2n} \frac{\int [dr^3][dr^{10}](r^3-ir^{10})^{2n}e^{-\frac{3}{4^3}r_a^3 r_a^3 - \frac{1}{4^3}r_a^{10}r_a^{10}}}{\int [dr^3][dr^{10}]e^{-\frac{3}{4^3}r_a^3 r_a^3 - \frac{1}{4^3}r_a^{10}r_a^{10}}} \\
&= N\left(\frac{-2^7}{3\Omega^2}\right)^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\,.
\end{aligned}
\tag{4.29}
$$

Both (4.27) and (4.29) agree exactly with the localization results when $\Omega \to 0$ (4.12), (4.13). Note that the same reasoning for Z can be applied to the pure IKKT partition function deep in the commuting modes valley once removing the Gaussian deformation ($S_0 \to 0$), setting $\Omega \to 1$, and replacing the expansion in Ω by an expansion in $|r_{ab}| \gg 1$. The equation (4.27) would then reduce to an undetermined expression $\int [dr^I][d\theta_\alpha]1 = \infty \cdot 0$. In fact, a careful treatment of obtaining an effective action for large commuting modes separation $|r_{ab}| \gg 1$ has been carried in [31], and the integral over the fermions θ_α in this case would reduce to a power-law $1/r^{\#}$ suppressed enough to ensure the convergence of the IKKT partition function. We see explicitly in (4.27) that this power-law decay is subleading, with the leading piece being due to the fermion mass term present in polarized IKKT. This leading piece is the cause of the divergence in the $\Omega \to 0$ limit of polarized IKKT.[20] This discussion is consistent with appendix B where we see the divergence coming from the Pfaffian mass terms, the original IKKT Pfaffian being subleading.

4.3 The free limit $\Omega \to \infty$

From the localization result: Rescaling $m_{si} \to m_{si}/\Omega^2$ and taking $\Omega \to \infty$ in the localization result (3.61), we obtain

$$
Z_{\mathcal{R}}^{(\Omega\to\infty)} = \mathcal{N}_{\mathcal{R}} \left(\frac{1}{\Omega^2}\right)^{\sum_s n_s^2} e^{\frac{9\Omega^4}{2^{15}}\sum_s n_s(N_s^3 - N_s)} \int \prod_s \prod_{i=1}^{n_s} \Delta(m) \exp\left(-\frac{3}{2^7}\sum_s\sum_{i=1}^{n_s} N_s m_{si}^2\right),
\tag{4.30}
$$

where

$$
\Delta(m) = \left(\prod_s \prod_{i\neq j}(m_{si}-m_{sj})^2\right)^{1/2},
\tag{4.31}
$$

$$
\mathcal{N}_{\mathcal{R}} = \frac{(2\pi)^{5N^2 + N/2 - \sum_s n_s/2}}{\prod_{k=1}^{N-1}k! \prod_s n_s!}\left(\frac{64}{3}\right)^{\sum_s n_s^2}\prod_{s,t}\prod_{\substack{J=|j_s-j_t|\\ J\neq 0}}^{j_s+j_t}\left[\frac{[(2+3J)^2]^3\,[(3J)^2]}{[(1+3J)^2]^3\,[(3+3J)^2]}\right]^{n_s n_t}.
\tag{4.32}
$$

---

[20] A similar analysis can be found in another attempt to deform IKKT in 4d [34, Sec.4.1].

This can be computed for any saddle $\mathcal{R}$, since we can write it as a matrix integral. We obtain

$$
\begin{aligned}
Z_{\mathcal{R}}^{(\Omega \to \infty)} = {} & \frac{(2\pi)^{5N^2+N/2}}{\prod_{k=1}^{N-1} k!} \prod_s \left[\prod_{k=1}^{n_s-1} (k!) \left(\frac{1}{\sqrt{2\pi}} \right)^{\sum_s n_s} \left(\frac{2^9}{3\sqrt{3}\sqrt{N}\Omega^2} \right)^{\sum_s n_s^2} \right] \\
& \cdot \prod_{s,t} \prod_{J=|N_s-N_t|/2}^{(N_s+N_t-2)/2} \left[\frac{\left[(2+3J)^2 \right]^3 \left[(3J)^2 \right]}{\left[(1+3J)^2 \right]^3 \left[(3+3J)^2 \right]} \right]^{n_s n_t} e^{\frac{9\Omega^4}{2^{15}} \sum_s n_s (N_s^3 - N_s)} .
\end{aligned}
\tag{4.33}
$$

In particular we obtain that

$$
Z_{U(N),\text{trivial}}^{(\Omega \to \infty)} = \left(\frac{2^{14}\pi^5}{3\sqrt{3}\Omega^2} \right)^{N^2} ,
\tag{4.34}
$$

$$
Z_{U(N),\text{irrep}} = \frac{(2\pi)^{5N^2+(N-1)/2}}{\prod_{k=1}^{N-1} k!} \frac{2^9}{3\sqrt{3}N\sqrt{N}\Omega^2} \prod_{J=1}^{N-1} \left(\frac{3J+2}{3J+1} \right)^3 e^{\frac{9\Omega^4}{2^{15}}(N^3-N)} ,
\tag{4.35}
$$

$$
Z_{SU(N),\text{irrep}} = \frac{(2\pi)^{(10N+11)(N-1)/2}}{N\sqrt{N}\prod_{k=1}^{N-1}(k!)} \prod_{J=1}^{N-1} \left(\frac{3J+2}{3J+1} \right)^3 e^{\frac{9\Omega^4}{2^{15}}(N^3-N)} .
\tag{4.36}
$$

Note that we wrote the irreducible result for the $SU(N)$ partition function using (3.66). We also dropped the $(\Omega \to \infty)$ superscript from $Z_{U(N),\text{irrep}}$ because this is the exact formula for any Ω (in this case $Z_{1-\text{loop}} = 1$).

The correlators $\langle \text{Tr}\phi^{2n} \rangle$ can also be easily computed since, after rescaling $M \to \frac{1}{\Omega^2}M$, the dominant term in (3.63) comes from the irreducible fuzzy sphere $\frac{3\Omega}{8}L_i$. We thus obtain

$$
\langle \text{Tr}\phi^{2n} \rangle_{\Omega \to \infty} = \langle \text{Tr}\phi^{2n} \rangle_{SU(N),\Omega \to \infty} = \left(\frac{3\Omega}{8} \right)^{2n} \text{Tr}L_3^{2n} = \left(\frac{3\Omega}{8} \right)^{2n} \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} m^{2n} ,
\tag{4.37}
$$

where the equality between $SU(N)$ and $U(N)$ correlators follows from using (3.67) at large Ω.

From direct methods: The limit $\Omega \to \infty$ around the trivial vacuum can be directly obtained from looking at the mass-deformation of the polarized IKKT action. In view of (4.16), we obtain

$$
Z_{\text{trivial}}^{(\Omega \to \infty)} = \left(\frac{1}{\Omega^2} \right)^{N^2} \int [dX^I][d\psi_\alpha] e^{-\text{Tr}\left(\frac{3}{4^3}X_i X_i + \frac{1}{4^3}X_p X_p - \frac{1}{8}\bar{\psi}\Gamma^{123}\psi \right)} = \left(\frac{2^{14}\pi^5}{3\sqrt{3}\Omega^2} \right)^{N^2} ,
\tag{4.38}
$$

which agrees with (4.34).

The limit $\Omega \to \infty$ around the irreducible vacuum was carefully treated in [12]. Their result is the same as (4.36) once changing the measure convention from ours ($\text{Tr}T^A T^B = \delta^{AB}$) to theirs ($\text{Tr}T^A T^B = 2\delta^{AB}$).

4.4 Recovering IKKT

In this section, we give evidence for the conjecture that the IKKT partition function can be recovered from (1.6) by not only taking $\Omega \to 0$, but also closing the contour of the moduli integrals on the upper complex plane, thus picking poles and making the result finite as $\Omega \to 0$. Let us first recall that

$$\lim_{\Omega \to 0} Z_{SU(N)} = \sum_{\mathcal{R}} \lim_{\Omega \to 0} \frac{Z_{\mathcal{R}}}{Z_{U(1)}} , \tag{4.39}$$

$$\lim_{\Omega \to 0} Z_{\mathcal{R}} = \lim_{\Omega \to 0} C_{\mathcal{R}} \int \prod_{s,i} dm_{si} Z_{1-\text{loop}} e^{-\frac{3\Omega^4}{27} \sum_{si} N_s m_{si}^2} . \tag{4.40}$$

We will give a prescription to make the ratio $Z_{\mathcal{R}}/Z_{U(1)}$ finite, such that

$$Z_{\mathcal{R}}^{\text{IKKT}} \equiv \lim_{\Omega \to 0} \left(\frac{Z_{\mathcal{R}}}{Z_{U(1)}} \right)_{\text{regulated}} < \infty . \tag{4.41}$$

The recipe we provide to define $Z_{\mathcal{R}}^{\text{IKKT}}$ is the following.

1. Parameterize the $(\sum_s n_s)$-dimensional integral $Z_{\mathcal{R}}$ in terms of its trace $m_{\text{com}} \equiv \sum_{s,i} m_{si}$ and in terms of $(\sum_s n_s - 1)$ relative variables $\Delta_{ij} \equiv m_{si} - m_{tj}$ (for some s, t) such that all the variables $\{\Delta_{ij}\}$ fully determine all differences $m_{si} - m_{tj}$.

2. Integrate out the trace $m_{\text{com}} = \sum_{si} m_{si}$ which does not appear in $Z_{1-\text{loop}}$. This trace will simplify with $Z_{U(1)}$ up to a constant.

3. The remaining $(\sum_s n_s - 1)$-dimensional integral over $\{\Delta_{ij}\}$ is divergent when $\Omega \to 0$ if integrated on the real axis. Here is the prescription: **Close the contour on the upper-half plane** as shown in figure 4. The integral over Δ_{ij} will pick contributions from poles at constant positions, but also from poles due to the other variables.

4. Iterate the pole-picking procedure up to having exhausted all integrations.

We conjecture that

$$Z_{\text{IKKT}} = \sum_{\mathcal{R}} Z_{\mathcal{R}}^{\text{IKKT}} . \tag{4.42}$$

In the following, we prove it for $N = 2$ and $N = 3$.

4.4.1 IKKT for $N = 2$

Let us consider the toy example $N = 2$. After removing the center of mass moduli integral, we're left with (4.2) which we repeat here for convenience

$$Z_{SU(2)} = \frac{2^{16} \pi^{29/2}}{3} \int_{-\infty}^{\infty} dx \left(\frac{(4 + x^2)^3 x^2}{(1 + x^2)^3 (9 + x^2)} \right) e^{-\frac{3\Omega^4}{2^{14}} x^2} + 2^8 5^3 \pi^{31/2} e^{\frac{27}{2^{14}} \Omega^4} . \tag{4.43}$$

Setting $\Omega \to 0$ and closing the x integral on the upper complex plane, we pick the residues at $x_1 = i$ and $x_2 = 3i$, giving

$$Z_{\text{trivial}}^{\text{IKKT}} = \frac{2^{16} \pi^{29/2}}{3} 2\pi i \sum_{m=1}^{2} \text{Res} \left(\frac{(4 + x^2)^3 x^2}{(1 + x^2)^3 (9 + x^2)}, x = x_m \right) = 2^8 35 \pi^{31/2} , \tag{4.44}$$
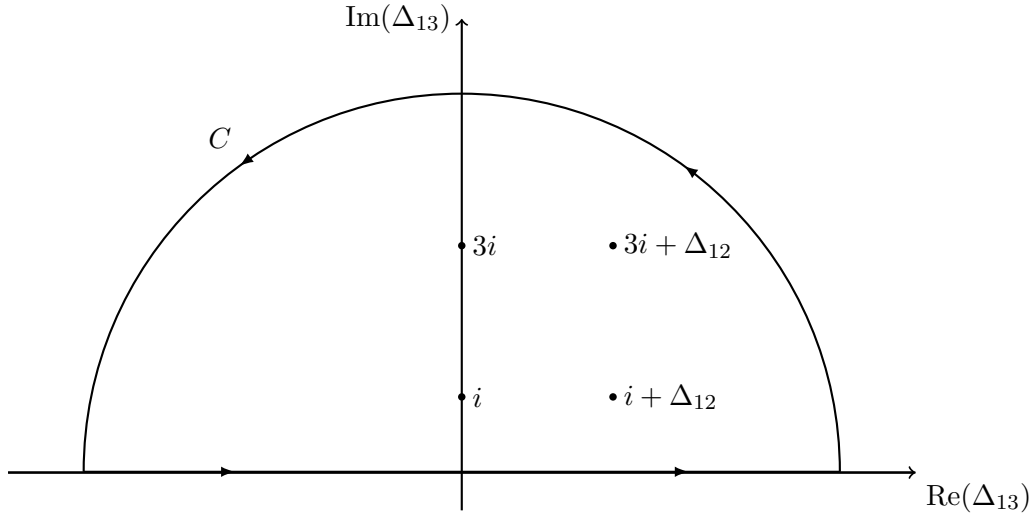
Figure 4: Contour integral displayed for the variable Δ_{13} in the trivial saddle of $N = 3$. The original polarized IKKT integral was taken on the real axis, and we propose to close it with a semi-circular arc in the upper half plane to regulate the divergence as $\Omega \to 0$.

$$Z_{\mathrm{irrep}}^{\mathrm{IKKT}} = 2^8 5^3 \pi^{31/2}. \tag{4.45}$$

They indeed sum to the $N = 2$ IKKT partition function (2.4)

$$Z_{\mathrm{trivial}}^{\mathrm{IKKT}} + Z_{\mathrm{irrep}}^{\mathrm{IKKT}} = 2^{13} 5 \pi^{31/2} = Z_{\mathrm{IKKT}}. \tag{4.46}$$

4.4.2 IKKT for $N = 3$

The number 3 has three partitions, $3 = 1 + 1 + 1$, the trivial saddle, which we will denote by $\mathcal{R} = (1, 1, 1)$, $3 = 3$, the irreducible saddle which we denote by $\mathcal{R} = (3)$ and $3 = 1 + 2$, corresponding to $n_s = 1, N_s = 1, n_t = 1, N_t = 2$, which we will denote by $\mathcal{R} = (1, 2)$. Let us start by computing the trivial saddle contribution $Z_{(1,1,1)}^{\mathrm{IKKT}}$. The first step is performed by changing the m_1, m_2, m_3 variables to $m_{\mathrm{com}} \equiv m_1 + m_2 + m_3$ and $\Delta_{12} \equiv m_1 - m_2$, $\Delta_{13} \equiv m_1 - m_3$. Note that $m_2 - m_3 = \Delta_{13} - \Delta_{12}$ is expressed in terms of them.

$$Z_{(1,1,1)} = C_{(1,1,1)} \frac{1}{3} \int d\Delta_{12} d\Delta_{13} dm_{\mathrm{com}} Z_{1-\mathrm{loop}}$$
$$\exp\left(-\frac{3\Omega^4}{2^7} \frac{2}{3}(\Delta_{12}^2 + \Delta_{13}^2 - \Delta_{12}\Delta_{13}) - \frac{\Omega^4}{2^7} m_{\mathrm{com}}^2\right). \tag{4.47}$$

Integrating out m_{com}, dividing by $Z_{U(1)}$ and rescaling $\Delta_{ij} \to \Delta_{ij}/8$, we obtain

$$Z_{(1,1,1)}^{\mathrm{IKKT}} = C_{(1,1,1)} \frac{1}{2^{16}\pi^{9/2}} \sqrt{\frac{3}{2}} \oint_C d\Delta_{12} d\Delta_{13} A(\Delta_{12}, \Delta_{13}), \tag{4.48}$$

where

$$A(x, y) = \frac{(4 + x^2)^3 x^2}{(1 + x^2)^3 (9 + x^2)} \frac{(4 + y^2)^3 y^2}{(1 + y^2)^3 (9 + y^2)} \frac{(4 + (x - y)^2)^3 (x - y)^2}{(1 + (x - y)^2)^3 (9 + (x - y)^2)}. \tag{4.49}$$

Let us consider the integral over Δ_{13} first. Our contour prescription, illustrated in figure 4, is such that it encircle the poles at $\Delta_{13}^{(1)} = i, \Delta_{13}^{(2)} = 3i, \Delta_{13}^{(3)} = \Delta_{12} + i$ and $\Delta_{13}^{(4)} = \Delta_{12} + 3i$, yielding

$$
\begin{aligned}
B(\Delta_{12}) &\equiv \oint_C d\Delta_{13} A(\Delta_{12}, \Delta_{13}) = \sum_{m=1}^{4} 2\pi i \operatorname{Res}(A(\Delta_{12}, \Delta_{13}), \Delta_{13} = \Delta_{13}^{(m)}) \\
&= \frac{\pi}{128} \frac{P(\Delta_{12})}{(\Delta_{12}^2 + 1)^3 (\Delta_{12}^2 + 4)^2 (\Delta_{12}^2 + 9)(\Delta_{12}^2 + 16)^3 (\Delta_{12}^2 + 36)},
\end{aligned} \tag{4.50}
$$

$$
\begin{aligned}
P(x) &\equiv 105 x^{20} + 10920 x^{18} + 457170 x^{16} + 10423560 x^{14} + 142736685 x^{12} + 1175466936 x^{10} \\
&\quad + 5477149824 x^8 + 12450112896 x^6 + 11571604224 x^4 + 22451613696 x^2.
\end{aligned} \tag{4.51}
$$

We now iterate by integrating $B(\Delta_{12})$ over Δ_{12} whose contour encircles the poles at $\Delta_{12}^{(1)} = i, \Delta_{12}^{(2)} = 2i, \Delta_{12}^{(3)} = 3i, \Delta_{12}^{(4)} = 4i$ and $\Delta_{12}^{(5)} = 6i$. We thus obtain

$$
Z_{(1,1,1)}^{\text{IKKT}} = C_{(1,1,1)} \frac{1}{2^{16}\pi^{9/2}} \sqrt{\frac{3}{2}} 2\pi i \sum_{m=1}^{5} \operatorname{Res}(B(\Delta_{12}), \Delta_{12} = \Delta_{12}^{(m)}) = \frac{2^{39} \cdot 83 \cdot 241 \pi^{41}}{3^2 5^2 7^3 \sqrt{3}}. \tag{4.52}
$$

The only other vacuum which is non-trivial to compute is $(1, 2)$. Integrating out the center of mass and rescaling variables we obtain

$$
Z_{(1,2)}^{\text{IKKT}} = C_{(1,2)} \frac{2^{35} 5^3 \pi^{40}}{3\sqrt{3}} \oint_C dx \frac{(\frac{49}{3} + x^2)^3 (\frac{9}{4} + x^2)}{(\frac{25}{4} + x^2)^3 (\frac{81}{4} + x^2)}. \tag{4.53}
$$

The contour integral picks the poles at $x = \frac{5}{2}i$ and $x = \frac{9}{4}i$ thus giving

$$
Z_{(1,2)}^{\text{IKKT}} = \frac{2^{39} 499 \pi^{41}}{3 \cdot 5^2 7^3 \sqrt{3}}. \tag{4.54}
$$

Finally, $Z_{(3)}^{\text{IKKT}}$ trivially follows from (4.36),

$$
Z_{(3)}^{\text{IKKT}} = \frac{2^{43} 5^3 \pi^{41}}{3 \cdot 7^3 \sqrt{3}}. \tag{4.55}
$$

The sum precisely agrees with the $N = 3$ IKKT partition function (2.4)

$$
Z_{(1,1,1)}^{\text{IKKT}} + Z_{(1,2)}^{\text{IKKT}} + Z_{(3)}^{\text{IKKT}} = \frac{2^{41} 5 \pi^{41}}{9\sqrt{3}} = Z_{\text{IKKT}}. \tag{4.56}
$$

5 Gravity from the matrix model

In this section we will show that, in the large N limit, the localization equations are equivalent to the electrostatic equations defining the gravity dual described in [5]. A similar analysis was carried out in [14, 15] for the case of the BMN model, whose vacua are dual to Lin-Maldacena geometries [35].

5.1 The electrostatic problem

The dual geometry is determined by the 4-dimensional electrostatic potential of a configuration of conducting 3d balls and a background potential [5], see the right of figure 1. We will write an equation for the charge density of those conductors.

Consider a 3d ball of radius R with a spherically symmetric charge density $\sigma(r)$, with $0 < r < R$. Let us slice this ball[21] in thin disks, and compute the linear charge density $f(u)$ of such a slice, at a distance u from the center of the ball, measured on the equatorial plane, see figure 5. We obtain
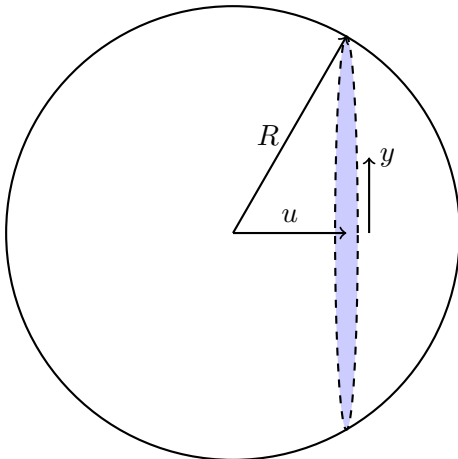

Figure 5: Slicing of a 3d ball into 2d disks (blue). The center of the disk is at a distance u from the center of the ball, and y is a radial coordinate on the disk. The linear charge density $f(u)$ measures the charge on the blue slice.

$$f(u) = 2\pi \int_0^{\sqrt{R^2-u^2}} dy\, y\, \sigma\left(r = \sqrt{u^2 + y^2}\right) = \pi \int_{u^2}^{R^2} dz\, \sigma(\sqrt{z}), \tag{5.1}$$

where y is a coordinate perpendicular to the equatorial plane, and the second equality comes from changing y to the variable $z = u^2 + y^2$. Taking a derivative with respect to u we get

$$f'(u) = -2\pi u \sigma(u). \tag{5.2}$$

Since this is the charge of a slice, we can simply compute the total charge of the ball by integrating over slices[22]

$$Q = \int_{-R}^{R} du\, f(u), \tag{5.4}$$

---

[21] At this point it might be surprising to consider the charge density of the slices instead of the one of the ball itself. In [14, 15], it was the charge density of such slices that reproduced the density of eigenvalues of the matrix side.

[22] Alternatively one can compute

$$Q = 4\pi \int_0^R du\, u^2 \sigma(u) = -2 \int_0^R du\, u f'(u) = -\int_{-R}^{R} du\, u f'(u) = \int_{-R}^{R} du\, f(u). \tag{5.3}$$

where we used $f(-u) = f(u)$.

The 4d electrostatic potential due to a ball of radius R_s, placed at position z_s and with charge density $\sigma_s(r)$, reads[23]

$$V_{\text{ball},s}(r, z) = \frac{1}{4\pi r} \int_0^{R_s} du\, u\sigma_s(u) \log\left(1 + \frac{4ru}{(r - u)^2 + (z - z_s)^2}\right). \qquad (5.5)$$

Writing it in terms of $f(u)$ and integrating by parts we obtain

$$V_{\text{ball},s}(r, z) = \frac{1}{8\pi^2 r} \int_{-R_s}^{R_s} du f_s(u) \left(\frac{r + u}{(r + u)^2 + (z - z_s)^2} + \frac{r - u}{(r - u)^2 + (z - z_s)^2}\right). \qquad (5.6)$$

Using $f(u) = f(-u)$, we can write

$$V_{\text{ball},s}(r, z) = \frac{1}{4\pi^2 r} \int_{-R_s}^{R_s} du f_s(u) \frac{r - u}{(r - u)^2 + (z - z_s)^2}. \qquad (5.7)$$

One may worry about evaluating this function at $z = z_s$ when $r < R_s$. In this case, we should take the limit[24]

$$V_{\text{ball},s}(r, z_s) = \frac{1}{4\pi^2 r} \lim_{\epsilon \to 0} \int_{-R_s}^{R_s} du f_s(u) \frac{r - u}{(r - u)^2 + \epsilon^2}. \qquad (5.8)$$

The total potential $V(r, z)$ is constant on the conducting ball s and takes the value $V(r, z_s) = V_s$ when $r < R_s$. Thus

$$V_s = V_b\left(r^2 z_s - z_s^3\right) + \sum_{t + \text{images}} V_{\text{ball},t}(r, z_s), \qquad (5.9)$$

where the sum runs over all the balls, including images[25], and the first term is the background potential. Multiplying by r and taking a derivative with respect to r, we obtain

$$-V_s + V_b(3r^2 z_s - z_s^3) + \sum_{t + \text{images}} \frac{1}{(2\pi)^2} \fint_{-R_t}^{R_t} dr' \frac{(z_s - z_t)^2 - (r - r')^2}{((z_s - z_t)^2 + (r - r')^2)^2} f_t(r') = 0 \qquad (5.10)$$

where we changed integration variable $u \to r'$ for later convenience and defined

$$\fint dy \frac{1}{y^2} f(y) \equiv \lim_{\epsilon \to 0} \int dy \frac{y^2 - \epsilon^2}{(y^2 + \epsilon^2)^2} f(y) = \lim_{\epsilon \to 0} \left[\int_{|y| > \epsilon} \frac{dy}{y^2} f(y) - \frac{2}{\epsilon} f(0)\right], \qquad (5.11)$$

which regulates the integral over the ball s. The linear charge densities $f_s(r)$, the size of the balls R_s and the potentials V_s are determined by (5.10) and by the conditions

$$Q_s = \int_{-R_s}^{R_s} dr f_s(r), \qquad\qquad f_s(R_s) = f_s'(R_s) = 0. \qquad (5.12)$$

The integral equations (5.10) and (5.12) are equivalent to the Laplace equation with boundary conditions set by the conducting balls. Hence those equations determine the 10 dimensional Euclidean geometry of the gravity dual [5]. In the next subsection we will show that the large N limit of the localization result gives exactly the same equations.

---

[23]This can be obtained simply by integrating the 4d electrostatic potential $V(x)$ of a single charge at position y ($V(x) = \frac{1}{4\pi^2} \frac{1}{|x - y|^2}$) over a ball with charge density σ_s.

[24]This is equivalent to taking the Cauchy principal value of the integral with $\epsilon = 0$.

[25]By inclusion of images, we mean that for each t, we also sum over $\tilde{t}$ where $f_{\tilde{t}} = -f_t$ and $z_{\tilde{t}} = -z_t$. This is the method of image charges in electrostatics, which ensures that $V = 0$ at $z = 0$.

5.2 The particle density

We now study the localization result (1.6) in the large n_s limit in which the particles form a continuum which can be described by continuous densities.

The localization result (1.6) can be written as

$$Z_{\mathcal{R}} = C_{\mathcal{R}} \int \prod_{s,i} dm_{si} e^{-S_{\text{eff}}} , \qquad (5.13)$$

where

$$S_{\text{eff}} = -\frac{9\Omega^4}{2^{15}} \sum_s n_s(N_s^3 - N_s) + \frac{3\Omega^4}{128} \sum_s N_s \sum_i m_{si}^2 + \sum_{(si,tj)} g_{st}(m_{si} - m_{tj}) , \qquad (5.14)$$

with

$$g_{st}(x) = \sum_{J=|N_s-N_t|/2}^{(N_s+N_t-2)/2} h_J(x) , \qquad (5.15)$$

and

$$h_J(x) = -\log \frac{\left[(2+3J)^2 + (8x)^2\right]^3 \left[(3J)^2 + (8x)^2\right]}{\left[(1+3J)^2 + (8x)^2\right]^3 \left[(3+3J)^2 + (8x)^2\right]} . \qquad (5.16)$$

It is useful to think of this as a statistical mechanics system with n_s particles of charge N_s in a 1D harmonic background potential and with two-body interactions described by $g_{st}(m_{si} - m_{tj})$. The two-body term depends on the charges N_s and N_t of the two particles and on the distance between them.

In order to connect with the electrostatic problem described in the previous subsection, it is convenient to write

$$g_{st}(x) = k_{st}(x) + k_{s\tilde{t}}(x) , \qquad (5.17)$$

with

$$k_{st}(x) = \sum_{J=|N_s-N_t|/2}^{\infty} h_J(x) , \qquad k_{s\tilde{t}}(x) = -\sum_{J=|N_s+N_t|/2}^{\infty} h_J(x) . \qquad (5.18)$$

As we shall see, the term $k_{s\tilde{t}}$ is similar to the interaction between the ball s and the image ball $\tilde{t}$ in the electrostatic problem. We can then write

$$S_{\text{eff}} = -\frac{9\Omega^4}{2^{15}} \sum_s n_s(N_s^3 - N_s) + \frac{3\Omega^4}{128} \sum_s N_s \sum_i m_{si}^2 + \sum_{\substack{(si,tj) \\ +\text{images}}} k_{st}(m_{si} - m_{tj}) , \qquad (5.19)$$

where for each pair (si, tj) we also sum over its image $(si, \tilde{t}j)$. The potential between particles can be written in closed form,

$$k_{st}(x) = 3\log \frac{\Gamma\left(\frac{2+3J+8ix}{3}\right) \Gamma\left(\frac{2+3J-8ix}{3}\right)}{\Gamma\left(\frac{1+3J+8ix}{3}\right) \Gamma\left(\frac{1+3J-8ix}{3}\right)} - \log\left(J^2 + \frac{(8x)^2}{3^2}\right) \qquad (5.20)$$

$$= \frac{1}{3}\left(\frac{1}{(3J+8ix)^2} + \frac{1}{(3J-8ix)^2}\right) + O\left(\frac{1}{(3J \pm 8ix)^4}\right) , \qquad (5.21)$$
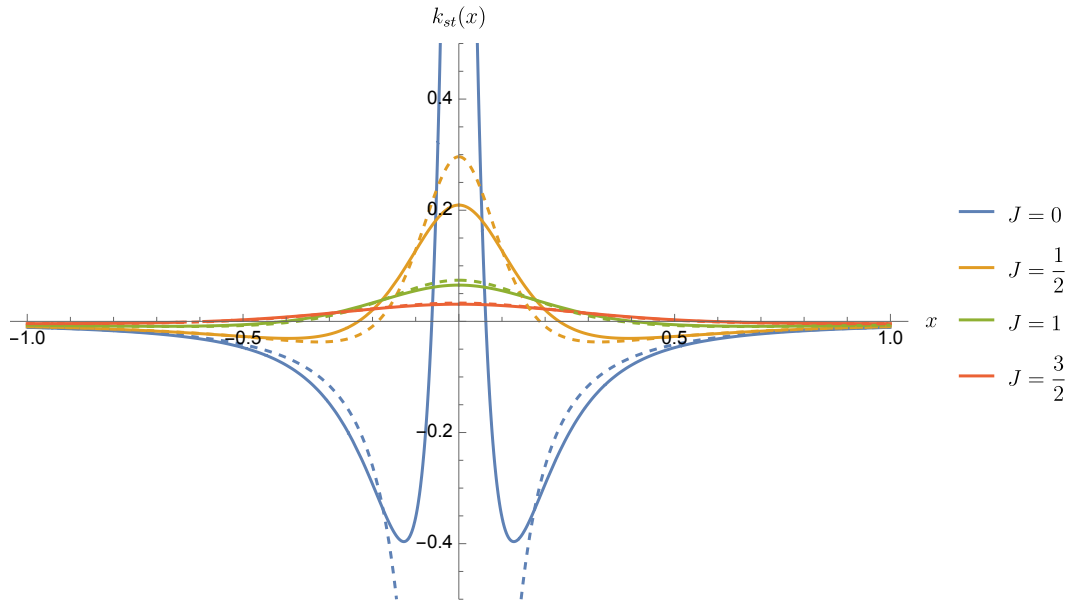
Figure 6: The two-body interaction potential $k_{st}(x)$ for $J = 0, \frac{1}{2}, 1, \frac{3}{2}$ given in (5.20). The dashed lines are the approximation in (5.21).

with $J = |N_s - N_t|/2$. In this formulation, each particle (si) interacts with every other particle (tj) and its image $(\tilde{t}j)$. From the plots of figure 6, we observe that the interaction is repulsive at short distances and attractive at large distances. This is not surprising because the basic building block $h_J(x)$ shows a similar behavior (in fact, it obeys $\int_{-\infty}^{\infty} dx h_J(x) = 0$ for any $J \geq 0$). Particles with the same charge have a short distance repulsion of infinite strength. At long distances, the two-body potentials decay to zero as the inverse of the fourth power of distance.

Let us now introduce the eigenvalue densities

$$\rho^{(s)}(x) = \sum_{i=1}^{n_s} \delta(x - m_{si}).\tag{5.22}$$

In the limit of large n_s, we expect that we can replace this exact eigenvalue density by a smooth function. We shall comment in more detail later on the validity of this approximation. This allows us to write

$$S_{\text{eff}} = -\frac{9\Omega^4}{2^{15}} \sum_s n_s(N_s^3 - N_s) + \frac{3\Omega^4}{2^7} \sum_s N_s \int dx x^2 \rho^{(s)}(x)\tag{5.23}$$

$$+ \frac{1}{2} \sum_{\substack{(s,t) \\ +\text{images}}} \int dx dy \rho^{(s)}(x) \rho^{(t)}(y) k_{st}(x - y) - \sum_s \mu_s \left(\int dx \rho^{(s)}(x) - n_s \right),$$

where we introduce μ_s as a Lagrange multiplier to enforce the constraint $\int dx \rho^{(s)}(x) = n_s$.

The saddle point equation for $\rho^{(s)}(x)$ is then

$$\frac{3\Omega^4}{2^7}N_s x^2 + \sum_{t+\text{images}} \int dy \, k_{st}(x-y)\rho^{(t)}(y) - \mu_s = 0 \,. \tag{5.24}$$

Finally, we approximate the exact two-body potential $k_{st}(x)$ by its large $J \sim x$ behavior given in (5.21),

$$\frac{3\Omega^4}{2^7}N_s x^2 - \mu_s + \frac{2}{3}\sum_{t+\text{images}} \mathcal{P}\!\!\int_{-x^{(t)}}^{x^{(t)}} dy \, \frac{\left(\frac{3}{2}|N_s - N_t|\right)^2 - (8(x-y))^2}{\left(\left(\frac{3}{2}|N_s - N_t|\right)^2 + (8(x-y))^2\right)^2}\rho^{(t)}(y) = 0 \,, \tag{5.25}$$

where we defined images $\tilde{t}$ for each t as

$$\rho^{(\tilde{t})}(x) = -\rho^{(t)}(x) \,, \qquad N_{\tilde{t}} = -N_t \,. \tag{5.26}$$

In addition, the term $t = s$ is defined by the limit $N_t \to N_s$ outside the integral. This is the meaning of the regulated integral symbol. Notice that this preserves the property $\int dx \, k_{ss}(x) = 0$ of the exact two-body potential. We also assumed that the densities $\rho^{(s)}(x)$ have support in an interval $[-x^{(s)}, x^{(s)}]$. Therefore, we have

$$n_s = \int_{-x^{(s)}}^{x^{(s)}} dx \, \rho^{(s)}(x) \,, \qquad \rho^{(s)}(x^{(s)}) = 0 \,. \tag{5.27}$$

Notice that the kernel in (5.25) can be written as a derivative of a simpler function that also decays at large distance,

$$\frac{\left(\frac{3}{2}|N_s - N_t|\right)^2 - (8(x-y))^2}{\left(\left(\frac{3}{2}|N_s - N_t|\right)^2 + (8(x-y))^2\right)^2} = -\frac{d}{dy}\frac{(x-y)}{\left(\frac{3}{2}|N_s - N_t|\right)^2 + (8(x-y))^2} \,. \tag{5.28}$$

Using this property and integrating by parts in (5.25) we find

$$\frac{3\Omega^4}{2^7}N_s x^2 - \mu_s + \frac{2}{3}\sum_{t+\text{images}} \mathcal{P}\!\!\int_{-x^{(t)}}^{x^{(t)}} dy \, \frac{(x-y)}{\left(\frac{3}{2}|N_s - N_t|\right)^2 + (8(x-y))^2}\frac{d\rho^{(t)}}{dy}(y) = 0 \,. \tag{5.29}$$

The new kernel for $t = s$ is proportional to the principal value of $\frac{1}{x-y}$ which is the usual kernel in Gaussian matrix models. Therefore, we expect the behavior close to $x^{(s)}$ to be of square root type,

$$\frac{d\rho^{(t)}}{dy}(y) \sim \sqrt{x^{(s)} - y} \qquad \Rightarrow \qquad \rho^{(t)}(y) \sim \left(x^{(s)} - y\right)^{\frac{3}{2}} \,. \tag{5.30}$$

Thus, we conclude that

$$n_s = \int_{-x^{(s)}}^{x^{(s)}} dx \, \rho^{(s)}(x) \,, \qquad \rho^{(s)}(\pm x^{(s)}) = \frac{d\rho^{(t)}}{dx}(\pm x^{(s)}) = 0 \,. \tag{5.31}$$

Validity of the approximations We made two approximations to arrive at (5.25). Firstly, we replaced discrete sums by integrals of continuous functions. Secondly, we replaced the two-particle potential $k_{st}(x)$ by its large $J \sim x$ behavior. In order to estimate the error in the first approximation, it is convenient to write

$$\rho^{(s)}(x) = \frac{n_s}{x^{(s)}} b\left(\frac{x}{x^{(s)}}\right) \tag{5.32}$$

with $b(w)$ an order one function with support in the interval $[-1, 1]$ and normalized by $\int_{-1}^{1} dw\, b(w) = 1$. In appendix D, we study the electrostatic problem with just one ball (and its image). There we show that

$$b(w) = \begin{cases} \frac{15}{16}(1 - w^2)^2 & \text{if } x^{(s)} \gg N_s \\ \frac{8}{3\pi^2}(1 - w^2)^{\frac{3}{2}} & \text{if } x^{(s)} \ll N_s \end{cases}. \tag{5.33}$$

In addition, we found that

$$\frac{x^{(s)}}{N_s} \sim \begin{cases} \xi^{\frac{1}{5}} & \text{if } \xi \gg 1 \\ \xi^{\frac{1}{4}} & \text{if } \xi \ll 1 \end{cases}, \qquad \xi \equiv \frac{n_s}{\Omega^4 N_s^5}. \tag{5.34}$$

We can then estimate the order of magnitude of the first 3 terms in the effective action (5.23), namely $\Omega^4 n_s N_s^3$, $\Omega^4 n_s N_s (x^{(s)})^2$ and $n_s^2/(x^{(s)})^2$. Notice that these 3 terms are of order n_s^2/N_s^2 if we keep ξ fixed.

Doing a Taylor expansion of $\rho^{(t)}(y)$ around x and referring to figure 6, the error introduced by approximating $k_{st}(x)$ can be estimated by

$$\int dx\, dy\, \rho^{(s)}(x) \rho^{(t)}(y) \left[k_{st}(x - y) - k_{st}^{\text{approx}}(x - y)\right] \sim \int dx\, \rho^{(s)}(x) \frac{d^2}{dx^2} \rho^{(t)}(x) \sim \frac{n_s^2}{(x^{(s)})^3}.$$

This is suppressed by a factor of $x^{(s)}$ relative to the third term in (5.23). Therefore, our approximation is valid as long as $x^{(s)} \gg 1$. The error introduced by the approximation of a continuous density of particles goes to zero as long as the density $\rho^{(s)} \sim n_s/x^{(s)}$ goes to infinity. Finally, in order to neglect the entropy term[26]

$$\sum_s \int dx\, \rho^{(s)}(x) \log \frac{\rho^{(s)}(x)}{n_s} \sim n_s \log x^{(s)}, \tag{5.35}$$

relative to the other terms in (5.23), we need $n_s/(x^{(s)})^2 \gg \log x^{(s)} \gg 1$. Putting everything together, we need

$$n_s \gg (x^{(s)})^2 \log(x^{(s)}) \gg 1. \tag{5.36}$$

This can be achieved in many ways. For example, we can keep ξ fixed (and therefore $x^{(s)}/N_s$ is also fixed) and take $n_s \gg N_s^2 \log N_s$. Alternatively, we can keep N_s and Ω fixed and send $n_s \to \infty$.

---

[26]The entropy term comes from converting the measure $\int \prod_i dm_{si} \to \int [d\rho^{(s)}(x)]$. For a review, refer to e.g. [36, App.C].

5.3 Matching with gravity

The localization equations (5.25) and (5.31) are exactly the same as the electrostatic equations (5.10) and (5.12) up to an identification of parameters. To find the correct identification we start by writing

$$x, y = ar, ar', \quad x^{(t)} = aR_t, \quad \rho^{(t)}(x = ar) = bf_t(r), \tag{5.37}$$

where we denote by x, y the variables of the densities of eigenvalues and r, r' the variables of the linear charge densities. Then, matching the kernels in (5.25) and (5.10), we find

$$\frac{3}{16a}N_s = z_s. \tag{5.38}$$

We can now rescale one of the equations so that the terms with the integral are identical. Then we match the r independent terms and the terms proportional to r^2, which gives

$$V_s + z_s^3 V_b = \frac{24a}{b\pi^2}\mu_s, \qquad 3V_b z_s = \frac{9a^3 N_s \Omega^4}{16b\pi^2}. \tag{5.39}$$

Finally we match the equations for the total charge (5.31) and (5.12) to get

$$abQ_s = n_s. \tag{5.40}$$

As explained in [5], quantizing the different fluxes in the gravity side leads to quantization of the heights of the disks z_s and their charges Q_s as

$$z_s = \frac{3\pi\mu\alpha'}{8}N_s, \qquad Q_s = \frac{\pi^4\mu^6\alpha'^3 g_s}{32}n_s, \tag{5.41}$$

where the constant μ is related to V_b as $V_b = \mu^5/2^7$. We can then solve the equations (5.38), (5.40) and the first one in (5.39) for the parameters a, b, μ_s and we find

$$a = \frac{1}{2\pi\mu\alpha'}, \quad b = \frac{64}{\pi^3\mu^5\alpha'^2 g_s}, \quad \mu_s = \frac{16}{3g_s\alpha'\mu^4}V_s + \frac{9}{2^{12}g_s}N_s^3\pi^3\alpha'^2\mu^4. \tag{5.42}$$

In addition, we still have the second equation in (5.39) that we did not use. We can solve it for μ and we find

$$\mu = \sqrt{g_{YM}}\Omega, \tag{5.43}$$

which exactly agrees with the matching derived in [5]. To summarize, the identification of localization parameters with gravity parameters is

$$x = \frac{1}{2\pi\mu\alpha'}r, \qquad \rho^{(s)}(x) = \frac{64}{\pi^3\mu^5\alpha'^2 g_s}f_s(r), \qquad \mu_s = \frac{16}{3g_s\alpha'\mu^4}V_s + \frac{9}{2^{12}g_s}N_s^3\pi^3\alpha'^2\mu^4. \tag{5.44}$$

5.4 The on-shell action

The action (5.23) evaluated on-shell reduces to

$$S_{\text{on-shell}} = -\frac{9\Omega^4}{2^{15}} \sum_s n_s N_s^3 + \frac{1}{2}\frac{3\Omega^4}{2^7} \sum_s N_s \int dx x^2 \rho^{(s)}(x) + \frac{1}{2}\sum_s \mu_s n_s. \tag{5.45}$$

Note that for a single ball, writing it in terms of the fixed quantity $\xi = n_s \Omega^{-4} N_s^{-5}$, we get

$$S_{\text{on-shell}} = -\frac{N^2}{\lambda^{2/3}} H(\xi), \qquad H(\xi) \approx \begin{cases} -\frac{5}{7 \cdot 2^7}(15\pi/8)^{2/5}\xi^{1/15} & \text{if } \xi \gg 1 \\ \frac{9}{2^{15}}\xi^{-1/3} & \text{if } \xi \ll 1 \end{cases}, \tag{5.46}$$

where $N = n_s N_s$ and $\lambda \equiv N/\Omega^4 = N g_{\text{YM}}^2/\mu^4$ is the dimensionless 't Hooft coupling. This aligns with the scaling similarity arguments in [37–39] and our scaling analysis from the supergravity perspective [5].

Using matching of parameters (5.44), (5.45) becomes

$$S_{\text{on-shell}} = \sum_s \left(-\frac{9\pi^3 \alpha'^2 \mu^4}{2^{13}} n_s N_s^3 + \frac{2^8}{3\pi^4 g_s^2 \alpha'^4 \mu^{10}} Q_s V_s + \frac{3}{4 g_s^2 \pi^3 \alpha'^3 \mu^4} N_s \int dr r^2 f_s(r) \right). \tag{5.47}$$

Remarkably, the first and the last term combine exactly to give the octopole[27] of the electrostatic configuration expressed as

$$q_{3,s} = 2z_s^3 Q_s - 6z_s \int_{-R_s}^{R_s} dr r^2 f_s(r) = \frac{27 g_s \alpha'^6 \mu^9 \pi^7}{2^{13}} n_s N_s^3 - \frac{9}{4}\pi\mu\alpha' N_s \int dr r^2 f_s(r). \tag{5.48}$$

We can then rewrite (5.47) as

$$S_{\text{on-shell}} = \sum_s \left(\frac{2^8}{3\pi^4 g_s^2 \alpha'^4 \mu^{10}} Q_s V_s - \frac{1}{3 g_s^2 \pi^4 \alpha'^4 \mu^5} q_{3,s} \right). \tag{5.49}$$

In the limit $z_s \gg R_s$, where we can compute $V_s \approx -\frac{\mu^5}{2^7} z_s^3$ and $q_3 \approx 2z_s^3 Q_s$ and we obtain

$$\frac{2^8}{3\pi^4 g_s^2 \alpha'^4 \mu^{10}} Q_s V_s \underset{N_i \to \infty}{\simeq} -\frac{9}{2^{16}} n_s N_s^3 \Omega^4, \qquad -\frac{1}{3 g_s^2 \pi^4 \alpha'^4 \mu^5} q_{3,s} \underset{N_i \to \infty}{\simeq} -\frac{9}{2^{16}} n_s N_s^3 \Omega^4, \tag{5.50}$$

which correctly reproduces the value of the potential at the minima.

It is interesting to compare equation (5.49) with the on-shell action found in the gravity dual [5]. There, we found a term proportional to $\sum_s Q_s V_s$ which matches the first term in (5.49) up to a factor of 2. We also found that boundary terms generically give something proportional to $\sum_s q_{3,s}$. It would be nice to carefully finish the gravitational computation and match the result (5.49).

---

[27]For more details on the octopole, refer to [5].

6 Discussion

Summary and main results In this paper, we studied the polarized IKKT matrix model and reduced the partition function, as well as a family of protected observables, to a sum over saddles where each term is parameterized by one block-diagonal matrix integral. Surprisingly, we found that the limit of vanishing mass-deformation did not reduce to the IKKT model, but instead led to a divergence. We made sense of the divergence by interpreting this limit as a strong coupling limit where the off-diagonal entries of the matrices can be integrated out perturbatively. In fact, we obtained that the structure exhibited by each saddle contribution is very similar to the picture in BMN, where each irreducible block forms a bound state which decouples from the others in the strong coupling limit. We also considered the limit where the mass-deformation dominates and obtained results consistent with [12]. Given that the limit of vanishing mass-deformation of the partition function diverges, one can ask whether there exists some natural regularization which would give rise to IKKT. We proposed an answer to this question by conjecturing a contour prescription of the moduli integrals and we verified that it works for $N = 2, 3$.

Then, we gave evidence that each saddle of the localization result was in one to one correspondence with the geometries we constructed in [5]. This was derived by analyzing the dynamics of the eigenvalues of the localization result in the limit where they form a continuum. In this limit, the partition function is dominated by some configurations of eigenvalues densities which obey a saddle point integral equation. In the gravity side, the metric is characterized by a potential which is obtained after solving an electrostatic problem. This problem can be reduced to an integral equation on the charge densities. We found exact agreement between the two sides once parameters were identified. We also obtained that the on-shell action of the eigenvalues densities is consistent with the bosonic potential evaluated at its minima, and that the structure agrees with the gravitational on-shell action we derived in [5].

Future directions

- It would be instructive to pursue the small Ω expansion of the effective action to subleading orders. The leading term derived in section 4.2 corresponds to free particles in an harmonic potential. The next terms will give rise to interactions between these particles. Can we understand how the distribution of particles changes when there are many interacting particles ? The dual gravitational description should emerge from this density field of interacting particles.

- It should be possible to obtain the S^6 partition function of k NS5-branes from a careful scaling limit of our result.

- Our conjectured prescription to obtain IKKT from polarized IKKT predicts that the sum over $SU(2)$ representations of the residues of the 1-loop determinant should somehow reduce to a sum over divisors of N, $\sum_{m|N} \frac{1}{m^2}$. Proving this conjecture might involve some non-trivial number theoretic relations.

- The (undeformed) IKKT partition function, which is given in terms of the divisor function $\sigma_2(N)$, does not have a uniform large N limit and instead behaves "erratically" as a function of N. The asymptotic behavior of σ_2 has been studied in the literature and it would be interesting to see if and how they can be reproduced from the gravity side. See [40] for discussions on the gravitational path integral and quantities that depend erratically on N.

A   Convention for the gamma matrices

To define the Euclidean $SO(10)$ gamma matrices, we first define $SO(9)$ Gamma matrices, inspired from [41, App.3.A.5],

$$
\begin{aligned}
\gamma^1 &\equiv \sigma^2 \otimes \sigma^2 \otimes \sigma^1 \otimes \mathbb{1}\,,\\
\gamma^2 &\equiv \sigma^2 \otimes \sigma^3 \otimes \mathbb{1} \otimes \sigma^2\,,\\
\gamma^3 &\equiv -\sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,,\\
\gamma^4 &\equiv -\sigma^2 \otimes \sigma^1 \otimes \mathbb{1} \otimes \sigma^2\,,\\
\gamma^5 &\equiv \sigma^2 \otimes \mathbb{1} \otimes \sigma^2 \otimes \sigma^3\,,\\
\gamma^6 &\equiv \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,,\\
\gamma^7 &\equiv \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2\,,\\
\gamma^8 &\equiv -\sigma^2 \otimes \mathbb{1} \otimes \sigma^2 \otimes \sigma^1\,,\\
\gamma^9 &\equiv \sigma^2 \otimes \sigma^2 \otimes \sigma^3 \otimes \mathbb{1}\,,
\end{aligned}
\tag{A.1}
$$

where σ^i are Pauli matrices, $\mathbb{1} = \mathbb{1}_2$ and we have reshuffled the indices and signs for convenience in our localization computations. The features of this representation are that

- The complex valued matrix σ^2 always appears an even number of times, so that $\gamma^{I'}$ are real and symmetric.

- We chose γ^3 diagonal.

- The matrices $\gamma^{I'=1,\dots,2}$ and $\gamma^{I'=4,\dots,9}$ take a block off-diagonal form, where the off-diagonal blocks are 8 by 8.

We then define the $SO(10)$ "Weyl" matrices, as

$$
\gamma^I = (\gamma^{I'}, -i\mathbb{1}_{16})\,, \qquad \bar\gamma^I = (\gamma^{I'}, i\mathbb{1}_{16})\,, \quad (I = 1, 2, \dots 10,\ I' = 1, 2, \dots, 9)\,.
\tag{A.2}
$$

This allows to define the $SO(10)$ gamma matrices as

$$\Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \bar{\gamma}^I & 0 \end{pmatrix},$$

$$\{\Gamma^I, \Gamma^J\} = \begin{pmatrix} 2\gamma^{(I}\bar{\gamma}^{J)} & 0 \\ 0 & 2\bar{\gamma}^{(I}\gamma^{J)} \end{pmatrix} = 2\delta^{IJ}\mathbb{1}_{32},$$

$$\Gamma^{IJ} \equiv \Gamma^{[I}\Gamma^{J]} = \begin{pmatrix} \gamma^{[I}\bar{\gamma}^{J]} & 0 \\ 0 & \bar{\gamma}^{[I}\gamma^{J]} \end{pmatrix} \equiv \begin{pmatrix} \bar{\gamma}^{IJ} & 0 \\ 0 & \gamma^{IJ} \end{pmatrix},$$

$$\Gamma^{IJK} \equiv \Gamma^{[I}\Gamma^J\Gamma^{K]} = \begin{pmatrix} 0 & \gamma^{[I}\bar{\gamma}^J\gamma^{K]} \\ \bar{\gamma}^{[I}\gamma^J\bar{\gamma}^{K]} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \gamma^{IJK} \\ \bar{\gamma}^{IJK} & 0 \end{pmatrix}, \tag{A.3}$$

$$\Gamma^{IJKL} = \begin{pmatrix} \gamma^{[I}\bar{\gamma}^J\gamma^K\bar{\gamma}^{L]} & 0 \\ 0 & \bar{\gamma}^{[I}\gamma^J\bar{\gamma}^K\gamma^{L]} \end{pmatrix} \equiv \begin{pmatrix} \bar{\gamma}^{IJKL} & 0 \\ 0 & \gamma^{IJKL} \end{pmatrix},$$

$$\Gamma_* \equiv (-i)^{10/2}\Gamma^1\Gamma^2\dots\Gamma^{10} = \begin{pmatrix} \mathbb{1}_{16} & 0 \\ 0 & -\mathbb{1}_{16} \end{pmatrix},$$

and higher rank gamma matrices also follow the same pattern as above. Our choice for the conjugation matrix $\mathcal{C}$ is

$$\mathcal{C} = \Gamma^{10} = \begin{pmatrix} 0 & -i\mathbb{1}_{16} \\ i\mathbb{1}_{16} & 0 \end{pmatrix}. \tag{A.4}$$

In this convention, the Majorana-Weyl condition is trivially satisfied by real left-handed spinors of the form $\psi = (\psi_{\alpha=1,\dots,16}, 0, \dots, 0)$ where ψ_α are real Grassmann numbers. For such spinors,

$$\mathcal{C}\Gamma^I\psi = -i\bar{\gamma}^I\psi, \qquad \Gamma^{IJ}\psi = \bar{\gamma}^{IJ}\psi, \qquad \mathcal{C}\Gamma^{123}\psi = -i\bar{\gamma}^{123}\psi,$$
$$\Gamma^{123}\Gamma^I\psi = \bar{\gamma}^{123}\bar{\gamma}^I\psi, \qquad \dots \tag{A.5}$$

where it is clear that we use the notation $\psi = (\psi, 0)$ implicitly both for the 32-component and 16-component spinor ψ.

B   The IKKT convergence argument, broken by fermion masses

In this appendix, we discuss convergence of the pure $SU(2)$ IKKT partition function, and show that the presence of massive fermions affects the convergence argument, leading to a divergence when $\Omega \to 0$. Let us start by looking at the bosonic part of the pure IKKT model

$$S_{\text{bos}}^{\text{IKKT}} = -\frac{1}{4}\text{Tr}[X_I, X_J]^2. \tag{B.1}$$

We define $X^I = Rx^I$ where $\sum_I \text{Tr}x^Ix^I = 1$. The divergent behaviour may arise from the region $R \gg 1$. However, it has been shown in [13, 20, 34] that correlators behave at large R as

$$\int [dX^I]\text{Tr}X^p e^{-S_{\text{bos}}^{\text{IKKT}}} \sim \int_{R \gg 1} \frac{dR}{R} R^{p-6}. \tag{B.2}$$

In particular the partition function of $SU(2)$ bosonic IKKT converges, as well as correlators with degree $p < 6$. Adding a bosonic mass deformation only gives an $e^{-\Omega^2 R^2}$ factor. Namely, with

$$S_{\text{bos}} = \text{Tr}\left[-\frac{1}{4}[X_I, X_J]^2 + \frac{3\Omega^2}{4^3}X_i X_i + \frac{\Omega^2}{4^3}X_p X_p + i\frac{\Omega}{3}\epsilon^{ijk}X_i X_j X_k \right], \tag{B.3}$$

correlators far in the commuting modes valley behave as

$$\int [dX^I]\text{Tr}X^p e^{-S_{\text{bos}}} \sim \int_{R \gg 1} \frac{dR}{R^6}R^p e^{-\Omega^2 R^2} \sim \Omega^{6-p}. \tag{B.4}$$

There are no surprises yet. Bosonic polarized IKKT has the same divergences when $\Omega \to 0$ as bosonic IKKT. The surprise comes when adding fermions.

When we add fermions to bosonic pure IKKT, we basically add

$$\int [d\psi_\alpha]e^{\frac{1}{2}\psi^\top \bar{\gamma}^I [X_I, \psi]} = \text{Pf}(X^I) \tag{B.5}$$

in the bosonic integral. One can integrate out the ψ's, giving a Pfaffian $\text{Pf}(X^I)$ which is a polynomial in the X matrices. For the case $N = 2$ that we consider here, it has been shown in [21] that

$$\text{Pf}(X^I) \propto \left(\text{Tr}[X_I, X_J][X_J, X_K][X_K, X_I]\right)^4. \tag{B.6}$$

To discuss the convergence behaviour of the $SU(2)$ IKKT partition function with fermions, we thus simply insert the Pfaffian in the left-hand-side of (B.2). Since $\text{Pf}(X^I)$ only involves commutators, it can only *help* the convergence of the partition function since it vanishes in the commuting modes valleys. In fact, the careful analysis of [13, 34] shows that correlators now converge up to $p < 14$. Let us now discuss what happens when we perform the mass deformation. In this case, we obtain the Pfaffian

$$\int [d\psi_\alpha]e^{\frac{1}{2}\bar{\psi}^\top \bar{\gamma}^I [X_I, \psi] - \frac{i}{8}\Omega \psi^\top \bar{\gamma}^{123}\psi} = \text{Pf}(X^I, \Omega) \tag{B.7}$$

which reduces to the previous one when $\Omega = 0$. One can check numerically that when $X^I \sim R\,\sigma^3$ where $\sigma^3 = \text{diag}(1, -1)$, i.e. when X^I are commuting matrices, with magnitude $\sim R$, the mass terms of the fermions create new contributions such that the Pfaffian goes as

$$\text{Pf}(X^I, \Omega) \sim \Omega^{24} + \Omega^{22}R^2 + \Omega^{20}R^4 + \cdots + \Omega^8 R^{16}. \tag{B.8}$$

Inserting this Pfaffian in (B.4), one obtains

$$\int [dX^I][d\psi_\alpha]e^{-S} = \int [dX^I]\text{Pf}(X^I, \Omega)e^{-S_{\text{bos}}}$$
$$\sim \int_{R \gg 1} \frac{dR}{R}R^{-6}(\Omega^8 R^{16} + \dots)e^{-\Omega^2 R^2} \sim \frac{1}{\Omega^2} + \dots \tag{B.9}$$

Therefore, we see a $1/\Omega^2$ divergence appearing, due to the massive terms of the Pfaffian. Formally, we see in this last integral why $\lim_{\Omega \to 0}$ and $\int [dX^I][d\psi_\alpha]$ don't commute, i.e.

$$0 = \int_{R \gg 1} \frac{dR}{R}\lim_{\Omega \to 0}\Omega^8 R^{10}e^{-\Omega^2 R^2} \neq \lim_{\Omega \to 0}\int_{R \gg 1}\frac{dR}{R}\Omega^8 R^{10}e^{-\Omega^2 R^2} = \infty. \tag{B.10}$$

C Technical details for localization

C.1 Constraints from SUSY closure

To impose the SUSY closure on gauge and global symmetry generators, we will need to use gamma matrices identities which are easier to express in 32-component notation. We thus begin by expressing the supersymmetries (3.5) as

$$\delta_\epsilon X^I = \bar\epsilon \Gamma^I \psi \,,$$

$$\delta_\epsilon \psi = \frac{i}{2}\Gamma^{IJ}\epsilon[X_I, X_J] + \frac{3\Omega}{8}\Gamma^{123}\Gamma^i \epsilon X_i + \frac{\Omega}{8}\Gamma^{123}\Gamma^p \epsilon X_p + i\nu_a K_a \,, \tag{C.1}$$

$$\delta_\epsilon K_a = -\bar\nu_a \Gamma^I [X_I, \psi] + i\frac{\Omega}{4}\bar\nu_a \Gamma^{123}\psi \,.$$

We then impose $\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}(\text{fields}) = \delta_B(\text{fields})$ on X^I, ψ_α and K_a where δ_B is some combination of $SU(N) \times SO(3) \times SO(7)$ generators.

Before diving in, let us introduce some useful identities. We will use Majorana flip relations, see e.g. [41, Sec. 3.2.1],

$$(\mathcal{C}\Gamma^A)^\top = -t_{r_A}(\mathcal{C}\Gamma^A) \,, \tag{C.2}$$

where $\Gamma^A = \Gamma^{\mu_1,\dots,\mu_{r_A}}$ is a rank-r_A gamma matrix, and in our convention $t_0 = t_3 = 1$ and $t_1 = t_2 = -1$. Recalling that we consider ϵ_i to be Grassmann-even (for Grassmann-odd fermions, invert the signs), this implies

$$\bar\epsilon_1 \Gamma^A \epsilon_2 = -\bar\epsilon_2 \Gamma^A \epsilon_1 \,, \qquad r_A = 0, 3, 4, 7, 8 \tag{C.3}$$

$$\bar\epsilon_1 \Gamma^A \epsilon_2 = \bar\epsilon_2 \Gamma^A \epsilon_1 \,. \qquad r_A = 1, 2, 5, 6, 9, 10 \tag{C.4}$$

Note also that for left-handed Weyl spinors ϵ_1, ϵ_2,

$$\bar\epsilon_1 \Gamma^A \epsilon_2 = 0 \,, \qquad r_A \text{ even} \tag{C.5}$$

which follows from $\bar\epsilon_1 \Gamma^A \epsilon_2 = \bar\epsilon_1 \Gamma^A \Gamma^* \epsilon_2 = (-1)^{r_A}\bar\epsilon_1 \Gamma^* \Gamma^A \epsilon_2 = (-1)^{r_A+1}\bar\epsilon_1 \Gamma^A \epsilon_2$. When considering anticommutators $\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}$, we will symmetrize $\delta_{\epsilon_1}\delta_{\epsilon_2}$ in $(\epsilon_1, \nu_a^1) \leftrightarrow (\epsilon_2, \nu_a^2)$, where ν_a^i is the "auxiliary" spinor that will be constrained in terms of ϵ_i. Thus, it is useful to note that the only non-vanishing symmetrized expressions are of the form

$$\bar\epsilon_1 \Gamma^A \epsilon_2 + (1 \leftrightarrow 2) \neq 0 \iff r_A = 1, 5, 9 \,. \tag{C.6}$$

All other gamma matrices vanish when symmetrized over ϵ_1, ϵ_2.

On X^I: We obtain

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}X^I = \frac{i}{2}\bar\epsilon_2 \Gamma^I \Gamma^{JK}\epsilon_1 [X_J, X_K] + \bar\epsilon_2 \Gamma^I \nu_a^1 K_a$$

$$+ \frac{3\Omega}{8}\bar\epsilon_2 \Gamma^I \Gamma^{123}\Gamma^j \epsilon_1 X_j + \frac{\Omega}{8}\bar\epsilon_2 \Gamma^I \Gamma^{123}\Gamma^q \epsilon_1 X_q + \{\epsilon_1, \nu_a^1\} \leftrightarrow \{\epsilon_2, \nu_a^2\} \,. \tag{C.7}$$

It is easy to notice that the auxiliary matrices K_a cannot appear in order to have algebra closure. Thus, we obtain the first constraint

$$\bar{\epsilon}_2\Gamma^I\nu_a^1 = \bar{\epsilon}_1\Gamma^I\nu_a^2 = 0\,. \tag{C.8}$$

Let us now consider $I = i = 1,2,3$ and $I = p = 4,5,...,10$ separately. Using that $\Gamma^I\Gamma^{JK} = \Gamma^{IJK} + \delta^{IJ}\Gamma^K - \delta^{IK}\Gamma^J$, where Γ^{IJK} do not contribute according to (C.6) because they are rank-3 gamma matrices, and that $\Gamma^i\Gamma^{123}\Gamma^q$, $\Gamma^p\Gamma^{123}\Gamma^j$ are rank-3 gamma matrices which do not contribute as well, we have

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}X^i = -2i\bar{\epsilon}_1\Gamma^J\epsilon_2[X^J, X^i] + \frac{3\Omega}{4}\bar{\epsilon}_1\Gamma^{123}\Gamma^{ij}\epsilon_2 X^j\,, \tag{C.9}$$

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}X^p = -2i\bar{\epsilon}_1\Gamma^J\epsilon_2[X^J, X^p] - \frac{\Omega}{4}\bar{\epsilon}_1\Gamma^{123}\Gamma^{pq}\epsilon_2 X^q\,, \tag{C.10}$$

where we recognize a sum of gauge generator and $SO(3) \times SO(7)$ rotations acting on X^I.

On ψ_α: Using (C.1), we have

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}\psi = -i\Gamma^{JI}\epsilon_2[\bar{\epsilon}_1\Gamma^I\psi, X^J] - i\nu_a^2\bar{\nu}_a^1\Gamma^I[X_I, \psi]$$
$$+ \frac{3\Omega}{8}\Gamma^{123}\Gamma^i\epsilon_2(\bar{\epsilon}_1\Gamma^i\psi) + \frac{\Omega}{8}\Gamma^{123}\Gamma^p\epsilon_2(\bar{\epsilon}_1\Gamma^p\psi) - \frac{\Omega}{4}\nu_a^2\bar{\nu}_a^1\Gamma^{123}\psi + \{\epsilon_1, \nu_a^1\} \leftrightarrow \{\epsilon_2, \nu_a^2\}\,. \tag{C.11}$$

Let us start by looking at the term independent of Ω. First, we separate $\Gamma^{JI} = \Gamma^J\Gamma^I - \delta^{IJ}$. Then, we use on the term containing $\Gamma^J\Gamma^I$ the Fierz identity [41]

$$\Gamma_{\alpha\beta}^I\Gamma_{\gamma\delta}^I = \frac{1}{2^5}\sum_A v_{r_A}\Gamma_{\alpha\delta}^A(\Gamma_A)_{\gamma\beta}$$
$$\equiv \frac{1}{2^5}\left(v_0\delta_{\alpha\delta}\delta_{\gamma\beta} + v_1\Gamma_{\alpha\delta}^I\Gamma_{\gamma\beta}^I + \frac{v_2}{2!}\Gamma_{\alpha\delta}^{IJ}\Gamma_{\gamma\beta}^{JI} + \cdots + \frac{v_{10}}{10!}\Gamma_{\alpha\delta}^{I_1...I_{10}}\Gamma_{\gamma\beta}^{I_{10}...I_1}\right)\,, \tag{C.12}$$

where

$$v_{r_A} \equiv (-1)^{r_A}(10 - 2r_A)\,. \tag{C.13}$$

Nicely, this vanishes for $r_A = 5$. Note that from condition (C.6), we only have contributions from $r_A = 1$ and its hodge dual $r_A = 9$. For Weyl spinors, the two give the same contribution. Altogether, we obtain

$$-i\Gamma^J\Gamma^I\epsilon_2[\bar{\epsilon}_1\Gamma^I\psi, X^J] + (1 \leftrightarrow 2) = \frac{i}{2}\bar{\epsilon}_1\Gamma^I\epsilon_2[\Gamma^J\Gamma^I\psi, X_J] + (1 \leftrightarrow 2)\,. \tag{C.14}$$

This identity allows to write the $\Omega = 0$ part of (C.11) as

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}\psi_\alpha|_{\Omega=0} = -i\bar{\epsilon}_1\Gamma^I\epsilon_2[X^I, \psi_\alpha] - \frac{i}{2}(\bar{\epsilon}_1\Gamma^I\epsilon_2)\Gamma_{\alpha\beta}^I[(\Gamma^J\psi)_\beta, X^J]$$
$$+ i(\epsilon_2)_\alpha(\bar{\epsilon}_1)_\beta[(\Gamma^I\psi)_\beta, X^I] + i(\nu_a^2)_\alpha(\bar{\nu}_a^1)_\beta[(\Gamma^I\psi)_\beta, X^I] + \{\epsilon_1, \nu_a^1\} \leftrightarrow \{\epsilon_2, \nu_a^2\}\,. \tag{C.15}$$

Only the first term is compatible with the algebra closure. This gives the constraint

$$\frac{1}{2}(\bar{\epsilon}_1\Gamma^I\epsilon_2)\Gamma^I_{\alpha\beta} = (\epsilon_2)_\alpha(\bar{\epsilon}_1)_\beta + (\nu^2_a)_\alpha(\bar{\nu}^1_a)_\beta\,. \tag{C.16}$$

Treating the Ω term in (C.11) is now straightforward by solving the ν dependence using the above constraint. We obtain

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}\psi = -2i\bar{\epsilon}_1\Gamma^I\epsilon_2[X^I, \psi] + \frac{3\Omega}{16}\bar{\epsilon}_1\Gamma^{123}\Gamma^{ij}\epsilon_2\Gamma^{ij}\psi - \frac{\Omega}{16}\bar{\epsilon}_1\Gamma^{123}\Gamma^{pq}\epsilon_2\Gamma^{pq}\psi\,. \tag{C.17}$$

On K_a: The anticommutator $\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a$ has three pieces, one of order 1, one of order Ω, and one of order Ω^2. Let us separate

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a = \{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_{\Omega=0} + \{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_\Omega + \{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_{\Omega^2}\,. \tag{C.18}$$

We start with the Ω-independent piece,

$$\begin{aligned}\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_{\Omega=0} &= -i\bar{\nu}^2_a\Gamma^I\nu^1_b[X^I, K_b] + i(\bar{\nu}^2_a\Gamma^I\psi)(\bar{\epsilon}_1\Gamma^I\psi) + i(\bar{\epsilon}_1\Gamma^I\psi)(\bar{\nu}^2_a\Gamma^I\psi) \\ &\quad - \frac{i}{2}\bar{\nu}^2_a\Gamma^I\Gamma^{JK}\epsilon_1[X^I, [X^J, X^K]] + \{\epsilon_1, \nu^1_a\} \leftrightarrow \{\epsilon_2, \nu^2_a\}\,.\end{aligned} \tag{C.19}$$

The first term $(+ \nu^1_a \leftrightarrow \nu^2_a)$ is exactly the gauge generator $-2i(\bar{\epsilon}_1\Gamma^I\epsilon_2)[X^I, \cdot]$ that we obtained in (C.9), (C.10), (C.17) provided that we impose

$$\bar{\nu}^1_a\Gamma^I\nu^2_b = \delta_{ab}\bar{\epsilon}_1\Gamma^I\epsilon_2\,, \tag{C.20}$$

whereas the rest of the first line is 0 because ψ are Grassmann-odd, and the second line vanishes thanks to the decomposition $\Gamma^I\Gamma^{JK} = \Gamma^{IJK} + \delta^{IJ}\Gamma^K - \delta^{IK}\Gamma^J$, and the use of Jacobi identity on Γ^{IJK}, and the condition $\bar{\nu}_a\Gamma^I\epsilon = 0$ on Γ^K, Γ^J.

Let us now look at the Ω-dependent piece of $\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a$. We obtain

$$\begin{aligned}\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_\Omega &= -\frac{3\Omega}{8}\bar{\nu}^2_a\Gamma^I[X^I, \Gamma^{123}\Gamma^i\epsilon_1 X_i] - \frac{\Omega}{8}\bar{\nu}^2_a\Gamma^I[X^I, \Gamma^{123}\Gamma^p\epsilon_1 X_p] \\ &\quad - \frac{\Omega}{8}\bar{\nu}^2_a\Gamma^{123}\Gamma^{IJ}\epsilon_1[X_I, X_J] + (\epsilon_1, \nu^1_a \leftrightarrow \epsilon_2, \nu^2_a) = 0\,.\end{aligned} \tag{C.21}$$

where in order to obtain the last equality, we separated the contributions from $I = i$, $I = p$ and $J = j$, $J = q$, anticommuted the matrices around and we used the constraint $\bar{\nu}\Gamma^I\epsilon = 0$. Finally,

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a|_{\Omega^2} = -i\frac{3\Omega^2}{32}\bar{\nu}^2_a\Gamma^i\epsilon_1 X_i - i\frac{\Omega^2}{32}\bar{\nu}^2_a\Gamma^i\epsilon_1 X_i + (\epsilon_1, \nu^1_a \leftrightarrow \epsilon_2, \nu^2_a) = 0\,, \tag{C.22}$$

where we used $(\Gamma^{123})^2 = -\mathbb{1}_{32}$ and the constraint $\bar{\nu}\Gamma^I\epsilon = 0$. Thus, we only obtain the gauge generator without $SO(3) \times SO(7)$ transformation, i.e.

$$\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}K_a = -2i\bar{\epsilon}_1\Gamma^I\epsilon_2[X^I, K_a]\,. \tag{C.23}$$

Identifying the symmetries: Let us define $SO(10)$ rotations by an antisymmetrized parameter ω_{IJ} (from which $SO(3)$ and $SO(7)$ are a subgroup) as

$$\delta_{SO(10),\omega} X^I = i\omega_{KL}(M_{KL})_{IJ}X^J\,, \qquad \delta_{SO(10),\omega}\psi = \frac{i}{2}\omega_{KL}\Gamma^{KL}\psi\,, \qquad \text{(C.24)}$$

where

$$(M_{KL})_{IJ} = i(\delta_{IK}\delta_{JL} - \delta_{KJ}\delta_{IL})\,. \qquad \text{(C.25)}$$

The $SO(3)$ and $SO(7)$ generators are a simple subgroup of the above with parameters ω_{ij} and ω_{pq} respectively. We also define the $SU(N)$ generator with parameter α (Hermitian matrix) as

$$\delta_{SU(N),\alpha} = i[\alpha,\cdot]\,. \qquad \text{(C.26)}$$

Then, from (C.9), (C.10), (C.17), (C.23), we derived[28]

$$\{\delta_{\epsilon_1},\delta_{\epsilon_2}\} = \delta_{SU(N),-2\bar{\epsilon}_1\Gamma^J\epsilon_2 X^J} + \frac{3\Omega}{8}\delta_{SO(3),-i\bar{\epsilon}_1\Gamma^{123}\Gamma^{ij}\epsilon_2} - \frac{\Omega}{8}\delta_{SO(7),-i\bar{\epsilon}_1\Gamma^{123}\Gamma^{pq}\epsilon_2}\,. \qquad \text{(C.27)}$$

Summary: We derived the constraints (C.8), (C.16) and (C.20), which are equivalent to (3.7) once written in 16-component notation. We also obtained along the way the action of the SUSY anticommutator on all fields (C.9), (C.10), (C.17), (C.23) which can be written as

$$\{\delta_{\epsilon_1},\delta_{\epsilon_2}\} = \delta_{SU(N)} + \delta_{SO(3)\times SO(7)}\,, \qquad \text{(C.28)}$$

as shown with explicit parameters in (C.27).

C.2 Solving the auxiliary constraints

We now solve the constraints on ν_a such that δ_s^2 closes off-shell. Here we will use the notation

$$M = 1,2,4,5,6,...,9 \qquad \text{(C.29)}$$

Note that we have a gamma matrix representation where $\bar{\gamma}^3$ and $\bar{\gamma}^{10}$ are proportional to the identity on both 8-component subspaces, but also the other $\bar{\gamma}^{I\neq3,10} = \bar{\gamma}^M$ matrices have 8×8 off-diagonal blocks (A.1), (A.2). In particular,

$$-\bar{\gamma}^3\epsilon = -i\bar{\gamma}^{10}\epsilon = \epsilon = \begin{pmatrix} \eta \\ 0 \end{pmatrix}\,, \qquad \bar{\gamma}^M\epsilon = \begin{pmatrix} 0 \\ \tilde{\eta}_M \end{pmatrix}\,, \qquad \text{(C.30)}$$

where we recall that $M \neq 3,10$. Note also that the eight $\tilde{\eta}_M$ form a basis of the 8-component spinor subspace since they are orthogonal to each other. Thus, it is easy to notice that the first constraint in (3.7) is satisfied by a set of spinors

$$\nu_{a=1,...,7} = \begin{pmatrix} \tilde{\nu}_a \\ 0 \end{pmatrix}\,, \qquad \tilde{\nu}_a^\top \eta = 0\,. \qquad \text{(C.31)}$$

---

[28]We also see that $\delta_{SO(3)}K_a = \delta_{SO(7)}K_a = 0$

Let us now look at the third constraint. Since $\bar{\gamma}^M$ are off-diagonal, this constraint is trivial for $I = M \neq 3, 10$. For $I = 3, 10$, $\bar{\gamma}^I$ is proportional to the identity on $\tilde{\eta}_a$ and ϵ, and it takes the same value on the left and right-hand-side. The constraint thus reduces to

$$\nu_a^\top \nu_b = \delta_{ab} \,. \tag{C.32}$$

This implies that ν_a have to be orthogonal to each other on the 8-component subspace, as well as being orthogonal to ϵ. In particular, this can only be satisfied with $a = 1, ..., 7$. Finally, the second constraint in (3.7) can be derived from the first and third constraints. This has been proven in [26]. Thus, it is unnecessary to discuss it, but it is not hard to see that it is satisfied by the above solution.

C.3 Constructing the BRST cohomology

In this section, we construct naturally the BRST cohomology described in the main text, justifying along the way why we need all the ghosts appearing in (3.33). This section only serves as some motivation for the construction (3.43).

Let us start by the naive gauge fixing action, with ghosts $(\tilde{C}, C, b)$, which has zero modes in general

$$S_{\rm gh} = {\rm Tr}(ibF + i\tilde{C}[L_j, [X_j, C]]) \,. \tag{C.33}$$

It is known from the BRST formalism that this can be written as

$$S_{\rm gh} = \delta_b V_{\rm gh} \,, \quad V_{\rm gh} = {\rm Tr}\tilde{C}F \,, \tag{C.34}$$

where

$$\delta_b{\rm boson} = [{\rm boson}, C] \,, \quad \delta_b{\rm fermion} = -\{{\rm fermion}, C\} \,, \tag{C.35}$$

$$\delta_b\tilde{C} = ib \,, \quad \delta_b C = -C^2 \,, \quad \delta_b b = 0 \,, \tag{C.36}$$

where "boson" and "fermion" are bosonic and fermionic physical fields. These transformations are constrained from requiring $\delta_b V_{\rm gh} = S_{\rm gh}$, $\delta_b^2 = 0$ and $\delta_b{\rm Tr}({\rm physical~fields}) = 0$.

However, we also want to get rid of the zero modes as discussed in section 3.4.1. Thus, we want the action with three new ghost zero-modes $(C_0, \tilde{C}_0, b_0)$ which commute with L_i,

$$S_{\rm gh} \to S_{\rm gh} = {\rm Tr}(ib(F + b_0) + i\tilde{C}[L_j, [X_j, C]]) + i\tilde{C}C_0 + iC\tilde{C}_0) \,. \tag{C.37}$$

This action is a valid gauge fixing action. We could stop here. However, we want to have a BRST formalism to treat this action on the same footing as the localization deformation. Let us try to write it as $QV_{\rm gh} = (\delta_s + \delta_b)V_{\rm gh}$.

Step 1: Add a ghost First, the new term ${\rm Tr}ibb_0$ is obtained from shifting $F \to F + b_0$ in $V_{\rm gh}$. By defining $\delta_b b_0 = -iC_0$, this also creates the term $\tilde{C}C_0$ along the way, namely

$$\mathrm{Tr}(ib(F + b_0) + i\tilde{C}[L_j, [X_j, C]] + i\tilde{C}C_0) = \delta_b\mathrm{Tr}\tilde{C}(F + b_0) \,, \tag{C.38}$$

with

$$\delta_b{\rm boson} = [{\rm boson}, C] \,, \quad \delta_b{\rm fermion} = -\{{\rm fermion}, C\} \,, \tag{C.39}$$

$$\delta_b\tilde{C} = ib \,, \quad \delta_b C = -C^2 \,, \quad \delta_b b = 0 \,, \quad \delta_b b_0 = -iC_0 \,. \tag{C.40}$$

To create the third term $iC\tilde{C}_0$, we need a zero-mode bosonic field such that $\delta_b(...) = \tilde{C}_0$, but we already used b_0. This forces us to introduce a **new ghost**, which we denote by $\tilde{a}_0$ such that

$$\delta_b\tilde{a}_0 = -i\tilde{C}_0\,. \tag{C.41}$$

Then, $iC\tilde{C}_0 \subset \delta_b(C\tilde{a}_0)$. We thus need to deform $V_{\rm gh}$ to

$$V_{\rm gh} \to V_{\rm gh} = {\rm Tr}(\tilde{C}(F + b_0) + C\tilde{a}_0)\,, \tag{C.42}$$

which is now such that

$$\delta_b V_{\rm gh} = {\rm Tr}(ib(F + b_0) + i\tilde{C}[L_j, [X_j, C]] + i\tilde{C}C_0 + iC\tilde{C}_0 - C^2\tilde{a}_0)\,. \tag{C.43}$$

Step 2: Add another ghost The above is not sufficient since $\tilde{a}_0$ doesn't appear quadratically multiplied to other bosons. The integral over $e^{-S_{\rm gh}}$ would be divergent. This is why we need to introduce yet **another bosonic ghost** a'_0 whose job in life is to multiply $\tilde{a}_0$.[29] Looking at $V_{\rm gh}$, a minimal way is to modify the BRST transformations by adding $ia'_0 \subset \delta_b C$ so that it comes multiplied to $\tilde{a}_0$. Thus, with the deformed BRST transformations,

$$\delta_b\text{boson} = [\text{boson}, C]\,, \quad \delta_b\text{fermion} = -\{\text{fermion}, C\}\,,$$
$$\delta_b\tilde{C} = ib\,, \quad \delta_b C = ia'_0 - C^2\,, \quad \delta_b b = 0\,, \quad \delta_b b_0 = -iC_0\,, \quad \delta_b\tilde{a}_0 = -i\tilde{C}_0\,, \tag{C.44}$$

we obtain

$$\delta_b V_{\rm gh} = {\rm Tr}(ib(F + b_0) + i\tilde{C}[L_j, [X_j, C]] + i\tilde{C}C_0 + iC\tilde{C}_0$$
$$+ i(a'_0 + iC^2)\tilde{a}_0)\,. \tag{C.45}$$

This is exactly the $\delta_b V_{\rm gh}$ part of (3.33). It is now, again, a valid gauge fixing action, because the integral over $\tilde{a}_0$ and a'_0 would reduce to a number. However, we also want to incorporate supersymmetry which is the next step. Before doing that, let us complete the algebra of δ_b. We did not yet write $\delta_b\tilde{C}_0$, $\delta_b C_0$ and $\delta_b a'_0$. Note that, from (C.44),

$$\delta_b^2(\text{physical field}) = i[(\text{physical field}), a'_0]\,. \tag{C.46}$$

Thus, we require $\delta_b^2 = i[\cdot, a'_0]$. This constrains $\delta_b\tilde{C}_0$, $\delta_b C_0$ to involve commutators with a'_0 and also modifies $\delta_b b$, yielding

$$\delta_b\text{boson} = [\text{boson}, C]\,, \quad \delta_b\text{fermion} = -\{\text{fermion}, C\}\,,$$
$$\delta_b C = ia'_0 - C^2\,,$$
$$\delta_b\tilde{C} = ib\,, \qquad \delta_b b = [\tilde{C}, a'_0]\,,$$
$$\delta_b\tilde{a}_0 = -i\tilde{C}_0\,, \qquad \delta_b\tilde{C}_0 = -[\tilde{a}_0, a'_0]\,,$$
$$\delta_b b_0 = -iC_0\,, \qquad \delta_b C_0 = -[b_0, a'_0]\,,$$
$$\delta_b a'_0 = 0\,. \tag{C.47}$$

This gives precisely (3.39).

---

[29]a'_0 is not yet a_0, because we will perform a shift as described in the last step.

Step 3: Include SUSY We now include δ_s by promoting

$$\delta_b V_{\mathrm{gh}} \to Q V_{\mathrm{gh}} \equiv (\delta_s + \delta_b) V_{\mathrm{gh}} \,, \tag{C.48}$$

to treat V_{gh} on the same footing as V_0. The operation of δ_s on the ghosts must be such that Q^2 acts on the same way on all matrices, reducing to some global $+$ gauge symmetry generator. Assuming that $\delta_s(\text{all ghosts}) = 0$ is inconsistent, because this would give $Q^2 \text{ghosts} = i[\text{ghosts}, a_0']$ whereas, for example $Q^2 X_{I'} = i[X_{I'}, a_0'] - i\Omega(\delta_\phi + \delta_{U(1)})X_{I'}$. A solution is to impose that

$$\delta_s C = \phi\,, \qquad \delta_s(\text{other ghosts}) = 0\,. \tag{C.49}$$

The resulting cohomology is consistent, as described in the main text, and is such that

$$\begin{aligned}
QV_{\mathrm{gh}} = \mathrm{Tr}\Big(ib(F[X] + b_0) + i\tilde{C}[L_j, [X_j, C]] + i\tilde{C}C_0 + iC\tilde{C}_0 \\
+ i\left(a_0' - i\phi + iC^2\right)\tilde{a}_0 + \tilde{C}[L_j, \psi_j]\Big)\,,
\end{aligned} \tag{C.50}$$

$$Q^2 = -i\Omega\delta_{U(1)} + i[\cdot, a_0']\,. \tag{C.51}$$

Step 4: Shift a_0' Finally, we choose to define $a_0' \equiv a_0 + i\frac{3\Omega}{8}L_3$ where $[a_0, L_i] = 0.$[30] To understand why we do this shift, note that a_0' and the fluctuations of X_{10} (the components of X_{10} which are orthogonal to M) appear in our localization computation through the variable

$$\tilde{\phi} = a_0' - i\phi + iC^2\,. \tag{C.52}$$

At the saddle point, $\phi = \frac{3\Omega}{8}L_3 - iM$ and the integral of $\tilde{a}_0$ sets $\tilde{\phi} = 0$. It is thus important that $a_0' = M + \frac{3\Omega}{8}L_3$ at the saddle point. This is only consistent with the shift we mentioned above.[31]

C.4 Explicit formulae for the localization deformation

Here, we write explicit formulae for V_0 and V, which are useful in order to understand the arguments underlying the claimed form of the deformation (3.55).

In the spinor basis (3.11) the supersymmetry acts as (3.15), implying

$$(\delta_s\psi_{I'})^\dagger = i[\bar{\phi}, X_{I'}] + \Omega\delta_{U(1)}X_{I'}\,, \qquad (\delta_s\chi_a)^\dagger = -iH_a^\dagger\,, \tag{C.53}$$

where $\bar{\phi} = X_3 + iX_{10}$. Thus,

$$V_0 = \mathrm{Tr}(\psi_{I'}(\delta_s\psi_{I'})^\dagger + \chi_a(\delta_s\chi_a)^\dagger) = \mathrm{Tr}\Big(\psi_{I'}(i[\bar{\phi}, X_{I'}] + \Omega\delta_{U(1)}X_{I'}) - iH_a^\dagger\chi_a\Big)\,. \tag{C.54}$$

In fact,

$$H_a^\dagger = H_a + 2is_a\,. \tag{C.55}$$

---

[30]Using (C.47), $[L_i, b_0] = 0$ and the Jacobi identity, one can show $[b_0, [a_0', L_i]] = 0$. Therefore, this choice is consistent with the BRST transformation (C.47).

[31]Also note that this choice is important for the property (C.60) to hold. If, at the saddle point, $a_0' = M$, this property would be spoiled and (3.55) would not be quadratic in the fluctuations.

Performing the change of variables (3.49) and introducing the ghost potential V_{gh}, we find that

$$
\begin{aligned}
V = V_0 + V_{\text{gh}} = \text{Tr}\bigg(& (\tilde{\psi}_{I'} - i[X_{I'}, C])(i[\bar{\phi}, X_{I'}] + \Omega \delta_{U(1)} X_{I'}) \\
& - i\tilde{H}_a \chi_a - \{\chi_a, C\}\chi_a + 2s_a \chi_a - i\tilde{C}[L_i, X_i] + \tilde{c}b_0 + c\tilde{a}_0 \bigg).
\end{aligned}
\tag{C.56}
$$

When expanding around the saddle, we need to make sure that there are no constant or linear piece in the fluctuation as claimed in arguing for (3.55). There are of course no constant terms because of the presence of fermions (which are all set to 0 at the saddle point). To make sure that there is no linear term we need to make sure that every term multiplying fermions does vanish at the saddle point. It is straight-forward to verify this. We indeed obtain that at the saddle, $X_i = \frac{3\Omega}{8} L_i$, $\tilde{H}_a = \tilde{a}_0 = b_0 = 0$, $\bar{\phi} = \frac{3\Omega}{8} L_3 + iM$,

$$
i[\bar{\phi}, X_i] + \Omega \delta_{U(1)} X_i|_{\text{saddle}} = 0,
\tag{C.57}
$$

$$
s_a|_{\text{saddle}} = 0.
\tag{C.58}
$$

Finally, let us verify the claim $RZ = R_0 \tilde{Z} + \text{subleading}$, where $\tilde{Z}$ is a notation for all the fluctuations and we remind the reader that

$$
R = -i\Omega \delta_{U(1)} + i\left[\cdot, a_0 + i\frac{3\Omega}{8} L_3\right], \qquad R_0 = -i\Omega \delta_{U(1)} + i\left[\cdot, M + i\frac{3\Omega}{8} L_3\right].
\tag{C.59}
$$

This claim is not obvious for matrices which are expanded around a non-trivial constant. This is only the case for X_i which are expanded around $\frac{3\Omega}{8} L_i$, that could generate a constant term. However, we have that

$$
\begin{aligned}
RX_i &= -i\left(\frac{3\Omega}{8}\right)^2 \epsilon_{3ij} L_j + i\left[\frac{3\Omega}{8} L_i, M + i\frac{3\Omega}{8} L_3\right] + R_0 \tilde{X}_i + \text{subleading} \\
&= -i\left(\frac{3\Omega}{8}\right)^2 \epsilon_{3ij} L_j - \left(\frac{3\Omega}{8}\right)^2 [L_i, L_3] + R_0 \tilde{X}_i + \text{subleading} = R_0 \tilde{X}_i + \text{subleading}.
\end{aligned}
\tag{C.60}
$$

C.5 Eigenvalues and the 1-loop determinant

In this section, we compute the 1-loop determinant[32]

$$
\tilde{Z}_{1-\text{loop}} = \Delta \left(\frac{\det_{\tilde{Z}_f} R_0}{\det_{\tilde{Z}_b} R_0}\right)^{1/2},
\tag{C.61}
$$

where Δ is the Vandermonde determinant (C.84) appearing from the diagonalization of M. As discussed in the main section, computing these determinants amounts to compute the eigenvalues of R_0 on the vector spaces spanned by the matrices $\tilde{Z}_b$ and $\tilde{Z}_f$, where we remind the reader that

$$
Z_b \equiv (X_{I'}, \tilde{a}_0, b_0), \quad Z_f \equiv (\chi_a, C, \tilde{C}).
\tag{C.62}
$$

---

[32] We reserve the notation $Z_{1-\text{loop}}$ for a nicely rescaled expression.

To do it, we first define

$$B_1 \equiv X_1 - iX_2\,, \qquad B_2 \equiv X_4 - iX_5\,,$$
$$B_3 \equiv X_6 - iX_7\,, \qquad B_4 \equiv X_8 - iX_9\,, \tag{C.63}$$
$$\xi_1 \equiv \chi_1 + i\chi_6\,, \qquad \xi_2 \equiv \chi_2 - i\chi_5\,, \qquad \xi_3 \equiv \chi_3 + i\chi_4\,.$$

We group the matrices in the following way,

$$
\begin{aligned}
(1) \; &: (B_2, B_3, B_4, B_2^\dagger, B_3^\dagger, B_4^\dagger), (\xi_1, \xi_2, \xi_3, \xi_1^\dagger, \xi_2^\dagger, \xi_3^\dagger)\,, \\
(2) \; &: (B_1, B_1^\dagger), (C, \tilde{C})\,, \\
(3) \; &: (X_3), (\chi_7)\,, \\
(4) \; &: (\tilde{a}_0, b_0)\,.
\end{aligned}
\tag{C.64}
$$

These are just a rewriting of the $9 + 2$ bosonic fields in Z_b and the $7 + 2$ fermions of Z_f. With this rewriting,

$$-i\Omega\delta_{U(1)}B_1 = \frac{3\Omega}{8}B_1\,, \quad -i\Omega\delta_{U(1)}B_{i=2,3,4} = \frac{\Omega}{8}B_i\,, \quad -i\Omega\delta_{U(1)}\xi_i = \frac{\Omega}{4}\xi_i\,, \tag{C.65}$$

$$-i\Omega\delta_{U(1)}(\text{other matrices}) = 0\,. \tag{C.66}$$

We recall that

$$R_0 \equiv -i\Omega\delta_{U(1)} - i\left[M + i\frac{3\Omega}{8}L_3, \cdot\right]\,. \tag{C.67}$$

To diagonalize $[L_3, \cdot]$, we use the fuzzy sphere harmonics expansion (see e.g. [42, Sec.4.3]) for all our (non-zero mode) matrices. More precisely, we split the matrices Φ into rectangular blocks $\Phi^{(s,t)}$, and expand each block as

$$\Phi^{(s,t)} = \sum_{J=|j_s - j_t|}^{j_s + j_t} \sum_{m=-J}^{J} \Phi_{Jm}^{(s,t)} \otimes Y_{Jm(j_s, j_t)}\,. \tag{C.68}$$

In terms of matrix dimension we have ($N_{s/t} = 2j_{s/t} + 1$, no sum in s, t)

$$\Phi^{(s,t)}: \; n_s N_s \times n_t N_t\,, \qquad \Phi_{Jm}^{(s,t)}: \; n_s \times n_t\,, \qquad Y_{Jm(j_s, j_t)}: \; N_s \times N_t\,. \tag{C.69}$$

The fuzzy sphere harmonics $Y_{Jm(j_s, j_t)}$ satisfy

$$
\begin{aligned}
\hat{L}_{\pm}Y_{Jm(j_s, j_t)} &= \sqrt{(J \mp m)(J \pm m + 1)}Y_{Jm(j_s, j_t)}\,, \\
\hat{L}_3 Y_{Jm(j_s, j_t)} &= mY_{Jm(j_s, j_t)}\,,
\end{aligned}
\tag{C.70}
$$

where $\hat{L}$'s denote operators rather than matrices (see [42] for more details). It follows that under the action of $[L_3, \cdot]$, $\Phi_{Jm}^{(s,t)} \to m\Phi_{Jm}^{(s,t)}$. Also note that for diagonal $M = \text{diag}(m_{si}, s = 1, ..., q, i = 1, ..., n_s)$,

$$([M, \Phi]_{Jm}^{(s,t)})_{ij} = (m_{si} - m_{tj})(\Phi_{Jm}^{(s,t)})_{ij}\,. \tag{C.71}$$

Finally, we wrote matrices in a diagonal form with respect to $-i\Omega\delta_{U(1)}$. Let us write

$$-i\Omega\delta_{U(1)}\Phi \equiv \Omega r_{U(1)}^{(\Phi)}\Phi\,. \tag{C.72}$$

Thus, R_0 acts diagonally on $\Phi_{Jm}^{(s,t)}$, and we obtain

$$(R_0\Phi)^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{m=-J}^{J} \left[\Omega \left(r_{U(1)}^{(\Phi)} + \frac{3}{8}m \right) \Phi_{Jm}^{(s,t)} - i[M, \Phi]_{Jm}^{(s,t)} \right] \otimes Y_{Jm(j_s,j_t)}. \quad \text{(C.73)}$$

Thus, all the components $\left(\Phi_{Jm}^{(s,t)} \right)_{ij}$ are eigenvectors with eigenvalue

$$\Omega \left(r_{U(1)}^{(\Phi)} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}).$$

Let us now consider $B_{i=2,\dots,4}, B_{i=2,\dots,4}^{\dagger}$. They have $r_{U(1)} = \frac{\Omega}{8}$ and $r_{U(1)} = -\frac{\Omega}{8}$ respectively. Thus, they contribute

$$\det{}_{B_{i=2,3,4}, B_{i=2,3,4}^{\dagger}} R_0$$
$$= \prod_{s,t} \prod_J \prod_m \prod_{i,j} \left(\Omega \left(\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(-\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3. \quad \text{(C.74)}$$

For the 6 fermions $\xi_{i=1,2,3}, \xi_{i=1,2,3}^{\dagger}$, since $r_{U(1)} = \pm\frac{\Omega}{4}$, we obtain

$$\det{}_{\xi_{i=1,2,3}, \xi_{i=1,2,3}^{\dagger}} R_0$$
$$= \left(\Omega \left(-\frac{1}{4} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(\frac{1}{4} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3. \quad \text{(C.75)}$$

Some nice cancellations of the m product happens when dividing the fermionic part by the bosonic, namely

$$\prod_{m=-J}^{J} \frac{\left(\Omega \left(-\frac{1}{4} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(\frac{1}{4} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3}{\left(\Omega \left(\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(-\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3}$$
$$= \prod_{m=-J}^{J} \frac{\left(\Omega \left(\frac{1}{8} + \frac{3}{8}(m-1) \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(-\frac{1}{8} + \frac{3}{8}(m+1) \right) - i(m_{si} - m_{tj}) \right)^3}{\left(\Omega \left(\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3 \left(\Omega \left(-\frac{1}{8} + \frac{3}{8}m \right) - i(m_{si} - m_{tj}) \right)^3}. \quad \text{(C.76)}$$

Note that the numerator is just a shift of the denominator, thus leading to many cancellations. The first numerator term and the first denominator term contribute respectively when $m = -J$ and $m = J$, other terms cancelling each other. Similarly, the second numerator term and the second denominator term contribute respectively when $m = J$ and $m = -J$. Thus, the first group contributes

$$\det{}^{(1)} \equiv \frac{\det_{\xi_{i=1,2,3}, \xi_{i=1,2,3}^{\dagger}} R_0}{\det_{B_{i=2,3,4}, B_{i=2,3,4}^{\dagger}} R_0} = \prod_{s,t} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{i,j} \left(\frac{\left(\frac{\Omega}{8}\right)^2 (2+3J)^2 + (m_{si} - m_{tj})^2}{\left(\frac{\Omega}{8}\right)^2 (1+3J)^2 + (m_{si} - m_{tj})^2} \right)^3. \quad \text{(C.77)}$$

For the second group,

$$\det_{B_1,B_1^\dagger} R_0$$
$$= \prod_{s,t} \prod_J \prod_m \prod_{i,j} \left(\Omega\left(\frac{3}{8} + \frac{3}{8}m\right) - i(m_{si} - m_{tj}) \right) \left(\Omega\left(-\frac{3}{8} + \frac{3}{8}m\right) - i(m_{si} - m_{tj}) \right) ,$$

(C.78)

$$\det_{C,\tilde{C}} R_0 = \prod_{s,t} \prod_J \prod_m \prod_{i,j} \left(\frac{3\Omega}{8}m - i(m_{si} - m_{tj}) \right) \left(\frac{3\Omega}{8}m - i(m_{si} - m_{tj}) \right) . \quad \text{(C.79)}$$

Again, we see that they only differ by $m \to m \pm 1$, leading to cancellations. We obtain

$$\det^{(2)} \equiv \frac{\det_{C,\tilde{C}} R_0}{\det_{B_1,B_1^\dagger} R_0} = \prod_{s,t} \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{i,j} \left(\frac{\left(\frac{\Omega}{8}\right)^2 (3J)^2 + (m_{si} - m_{tj})^2}{\left(\frac{\Omega}{8}\right)^2 (3+3J)^2 + (m_{si} - m_{tj})^2} \right) . \quad \text{(C.80)}$$

For the last two groups the story is even simpler. Since X_3 and χ_7 do not transform under the $U(1)$, they have the same R_0 eigenvalues, thus the determinants cancel each other.

$$\det^{(3)} \equiv \frac{\det_{\chi_7} R_0}{\det_{X_3} R_0} = 1 . \quad \text{(C.81)}$$

Finally, the group (4) is a group of bosonic ghost zero modes. They need to commute with L_i, thus they have the same form as M and have components $(\tilde{a}_0)_{s;ij}$, $(b_0)_{s;ij}$ in the diagonal block (s,s), and their R_0 value is

$$(R_0\tilde{a}_0)_{s;ij} = -i(m_{si} - m_{sj})(\tilde{a}_0)_{s;ij} , \qquad (R_0 b_0)_{s;ij} = -i(m_{si} - m_{sj})(b_0)_{s;ij} . \quad \text{(C.82)}$$

Thus,

$$\det^{(4)} \equiv \frac{1}{\det_{\tilde{a}_0,b_0} R_0} = (-1)^{\sum_s n_s^2} \prod_s \prod_{i,j} \frac{1}{(m_{si} - m_{sj})^2} . \quad \text{(C.83)}$$

This precisely cancels the zero-mode $J = 0$ ($s = t$) term in the numerator of $\det^{(2)}$.[33] We also need the Vandermonde determinant from the diagonalization of M. It reads

$$\Delta = \left(\prod_s \prod_{i \neq j} (m_{si} - m_{sj})^2 \right)^{1/2} . \quad \text{(C.84)}$$

This allows again $J = 0$ in the numerator of $\det^{(2)}$, provided $i \neq j$. Putting everything together,

$$\tilde{Z}_{1-\text{loop}} = \Delta \, \det^{(1)}\det^{(2)}\det^{(3)}\det^{(4)}$$

$$= (-1)^{\sum_s n_s^2} \left(\frac{64}{3\Omega} \right)^{\sum_s n_s} \prod_{s,t=1}^q \prod_{J=|j_s-j_t|}^{j_s+j_t} \prod_{\substack{i,j=1 \\ J=0: i\neq j}}^{n_s,n_t}$$

$$\left[\frac{\left[\left(\frac{\Omega}{8}\right)^2 (2+3J)^2 + (m_{si} - m_{tj})^2\right]^3 \left[\left(\frac{\Omega}{8}\right)^2 (3J)^2 + (m_{si} - m_{tj})^2\right]}{\left[\left(\frac{\Omega}{8}\right)^2 (1+3J)^2 + (m_{si} - m_{tj})^2\right]^3 \left[\left(\frac{\Omega}{8}\right)^2 (3+3J)^2 + (m_{si} - m_{tj})^2\right]} \right]^{1/2} .$$

(C.85)

---

[33]One could avoid these 0/0's, by not integrating over the $J = 0$ $i = j$ components of C, $\tilde{C}$, $\tilde{a}_0$ and b.

where the product is not taken over $i = j$ when $J = 0$. Separating $i = j$ and $i \neq j$ contributions,

$$
\tilde{Z}_{1-\text{loop}} = (-1)^{\sum_s n_s^2} \left(\frac{64}{3\Omega} \right)^{\sum_s n_s} \prod_{s=1}^{q} \left(\frac{1}{N_s} \prod_{J=1}^{N_s-1} \frac{(2+3J)^3}{(1+3J)^3} \right)^{n_s} \prod_{s,t=1}^{q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n_s,n_t} \prod_{J=|j_s-j_t|}^{j_s+j_t}
$$

$$
\left[\frac{\left[\left(\frac{\Omega}{8}\right)^2 (2+3J)^2 + (m_{si} - m_{tj})^2 \right]^3 \left[\left(\frac{\Omega}{8}\right)^2 (3J)^2 + (m_{si} - m_{tj})^2 \right]}{\left[\left(\frac{\Omega}{8}\right)^2 (1+3J)^2 + (m_{si} - m_{tj})^2 \right]^3 \left[\left(\frac{\Omega}{8}\right)^2 (3+3J)^2 + (m_{si} - m_{tj})^2 \right]} \right]^{1/2}.
$$
(C.86)

The partition function around the saddle $\mathcal{R}$ is thus given by

$$
Z_{\mathcal{R}} \propto e^{\frac{\Omega^4}{g_{\text{YM}}^2} \frac{9}{8192} \text{Tr} L_i^2} \int \prod_{s=1}^{q} \prod_{i=1}^{n_s} dm_{si} \tilde{Z}_{1-\text{loop}} \exp\left(-\frac{3\Omega^2}{2^7} \sum_{s=1}^{q} \sum_{i=1}^{n_s} (2j_s+1) m_{si}^2 \right). \quad \text{(C.87)}
$$

Note that we can get rid of the Ω-dependence in $\tilde{Z}_{1-\text{loop}}$ by the transformation $m_{si} \to \Omega m_{si}$. The full partition function is thus given by

$$
Z = \sum_{\mathcal{R}} C_{\mathcal{R}} e^{\frac{9\Omega^4}{2^{13}} \text{Tr} L_i^2} \int \prod_s \prod_{i=1}^{n_s} dm_{si} Z_{1-\text{loop}} \exp\left(-\frac{3\Omega^4}{2^7} \sum_s \sum_{i=1}^{n_s} N_s m_{si}^2 \right), \quad \text{(C.88)}
$$

where

$$
Z_{1-\text{loop}} = \prod_{s,t=1}^{q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n_s,n_t} \prod_{J=|j_s-j_t|}^{j_s+j_t}
$$

$$
\left[\frac{\left[\left(\frac{1}{8}\right)^2 (2+3J)^2 + (m_{si} - m_{tj})^2 \right]^3 \left[\left(\frac{1}{8}\right)^2 (3J)^2 + (m_{si} - m_{tj})^2 \right]}{\left[\left(\frac{1}{8}\right)^2 (1+3J)^2 + (m_{si} - m_{tj})^2 \right]^3 \left[\left(\frac{1}{8}\right)^2 (3+3J)^2 + (m_{si} - m_{tj})^2 \right]} \right]^{1/2}.
$$
(C.89)

The proportionality constant $C_{\mathcal{R}}$ is determined in the following appendix C.6, where we have absorbed the factor $(-1)^{\sum_s n_s^2} \left(\frac{64}{3\Omega} \right)^{\sum_s n_s} \prod_{s=1}^{q} \left(\frac{1}{N_s} \prod_{J=1}^{N_s-1} \frac{(2+3J)^3}{(1+3J)^3} \right)^{n_s}$.

C.6 Normalization factors

For bosonic X and fermionic ψ matrices we use the measures

$$
[dX] \equiv \prod_{i,j} d\tilde{X}_{ij} = 2^{\frac{N^2-N}{2}} \prod_{i \leq j} d\text{Re} X_{ij} \prod_{i < j} d\text{Im} X_{ij},
$$

$$
[d\psi] \equiv \prod_{i,j} d\tilde{\psi}_{ij} = 2^{-\frac{N^2-N}{2}} \prod_{i \leq j} d\text{Re}\psi_{ij} \prod_{i < j} d\text{Im}\psi_{ij},
$$
(C.90)

where $\tilde{X}$ and $\tilde{\psi}$ are matrices with N^2 real entries which contain all the information about the Hermitian matrices.

$$\tilde{X}_{ij} = \mathrm{Re}X_{ij} + \mathrm{Im}X_{ij}\,, \tag{C.91}$$
$$\tilde{\psi}_{ij} = \mathrm{Re}\psi_{ij} + \mathrm{Im}\psi_{ij}\,.$$

Indeed, since $\mathrm{Re}X_{ij}$ is symmetric and $\mathrm{Im}X_{ij}$ is antisymmetric, they can be obtained from $\tilde{X}_{ij}$ by taking its symmetric and antisymmetric parts. This measure satisfies the nice property that

$$\int [dX]e^{-\lambda \mathrm{Tr}X^2} = \int \prod_{i,j} d\tilde{X}_{ij} e^{-\lambda \sum_{i,j} \tilde{X}_{ij}^2} = \left(\frac{\pi}{\lambda}\right)^{N^2/2}\,, \tag{C.92}$$

$$\int [d\eta][d\psi]e^{-\mathrm{Tr}\eta\psi} = 1\,. \tag{C.93}$$

Note that it is the same as the $U(N)$ measure, namely, if we expand the matrices X and ψ in $U(N)$ generator components $X = X^A T^A$, $\psi = \psi^A T^A$ such that $\mathrm{Tr}T^A T^B = \delta^{AB}$, the measure is

$$[dX] = \prod_{A=1}^{N^2} dX^A\,, \qquad [d\psi] = \prod_{A=1}^{N^2} d\psi^A\,. \tag{C.94}$$

Indeed, this measure has the same Gaussian integrals (C.92), (C.93). Let us consider a more complicated example of Grassmann integrals, where we have an interpolating matrix M_{ab}^{ij} which is antisymmetric in the exchange $(a,i) \leftrightarrow (b,j)$. One obtains

$$\int \prod_{i=1}^{n_f} [d\psi^i] e^{-\sum_{a,b,i,j} \psi_{ab}^i M_{ab}^{ij} \psi_{ba}^j} = \prod_{a,b} \mathrm{Pf}(2M_{ab}) = 2^{\frac{n_f}{2}N^2} \prod_{a,b} (\det M_{ab})^{1/2}\,. \tag{C.95}$$

where the Pfaffian and determinant of M_{ab}^{ij} is taken over the i,j indices with a,b fixed. Note that we also had zero-mode ghosts $a_0, b_0, \ldots$. They are all expressed in terms of n_s by n_s Hermitian matrices. We thus define the measure similarly for each s. The result is the same by replacing $N^2 \to \sum_s n_s^2$. Now we get the following factors.

First, we find that

$$\int [dC][d\tilde{C}][db][dC_0][d\tilde{C}_0][db_0][da_0][d\tilde{a}_0] \exp\Bigg\{ \mathrm{Tr}\Big(ib(F + b_0) + i\tilde{C}[L_i, [X_i, C]]$$
$$+ i\tilde{C}C_0 + iC\tilde{C}_0 + i\left(a_0 + i\frac{3\mu}{8}L_3 - i\phi + iC^2\right)\tilde{a}_0 \Big) \Bigg\} \tag{C.96}$$
$$= (-1)^{\sum_s n_s^2}(2\pi)^{N^2}(2\pi)^{\sum_s n_s^2}\det'\left(\frac{\delta F}{\delta \alpha}\right)\Bigg|_{\alpha=0}\delta'(F)\,.$$

The factor $(-1)^{\sum_s n_s^2}$ comes from integration over $C_0, \tilde{C}_0$. The factor $(2\pi)^{N^2}$ comes from the integral over b and b_0, whereas the last factor comes from $\int[da_0][d\tilde{a}_0]e^{i\mathrm{Tr}(a_0+\ldots)\tilde{a}_0}$. Thus, when using the Fadeev-Popov trick (3.47), one should divide by these factors

$$\det'\left(\frac{\delta F}{\delta \alpha}\right)\delta'(F) = \frac{(-1)^{\sum_s n_s^2}}{(2\pi)^{N^2}(2\pi)^{\sum_s n_s^2}}\int [d\mathrm{ghosts}]e^{-tS_{\mathrm{gh}}}\,. \tag{C.97}$$

Secondly, we find using (C.92), (C.95) that

$$\int [dZ_b][dZ_b'][dZ_f][dZ_f']e^{-tQ(V_0+V_{\text{gh}})} = \pi^{9N^2}\pi^{\sum_s n_s^2}2^{9N^2}2^{\sum_s n_s^2}\left(\frac{\det_{Z_f}R_0}{\det_{Z_b}R_0}\right)^{1/2}. \quad \text{(C.98)}$$

Thirdly, we have two factors from "gauge fixing". First, since $\mathcal{G}_{\mathcal{R}} = U(N)/\otimes_s U(n_s)$, we get the volume factor appearing in the gauge fixing procedure

$$\text{Vol }\mathcal{G}_{\mathcal{R}} = \frac{\text{Vol }U(N)}{\prod_s \text{Vol }U(n_s)}, \quad \text{(C.99)}$$

where

$$\text{Vol}(U(N)) = \frac{(2\pi)^{(N^2+N)/2}}{\prod_{k=1}^{N-1} k!}. \quad \text{(C.100)}$$

We are also diagonalizing M. Thus, we are basically gauge fixing each of its blocks $s = 1, ..., q$ using $SU(n_s)$, up to $U(1)^{n_s-1}$ remaining gauge redundancy. However, we don't gauge fix the permutation symmetry. This would correspond to gauge fixing, for example, $m_{si} < m_{sj} \,\forall i < j$. This means we need to divide by $n_s!$. Thus, the diagonalization of M reads

$$\int [dM] = \prod_s \left(\frac{1}{n_s!}\frac{\text{Vol }U(n_s)}{\text{Vol }U(1)^{n_s}}\right)\int \prod_{s,i} dm_{si}\Delta(m). \quad \text{(C.101)}$$

Combining these two volume factors we obtain

$$\frac{(2\pi)^{(N^2+N)/2-\sum_s n_s}}{\prod_{k=1}^{N-1} k!\prod_s n_s!}. \quad \text{(C.102)}$$

Now note that this is the path integral including the auxiliary fields K_a. The auxiliary fields contribute $(2\pi)^{7N^2/2}$ to the path integral. Thus,

$$\int [dX_a][d\psi_\alpha]e^{-S} = (2\pi)^{-7N^2/2}\int [dX_a][d\psi_\alpha][dK_a]e^{-S|_{\text{with auxiliaries}}}. \quad \text{(C.103)}$$

Combining all these factors (C.97), (C.98), (C.102), (C.103), as well as absorbing the factor $(-1)^{\sum_s n_s^2}(64/3)^{\sum_s n_s}\prod_{s=1}^q \left(\frac{1}{N_s}\prod_{J=1}^{N_s-1}\frac{(2+3J)^3}{(1+3J)^3}\right)^{n_s}$ in (C.85) we obtain

$$\begin{aligned}
C_{\mathcal{R}} &= \frac{(2\pi)^{9N^2/2+(N^2+N)/2-\sum_s n_s}}{\prod_{k=1}^{N-1} k!\prod_s n_s!}\left(\frac{64}{3}\right)^{\sum_s n_s}\prod_{s=1}^q \left(\frac{1}{N_s}\prod_{J=1}^{N_s-1}\frac{(2+3J)^3}{(1+3J)^3}\right)^{n_s} \\
&= \frac{(2\pi)^{5N^2+N/2}}{\prod_{k=1}^{N-1} k!\prod_s n_s!}\left(\frac{32}{3\pi}\right)^{\sum_s n_s}\prod_{s=1}^q \left(\frac{1}{N_s}\prod_{J=1}^{N_s-1}\frac{(2+3J)^3}{(1+3J)^3}\right)^{n_s},
\end{aligned} \quad \text{(C.104)}$$

as claimed in the introduction (1.8).

D Approximate solutions to the electrostatic problem

In this appendix we solve the electrostatic problem (5.25) and (5.31) (or equivalently (5.10) and (5.12)) with a single disk in the limits $z_s/R_s \gg 1$ and $z_s/R_s \ll 1$. We also solve the general problem with one disk numerically.

D.1 Analytics

As usual in matrix models, it is convenient to introduce resolvents

$$G_s(z) \equiv \int_{-R_s}^{R_s} dr \frac{f_s(r)}{z - r} = \frac{Q_s}{z} + O\left(\frac{1}{z^3}\right) . \tag{D.1}$$

Notice that $G_s(-z) = -G_s(z)$ because $f_s(r) = f_s(-r)$. Equation (5.10) can be written as

$$
\begin{aligned}
\tilde{V}_s &= V_b 3r^2 z_s - \sum_{t+\text{images}} \frac{1}{8\pi^2} \mathcal{P}\!\!\int_{-R_t}^{R_t} dr' f_t(r') \left[\frac{1}{(r'-r-i(z_s-z_t))^2} + \frac{1}{(r'-r+i(z_s-z_t))^2} \right] \\
&= V_b 3r^2 z_s - \sum_{t+\text{images}} \frac{1}{8\pi^2} \oint \frac{dr'}{2\pi i} G_t(r') \left[\frac{1}{(r'-r-i(z_s-z_t))^2} + \frac{1}{(r'-r+i(z_s-z_t))^2} \right] \\
&= V_b 3r^2 z_s + \sum_{t+\text{images}} \frac{1}{8\pi^2} \left[G'_t(r+i(z_s-z_t)) + G'_t(r-i(z_s-z_t)) \right] , \tag{D.2}
\end{aligned}
$$

where $\tilde{V}_s = V_s + V_b z_s^3$, $\oint$ denotes the contour around the cut of G_t and in the last line we opened the contour and picked up the poles of the integrand.

The last equation tell us that $G'_s(z+i\epsilon) + G'_s(z-i\epsilon)$ is analytic in a region around the branch point of $G_s(z)$ at $z = R_s$. Therefore, we must have an expansion in half integers powers, i.e. $G_s(z) = \sum_{n=0}^{\infty} c_n (z - R_s)^{\frac{n}{2}}$ with $c_n \in \mathbb{R}$. It is then convenient to change variable to remove the branch points at $z = \pm R_s$. More precisely, let us map the z complex plane minus the cut $[-R_s, R_s]$ to the interior of the unit disk using the Zhukovsky map[34]

$$\frac{2z}{R_s} = w + \frac{1}{w} , \qquad H_s(w) \equiv G'_s(z) . \tag{D.3}$$

By construction, $H_s(w)$ is analytic for $|w| < 1$. In addition, it is an even function, obeying $H_s(w) = H_s(-w) = -4Q_s/R_s^2 w^2 + O(w^4)$ Equation (D.2), can then be written as

$$\tilde{V}_s = V_b \frac{3}{4} R_s^2 z_s \left(w + \frac{1}{w}\right)^2 + \sum_{t+\text{images}} \frac{1}{8\pi^2} \left[H_t(w_t^+) + H_t(w_t^-) \right] , \tag{D.4}$$

where

$$2\frac{z}{R_s} = w + \frac{1}{w} , \qquad 2\frac{z \pm i(z_s - z_t)}{R_t} = w_t^{\pm} + \frac{1}{w_t^{\pm}} . \tag{D.5}$$

For $|w| = 1$ we choose the solutions obeying $|w_t^{\pm}| < 1$. For the special case $t = s$, we choose $w_s^+ = w$ and $w_s^- = 1/w$. Equation (D.4) is valid for $|w| = 1$ but since it is analytic in w we can analytically continue it. Moving w slightly outside the unit disk, we can express $H_s(w)$ in terms of analytic functions and a linear combination of $H_t(w_t^{\pm})$ evaluated at $|w_t^{\pm}| < 1$. This allows us to expand the region of analyticity of $H_s(w)$ from the unit disk to a strictly bigger region. Therefore, the series expansion of $H_s(w)$ around $w = 0$ has a radius of convergence larger than 1.

---

[34] We choose the solution $w = z/R_s - \sqrt{(z/R_s)^2 - 1}$ so that w is inside the unit disk.

Let us consider the special case of only one ball. Then

$$\tilde{V}_s = V_b \frac{3}{4} R_s^2 z_s \left(w + \frac{1}{w} \right)^2 + \frac{1}{8\pi^2} \left[H_s(w) + H_s(1/w) - H_s(w_{\tilde{s}}^+) - H_s(w_{\tilde{s}}^-) \right] , \qquad (D.6)$$

with

$$w_{\tilde{s}}^\pm + \frac{1}{w_{\tilde{s}}^\pm} = w + \frac{1}{w} \pm i\frac{4z_s}{R_s} . \qquad (D.7)$$

We don't know how to solve this equation analytically in full generality but the limits of $z_s/R_s \gg 1$ and $z_s/R_s \ll 1$ are tractable. Let us start with the simpler case $z_s/R_s \gg 1$. Then, $w_{\tilde{s}}^\pm = \mp i\frac{R_s}{4z_s} + O(R_s/z_s)^2 \ll 1$ and we can neglect the effect of the image ball. In this limit, we find

$$\tilde{V}_s = V_b \frac{3}{4} R_s^2 z_s \left(w + \frac{1}{w} \right)^2 + \frac{1}{8\pi^2} \left[H_s(w) + H_s(1/w) \right] , \qquad (D.8)$$

whose solution is simply

$$H(w) = -6\pi^2 V_b R_s^2 z_s w^2 , \qquad \tilde{V}_s = V_b \frac{3}{2} R_s^2 z_s \qquad Q_s = \frac{3}{2}\pi^2 V_b R_s^4 z_s . \qquad (D.9)$$

In this approximation we can also compute the octopole

$$q_{3,s} = 2z_s^3 Q_s - 6z_s \int_{-R_s}^{R_s} dr\, r^2 f_s(r) \approx 2Q_s z_s^3. \qquad (D.10)$$

Let us now consider the limit $z_s/R_s \ll 1$. In this limit, we expect G_s to vary slowly on the length scale z_s. Therefore, we can approximate the exact equation

$$\tilde{V}_s = V_b 3r^2 z_s + \frac{1}{8\pi^2} \left[G_s'(r + i\epsilon) + G_s'(r - i\epsilon) - G_s'(r + 2iz_s) - G_s'(r - 2iz_s) \right] ,$$

with

$$\tilde{V}_s = V_b 3r^2 z_s - \frac{2iz_s}{8\pi^2} \left[G_s''(r + i\epsilon) - G_s''(r - i\epsilon) + O(z_s) \right] , \qquad (D.11)$$

which gives us the discontinuity of G_s''. From the definition, we find that

$$G_s''(z) = \int_{-R_s}^{R_s} dr\, f_s(r) \frac{d^2}{dz^2} \frac{1}{z - r} = \int_{-R_s}^{R_s} dr\, f_s(r) \frac{d^2}{dr^2} \frac{1}{z - r} = \int_{-R_s}^{R_s} dr\, f_s''(r) \frac{1}{z - r} , \quad (D.12)$$

where we used that $f_s(\pm R_s) = f_s'(\pm R_s) = 0$. Therefore, equation (D.11) gives

$$f_s''(r) \approx 2\pi \left(V_b 3r^2 - \frac{\tilde{V}_s}{z_s} \right) , \qquad (D.13)$$

which leads to

$$f_s(r) \approx -\pi \frac{\tilde{V}_s}{z_s} r^2 + \frac{\pi}{2} V_b r^4 + const. \qquad (D.14)$$

$$= \frac{\pi}{2} V_b \left(R_s^2 - r^2 \right)^2 , \qquad (D.15)$$

where we imposed $f_s(\pm R_s) = f'_s(\pm R_s) = 0$ to determine the integration constant and $\tilde{V}_s = V_b z_s R_s^2$. Notice that this approximation fails for r close to R_s because it does not give the expected behavior $f_s(r) \sim (R_s - r)^{\frac{3}{2}}$. However, this is a small effect at the tip that does not affect the leading behavior of the macroscopic quantities. The total charge is then

$$Q_s = \int_{-R_s}^{R_s} dr f_s(r) = \frac{8\pi}{15} V_b R_s^5. \tag{D.16}$$

The octopole is given by

$$q_{3,s} = 2z_s^3 Q_s - 6z_s \int_{-R_s}^{R_s} dr r^2 f_s(r) \approx -\frac{16\pi}{35} V_b z_s R_s^7. \tag{D.17}$$

The on-shell action is then

$$S_{\text{on-shell}} = \frac{1}{3\pi^4 g_s^2 \alpha'^4 \mu^5} \sum_s \left(\frac{2}{V_b} Q_s V_s - q_{3,s} \right) = \frac{5}{7 \cdot 2^7} \left(\frac{15\pi}{8N_s} \right)^{2/5} \Omega^{12/5} N^{7/5}, \tag{D.18}$$

where we used $V_b = \mu^5/2^7$, $N = N_s n_s$, $\mu = \Omega\sqrt{g_{\text{YM}}}$, $g_s = (2\pi)^3 \alpha'^2 g_{\text{YM}}^2$ and the relations (5.41).

D.2 Numerics

We want to solve the equation

$$V_{\text{ball},s}(r, z_s) + V_{\text{image},s}(r, z_s) + V_{bg}(r, z_s) = V_s, \qquad 0 \leq r \leq R_s. \tag{D.19}$$

The potential due to the ball reads

$$V_{\text{ball},s}(r, z) = \frac{1}{4\pi r} \int_0^{R_s} du u \sigma_s(u) \log \left(1 + \frac{4ru}{(r-u)^2 + (z-z_s)^2} \right). \tag{D.20}$$

We can combine the contributions from the ball and its image, giving

$$V_{\text{ball},s}(r, z_s) + V_{\text{image},s}(r, z_s) = \frac{1}{4\pi r} \text{ p. v.} \int_0^{R_s} du u \sigma_s(u) \log \left(\frac{(r+u)^2}{(r-u)^2 \left(1 + \frac{4ru}{(r-u)^2 + 4z_s^2} \right)} \right). \tag{D.21}$$

To solve the equation numerically we can write $\sigma(r)$ as an expansion into a complete basis of functions $f_n(r)$ on $[0, R_s]$. A convenient choice are the shifted Legendre polynomials. We can then expand

$$\sigma_s(r) = \sum_{n=0}^{N_{max}} c_n P_n \left(2\frac{r}{R_s} - 1 \right). \tag{D.22}$$

From the discussion of the previous section (see also the near-tip expansion in [35]) we expect the charge density to behave as $\sqrt{R_s^2 - r^2}$ close to the tip. This is hard to reproduce with a finite number of polynomials, so to improve the convergence we consider

$$\sigma_s(r) = \sum_{n=0}^{N_{max}} c_n P_n \left(2\frac{r}{R_s} - 1 \right) + \tilde{c}\sqrt{R_s^2 - r^2}, \tag{D.23}$$

where $\tilde{c}$ is a new free parameter. To impose the equations of motion (D.19) we take a uniform grid with N_{grid} points on the segment $[0, R_s]$. We call r_i the points of that grid. We construct the constraint function

$$\text{constr}^{(EOM)} = \sum_{i=1}^{N_{grid}} (\text{EOM}(r_i))^2, \tag{D.24}$$

which is always positive and vanishes on the physical solution. Our strategy is to do this minimization numerically. In addition we want to impose that σ vanishes at $r = R_s$ and has zero derivative at $r = 0$. We can easily implement this by modifying the constraint function. The numerical problem we solve is then[35]

$$\underset{\{c_i, \tilde{c}, V_s\}}{\text{Minimize}} \left[\sum_{i=1}^{N_{grid}} (\text{EOM}(r_i))^2 + \sigma_s(R_s)^2 + \sigma_s'(0)^2 \right] \tag{D.25}$$

For the value of N_{grid} we choose $2N_{max}$ so that we always have more constraints than free parameters. We also set $V_b = 1$.

On figure 7 we study the convergence of the result with N_{max}. We see that the convergence is really fast.
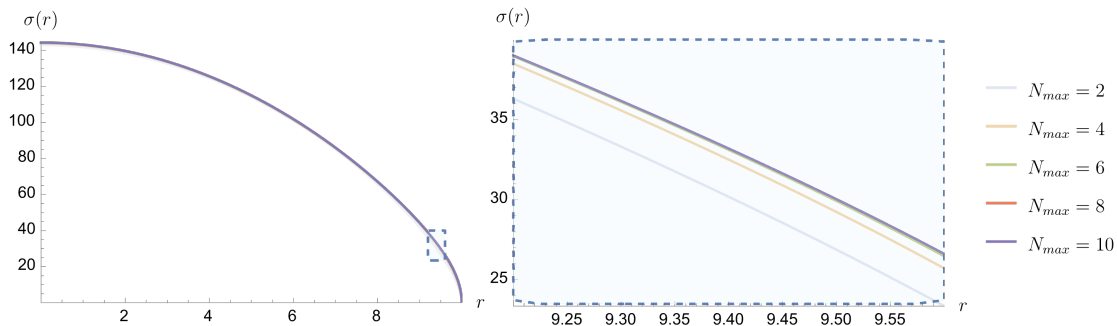

Figure 7: Convergence study of the numerical problem (D.25) with N_{max}. We solved for the charge density with $z_s = 1$ and $R_s = 10$.

On figure 8 we study the shape of the charge density in different regimes of the dimensionless ratio z_s/R_s. The analytic solutions in both limits $z_s/R_s \to 0$ and $z_s/R_s \to \infty$ are derived in (D.15) and the appendix of [5] respectively. We see that the numerical solution nicely interpolates between those results.

We also study the total charge, the potential and to octopole, shown on figures 9 and 10. We also observe a clean transition from the asymptotics $z_s/R_s \to 0, \infty$ where the numerics agree with the analytical results.

Finally we can measure numerically the function $H(\xi)$ defined in (5.46) by plotting the on-shell action as function of the parameter $\xi \equiv n_s \Omega^{-4} N_s^{-5}$. In terms of the electrostatic

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[35]One could worry that the result can be unphysical if the minimum is not zero. However in all our results we find that the minimum is zero (up to numerical accuracy).
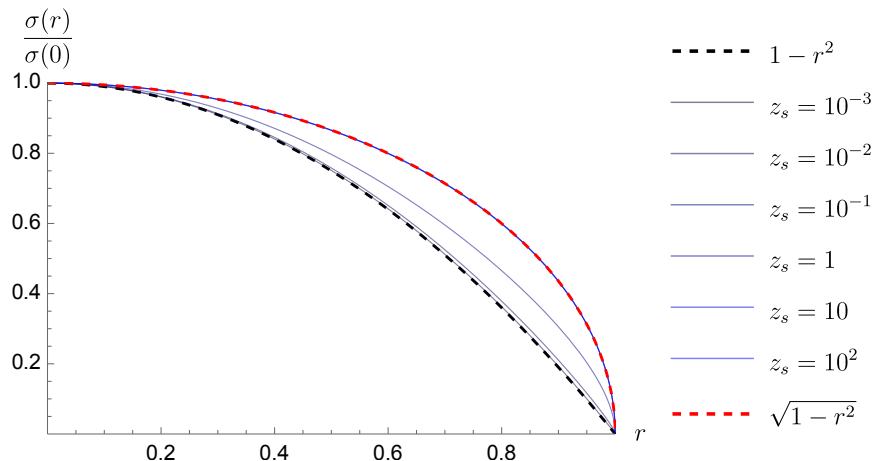
Figure 8: Numerical results for the (normalized) charge density for different heights z_s when $R_s = 1$. We used $N_{max} = 4$. The black and red dashed lines are the analytic approximations when $z_s \ll R_s, z_s \gg R_s$ given by (D.15) and (D.9).
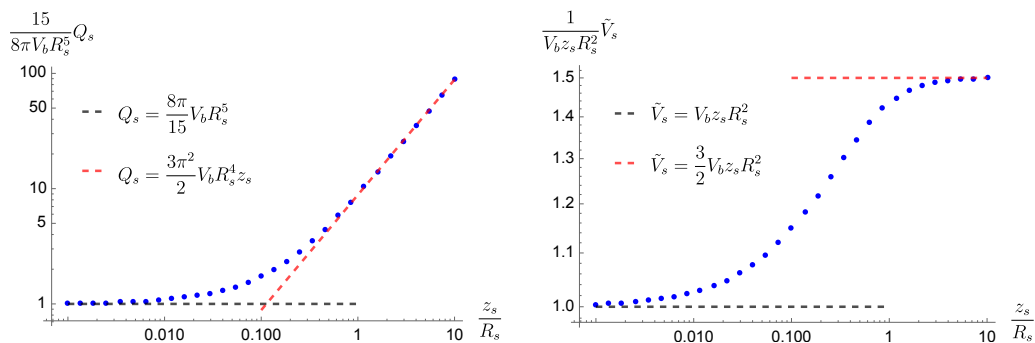

Figure 9: Numerical results (blue dots) for the total charge Q_s and the potential V_s. We used $N_{max} = 4$. The black and red dashed lines are the analytic approximations when $z_s \ll R_s, z_s \gg R_s$ given by (D.16) and (D.9).

variables we have

$$\xi = \frac{3^5}{2^{20}\pi^2} \frac{Q_s}{z_s^5 \mu^5}, \tag{D.26}$$

which can be computed with our numerics. The result for $H(\xi) = -S_{\text{on-shell}}\lambda^{2/3}/N^2$ is shown on figure 11.

$$\frac{35}{16\pi V_b R_s^7 z_s} q_{3,s}$$

$$q_{3,s} = -\frac{16\pi}{35} V_b R_s^7 z_s$$
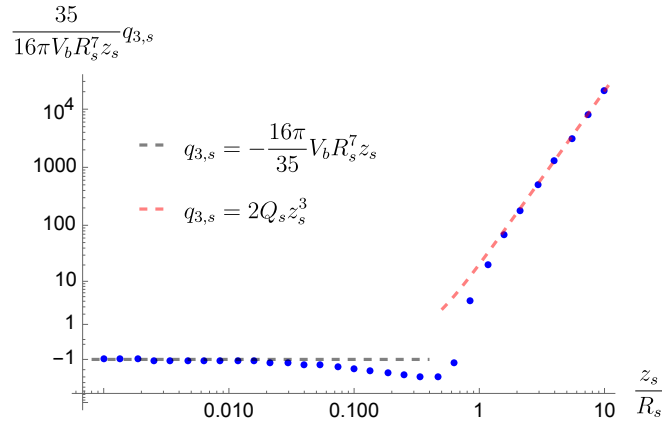
$$q_{3,s} = 2Q_s z_s^3$$

Figure 10: Numerical results (blue dots) for the octopole $q_{3,s}$. We used $N_{max} = 4$. The black and red dashed lines are the analytic approximations when $z_s \ll R_s, z_s \gg R_s$ given by (D.17) and (D.10).
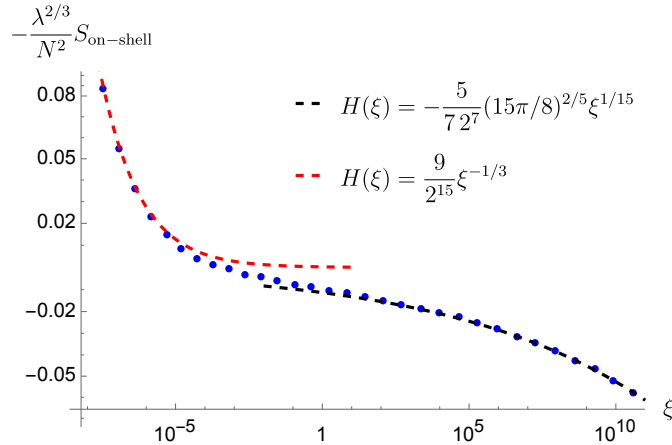

$$-\frac{\lambda^{2/3}}{N^2} S_{\text{on-shell}}$$

$$H(\xi) = -\frac{5}{7\,2^7}(15\pi/8)^{2/5}\xi^{1/15}$$

$$H(\xi) = \frac{9}{2^{15}}\xi^{-1/3}$$

Figure 11: Numerical results (blue dots) for the on shell action (5.46). We used $N_{max} = 4$. The black and red dashed lines are the analytic approximations when $\xi \ll 1$ and $\xi \gg 1$ respectively.

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