

What is the Geometry of Effective Field Theories?

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We elaborate on a recently proposed geometric framework for scalar effective field theories. Starting from the action, a metric can be identified that enables the construction of geometric quantities on the associated functional manifold. These objects transform covariantly under general field redefinitions that relate different operator bases, including those involving derivatives. We present a novel geometric formula for the amplitudes of the theory, where the vertices in Feynman diagrams are replaced by their geometrized counterparts. This makes the on-shell covariance of amplitudes manifest, providing the link between functional geometry and effective field theories.

INTRODUCTION

One of the fundamental difficulties in the Lagrangian formulation of effective field theories (EFTs) is that one can redefine the fields in a theory without changing the physical predictions, *e.g.* scattering amplitudes. The freedom to perform field redefinitions to change the form of the Lagrangian can often obscure the physical content of an EFT. This is in fact a familiar situation in any physical theory, where one can pick different coordinate systems for the dynamical degrees of freedom. In the case of classical Hamiltonian mechanics, this corresponds to choosing different coordinates for the same symplectic geometry. What is the analogous geometric picture for EFTs? Is it possible to characterize and classify EFTs by the geometric properties of their associated manifolds? In this letter, we make significant progress towards answering these ambitious questions.

The notion that fields in an EFT exist on a field manifold has a long history [1–12]. This idea, which is often called ‘field space geometry,’ has recently experienced a renaissance, as many new exciting applications have been developed, *e.g.* to the Higgs sector of the Standard Model [13–24], EFTs with fermions [25–27], gauge bosons and higher-spin fields [28, 29], geometric soft theorems [30, 31], geometry-kinematics duality [32], and renormalization group evolution equations [33–37].

One of the primary goals of the field space geometry program is to express the amplitudes in terms of geometric quantities defined on the field manifold. In this language, field redefinitions are recast as coordinate transformations, and the invariance of on-shell amplitudes can be made manifest by writing them explicitly in terms of geometric quantities. This approach has led to many insights for understanding the theory and phenomenology of EFTs. However, it has a significant limitation in that it only accommodates field redefinitions without deriva-

tives:

$$\phi = f(\tilde{\phi}), \quad (1)$$

where f is a real analytic function that relates the fields ϕ to the fields in the transformed basis $\tilde{\phi}$, and we are suppressing the flavor index i of the scalar fields ϕ^i . On the other hand, it is known that amplitudes are invariant under a much broader class of redefinitions [38–46]:

$$\phi = f(\tilde{\phi}, \partial_\mu \tilde{\phi}, \partial_\mu \partial_\nu \tilde{\phi}, \dots) = F[\tilde{\phi}], \quad (2)$$

where F is a functional of $\tilde{\phi}$. In other words, the coordinate transformation in Eq. (1) must be upgraded to a functional transformation. The set of field redefinitions in Eq. (2) are the origin of operator redundancies in EFTs. In this letter, we elaborate on a recently proposed geometry [32] that accommodates these more generally allowed field redefinitions [47]. We refer to this framework as ‘functional geometry.’

Some initial explorations towards understanding amplitudes on the functional manifold were recently performed in Refs. [48, 49], where two of us with collaborators showed that the off-shell amplitudes that have been stripped of their external wavefunction factors (see Eq. (14) below) transform covariantly up to a set of terms that vanish at the physical vacuum with the on-shell conditions enforced. This ‘on-shell covariance’ of off-shell amplitudes must be accommodated by any geometric picture that describes a non-trivial manifold away from the point that corresponds to the physical vacuum. We emphasize that performing a field redefinition is inherently an off-shell operation, so understanding the true underlying geometry of EFTs requires exploring how off-shell amplitudes transform.

In this work, we build on the metric and the associated geometry introduced in Ref. [32] in the context of geometry-kinematics duality to elucidate the covari-

ant properties of EFT amplitudes in functional geometry. Starting from this metric, one can define covariant quantities such as curvature tensors, and covariant derivatives thereof. However, since the amplitudes are only on-shell covariant [48, 49], it must be the case that they are not constructed entirely from these tensors. A main result in this letter is that off-shell amplitudes can be constructed from the standard Feynman rules using a geometrized version of the vertices. These new vertices are not tensors on the functional manifold, but are on-shell covariant. Expressing off-shell amplitudes in terms of these novel building blocks makes their on-shell covariance manifest. As a corollary, we will show that our construction reproduces the geometry-kinematics duality [32] in the massless limit, and leads to a generalization of this duality to massive theories.

The introduction of functional geometry sets the stage to significantly impact our general understanding of EFTs. By reframing results based on field space geometry in terms of functional geometry, we can explore the robustness of conclusions drawn when using the more limited class of field redefinitions expressed by Eq. (1). There are many applications to the theory and phenomenology of EFTs, some of which we highlight in the Outlook section below.

GEOMETRY OF THE FUNCTIONAL MANIFOLD

Field redefinitions of the form Eq. (2) correspond to coordinate changes on the functional manifold, $\phi = F[\tilde{\phi}]$. A key property of these field redefinitions is that they commute with spacetime translations,

$$\mathcal{T}_\epsilon F[\phi] = F[\mathcal{T}_\epsilon \phi], \quad (3)$$

where $\mathcal{T}_\epsilon \phi(x) = \phi(x + \epsilon) = \phi(x) + \epsilon^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2)$. As a result, we have

$$\partial_\mu \phi^i(x) = \int d^4 y \frac{\delta \phi^i(x)}{\delta \tilde{\phi}^j(y)} \partial_\mu \tilde{\phi}^j(y), \quad (4)$$

where i, j are flavor indices, and ‘ δ ’ denotes a functional derivative in the usual sense. Eq. (4) shows that $\partial_\mu \phi^i$ transforms as a vector under field redefinitions that can include derivatives.

It is more convenient to use momentum space fields $\phi^i(p) = \int d^4 x e^{ip \cdot x} \phi^i(x)$ to chart the functional manifold. Field redefinitions in Eq. (2) can then be written as (suppressing flavor indices):

$$\begin{aligned} \phi(p) &= c_0(p) + \int \frac{d^4 p_1}{(2\pi)^4} c_1(p; p_1) \tilde{\phi}(p_1) \\ &+ \frac{1}{2} \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} c_2(p; p_1, p_2) \tilde{\phi}(p_1) \tilde{\phi}(p_2) + \dots, \end{aligned} \quad (5)$$

with

$$c_n(p; p_1, \dots, p_n) \propto (2\pi)^4 \delta^4\left(p - \sum_{a=1}^n p_a\right). \quad (6)$$

From Eqs. (5) and (6), we find the momentum space version of Eq. (4):

$$p_\mu \phi^i(p) = \int \frac{d^4 q}{(2\pi)^4} \frac{\delta \phi^i(p)}{\delta \tilde{\phi}^j(q)} q_\mu \tilde{\phi}^j(q), \quad (7)$$

which implies that $p_\mu \phi^i(p)$ transforms as a vector. We emphasize that the vector transformation property in Eqs. (4) and (7) is a consequence of translation invariance of the field redefinitions in Eq. (2).

The EFT action $S[\phi]$ is a scalar on the functional manifold. Therefore, if we can write [32]

$$S = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} g_{ij}(p, q) p_\mu \phi^i(p) q^\mu \phi^j(q), \quad (8)$$

with $g_{ij}(p, q) = g_{ji}(q, p)$, then $g_{ij}(p, q)$ can be identified as a metric on the functional manifold, and as such it transforms as a $(0, 2)$ -tensor. For example, $g_{ij}(p, q) = (2\pi)^4 \delta^4(p + q) \delta_{ij} (1 - m_i^2/p^2)$ corresponds to a theory of free scalars. Note that for a given action, Eq. (8) does not uniquely determine the metric [32]; two viable metrics can differ by a tensor $h_{ij}(p, q)$ that satisfies $\int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} h_{ij}(p, q) p_\mu \phi^i(p) q^\mu \phi^j(q) = 0$. We will come back to discuss this ambiguity in the Outlook section.

Starting from a metric, we can construct the Christoffel connection, covariant derivative, and Riemann curvature tensor in the usual way. In what follows, we abbreviate $\phi^{i_a}(p_a) \equiv \phi^a$, $g_{i_1 i_2}(p_1, p_2) \equiv g_{12}$, and similarly for all the other objects. In this notation, derivatives on the functional manifold are normalized as

$$,_a \equiv (2\pi)^4 \frac{\delta}{\delta \phi^a}, \quad (9)$$

and index contraction means

$$^b{}_b \equiv \sum_{i_b} \int \frac{d^4 p_b}{(2\pi)^4}. \quad (10)$$

The Christoffel connection is given by

$$\Gamma_{12}^a = \frac{1}{2} g^{ab} (g_{b1,2} + g_{b2,1} - g_{12,b}), \quad (11)$$

where the inverse metric g^{ab} is defined from $g^{ab} g_{bc} = \delta_a^c \equiv \delta_a^{i_c} (2\pi)^4 \delta^4(p_a - p_c)$, and the Riemann curvature tensor is

$$R_{1234} = \frac{1}{2} (g_{14,23} - g_{24,13}) - g_{ab} \Gamma_{13}^a \Gamma_{24}^b - (3 \leftrightarrow 4). \quad (12)$$

We are often interested in the values of geometric quantities at the special point on the functional man-

ifold corresponding to the physical vacuum $\bar{\phi}$. Using overline to denote quantities evaluated at $\bar{\phi}$, we have $\bar{S}_{;a} = \bar{S}_{,a} = 0$ at tree level, where semicolon denotes the covariant derivative. Without loss of generality, we focus on coordinate choices where $\bar{\phi} = 0$. From the definition of the metric in Eq. (8), one can readily show:

$$\bar{S}_{;123} = 0, \quad (13a)$$

$$\bar{S}_{;(1234)} = -\frac{2}{3} \left[s_{12} \bar{R}_{1(34)2} + s_{13} \bar{R}_{1(24)3} + s_{14} \bar{R}_{1(23)4} \right], \quad (13b)$$

where $s_{ab} \equiv (p_a + p_b)^2$ are Mandelstam variables and parentheses denote symmetrization of the indices, *e.g.* $\bar{R}_{1(34)2} = \frac{1}{2}(\bar{R}_{1342} + \bar{R}_{1432})$.

GEOMETRIZING AMPLITUDES

Given an EFT action, one can recursively construct the off-shell amplitudes $\mathcal{M}_{1\dots n}$ [50] using standard Feynman rules [48, 49]:

$$\mathcal{M}_{123} = V_{123}, \quad (14a)$$

$$\mathcal{M}_{1\dots n(n+1)} = \mathcal{M}_{1\dots n, n+1} - \sum_{k=1}^n \Delta^{ab} V_{b(n+1)k} \mathcal{M}_{1\dots \cancel{k} \dots n}. \quad (14b)$$

We use ‘ \cancel{k} ’ to denote the absence of index k . In these equations, $V_{1\dots k}$ are one-particle-irreducible (1PI) vertices and Δ^{ab} is the full propagator. At tree level,

$$V_{1\dots k} = S_{,1\dots k}, \quad \text{and} \quad \Delta^{ab} S_{,bc} = \delta_c^a. \quad (15)$$

At loop level, one replaces the classical action S in Eq. (15) by the 1PI effective action. From Eq. (14), we can write down the n -point off-shell amplitude as a function of propagators Δ^{ab} and vertices $V_{a_1\dots a_k}$ with $k \leq n$:

$$\mathcal{M}_{1\dots n} = \mathcal{F}_{1\dots n} \left(\Delta^{ab}, \{V_{a_1\dots a_k}\} \right). \quad (16)$$

For example,

$$\mathcal{M}_{123} = V_{123}, \quad (17a)$$

$$\mathcal{M}_{1234} = V_{1234} - \Delta^{ab} (V_{b41} V_{a23} + V_{b42} V_{1a3} + V_{b43} V_{12a}). \quad (17b)$$

On-shell amplitudes are then obtained via the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [51, 52] by evaluating $\mathcal{M}_{1\dots n}$ at the physical vacuum $\bar{\phi}$ for on-shell external momenta $p_i^2 = m_i^2$, and dressing it with

wavefunction factors:

$$(2\pi)^4 \delta^4(p_1 + \dots + p_n) \mathcal{A}_{i_1\dots i_n}(p_1, \dots, p_n) = \left(r_1^{1/2} \dots r_n^{1/2} \right) \left(\bar{\mathcal{M}}_{1\dots n} \Big|_{\text{on-shell}} \right). \quad (18)$$

Under generic field redefinitions, the off-shell amplitudes do not transform covariantly:

$$\widetilde{\mathcal{M}}_{a_1\dots a_n} = \left(\frac{\delta\phi^{b_1}}{\delta\tilde{\phi}^{a_1}} \dots \frac{\delta\phi^{b_n}}{\delta\tilde{\phi}^{a_n}} \right) \mathcal{M}_{b_1\dots b_n} + X_{a_1\dots a_n}, \quad (19)$$

with $X_{a_1\dots a_n} \neq 0$. However, one can show recursively that [49]:

$$\bar{X}_{a_1\dots a_n} \Big|_{\text{on-shell}} = 0. \quad (20)$$

In other words, $\mathcal{M}_{1\dots n}$ are ‘on-shell covariant’ [53]. Meanwhile, the $r_i^{1/2}$ factors in Eq. (18) transform as vielbeins [30, 49], so on-shell amplitudes $\mathcal{A}_{i_1\dots i_n}(p_1, \dots, p_n)$ for any given set of particle species and momenta are invariant under field redefinitions.

The main new result of this work is that, starting from Eq. (14), it can be shown recursively (see Ref. [54] for details) that $\mathcal{M}_{1\dots n}$ can in fact be constructed from a set of on-shell-covariant building blocks:

$$\boxed{\mathcal{M}_{1\dots n} = \mathcal{F}_{1\dots n} \left(\Delta^{ab}, \{V_{a_1\dots a_k}\} \right)}. \quad (21)$$

Here $\mathcal{F}_{1\dots n}$ is the same function as in Eq. (16), but the arguments $V_{a_1\dots a_k}$ are replaced by a new set of vertices $\mathcal{V}_{a_1\dots a_k}$. At tree level, these new vertices are given by

$$\mathcal{V}_{1\dots k} \equiv S_{;1\dots k} + \Gamma_{1\dots k}^a S_{,a} + \sum_{b \in \text{external}} \Gamma_{1\dots \cancel{b} \dots k}^a S_{,ab}, \quad (22)$$

where the sum is over the subset of $\{1, \dots, k\}$ that correspond to external legs, and $\Gamma_{1\dots k}^a$ are the generalized Christoffel symbols defined recursively via

$$\Gamma_{1\dots k(k+1)}^a \equiv \Gamma_{1\dots k, k+1}^a - \sum_{b=1}^k \Gamma_{b(k+1)}^c \Gamma_{1\dots \cancel{b} \dots k}^a. \quad (23)$$

At loop level, one again replaces the action S in Eq. (22) by the 1PI effective action. Note from Eq. (22) that when the Christoffel connection vanishes, $\mathcal{V}_{1\dots k}$ reduce to $V_{1\dots k}$. Therefore, $\mathcal{V}_{1\dots k}$ are a geometric version of the standard Feynman vertices $V_{1\dots k}$, and Eq. (21) can be viewed as a geometrization of Eq. (16). Importantly, the building blocks in Eq. (21), Δ^{ab} and $\mathcal{V}_{1\dots k}$, are all on-shell covariant; on-shell covariance of Δ^{ab} follows from $\bar{S}_{,ab} = \bar{S}_{;ab}$, while the on-shell covariance of $\mathcal{V}_{1\dots k}$ can be proved by induction, which we detail in a follow-up paper [54]. Since sums and products of on-shell-covariant objects are also on-shell covariant, Eq. (21) expresses EFT amplitudes in

a manifestly on-shell covariant form!

In the following sections, we specialize to the cases of massless and massive theories, respectively. A simple example to illustrate the application of our formalism is included in the Supplemental Material.

MASSLESS THEORIES

In massless theories, $\bar{\Gamma}_{1\dots\check{b}\dots k}^a \bar{S}_{,ab} = p_b^2 \bar{g}_{ab} \bar{\Gamma}_{1\dots\check{b}\dots k}^a$ vanishes as the external momentum p_b goes on-shell, as long as $\bar{g}_{ab} \bar{\Gamma}_{1\dots\check{b}\dots k}^a|_{\text{on-shell}}$ is finite. This implies

$$\bar{V}_{1\dots k}|_{\text{on-shell}} = \bar{S}_{;1\dots k}|_{\text{on-shell}} \quad (\text{massless}). \quad (24)$$

The only possible exception is the $k = 3$ case with 1, 2, 3 all external, because $\bar{g}_{a1} \bar{\Gamma}_{23}^a$ (and permutations) may diverge on-shell due to the special 3-point kinematics [54]. Consequently,

$$\bar{\mathcal{M}}_{1\dots n}|_{\text{on-shell}} = \mathcal{F}_{1\dots n}(\bar{\Xi}^{ab}, \{\bar{S}_{;a_1\dots a_k}\})|_{\text{on-shell}} \quad (25)$$

for all $n \geq 4$, where $\bar{\Xi}^{ab}$ is the covariant propagator satisfying $\bar{\Xi}^{ab} S_{;bc} = \delta_c^a$ (the covariant version of $\Delta^{ab} S_{;bc} = \delta_c^a$), and we have $\bar{\Delta}^{ab} = \bar{\Xi}^{ab}$ since $\bar{S}_{,bc} = \bar{S}_{;bc}$. Eq. (25) tells us that in massless theories, $\bar{\mathcal{M}}_{1\dots n}$ are actually on-shell equivalent to a set of tensors built from $\bar{\Xi}^{ab}$ and $\{\bar{S}_{;a_1\dots a_k}\}$.

An interesting special case of massless theories is the nonlinear sigma model (NLSM):

$$S_{\text{NLSM}} \equiv \frac{1}{2} \int d^4x \mathbf{g}_{ab}(\phi) (\partial_\mu \phi^a) (\partial^\mu \phi^b), \quad (26)$$

where \mathbf{g}_{ab} is the metric in field space geometry. Comparing with Eq. (8) we see that

$$g_{ij}(p, q) = \int d^4x e^{-i(p+q)\cdot x} \mathbf{g}_{ij}(\phi(x)). \quad (27)$$

Taking successive functional derivatives and setting $\phi = \bar{\phi}$, we obtain

$$\bar{g}_{ab,1\dots k} = (2\pi)^4 \delta^4(p_a + p_b + p_1 + \dots + p_k) \bar{\mathbf{g}}_{i_a i_b, i_1 \dots i_k}. \quad (28)$$

An immediate consequence of Eq. (28) is that in the NLSM, any expression in functional geometry that is a sum of contractions of $\bar{g}_{ab,1\dots k}$ with \bar{g}^{ab} (potentially also multiplied by p_a^μ factors) will be equal to the corresponding expression in field space geometry in terms of $\bar{\mathbf{g}}_{i_a i_b, i_1 \dots i_k}$ and $\bar{\mathbf{g}}^{i_a i_b}$, multiplied by an overall momentum-conserving δ -function. Off-shell amplitudes evaluated at the physical vacuum $\bar{\mathcal{M}}_{1\dots n}$ are such expressions. Importantly, from Eqs. (8) and (21) we see that a universal set of such expressions in terms of $\bar{g}_{ab,1\dots k}$ and \bar{g}^{ab} will give $\bar{\mathcal{M}}_{1\dots n}$ for *all* theories, while Eq. (28) implies that the same expressions in terms of $\bar{\mathbf{g}}_{i_a i_b, i_1 \dots i_k}$ and $\bar{\mathbf{g}}^{i_a i_b}$ will give the correct $\bar{\mathcal{M}}_{1\dots n}$ in the NLSM. In other words,

to find the universal expressions for $\bar{\mathcal{M}}_{1\dots n}$ in terms of $\bar{g}_{ab,1\dots k}$ and \bar{g}^{ab} , one can simply take the field space geometry expressions for $\bar{\mathcal{M}}_{1\dots n}$ in the NLSM, and make the replacement $\bar{\mathbf{g}} \rightarrow \bar{g}$. This is the geometry-kinematics duality proposed for massless theories in Ref. [32]. From our argument above, it is clear that the same strategy of replacing $\bar{\mathbf{g}} \rightarrow \bar{g}$ should yield the correct amplitudes in both massless and massive theories. The special feature in the massless case is that 4- and higher-point *on-shell* amplitudes can be written entirely in terms of tensors on the functional manifold as we saw above. They can therefore be obtained from the tensorial expressions of on-shell NLSM amplitudes in field space geometry, by upgrading the curvature tensors and their covariant derivatives to their counterparts in functional geometry. See Ref. [32] for many explicit examples. On the other hand, to obtain 3-point amplitudes in massless theories, as well as amplitudes in general massive theories, one must perform the aforementioned replacement in the non-tensorial expressions of *off-shell* NLSM amplitudes.

MASSIVE THEORIES

For massive theories, $\bar{\Gamma}_{1\dots\check{b}\dots k}^a \bar{S}_{,ab} = p_b^2 \bar{g}_{ab} \bar{\Gamma}_{1\dots\check{b}\dots k}^a$ does not vanish as p_b goes on-shell, so

$$\bar{V}_{1\dots k}|_{\text{on-shell}} \neq \bar{S}_{;1\dots k}|_{\text{on-shell}} \quad (\text{massive}), \quad (29)$$

and $\bar{\mathcal{M}}_{1\dots n}$ are generically not on-shell equivalent to the tensors $\mathcal{F}_{1\dots n}(\bar{\Xi}^{ab}, \{S_{;a_1\dots a_k}\})$, *i.e.*, Eq. (25) does not hold generically. As discussed above, this does not invalidate a geometry-kinematics duality, as long as we implement the replacement $\bar{\mathbf{g}} \rightarrow \bar{g}$ in the correct non-tensorial expressions. However, we emphasize that our main result in Eq. (21) organizes these non-tensorial expressions into on-shell-covariant building blocks, making the on-shell covariance of $\bar{\mathcal{M}}_{1\dots n}$ manifest.

Another perspective on the presence of non-tensorial $p_b^2 \bar{g}_{ab} \bar{\Gamma}_{1\dots\check{b}\dots k}^a$ terms can be gained by going to Riemann normal coordinates η^a :

$$\phi^a = \eta^a - \sum_{k=2}^{\infty} \frac{1}{k!} \bar{\Gamma}_{(b_1 \dots b_k)}^a \eta^{b_1} \dots \eta^{b_k}. \quad (30)$$

In the η^a basis, Eqs. (24) and (25) are satisfied since covariant derivatives are the same as ordinary derivatives in normal coordinates, and there are no non-tensorial terms. However, the nonlocal field redefinition in Eq. (30) does not belong to the set of field redefinitions in Eq. (2), so naively applying the LSZ reduction formula Eq. (18) in the η^a basis would not give the correct on-shell amplitudes. The difference can be systematically calculated by working out modifications to Eq. (18) due to the introduction of additional single-particle poles, and we have checked that they exactly reproduce the non-tensorial terms; see Ref. [54] for details.

We emphasize that the discussion above does not preclude the possibility that $\mathcal{M}_{1\dots n}$ in massive theories are on-shell equivalent to some other tensors constructed differently from those in massless theories. For example, one might construct an alternative connection Γ_{12}^a which, unlike the metric-compatible connection Γ_{12}^a , does not have a pole at $p_a^2 = m_a^2$. Such a connection would ensure $\bar{\Gamma}_{1\dots\hat{b}\dots k}^a \bar{S}_{,ab}$ vanishes on-shell, thereby resurrecting Eq. (25) when covariant derivatives are taken under Γ_{12}^a . We leave an investigation of this possibility to future work.

OUTLOOK

In this letter, we have elaborated on the construction of ‘functional geometry’ for scalar EFTs. We have shown how to write off-shell amplitudes as functions of on-shell-covariant objects on the functional manifold. These functions (Eq. (21)) are identical to those constructed using the usual Feynman propagators and vertices (Eq. (16)), but with the vertices $V_{a_1\dots a_k}$ replaced by their geometrization $\mathcal{V}_{a_1\dots a_k}$ given in Eq. (22). In this way, the on-shell covariance of amplitudes is made manifest. We have also discussed how our formalism leads to a more general version of the geometry-kinematics duality [32], which applies to all amplitudes in both massless and massive theories.

The discussion in this letter focused on scalar EFTs. It would be interesting to extend the same approach to accommodate higher-spin fields, which we leave to future work. Another avenue worth further investigation is the ambiguity associated with metric choice: Eq. (8) does not uniquely determine $g_{ij}(p, q)$. Interestingly, the values of $R_{abcd;1\dots k}$ and $S_{;1\dots k}$, as well as our new geometrized vertices $\mathcal{V}_{1\dots k}$ generally vary with the choice of the metric. This is even the case for their values at the physical vacuum with on-shell momenta [55]. The metric choice dependence drops only after these building blocks are assembled into the off-shell amplitudes as in Eq. (21). Note that this does not invalidate Eq. (21) as a manifest proof of the on-shell covariance of $\mathcal{M}_{1\dots n}$, because for the new Lagrangian obtained by a field redefinition, we know that there exists one choice of metric that would correspond to the tensor transformation of the old one. Sticking to this choice, each building block will be on-shell covariant. On the other hand, the meaning of the geometrized vertices $\mathcal{V}_{1\dots k}$ is somewhat obscured by this metric choice dependence. We know that in field space geometry, the covariantized vertices correspond to vertices in normal coordinates which are directly related to the contact terms in amplitudes. It would be interesting to see if there is an analogous interpretation of $\mathcal{V}_{1\dots k}$ in functional geometry.

A set of more ambitious goals are to probe the nature of different EFTs and relations between their amplitudes with the functional geometry framework. Concretely,

there are established criteria in the language of field space geometry that characterize if an EFT has global symmetries [30], is renormalizable or non-renormalizable (e.g. SM vs. SMEFT or HEFT) [13, 15], whether a symmetry is linearly or nonlinearly realized (e.g. SMEFT vs. HEFT) [1, 2, 13, 15, 18], etc. Upgrading these results with functional geometry will take us closer to establishing rigorous criteria for classifying EFTs. Meanwhile, the fact that amplitudes in all scalar EFTs can be constructed from on-shell-covariant building blocks may be exploited to shed new light on the relations between amplitudes in different EFTs. We have only begun to understand the broad implications of the new perspective that functional geometry brings to the dynamics of EFTs.

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SUPPLEMENTAL MATERIAL

Geometry of the $\phi^3 + \phi^4$ theory

In this supplemental material we apply the general formalism discussed in the main text to a concrete example. We take the renormalizable Lagrangian of a single flavor scalar field ϕ :

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{6} \mu \phi^3 - \frac{1}{24} \lambda \phi^4. \quad (\text{A.1})$$

The corresponding action is

$$\begin{aligned} S = & - \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} (2\pi)^4 \delta^4(q_{12}) \phi(q_1) \phi(q_2) \frac{1}{2} (q_1 q_2 + m^2) \\ & - \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} (2\pi)^4 \delta^4(q_{123}) \phi(q_1) \phi(q_2) \phi(q_3) \frac{1}{6} \mu \\ & - \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} \frac{d^4 q_4}{(2\pi)^4} (2\pi)^4 \delta^4(q_{1234}) \phi(q_1) \phi(q_2) \phi(q_3) \phi(q_4) \frac{1}{24} \lambda, \end{aligned} \quad (\text{A.2})$$

where we have used the abbreviation $q_{1\dots k} \equiv q_1 + \dots + q_k$, and $q_1 q_2 \equiv q_1 \cdot q_2$.

One choice of the metric that satisfies Eq. (8) is

$$\begin{aligned} g_{\phi\phi}(p_1, p_2) = & (2\pi)^4 \delta^4(p_{12}) \left(1 + \frac{m^2}{p_1 p_2} \right) + \frac{1}{3} \mu \frac{1}{p_1 p_2} \phi(-p_{12}) \\ & + \frac{1}{12} \lambda \frac{1}{p_1 p_2} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} (2\pi)^4 \delta^4(p_{12} + q_{12}) \phi(q_1) \phi(q_2). \end{aligned} \quad (\text{A.3})$$

The first and second derivatives of the metric are

$$g_{\phi\phi,\phi}(p_1, p_2, p_3) = \left[(2\pi)^4 \delta^4(p_{123}) \frac{1}{3} \mu + \frac{1}{6} \lambda \phi(-p_{123}) \right] \frac{1}{p_1 p_2}, \quad (\text{A.4a})$$

$$g_{\phi\phi,\phi\phi}(p_1, p_2, p_3, p_4) = (2\pi)^4 \delta^4(p_{1234}) \frac{1}{6} \lambda \frac{1}{p_1 p_2}. \quad (\text{A.4b})$$

The inverse metric and the Christoffel connection evaluated at the physical vacuum $\bar{\phi} = 0$ are

$$\bar{g}^{ab} = (2\pi)^4 \delta^4(p_{ab}) \frac{p_a^2}{p_a^2 - m^2}, \quad (\text{A.5})$$

$$\bar{\Gamma}_{12}^a = -(2\pi)^4 \delta^4(p_a - p_{12}) \frac{p_a^2}{p_a^2 - m^2} \frac{1}{6} \mu \left(\frac{1}{p_a p_1} + \frac{1}{p_a p_2} + \frac{1}{p_1 p_2} \right). \quad (\text{A.6})$$

Here we have started using our abbreviation $\phi^{i_a}(p_a) \rightarrow \phi^a$. The Riemann curvature tensor (evaluated at $\bar{\phi} = 0$) can then be computed:

$$\begin{aligned} \bar{R}_{1234} = & -\frac{1}{2} (\bar{g}_{13,24} - \bar{g}_{23,14}) - \bar{g}_{ab} \bar{\Gamma}_{13}^a \bar{\Gamma}_{24}^b - (3 \leftrightarrow 4) \\ = & (2\pi)^4 \delta^4(p_{1234}) \left[\frac{1}{12} \lambda \left(\frac{1}{p_1 p_4} - \frac{1}{p_2 p_4} \right) \right. \\ & \left. - \frac{1}{36} \mu^2 \frac{p_{13}^2}{p_{13}^2 - m^2} \left(\frac{1}{p_1 p_{13}} + \frac{1}{p_3 p_{13}} + \frac{1}{p_1 p_3} \right) \left(\frac{1}{p_2 p_{24}} + \frac{1}{p_4 p_{24}} + \frac{1}{p_2 p_4} \right) \right] - (p_3 \leftrightarrow p_4). \end{aligned} \quad (\text{A.7})$$

The regular functional derivatives of the action S at $\phi = \bar{\phi} = 0$ are:

$$\bar{S}_{,12} = (2\pi)^4 \delta^4(p_{12}) (p_1^2 - m^2), \quad (\text{A.8a})$$

$$\bar{S}_{,123} = -(2\pi)^4 \delta^4(p_{123}) \mu, \quad (\text{A.8b})$$

$$\bar{S}_{,1234} = -(2\pi)^4 \delta^4(p_{1234}) \lambda. \quad (\text{A.8c})$$

The covariant derivatives then follow from the relations

$$S_{;12} = S_{,12} - \Gamma_{12}^a S_{,a}, \quad (\text{A.9a})$$

$$S_{;123} = S_{,123} - (\Gamma_{12}^a S_{,a3})_{3 \text{ terms}} - \Gamma_{123}^a S_{,a}, \quad (\text{A.9b})$$

$$S_{;1234} = S_{,1234} - (\Gamma_{12}^a S_{,a34})_{6 \text{ terms}} + (\Gamma_{12}^a \Gamma_{34}^b S_{,ab})_{3 \text{ terms}} - (\Gamma_{123}^a S_{,a4})_{4 \text{ terms}} - \Gamma_{1234}^a S_{,a}. \quad (\text{A.9c})$$

Specifically, we obtain

$$\bar{S}_{;12} = \bar{S}_{,12} = (2\pi)^4 \delta^4(p_{12}) (p_1^2 - m^2), \quad (\text{A.10a})$$

$$\bar{S}_{;123} = 0, \quad (\text{A.10b})$$

$$\bar{S}_{;1234} = (p_4^2 + 2p_2 p_4) \bar{R}_{1234} + (2p_3 p_4) \bar{R}_{1324}, \quad (\text{A.10c})$$

where the curvature is given in Eq. (A.7). One can check that these explicit results satisfy the general relations in Eq. (13).

Now let us move on to the amplitudes. The usual Feynman propagator and vertices are (recall $\Delta^{ab} S_{,bc} = \delta_c^a$ and $V_{,1\dots k} = S_{,1\dots k}$)

$$\bar{\Delta}^{ab} = (2\pi)^4 \delta^4(p_{ab}) \frac{1}{p_a^2 - m^2}, \quad (\text{A.11a})$$

$$\bar{V}_{123} = -(2\pi)^4 \delta^4(p_{123}) \mu, \quad (\text{A.11b})$$

$$\bar{V}_{a12} = \bar{V}_{1a2} = \bar{V}_{12a} = -(2\pi)^4 \delta^4(p_{a12}) \mu, \quad (\text{A.11c})$$

$$\bar{V}_{1234} = -(2\pi)^4 \delta^4(p_{1234}) \lambda. \quad (\text{A.11d})$$

With these, one can compute the 3-point and 4-point amplitudes from Eq. (16) (more specifically Eq. (17)):

$$(2\pi)^4 \delta^4(p_{123}) \mathcal{A}_{123} = \bar{\mathcal{M}}_{123}|_{\text{on-shell}} = \bar{V}_{123}|_{\text{on-shell}} = -(2\pi)^4 \delta^4(p_{123}) \mu, \quad (\text{A.12a})$$

$$\begin{aligned} (2\pi)^4 \delta^4(p_{1234}) \mathcal{A}_{1234} &= \bar{\mathcal{M}}_{1234}|_{\text{on-shell}} = \left[\bar{V}_{1234} - \bar{\Delta}^{ab} (\bar{V}_{b41} \bar{V}_{a23} + \bar{V}_{b42} \bar{V}_{1a3} + \bar{V}_{b43} \bar{V}_{12a}) \right] \Big|_{\text{on-shell}} \\ &= -(2\pi)^4 \delta^4(p_{1234}) \left(\lambda + \frac{\mu^2}{p_{12}^2 - m^2} + \frac{\mu^2}{p_{13}^2 - m^2} + \frac{\mu^2}{p_{14}^2 - m^2} \right). \end{aligned} \quad (\text{A.12b})$$

On the other hand, our geometrized vertices defined in Eq. (22) are

$$\bar{V}_{123} = \bar{S}_{,123} = -(2\pi)^4 \delta^4(p_{123}) \mu, \quad (\text{A.13a})$$

$$\bar{V}_{a12}|_{\text{on-shell}} = \bar{V}_{1a2}|_{\text{on-shell}} = \bar{V}_{12a}|_{\text{on-shell}} = \left(\bar{S}_{,a12} - \bar{\Gamma}_{12}^b \bar{S}_{,ab} \right) \Big|_{\text{on-shell}} = (2\pi)^4 \delta^4(p_{a12}) \mu \frac{m^2}{3p_1 p_2}, \quad (\text{A.13b})$$

$$\begin{aligned} \bar{V}_{1234}|_{\text{on-shell}} &= \left[\bar{S}_{,1234} - (\bar{\Gamma}_{12}^a \bar{S}_{,a34})_{6 \text{ terms}} + (\bar{\Gamma}_{12}^a \bar{\Gamma}_{34}^b \bar{S}_{,ab})_{3 \text{ terms}} \right] \Big|_{\text{on-shell}} \\ &= -(2\pi)^4 \delta^4(p_{1234}) \left\{ \lambda + \left[\frac{\mu^2}{p_{12}^2 - m^2} \left[1 - \left(\frac{m^2}{3p_1 p_2} \right)^2 \right] \right]_{3 \text{ terms}} \right\}. \end{aligned} \quad (\text{A.13c})$$

We see that they are different from standard vertices in Eq. (A.11). However, upon assembling them with the same functions in Eqs. (16) and (21) (specifically Eq. (17) for 3-point and 4-point amplitudes), they do give the same amplitudes:

$$\overline{\mathcal{M}}_{123}|_{\text{on-shell}} = \overline{\mathcal{V}}_{123}|_{\text{on-shell}}, \quad (\text{A.14a})$$

$$\overline{\mathcal{M}}_{1234}|_{\text{on-shell}} = \left[\overline{\mathcal{V}}_{1234} - \overline{\Delta}^{ab} (\overline{\mathcal{V}}_{b41} \overline{\mathcal{V}}_{a23} + \overline{\mathcal{V}}_{b42} \overline{\mathcal{V}}_{1a3} + \overline{\mathcal{V}}_{b43} \overline{\mathcal{V}}_{12a}) \right] \Big|_{\text{on-shell}}. \quad (\text{A.14b})$$

In this example, everything we have demonstrated has been done using on-shell quantities. But we know from the general formalism [54] that the agreement also holds off-shell:

$$\mathcal{M}_{123} = V_{123} = \mathcal{V}_{123}, \quad (\text{A.15a})$$

$$\mathcal{M}_{1234} = V_{1234} - \Delta^{ab} (V_{b41} V_{a23} + V_{b42} V_{1a3} + V_{b43} V_{12a}) = \mathcal{V}_{1234} - \Delta^{ab} (\mathcal{V}_{b41} \mathcal{V}_{a23} + \mathcal{V}_{b42} \mathcal{V}_{1a3} + \mathcal{V}_{b43} \mathcal{V}_{12a}). \quad (\text{A.15b})$$

The geometrization $V \rightarrow \mathcal{V}$ is just a reorganization of terms in the amplitudes. The geometrized vertices $\mathcal{V}_{1\dots k}$ are on-shell covariant under field redefinitions. This demonstrates the main point of the letter.

To complete this example, let us also check the relation between the geometrized vertices $\mathcal{V}_{1\dots k}$ and the covariant derivatives $S_{;1\dots k}$. For 3-point vertices with all external indices, we see from Eqs. (A.10b) and (A.13a) that

$$\overline{\mathcal{V}}_{123}|_{\text{on-shell}} \neq \overline{S}_{;123}|_{\text{on-shell}} = 0. \quad (\text{A.16})$$

So the 3-point amplitude is not on-shell equivalent to the tensor $S_{;123}$. However, for the geometrized 3-point vertex with one internal index, we see from Eq. (A.13b) that

$$\overline{\mathcal{V}}_{a12}|_{\text{on-shell}} = \overline{S}_{;a12}|_{\text{on-shell}} = 0 \quad \text{when} \quad m^2 = 0. \quad (\text{A.17})$$

So the 3-point vertex actually does not enter the geometrized expressions of on-shell higher-point amplitudes in massless theories.

To check the 4-point vertex, we compute $\overline{S}_{;1234}|_{\text{on-shell}}$ from Eq. (A.10c) and obtain

$$\begin{aligned} \overline{S}_{;1234}|_{\text{on-shell}} = & -(2\pi)^4 \delta^4(p_{1234}) \left\{ \lambda \left[1 + \frac{m^2}{6} \left(\frac{1}{p_1 p_3} + \frac{1}{p_1 p_4} \right) \right] \right. \\ & \left. + \mu^2 \left[\frac{p_{12}^2 - 2m^2}{p_{12}^2 (p_{12}^2 - m^2)} \left(1 + \frac{m^2}{3p_1 p_2} \right)^2 + \frac{1}{p_{13}^2} \left(1 + \frac{m^2}{3p_1 p_3} \right)^2 + \frac{1}{p_{14}^2} \left(1 + \frac{m^2}{3p_1 p_4} \right)^2 \right] \right\}. \quad (\text{A.18}) \end{aligned}$$

Comparing this with Eq. (A.13c), we see that

$$\overline{\mathcal{V}}_{;1234}|_{\text{on-shell}} = \overline{S}_{;1234}|_{\text{on-shell}} \quad \text{when} \quad m^2 = 0, \quad (\text{A.19})$$

This explicitly demonstrates the relation Eq. (24) in the $k = 4$ case in our example.