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ABSTRACT

Using the Karabali and Schnitzer formulation of the G/H gauged WZW theories, we develop the quantum WZW fields realization of cosets of type $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$. The quantum constraints obtained from the gauging of the anomaly free subgroup G of the direct product $G \otimes G$ are solved explicitly for the case $G = SU(2)$. Our results are shown to be in agreement with those obtained by standard methods, in particular the algebraic ones. Special features like the realization of the primary fields in terms of WZW fields as well as links with the topological G/G theory are discussed. Other properties are also given.

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1 Introduction

Many of two dimensional conformal field theories (2d CFT) are classified by the GKO construction due to Goddard, Kent and Olive [1]. This construction is algebraic in nature since it is based essentially on the theory of Lie algebras and their representations. Quantum field theoretically, the GKO construction corresponds to gauged WZW (GWZW) theories [2, 3, 4, 5]. In a first publication [4], Karabali, Schnitzer and collaborators, have shown that the conformal central charges corresponding to the G/H, GWZW theory and the G/H, algebraic GKO construction are the same. They have shown also that the energy momentum tensor T for the GWZW theory and that of the GKO, T^{GKO}, are equal in the weak sense, i.e. the equality holds between physical states:

$$\langle Phys|T|Phys\rangle = \langle Phys|T^{GKO}|Phys\rangle \ . \qquad (1.1)$$

In a subsequent paper, the above mentioned authors [6], see also [7], have shown that the analytic energy momentum tensor $T(z)$ of the G/H GWZW model splits into T^{GKO} and a Q-BRST exact operator $T' = \{Q, X\}$ having a zero conformal central charge ($c' = 0$) and acting trivially on the physical states;

$$Q|Phys\rangle = 0 \ . \qquad (1.2)$$

On the other hand, in the last few years, there has been much interest in the study of integrable deformations of two dimensional conformal theories, especially minimal models [8, 9, 10]. The methods used in these studies are mostly algebraic [1, 11] or a mixture of algebraic methods and quantum fields [12]. Other methods use the Feigin–Fuchs approach [13] or the scattering S matrix techniques [14, 15]. However, very few works, dealing with this subject, use the pure quantum field theoretical approach [16], see also [17]. One of the problems one encounters in using the field theoretical method is basically the lack of the quantum conformal field action S_0 describing the critical theory. Apart from some models like the Ising model [18, 17], the gaussian model [18], the Liouville theory and generalizations [19, 20], there is no quantum field action that describe all models of the minimal series and its extensions. One of the results of this work is to show that such action S can be constructed indeed by using the GWZW as formulated by Karabali and Schnitzer.

Therefore, the aim of this paper is to exploit the recent developments of the GWZW theory in connection with the algebraic coset model in order to build up the quantum field theory of models, of the type of the unitary minimal series. More precisely, we consider cosets of type $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ and study their quantum field representations. Among the results that we have obtained, we mention the following:

(i) The quantum field action S describing the conformal $G_{k_1}\otimes G_{k_2}/G_{k_1+k_2}$ coset models, for simple groups G, reads as:

$$S[k_1,k_2,g_1,g_2,h,b,c,\bar{b},\bar{c}] = k_1 I[g_1] + k_2[g_2] - (k_1+k_2+c_v)I[h] + \int d^2z\ tr(b\bar{\partial}c + \bar{b}\partial\bar{c}) \ , \quad (1.3)$$

where $kI[g]$ is the usual level k, WZW action, see Eq.(2.1); $b,c,\bar{b},\bar{c}$ are the ghost fields and c_v is the quadratic casimir of the group G. g_1,g_2 and h are the WZW fields belonging to the fundamental representation of G and parametrizing respectively the Kac Moody groups G_{k_1}, G_{k_2} and $G_{k_1+k_2}$. Choosing $G = SU(2)$, $k_2 = 1$ and $c_v = 2$, one obtains the conformal field action describing the models of the minimal series of central charge $c = 1 - 6/[(k_1+2)(k_1+3)]$. Note that Eq.(1.3), which reduces to the topological G_k/G_k theory when we set $k_2 = 0$, can also be extended to more than two factors $G_{k_1}\otimes G_{k_2}$.

(ii) The total Kac Moody (KM) analytic current $J(z)$ generating the KM symmetry of Eq.(1.3) is found to be a Q BRST exact operator,

$$J(z) = \{Q, b(z)\} \ , \quad (1.4)$$

exactly as in the topological G/G GWZW theories [21]. Note that this identity can also be viewed as a natural definition of the BRST charge operator Q. Its non degenerate solution coincides with the Karabali Schnitzer BRST operator [5], see also Eq.(4.5). Moreover, projecting Eq.(1.4) on the physical states of the $G_{k_1}\otimes G_{k_2}/G_{k_1+k_2}$ GWZW theory, one obtains constraint equations which by solving them give the physical spectrum of the conformal coset obtained by the standard methods [22]. Also we recover the relation existing between the energy momentum tensor T of the $G_{k_1}\otimes G_{k_2}/G_{k_1+k_2}$ GWZW model and T^{GKO}.

(iii) The full Fock space of the $SU_{k_1}(2)\otimes SU_{k_2}(2)/SU_{k_1+k_2}(2)$ GWZW model is shown to be completely characterized by four quantum numbers namely the energy, the isospin, its projection and the ghost number. It is also shown that the Virasoro primary fields obtained from the algebraic method correspond to the ground states of the quantum field theoretical approach.

(iv) For the interesting example of critical models of the minimal series, we have made contact between the known formula for the conformal weight $\Delta_{r,s}$ and the one obtained by using the $SU_k(2)\times SU_1(2)/SU_{k+1}(2)$ GWZW theory. Among the consequences of this analysis we mention: First, the derivation of the WZW fields realizations for the first primary fields. The latter can be used in the study of the deformation of minimal models, see Eq.(6.13). Second, by setting $k_2 = 0$, we recover the $SU_k(2)/SU_k(2)$ topological WZW model, whose ground states correspond to primary fields of the type $\phi_{r,r}$, $0 \leq r \leq k+1$.

The presentation of this paper is as follows. In Section 2, we analyse the WZW theory for groups of type $G_{k_1} \otimes G_{k_2}$ and write down the corresponding action. In Section 3 we study in detail the $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ GWZW theory. The analysis of the quantum constraints, obtained from the gauging of the anomaly free subgroup G of $G \otimes G$, is made in Section 4. In Section 5 we derive the physical spectrum of the $SU_{k_1}(2) \otimes SU_{k_2}(2)/SU_{k_1+k_2}(2)$ GWZW theory. Interesting consequences at the level of the minimal series are exhibited. Discussions and conclusions are given in Section 6.

2 The $G_{k_1} \otimes G_{k_2}$ WZW theory

We start by considering a general WZW model on a Riemann surface $\mathcal{M}$ with fields $g = g(\xi), \xi = (\xi^1, \xi^2)$, taking values in a representation $R(\lambda_1) \otimes R(\lambda_2)$ of a compact Lie group of type $G = G_1 \otimes G_2$. λ_1 and λ_2 are highest weights associated with the representations $R_1 = R(\lambda_1)$ and $R_2 = R(\lambda_2)$ of the corresponding groups. Before writing down the action $S[G_{k_1} \otimes G_{k_2}]$ of the $G_{k_1} \otimes G_{k_2}$ WZW model that we will consider, let us first recall the action $S[G_k] = S[k, g]$ for the usual G_k WZW theory [23]:

$$\begin{aligned} S[k,g] &= \frac{k}{16\pi} \int_{\mu} d^2\xi \ \operatorname*{tr}_R(\partial_\mu g^{-1} \partial^\mu g) \\ &+ \frac{k}{24\pi} \int_{\mathcal{B}} d^3x \ \epsilon^{\alpha\beta\gamma} \ \operatorname*{tr}_R(\tilde{g}^{-1}\partial_\alpha \tilde{g} \ \tilde{g}^{-1}\partial_\beta \tilde{g} \ \tilde{g}^{-1}\partial_\gamma \tilde{g}) , \end{aligned} \tag{2.1}$$

where $\tilde{g}(x) = \tilde{g}(\xi^0, \xi^1, \xi^2)$ is assumed to be well defined on a three dimensional manifold $\mathcal{B}$ of boundary $\mathcal{M}$. The symbol tilda carried by $\tilde{g}$ will be disregarded in the following. The action (2.1) is invariant under the gauge transformations

$$g(\xi) \longrightarrow \Omega(z) \, g(\xi) \, \bar{\Omega}^{-1}(\bar{z}) , \tag{2.2}$$

where Ω and $\bar{\Omega}$ are R-valued matrices analytically depending on $z = \xi^1 + i\xi^2$ and $\bar{\xi} = \xi^1 - i\xi^2$ respectively. The symmetry (2.2) is generated by conserved currents $J^a_z = J^a$ and $J^a_{\bar{z}} = \bar{J}^a$ satisfying

$$\bar{\partial} J^a = 0, \quad \partial \bar{J}^a = 0, \quad a = 1, 2, \ldots D = \dim G , \tag{2.3}$$

and realized in terms of the WZW field g as

$$\begin{aligned} J^a &\sim k \; tr(t^a g^{-1} \partial g) \\ \bar{J}^a &\sim k \; tr(t^a \bar{\partial} g \; g^{-1}) . \end{aligned} \tag{2.4}$$

In these equations $\partial = \partial/\partial z, \bar{\partial} = \partial/\partial \bar{z}$ and the t^a's are the generators of the R-representation of the compact Lie group G obeying the finite dimensional Lie algebra

$$[t^a, t^b] = i f^{ab}{}_c \, t^c . \tag{2.5}$$

Note that the currents (2.4) obey the O.P.E of the G_K Kac Moody (KM) algebra:

$$\begin{aligned} J^a(z).J^b(w) &= \frac{k/2}{(z-w)}\ \delta^{ab} + i f^{ab}{}_c\ \frac{J^c(w)}{(z-w)} \\ \bar{J}^a(\bar{z}).\bar{J}^b(\bar{w}) &= \frac{k/2}{(\bar{z}-\bar{w})}\ \delta^{ab} + i f^{ab}{}_c\ \frac{\bar{J}^c(w)}{(\bar{z}-\bar{w})}\ . \end{aligned} \tag{2.6}$$

Using the Laurent modes $J_n^a = \oint \frac{dz}{2i\pi}\ z^n\ J^a(z)$, Eqs.(2.6) read as

$$[J_n^a, J_m^b] = i f^{ab}{}_c\ J_{n+m}^c + \frac{k}{2}\ n\ \delta^{ab}\delta_{n+m,0} \tag{2.7}$$

and a simlar relation for the antianalytic sector. Also note that the KM gauge symmetries (2.2) induce a conformal symmetry generated by the Sugawara energy momentum tensor $T(z)$ [24]

$$T(z) = \frac{:J^a(z)\ J^a(z):}{(k+c_v)}\ , \tag{2.8}$$

where c_v is the quadratic Casimir of the adjoint representation of G. Using the relation

$$J^a(z)\ g(w,\bar{w}) = \frac{t^a}{(z-w)}\ g(w,\bar{w})\ , \tag{2.9}$$

together with Eq.(2.8), one can compute the conformal weight Δ_g of the WZW field g. Straightforward algebra shows that:

$$T(z)\ g(w,\bar{w}) = \frac{\Delta_g}{(z-w)^2}\ g(w,\bar{w}) + \frac{1}{(z-w)}\ \partial_w\ g(w,\bar{w}) \tag{2.10}$$

where

$$\Delta_g = C_R/(k+c_v) \tag{2.11}$$

and C_R is the Casimir of the representation R of G in which the field g lives. In the case $G = SU(2)$ and $R = \underline{2}$ for example, the conformal weight Δ_g reads:

$$\Delta_g = j(j+1)/(k+2) = \frac{3}{4(k+2)}\ . \tag{2.12}$$

Having given the basic facts for the G_K WZW theory, we come now to the derivation of the action $S[G_{k_1} \otimes G_{k_2}]$ of the $G_{k_1} \otimes G_{k_2}$ WZW theory. The following construction generalizes the previous review and can be extended to direct products of more than two factors.

Consider first the case $k_1 = k_2 = k$ and parametrize the WZW field $g(\xi)$ as:

$$\begin{aligned} g(\xi) &= g_1(\xi) \otimes g_2(\xi) \\ g^{-1}(\xi) &= g_1^{-1}(\xi) \otimes g_2^{-1}(\xi) \end{aligned} \tag{2.13}$$

where g_1 and g_2 take values respectively in the representations R_1 and R_2 of the groups G_1 and G_2. $g(\xi)$ is a tensor field of $2+2=4$ indices. By inserting the decomposition (2.13) in Eq.(2.1), one obtains

$$\begin{aligned} S[k,g_1,g_2] &= \frac{k}{16\pi}\int_{\mathcal{M}} d^2\xi \ \operatorname*{tr}_R[\partial_\mu(g_1^{-1}\otimes g_2^{-1})\partial^\mu(g_1\otimes g_2)] \\ &+\frac{k}{24\pi}\int_{\mathcal{B}} d^3x \in^{\alpha\beta\gamma} \ \operatorname*{tr}_R[g_1^{-1}\otimes g_2^{-1})\partial_\alpha(g_1\otimes g_2)(g_1^{-1}\otimes g_2^{-1})\partial_\beta(g_1\otimes g_2)\cdot \\ &\qquad (g_1^{-1}\otimes g_2^{-1})\partial_\gamma(g_1\otimes g_2)]\ . \end{aligned} \tag{2.14}$$

where the trace is taken over the tensor product representation $R=R_1\otimes R_2$. Then using the following relations

$$\begin{aligned} \operatorname*{tr}_R[A_1\otimes A_2] &= \operatorname*{tr}_{R_1}[A_1]\cdot\operatorname*{tr}_{R_2}[A_2] \\ \partial_\mu[g_1\otimes g_2] &= (\partial_\mu g_1\otimes g_2)+(g_1\otimes\partial_\mu g_2)\ , \end{aligned} \tag{2.15}$$

one finds the $G_k\otimes G_K$ action:

$$\begin{aligned} S[k,g_1,g_2] &= S[k_1',g_1]+S[k_2',g_2] \\ &+\frac{k}{16\pi}\int_{\mathcal{M}} d^2\xi\ \mathcal{L}_0+\frac{k}{24\pi}\int_{\mathcal{B}} d^3x\ \varepsilon^{\alpha\beta\gamma}\ \mathcal{L}_{\alpha\beta\gamma}\ . \end{aligned} \tag{2.16}$$

$S[k_i',g_i]$ $i=1,2$ is similar to Eq.(2.1) with $k_1'=k\dim R_2, k_2'=k\dim R_1$ and where $\mathcal{L}_0$ and $\mathcal{L}_{\alpha\beta\gamma}$ are given by

$$\mathcal{L}_0=\operatorname*{tr}_{R_1}[\partial_\mu g_1^{-1}g_1]\cdot\operatorname*{tr}_{R_2}[g_2^{-1}\partial^\mu g_2]+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\mu g_1]\ \operatorname*{tr}_{R_2}[\partial^\mu g_2^{-1}g_2] \tag{2.17}$$

$$\begin{aligned} \mathcal{L}_{\alpha\beta\gamma} = &\ \operatorname*{tr}_{R_1}[g_1^{-1}\partial_\alpha g_1 g_1^{-1}\partial_\beta g_1]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\gamma g_2] \\ &+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\alpha g_1 g_1^{-1}\partial_\gamma g_1]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\beta g_2] \\ &+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\beta g_1 g_1^{-1}\partial_\gamma g_1]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\alpha g_2] \\ &+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\alpha g_1]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\beta g_2\ g_2^{-1}\partial_\gamma g_2] \\ &+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\beta g_1]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\alpha g_2\ g_2^{-1}\partial_\gamma g_2] \\ &+\operatorname*{tr}_{R_1}[g_1^{-1}\partial_\gamma g_n]\operatorname*{tr}_{R_2}[g_2^{-1}\partial_\alpha g_2\ g_2^{-1}\partial_\beta g_2]\ . \end{aligned} \tag{2.18}$$

For simple groups G_1 and G_2 for which the following relations

$$\operatorname*{tr}_{R_1}(g_1^{-1}\partial_\mu g)=0,\qquad \operatorname*{tr}_{R_2}[g_2^{-1}\partial_\mu g_2]=0 \tag{2.19}$$

hold, the $G_k\otimes G_k$ WZW action (2.16–18) reduces to the two terms:

$$S[k,g_1,g_2]=S[k_1',g_1]+S[k_2',g_2]\ . \tag{2.20}$$

Here we would like to note the following facts: (i) By setting the WZW field $g_2(\xi)$ to the identity, one recovers the usual G_K WZW model described by the action Eq.(2.1)

$$S[k_1,g_1]=k_1 I[g_1]\ . \tag{2.21}$$

(ii) If the fields g_1 and g_2 belong to representations R_1 and R_2 such that $\dim R_1 = \dim R_2$, then the two terms of Eq.(2.20) are formally the same. Moreover, if the two groups G_1 and G_2 are identical as in the coset models, then the two fields g_1 and g_2 can be used to build up relevant operators involved in the study of the integrable deformations of conformal coset models $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$. For the $SU_k(2) \otimes SU_K(2)/SU_{2k}(2)$ coset model of Ref.[25], the $\phi_{2,1}$ deformation is given by

$$\delta S \sim \int_{\mathcal{M}} d^2\xi \ \underset{R}{\mathrm{tr}}(g_1^{-1} g_2 + g_2^{-1} g_1) \ . \tag{2.22}$$

Other field realizations of conformal primary fields $\phi_{r,s}$ will be given in Section 6 (see Eqs.(6.13)). (iii) Finally, the action $S[k_1, k_2, g_1, g_2]$ of the $G_{k_1} \otimes G_{k_2}$ WZW theory for $k_1 \neq k_2$ is also given by Eq.(2.20) namely:

$$S[k_1, k_2, g_1, g_2] = k_1 I[g_1] + k_2[g_2] \ . \tag{2.23}$$

Varying $S[k_1, k_2, g_1, g_2]$ with respect to g_1 and g_2, one obtains the following two equations of motion

$$\begin{aligned} \bar\partial J_1^a &= 0, \quad \bar\partial J_2^a = 0 \\ \partial \bar J_1^a &= 0, \quad \partial \bar J_2^a = 0 \ , \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} J_1^a &\sim k_1 \ \mathrm{tr}(t^a g_1^{-1} \partial g_1), \quad J_2^a \sim k_2 \ \mathrm{tr}(t^a g_2^{-1} \partial g_2) \\ \bar J_1^a &\sim k_1 \ \mathrm{tr}(t^a \bar\partial g_1 g_1^{-1}), \quad \bar J_2^a \sim k_2 \ \mathrm{tr}(t^a \bar\partial g_2 g_2^{-1}) \ . \end{aligned} \tag{2.25}$$

By adding the two currents J_1^a, J_2^a and using Eqs.(2.6), one finds the O.P.E. algebra

$$J_{12}^a(z) \ J_{12}^b(w) = \frac{(k_1 + k_2)/2}{(z-w)^2} \ \delta^{ab} + i \ f^{ab}{}_c \ \frac{\partial J_{12}^c(w)}{(z-w)} \tag{2.26}$$

where $J_{12}^a = J_1^a + J_2^a$. Note that there are also similar relations for the antianalytic currents $\bar J_{12}^a = \bar J_1^a + \bar J_2^a$ that generate the diagonal subgroup of the $(G_{k_1} \otimes G_{k_2})_R$ right symmetry.

3 The gauged $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ WZW model

Following the standard method used in the study of the gauged G/H WZW theory [6, 26], the action that describes the gauging anomaly free vector subgroup G of the global $(G \otimes G)_L \otimes (G \otimes G)_R$ symmetry is:

$$S[k_1, k_2, g_1, g_2, A, \bar A] = k_1 I[g_1, A, \bar A] + k_2 I[g_2, A, \bar A] \tag{3.1}$$

where $I[g_i, A, \bar A]$ ($i = 1, 2$) are given by:

$$\begin{aligned} k_i I[g_i, A, \bar A] &= k_i I[g_i] + \tfrac{k_i}{4\pi} \ \underset{R_i}{\mathrm{tr}} \int_{\mathcal{M}} d^2 z \ \mathcal{L}(g_i, A, \bar A), \\ \mathcal{L}(g_i, A, \bar A) &= A[\bar\partial g_i g_i^{-1}] - \bar A [g_i^{-1} \partial g_i] + [A g_i \bar A g_i^{-1}] - \bar A A \ . \end{aligned} \tag{3.2}$$

The light cone components A and $\bar{A}$ of the gauge field A_μ which belong to the adjoint representation of G are parametrized as:

$$A = \partial F F^{-1}, \qquad \bar{A} = \bar{\partial} H H^{-1} , \tag{3.3}$$

where F and H are group elements of G. The dependence of Eqs.(3.1-2) on these non propagating fields induces constraints which classically reads

$$\begin{aligned} k_1 g_1 \bar{\partial} g_1^{-1} + k_2 g_2 \bar{\partial} g_2^{-1} + g_1 \bar{A} g_1^{-1} + g_2 \bar{A} g_2^{-1} - \bar{A} &= 0 \\ k_1 g_1^{-1} \partial g_1 + k_2 g_2^{-1} \partial g_2 + g_1^{-1} A g_1 + g_2^{-1} A g_2 - A &= 0 . \end{aligned} \tag{3.4}$$

At the quantum level there is analogous constraints which play a crucual role in the identification of the spectrum of the conformal $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ coset models. For details see Sections 4 and 5. To write down these constraints, one should first calculate the partition function Z associated with the action (3.1-2) and then derive the Green functions for the Kac Moody currents. Let us give the main lines of the computation for the derivation of the quantum constraints. The $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ partition function Z is,

$$Z = \int \mathcal{D}g_1 \mathcal{D}g_2 \mathcal{D}A \mathcal{D}\bar{A} \exp\left[-k_1, I_1[g_1, A, \bar{A}] - k_2 I[g_2, A, \bar{A}]\right] . \tag{3.5}$$

Using the Polyakov and Weigmann identity [27, 6, 26]

$$I[g_i, F] = I[g_i] + I[F] - \frac{1}{4\pi} \int_{\mathcal{M}} d^2z \ \mathop{\mathrm{tr}}_{R_i}(g_i^{-1} \partial g_i \bar{\partial} F F^{-1}) , \tag{3.6}$$

we can rewrite the action (3.1-2) as

$$S[k_1, k_2, g_1, g_2, A, \bar{A}] = k_1 I[F^{-1} g_1 H] + k_2 I[F^{-1} g_2 H] - (k_1 + k_2) I[F^{-1} H] . \tag{3.7}$$

Then using the change of variables Eqs.(3.3) and taking into account Eq.(3.7), the partition function Z becomes

$$\begin{aligned} Z = \int \mathcal{D}g_1 \mathcal{D}g_2 \mathcal{D}F \mathcal{D}H \det D \det \bar{D} \ \ &\exp\ \ [-k_1 I(F^{-1} g_1 H) \exp[-k_2 I(F^{-1} g_2 H)] \times \\ &\exp\ \ [+(k_1 + k_2) I(F^{-1} H)] . \end{aligned} \tag{3.8}$$

where the Jacobians $\det D$ and $\det \bar{D}$ corresponding to the change of variables (3.3) are as follows

$$\begin{aligned} \det D &= \det(\partial - [A,]) \\ \det \bar{D} &= \det(\bar{\partial} - [\bar{A},]) \\ \det D \det \bar{D} &= \exp(2c_v I[F^{-1} H]) \det \partial \det \bar{\partial} . \end{aligned} \tag{3.9}$$

$\det \partial$ and $\det \bar{\partial}$ are free chiral determinants which respectively can be represented by ghost fields b, c and $\bar{b}, \bar{c}$ belonging to the adjoint representation of G. As usual the ghost c and $\bar{c}$ have conformal weight zero while b and $\bar{b}$ have conformal weight one.

Moreover, using the vector invariance of the Haar measure $\mathcal{D}g_1\mathcal{D}g_2\mathcal{D}F\mathcal{D}H\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}\bar{b}$, one can make the following convenient change of variables

$$\begin{aligned} \tilde{g}_1 &= F^{-1}g_1H, \quad \mathcal{D}\tilde{g}_1 = \mathcal{D}g_1 \\ \tilde{g}_2 &= F^{-1}g_2H, \quad \mathcal{D}\tilde{g}_2 = \mathcal{D}g_2 \ , \end{aligned} \tag{3.10}$$

leading to

$$Z = \int \mathcal{D}g_1\mathcal{D}g_2\mathcal{D}F\mathcal{D}H\mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c} \ \exp\ [-k_1I[(g_1) - k_2I(g_2)] \exp[(k_1+k_2+2c_v)I(F^{-1}H)] \times \\ \exp\ [-\mathrm{tr}\int_{\mathcal{M}} d^2z[b\bar{\partial}c + \bar{b}\partial\bar{c}] \ . \tag{3.11}$$

Finally, choosing the gauge $A = 0$ or equivalently $H = 1$, one obtains the following gauge fixed partition function

$$Z = \int \mathcal{D}g_1\mathcal{D}g_2\mathcal{D}F\mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c} \ \exp\ [-k_1I(g_1) - k_2I(g_2)] \exp[(k_1+k_2+2C_v)I[F^{-1}]] \times \\ \exp\ [-\mathrm{tr}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c})] \ , \tag{3.12}$$

where F^{-1} has been replaced by F.

To identify the quantum constraints that we have mentioned earlier, one couples the action $S = S[k_1, k_2, g_1, g_2, F, b, c, \bar{b}, \bar{c}]$,

$$S = k_1I[g_1] + b_2I[g_2] - (k_1+k_2+2C_v)I[F] + \int d^2z \ \mathrm{tr}(b\bar{\partial}c + \bar{b}\partial c) \tag{3.13}$$

to the external gauge fields B and $\bar{B}$ either as we did for Eq.(3.1) or equivalently by means of the minimal substitution $\partial \to \nabla = \partial - B$ and $\bar{\partial} = \bar{\partial} - \bar{B}$,

$$B = U^{-1}\partial U; \qquad \bar{B} = V^{-1}\bar{\partial}V \ , \tag{3.14}$$

where U and V are group elements of G. The resulting action $I[B, \bar{B}]$ describing couplings to the external fields is then:

$$S[B, \bar{B}] = k_1I[g_1, B, \bar{B}] + k_2I[g_2, B, \bar{B}] - (k_1+k_2+2c_v)I[F, B, \bar{B}] - \int d^2z \ \mathrm{tr}(b\bar{\nabla}c + \bar{b}\nabla\bar{c}) \ . \tag{3.15}$$

The corresponding partition $Z[B, \bar{B}]$ reads,

$$Z[B, \bar{B}] = \int \mathcal{D}g_1\mathcal{D}g_2\mathcal{D}F\mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c} \exp(-S[B, \bar{B}]) \ . \tag{3.16}$$

Using now Eqs.(3.14) and (3.6), one may check that the following relations hold

$$\begin{aligned} k_1I[g_1, B, \bar{B}] &= k_1I[U^{-1}g_1V] - k_2I[U^{-1}V] && (a) \\ k_2I(g_2, B, \bar{B}] &= k_2I[U^{-1}g_2V] - k_2I[U^{-1}V] && (b) \\ -(k_1+k_2+2c_v)I[F, B, \bar{B}] &= -(k_1+k_2+2c_v)I[U^{-1}FV] + (k_1+k_2+2c_v)I[U^{-1}V] && (c) \\ \textstyle\int d^2z \ \mathrm{tr}(b\bar{\nabla}c + \bar{b}\nabla\bar{c}) &= \textstyle\int d^2z \ \mathrm{tr}(b\bar{\partial}c + \bar{b}\partial c) - 2c_vI[U^{-1}V] \ . && (d) \end{aligned} \tag{3.17}$$

Note that in establishing Eq.(3.17-d), one should do the chiral rotations

$$
\begin{array}{ll} c' = V^{-1}c, & \bar{c}' = U^{-1}\bar{c} \\ b' = bV, & \bar{b}' = \bar{b}U \ , \end{array} \tag{3.18}
$$

which gives the free ghost system and a WZW term, $2c_v I[U^{-1}V]$, coming from the Jacobian of the transformation of the Haar measure [26]. Putting Eqs.(3.17) back into Eqs.(3.15–16), one discovers that the partition function (3.16) does not depend on the external fields B and $\bar{B}$. Consequently, functional derivaties of $Z[B,\bar{B}]$ with respect to B and $\bar{B}$ vanish identically. In particular, we have the following quantum constraints

$$
\begin{aligned}
\left.\frac{\delta Z[B,\bar{B}]}{\delta \bar{B}}\right|_{B=\bar{B}=0} &= \langle Phys|J_{tot}|Phys\rangle = 0 \quad (a) \\
\left.\frac{\delta Z[B,\bar{B}]}{\delta B}\right|_{B=\bar{B}=0} &= \langle Phys|\bar{J}_{tot}|Phys\rangle = 0 \quad (b)
\end{aligned} \tag{3.19}
$$

where $T_{tot} = J_1 + J_2 + \tilde{J}_{12} + J_{gh}$ and $\bar{J}_{tot} = \bar{J}_1 + \bar{J}_2 + \bar{\tilde{J}}_{12} + \bar{J}_{gh}$ are the total left and right Kac Moody currents given by

$$
\begin{aligned}
J_1 &= k_1 g_1^{-1}\partial g_1 \\
J_2 &= k_2 g_2^{-1}\partial g_2 \\
\tilde{J}_{12} &= -(k_1 + k_2 + 2c_v)F^{-1}\partial F \\
J_{gh} &= \ :b(z)c(z):
\end{aligned} \tag{3.20}
$$

and similar relations for the antianalytic sector.

4 More on the quantum constraints

Using the following O.P.E. algebras

$$
\begin{aligned}
J_1^a(z)J_1^b(w) &= (z-w)^{-2}\left[\tfrac{k_1}{2}\delta^{ab} + i f^{ab}{}_c (z-w)\partial J_1^c(w)\right] \\
J_2^a(z)J_2^b(w) &= (z-w)^{-2}\left[\tfrac{k_2}{2}\delta^{ab} + i f^{ab}{}_c (z-w)\partial J_2^c(w)\right] \\
\tilde{J}_{12}^a(z)\tilde{J}_{12}^b(w) &= (z-w)^{-2}\left[-\tfrac{(k_1+k_2+2c_v)}{2}\delta^{ab} + i f^{ab}{}_c (z-w)\partial \tilde{J}_{12}^c(w)\right] \\
J_{gh}^a(z)J_{gh}^b(w) &= (z-w)^{-2}[c_v\delta^{ab} + i f^{ab}{}_c (z-w)\partial J_{12}^c(w)]
\end{aligned} \tag{4.1}
$$

one finds: (i) The central charge k of the total current J_{tot}^a is identically zero:

$$
J_{tot}^a(z)J_{tot}^b(w) = i f^{ab}{}_c \partial J^c(w)/(z-w) \ . \tag{4.2}
$$

(ii) Writing J_{tot}^a as

$$
J_{tot}^a = (J_1^a + J_2^a - J_{12}^a) + (J_{12}^a + \tilde{J}_{12}^a + J_{gh}^a) \ , \tag{4.3}
$$

one sees that the two terms of the right hand side of this equation correspond to the GKO $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ coset current and a topological term of zero KM central charge. Note that for the cosets we are considering in this study, $J^a_{GKO} = 0$ as shown by Eq.(2.26). Using this property and dimensional arguments, one sees that the total KM currents J^a_{tot} are BRST exact operators of the form

$$J^a(z) = \{Q, b^a(z)\} \ , \tag{4.4}$$

where Q is the BRST charge operator of the GWZW theory. Actually Eq.(4.4) may be viewed as a natural definition of the BRST charge. A non degenerate solution of this equation is given by the BRST nilpotent charge operator:

$$Q = \oint \frac{dz}{2i\pi} \left\{ : c^a(z) \left[J_1^a(z) + J_2^b(z) + \tilde{J}_{12}(z) \right] : -\frac{i}{2} f^{ab}{}_c : c^a(z) b^b(z) c^c(z) : \right\} . \tag{4.5}$$

Expanding J_{tot}, b and c in Laurent series as

$$\begin{aligned} J^a_{tot} &= \sum_{n=-\infty}^{\infty} z^{-m-1} \ J^a_{tot,n} \\ b^a(z) &= \sum_{n=-\infty}^{\infty} z^{-n-1} \ b^a_m, \\ c^a(z) &= \sum_{n=-\infty}^{\infty} z^{-n} \ c_n \ , \end{aligned} \tag{4.6}$$

and using the canonical anticommutation relations

$$\begin{aligned} \{c^a_n, c^b_m\} &= \{b^a_n, b^b_m\} = 0 \\ \{b^a_n, c^b_m\} &= \delta^{ab}\delta_{n+m,0} \ , \end{aligned} \tag{4.7}$$

together with

$$\begin{aligned} J^a_n &= \{Q, b^a_n\} & (a) \\ Q &= \sum_{n=-\infty}^{\infty} : c^a_{-n}[J^a_{1,n} + J^a_{2,n} + \tilde{J}^a_{12,n}] : -\tfrac{1}{2} f^{ab}{}_c \sum_{n,m=-\infty}^{\infty} : c^a_{-n} b^b_{-n} c^c_{n+m} : , & (b) \end{aligned} \tag{4.8}$$

the quantum constraint Eqs.(3.19) read as:

$$\langle Phys|J^a_{tot,n}|Phys\rangle = \langle Phys|Qb^a_n + b^a_n Q|Phys\rangle = 0 \ . \tag{4.9}$$

Physical states of the $G_{k_1} \otimes G_{k_2}/G_{k_1+k_2}$ GWZW model are then defined as

$$\begin{aligned} J^a_{tot,n}|Phys\rangle &= 0; \quad n > 0, a = 1, \ldots, D & (a) \\ J^\alpha_{tot,0}|Phys\rangle &= 0; \quad \alpha \text{ positive roots of } G & (b) \\ J^i_{tot,0}|Phys\rangle &= 0; \quad i = 1, \ldots, r = \text{rank } G \ . & (c) \end{aligned} \tag{4.10}$$

An equivalent statement is

$$Q|Phys\rangle = 0 \ . \tag{4.11}$$

Note that Eqs.(4.10, b-c) show that physical states behave as scalars only with respect to the algebra generated by the modes $J^a_{tot,o}$:

$$[J^a_{tot,0}, J^b_{tot,0}] = i \ f^{ab}{}_c \ J^c_{tot,0} \ . \tag{4.12}$$

The holomorphic total energy momentum tensor $T(z)$ generating the conformal symmetry of the GWZW action (3.13) reads as:

$$T(z) = T_1(z) + T_2(z) + \tilde{T}_{12}(z) + T_{gh}(z) \ , \tag{4.13}$$

where T_1, T_2 and T_{12} are given by the Sugawara construction

$$\begin{aligned} (k_1 + c_v)T_1(z) &= \ : J_1^a(z) J_1^a(z) : \\ (k_2 + c_v)T_2(z) &= \ : J_2^a(z) J_2^a(z) : \\ [-(k_1 + k_2 + 2c_v) + c_v]\tilde{T}_{12}(z) &= \ : \tilde{J}_{12}(z) \tilde{J}_{12}(z) : \ , \end{aligned} \tag{4.14}$$

and the ghost term is given by the usual free fermionic realization

$$T_{gh}(z) =: b^a \partial c^a : \ . \tag{4.15}$$

These conserved currents obey the following O.P.E. algebra

$$\begin{aligned} T_1(z)T_1(0) &= \ z^{-4}\left[\tfrac{C(G,k_1)}{2} + 2z^2 T_1(0) + z\partial T_1(0)\right] \\ T_2(z)T_2(0) &= \ z^{-4}\left[\tfrac{C(G,k_2)}{2} + 2z^2 T_2(0) + z\partial T_2(0)\right] \\ \tilde{T}_{12}(z)\tilde{T}_{12}(0) &= \ z^{-4}\left[\tfrac{C(G,-k_1-k_2-2c_v)}{2} + 2z^2 T_{12}(0) + z\partial \tilde{T}_{12}(0)\right] \\ T_{gh}(z)T_{gh}(0) &= \ z^{-4}[-2D/2 + 2z^2 T_{gh}(0) + z\partial T_{gh}(0)] \ , \end{aligned} \tag{4.16}$$

where the conformal anomaly $C[G,x]$ is given by

$$C[G,x] = \frac{xD}{x + c_v} \ . \tag{4.17}$$

Using the above relations, one finds that

$$T(z)T(0) = z^{-4}[C_{tot}/2 + 2z^2 T(0) + z\partial T(0)] \ , \tag{4.18}$$

where the total central charge is

$$C_{tot} = \frac{k_1 D}{k_1 + c_v} + \frac{k_2 D}{k_2 + c_v} + \frac{(k_1 + k_2 + 2c_v)D}{k_1 + k_2 + c_v} - 2D \ . \tag{4.19}$$

Similarly as for Eqs.(4.3-4), the total energy momentum tensor can be decomposed as the sum of the GKO term T^{GKO} plus a topological term T' where

$$\begin{aligned} T^{GKO} &= \ T_1 + T_2 - T_{12}, \\ T_{12} &= \ \tfrac{:J_{12}^a(z) J_{12}^a(z):}{(k_1 + k_2 + c_v)} \ . \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} T' &= T_{12} + \tilde{T}_{12} + T_{gh} \\ &= \frac{\{: J_{12}^a(z) J_{12}^a(z) : - : \tilde{J}_{12}^a(z) \tilde{J}_{12}(z) :\}}{(k_1 + k_2 + c_v)} - : b^a(z)\partial c^a(z) : . \end{aligned} \tag{4.21}$$

As observed by Karabali and Schnitzer, the two above currents obey the following O.P.E. algebra

$$\begin{aligned} T^{GKO}(z)T^{GKO}(0) &= z^{-4}[C_{GKO}/2 + 2z^2T^{GKO}(0) + z\partial T^{GKO}(0)] \\ T^{GKO}(z)T'(0) &= \text{Regular terms} \\ T'(z)T'(0) &= z^{-4}[0 + 2z^{-2}T'(0) + z\partial T'(0)] \ . \end{aligned} \tag{4.22}$$

Such O.P.E. algebra may also be seen at the level of the conformal anomaly Eq.(4.19) which decomposes as:

$$\begin{array}{lll} C_{tot} & = C_{GKO} + C', & (a) \\ C_{GKO} & = C(G, k_1) + C(G, k_2) - C(G, k_1 + k_2) & (b) \\ C' & = C(G, k_1 + k_2) + C(G, -k_1 - k_2 - 2c_v) + C_{gh} = 0 \ . & (c) \end{array} \tag{4.23}$$

Eqs.(4.21) and (4.23-c) together with the dimensional arguments show that T' is a topological quantity given by

$$T' = \{Q, : b^a(z)[J^a_{12} - \tilde{J}^a_{12}] :\}/(k_1 + k_2 + c_v) \ . \tag{4.24}$$

Expanding the analytic currents $T(z)$, T^{GKO} and T' in Laurent series as,

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} z^{-n-2} L_m \\ T^{GKO}(z) &= \sum_{n=-\infty}^{\infty} z^{-n-2} L^{GKO}_m \\ T'(z) &= \sum_{n=-\infty}^{\infty} z^{-n-2} L'_m \ , \end{aligned} \tag{4.25}$$

and parametrizing the physical states $|Phys\rangle$ in terms of the energies Δ_i's and the highest weights λ_i's as:

$$|Phys\rangle = |\Delta_1, \lambda_1\rangle|\Delta_2, \lambda_2\rangle|\tilde{\Delta}_{12}, \tilde{\lambda}_{12}\rangle|\Delta_{gh}, \lambda_{gh}\rangle \ , \tag{4.26}$$

one can compute the conformal weights Δ, Δ_{GKO} and Δ' corresponding to the eigenvalues of L_0, L_0^{GKO} and L'_0 respectively. We find

$$\Delta = \Delta_1 + \Delta_2 - \tilde{\Delta}_{12} + \Delta_{gh} \tag{4.27}$$

where Δ_1, Δ_2 and $\tilde{\Delta}_{12}$ are given by:

$$\begin{aligned} \Delta_1 &= [cas(\lambda_1)/(k_1 + c_v)] + n_1, \\ \Delta_2 &= [cas(\lambda_2)/(k_2 + c_v)] + n_2, \\ \tilde{\Delta}_{12} &= [cas(\tilde{\lambda}_{12})/(k_1 + k_2 + c_v)] + \tilde{n}_{12} \end{aligned} \tag{4.28}$$

$$n_1, n_2, \tilde{n}_{12} \quad \text{are positive integers.}$$

Δ_{GKO} is computed by using the constraint $\Delta' = 0$ corresponding to the vanishing of the topological energy i.e.

$$\begin{aligned} \Delta_{12} &= \tilde{\Delta}_{12} - \Delta_{gh} \\ \Delta &= \Delta_{GKO} = \Delta_1 + \Delta_2 - \Delta_{12} \ . \end{aligned} \tag{4.29}$$

Therefore the field theoretical and the algebraic coset methods lead to the same conformal weights.

5 Spectrum of the $[SU(2)_{k_1} \otimes SU(2)_{k_2}/SU_{k_1+k_2}(2)]$ WZW models

In this section we analyse the spectrum of unitary representations of the $[SU(2)_{k_1} \otimes SU(2)_{k_2}/SU_{k_1+k_2}(2)]$ GWZW theory by using the methods of quantum field theory. Apart from the works of [16, 17] and a few others, most studies dealing with this subject use algebraic methods based essentially on the GKO coset construction [1]. As ghost states play a crucial role in constructing the physical spectrum of the GWZW theory, we start first by considering this sector.

5.1 The ghost sector

The first fact we want to show here is that the ghost Fock space $\mathcal{F}_{gh}$ can be decomposed into orthogonal subspaces $\mathcal{F}_{gh,m}^{(n,j,\eta)}$ completely characterized by the quantum numbers n, j, m and η corresponding respectively to the energy, the $SU(2)$ ghost isospin, its projection and the ghost number. In symbols

$$\mathcal{F}_{gh} = \bigoplus_{\eta=-\infty}^{\infty} \bigoplus_{j=0}^{\infty} \bigoplus_{m=-j}^{j} \bigoplus_{n=0}^{\infty} \mathcal{F}_{gh,m}^{(n,j,\eta)} . \tag{5.1}$$

Secondly, as a consequence of the $SU_2(2)$ Kac Moody symmetry Eq.(4.1) of the ghost sector of the GWZW theory:

$$\begin{aligned} [J^0_{gh,n}, J^{\pm}_{gh,m}] &= \pm J^{\pm}_{gh,n+m} \\ [J^{+}_{gh,n}, J^{-}_{gh,m}] &= J^0_{gh,n+m} + 2n\delta_{n+m,0} \\ [J^0_{gh,n}, J^0_{gh,m}] &= 2n\delta_{n+m,0} , \end{aligned} \tag{5.2}$$

there are five integrable representations $|\phi_{j_{gh}}\rangle$ having $j_{gh} = 0, 1/2, 1, 3/2$ and $j_{gh} = c_v = 2$.

To establish the decomposition Eq.(5.1), consider first the ghost operator number N of the $[SU_{k_1}(2) \otimes SU_{k_2}(2)/SU_{k_1+k_2}(2)]$ GWZW theory and write it as

$$N = - \sum_{m=-\infty}^{\infty} : (b^+_m c^-_{-m} - c^+_m b^-_{-m} + b^0_m c^0_{-m}) : . \tag{5.3}$$

Because of the identities

$$\begin{aligned} [N, c^a_n] &= c^a_n ; a = 0, +, - \\ [N, b^a_n] &= -b^a_n ; n \in \boldsymbol{Z} , \end{aligned} \tag{5.4}$$

ghost states that are created from the ghost vacuum $|0\rangle_{gh}$ for example, i.e.,

$$|ghost\rangle = \prod_{i=1}^{\alpha^+} c^+_{-n_i} \prod_{j=1}^{\alpha^-} c^-_{-n_j} \prod_{k=1}^{\alpha^0} c^0_{-m_{0,k}} \prod_{\ell=1}^{\beta^+} b^+_{-m_\ell} \prod_{r=1}^{\beta^-} b^-_{-m_r} \prod_{s=1}^{\beta^0} b^0_{-m_{0,s}} |0\rangle_{gh} , \tag{5.5}$$

are eigenstates of N with eigenvalue $\eta = (\alpha^+ - \beta^+) + (\alpha^- - \beta^-) + (\alpha^0 - \beta^0)$. This means that the corrresponding ghost Fock space $\mathcal{F}_{gh}$ can be decomposed into orthogonal subspaces $\mathcal{F}^{\eta}_{gh}$ whose states have all of them the same ghost number η. We have

$$\begin{aligned} \mathcal{F}_{gh} = & \dots \oplus \mathcal{F}^{-1}_{gh} \oplus \mathcal{F}^{0}_{gh} \oplus \mathcal{F}^{-1}_{gh} \oplus \dots \\ = & \bigoplus_{\eta=-\infty}^{\infty} \mathcal{F}^{\eta}_{gh} \ . \end{aligned} \tag{5.6}$$

Ghost states having $\eta = 0$ are the vacuum $|0\rangle_{gh}, b^{a}_{-n}c^{b}_{-m}|0\rangle_{gh}$, n, m positive integers, and more generally ghost states having as many c creation operators as b operators; like

$$\prod_{i} J^{a_i}_{-n_i}|0\rangle_{gh} \in \mathcal{F}^{0}_{gh} \ . \tag{5.7}$$

The next step in the proof of Eq.(5.1) is that given a Fock subspace $\mathcal{F}^{\eta}_{gh}$ with given ghost number η, one can split it into subsets $\mathcal{F}^{(j,\eta)}_{gh}$ of different $SU(2)$ integer isospins j. This structure follows from the property

$$[N, J^{a}_{gh,n}] = 0; \quad a = 0, +, -; n \in \mathbb{Z} \ . \tag{5.8}$$

where $J^{a}_{gh,n}$ are the modes of the ghost KM currents

$$\begin{aligned} J^{\pm}_{gh,n} &= \sum_{m=-\infty}^{\infty} :(b^{\pm}_{n}c^{0}_{n-m} + b^{0}_{m}c^{\pm}_{n-m}): \\ J^{0}_{gh,n} &= \sum_{m=-\infty}^{\infty} :(b^{+}_{m}c^{-}_{n-m} + c^{+}_{m}b^{-}_{n-m}): \ . \end{aligned} \tag{5.9}$$

Note that ghost states of Eq.(5.6) can be constructed by using the above KM modes. We have for example

$$\begin{aligned} \prod_{i\geq 1} J^{a_i}_{-n_i} c^{a_0}_{-n_0}|0\rangle_{gh} &\in \mathcal{F}^{1}_{gh} \\ \prod_{i\geq 1} J^{a_i}_{-n_i} b^{a_0}_{-n_0}|0\rangle_{gh} &\in \mathcal{F}^{-1}_{gh} \end{aligned} \tag{5.10}$$

Eqs.(5.10) give ghost states of a definite ghost number η, but arbitrary isospin j. This degeneracy is exactly what Eq.(5.8) means. Accordingly, each set $\mathcal{F}^{\eta}_{gh}$ of a given ghost number η can be decomposed into orthogonal subspaces $\mathcal{F}^{(j,\eta)}_{gh}$ of definite ghost isospin j:

$$\begin{aligned} \mathcal{F}^{\eta}_{gh} &= \bigoplus_{j\geq 0} \mathcal{F}^{(j,\eta)}_{gh} \\ \mathcal{F}^{(j,\eta)}_{gh} &= \bigoplus_{m=-j}^{j} \mathcal{F}^{(j,\eta)}_{gh,m} \ . \end{aligned} \tag{5.11}$$

where $\mathcal{F}^{(j,\eta)}_{gh,m}$ is the set of ghost states of ghost number η, isospin j and isospin projection m. For example

$$c^{+}_{-n}|0\rangle_{gh} \in \mathcal{F}^{1,1}_{gh,1} \ . \tag{5.12}$$

The last step for the proof of Eq.(5.1) is that both the operators N and $J^a_{gh,0}$ commute with the ghost hamiltonian $L_{gh,0}$:

$$L_{gh,0} = \sum_{m=-\infty}^{\infty} m : (c^+_m b^-_{-m} - b^+_m c^-_{-m} - b^0_m c^0_{-m}) : \quad . \tag{5.13}$$

This property in turn implies that given a set $\mathcal{F}^{(j,\eta)}_{gh,m}$, we have the following decomposition:

$$\mathcal{F}^{(j,\eta)}_{gh,m} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n,j,\eta)}_{gh,m} \quad . \tag{5.14}$$

This shows that the ghost state of Eq.(5.12) belongs to $\mathcal{F}^{(n,1,1)}_{gh,1}$.

5.2 The matter sector

There are two matter sectors: the positive matter sector whose states are denoted as $|\text{matter}\rangle_+$ and the auxiliary sector denoted by $|\text{matter}\rangle_-$. For the positive matter sector of the $SU_{k_1}(2) \otimes SU_{k_2}(2)/SU_{k_1+k_2}(2)$ GWZW quantum field theory, the Fock space $\mathcal{F}_+$ is a tensor product of the Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ associated with the $SU_{k_1}(2)$ and $SU_{k_2}(2)$ KM symmetries (see Eq.(3.13)):

$$\mathcal{F}_+ = \mathcal{F}_1 \otimes \mathcal{F}_2 \ . \tag{5.15}$$

Moreover, using the $SU_{k_i}(2)$ symmetries,

$$\begin{aligned} [J^0_{i,n}, J^\pm_{i,m}] &= \pm J^\pm_{i,n+m} \\ [J^+_{i,n}, J^-_{i,m}] &= \pm J^0_{i,n+m} + \frac{k_i}{2}\, n\delta_{n+m} \\ [J^0_{i,n}, J^0_{i,m}] &= \frac{k_i}{2}\, n\delta_{n+m,0} \ , \end{aligned} \tag{5.16}$$

which show that the two subsectors, $i = 1, 2$, of the positive matter sector, have $k_i + 1$ integrable representations $|\phi_{j_i}\rangle$ of isospins $j_i = 0, 1/2, \ldots, k_i/2$ and conformal weights Δ_i

$$\Delta_i = \frac{j_i(j_i + 1)}{k_i + 2} \ . \tag{5.17}$$

One can see that the Fock space $\mathcal{F}_+$ of the primary states $|\phi_{j_i}\rangle$ can be expressed as:

$$\mathcal{F}_+ = \left[\bigoplus_{j_1=0}^{\frac{k_1}{2}} \mathcal{F}_{j_1}\right] \otimes \left[\bigoplus_{j_2=0}^{\frac{k_2}{2}} \mathcal{F}_{j_2}\right] \ . \tag{5.18}$$

Furthermore, each set $\mathcal{F}_{j_i}$ in turn can be decomposed into direct sum of subspaces $\mathcal{F}^{(\Delta_i+n_i,\ell_i,0)}_{j_i,m}$ in the same way as we did for $\mathcal{F}_{gh}$,

$$\mathcal{F}_{j_i} = \bigoplus_{n_i \geq 0} \bigoplus_{\ell_i \geq 0} \bigoplus_{\ell_i \leq m_i \leq \ell_i} \mathcal{F}^{(\Delta_i+n_i,\ell_i,0)}_{j_i,m_i} \ . \tag{5.19}$$

The only difference with the decomposition Eq.(5.1) is that in the matter sector the ghost number is always zero. For example

$$J^{\pm,0}_{i,-n}|\phi_{j_i}\rangle \in \mathcal{F}^{(\Delta_1+n,1,0)}_{j_i,\pm,0} \ . \tag{5.20}$$

Similarly, the Fock space $\tilde{\mathcal{F}}_{12}$ of the auxiliary, $|\mathrm{matter}\rangle_-$, decomposes into a direct sum of subspace $\mathcal{F}^{(\tilde{\Delta}_{12}+n,\ell m,0)}_{12,m}$ as:

$$\begin{aligned} \mathcal{F}_{12} &= \bigoplus_{j_{12}=0}^{(k_1+k_2+2)/2} \mathcal{F}_{j12} \\ \mathcal{F}_{\tilde{j}_{12}} &= \bigoplus_{n_{12}\geq 0} \bigoplus_{\ell\geq 0} \bigoplus_{-\ell\leq m\leq \ell} \mathcal{F}^{(\tilde{\Delta}_{12}+\tilde{n}_{12},\ell,0)}_{\tilde{j}_{12},m} \end{aligned} \tag{5.21}$$

where $\tilde{\Delta}_{12} = [\tilde{j}_{12}(\tilde{j}_{12}+1)/(k_1+k_2+2)]$.

To summarize the previous analysis, physical states

$$|Phys\rangle = |matter\rangle_+|matter\rangle_-|ghost\rangle \ , \tag{5.22}$$

of the $SU_{k_1}(2)\times SU_{k_2}(2)/SU_{k_1+k_2}(2)$ GWZW models are completely specified by the set $S_1 \cup S_2$ of quantum numbers corresponding to the primary states and their discendent respectively:

$$\begin{aligned} S_1 &= \{k_1 k_2; \Delta_1, j_1, m_1; \Delta_2, j_2, m_2; \tilde{\Delta}_{12}, \tilde{j}_{12}, \tilde{m}_{12}; \Delta_{gh}, j_{gh}, m_{gh}\} \\ S_2 &= \{n_1, j'_1, m'_1; n_2, j'_2, m'_2; \tilde{n}_{12}, \tilde{j}'_{12}, \tilde{m}'_{12}, n_g, j'_{gh}, m'_{gh}\} \ , \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} &0 \leq j_1 \leq \tfrac{k_1}{2}, \quad -j_1 \leq m_1 \leq j_1 \\ &0 \leq j_2 \leq \tfrac{k_2}{2}, \quad -j_2 \leq m_2 \leq j_2 \\ &0 \leq j_{gh} \leq 2; \quad -j_{gh} \leq m_2 \leq j_{gh} \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} &j'_1 \geq 0; \quad -j'_1 \leq m'_1 \leq j'_1 \\ &j'_2 \geq 0; \quad -j'_2 \leq m'_2 \leq j'_2 \\ &\tilde{j}'_{12} \geq 0; \quad -\tilde{j}'_{12} \leq m'_{12} \leq \tilde{j}'_{12} \\ &j'_{gh} \geq 0; \quad -j'_{gh} \leq m'_{gh} \leq j'_{gh} \\ &n_1, n_2, \tilde{n}_{12}, n_{gh} \quad \text{positive integers.} \end{aligned} \tag{5.25}$$

Imposing the quantum constraints Eqs.(4.10) which in our case read as:

$$\begin{aligned} (J^+_{1,0} + J^+_{2,0} - \tilde{J}^+_{12,0} + J^+_{gh,0})|Phys\rangle &= 0 \\ (J^0_{1,0} + J^0_{2,0} - \tilde{J}^0_{12,0} + J^0_{gh,0})|Phys\rangle &= 0 \end{aligned} \tag{5.26}$$

one obtains the following relations

$$\begin{aligned} \sum_{\alpha=1,2,gh} [j_\alpha(j_\alpha+1) - m_\alpha(m_\alpha+1)]^{1/2} - [\tilde{j}_{12}(\tilde{j}_{12}+1) - \tilde{m}_{12}(\tilde{m}_{12}+1)] = 0 \\ m_1 + m_2 - \tilde{m}_{12} + m_{gh} = 0 \ , \end{aligned} \tag{5.27}$$

where the unknown quantities are $\tilde{j}_{12}$ and $\tilde{m}_{12}$. In the next section we will solve these constraint equations for the minimal models by using the techniques of the Clebsh–Gordon decomposition of the isospin.

6 Discussions and Conclusion

One of the consequences of the study carried out in this paper is that unitary minimal models of conformal central charge $c(k) = 1-6/[(k+2)(k+3)]$, $k = 1, 2, \ldots$ are described by the effective action

$$S = kI[g_1] + I[g_2] - (k+3)I[h] + \int d^2z \ \mathrm{tr}(b\bar{\partial}c + \bar{b}\partial\bar{c}) \ , \tag{6.1}$$

where g_1, g_2 and h are 2×2 matrix elements of the $SU(2)$ groups. The $I[f]$'s, $f = g_1, g_2, h$, are obviously the WZW actions given by Eq.(3.13). The conformally invariant action (6.1) describes the universality classes of critical models of two dimensional statistical mechanics such as the $2d$, $c = 1/2$ critical and $c = 7/10$ tricritical Ising models corresponding to $k = 1$ and $k = 2$ respectively. Moreover for a given value of k, the primary fields $\phi_{rs}; 1 \le r \le k+1, 1 \le s \le k+2$ whose conformal weight $\Delta_{r,s}(k)$

$$\Delta_{rs}(k) = \frac{[r[k+3] - s[k+2]]^2 - 1}{4(k+2)(k+3)} \ , \tag{6.2}$$

correspond, in the language of gauged WZW theory, to the ground states $\phi(j_1, j_2, j_{12})$ of the effective action (6.1). Recall that according to Eqs.(4.9-10) and (4.27-29), these constrained vacua lead to the following $SU_k(2) \otimes SU_1(2)/SU_{k+1}(2)$ conformal weights $\Delta_{j_1,j_{12}}(k) = \Delta(k, j_1, j_2, j_{12})$:

$$\Delta_{j_1,j_{12}}(k) = \frac{j_1(j_1+1)}{k+2} + \frac{j_2(j_2+1)}{3} - \frac{j_{12}(j_{12}+1)}{k+3} + \delta_{j_1,1}\delta_{j_2,0}\delta_{j_{12},0} \ , \tag{6.3}$$

where

$$\begin{aligned} &0 \le j_1 \le k/2; 0 \le j_2 \le 1/2 \\ &|j_1 - j_2| \le j_{12} \le j_1 + j_2 \ . \end{aligned} \tag{6.4}$$

The following two tableaux give the Kac tables for the $c = 1/2$ and $c = 7/10$ Ising models calculated from the coset formula Eq.(6.3)

(i) $c = 1/2$ critical Ising ($k = 1$)

In this case, the last term of the right hand side of Eq.(6.3) namely, the term involving the Kronechers does not contribute since j_1 is always less than one.

j_1	j_2	j_{12}	$\phi_{j_1,j_{12}}$	$\Delta_{j_1,j_{12}}$	ϕ_{rs}
0	0	0	$\phi_{0,0}$	0	ϕ_{11}
0	1/2	1/2	$\phi_{0,1/2}$	1/16	ϕ_{12}
1/2	0	1/2	$\phi_{1/2,1/2}$	1/16	ϕ_{22}
1/2	1/2	0	$\phi_{1/2,0}$	1/2	ϕ_{21}
1/2	1/2	1	$\phi_{1/2,1}$	0	ϕ_{23}

(6.5)

(ii) $c = 7/10$ tricritical Ising ($k = 2$)

In this case, the term involving the product of Kroneckers Eq.(6.3) contribute since $k \geq 2$.

j_1	j_2	j_{12}	$\phi_{j_1,j_{12}}$	$\Delta_{j_1,j_{12}}$	ϕ_{rs}
0	0	0	$\phi_{0,0}$	0	$\phi_{1,1}$
0	1/2	1/2	$\phi_{0,1/2}$	1/10	$\phi_{1,2}$
1/2	0	1/2	$\phi_{1/2,1/2}$	3/80	$\phi_{2,2}$
1/2	1/2	0	$\phi_{1/2,0}$	7/16	$\phi_{2,1}$
1/2	1/2	1	$\phi_{1/2,1}$	3/80	$\phi_{2,3}$
1	0	1	$\phi_{1,1}$	1/10	$\phi_{3,3}$
1	1/2	1/2	$\phi_{1,1/2}$	3/5	$\phi_{3,2}$
1	1/2	3/2	$\phi_{1,3/2}$	0	$\phi_{3,4}$
1	0	0	$\phi_{1,0}$	1/2 + 1	$\phi_{3,1}$

(6.6)

Here we want to make a couple of remarks: The first one is that the formula (6.3) leads to the same Kac table obtained if one were to use Eq.(6.2). This property which is explicitly shown in Eqs.(6.5-6) has been checked also for higher values of k. The second remark we want to make is that the correspondence between Eqs.(6.2) and (6.3) is given by

$$r = 2j_1 + 1, \quad s = 2j_{12} + 1 \; . \tag{6.7}$$

Putting back this change into Eq.(6.2), one gets:

$$\Delta'_{j_1,j_{12}}(k) = \frac{[2j_1[k+3] - 2j_{12}[k+2] + 1]^2 - 1}{4(k+2)(k+3)} \tag{6.8}$$

where $0 \leq j_1 \leq k/2$ and $0 \leq j_{12} \leq (k+1)/2$. Note that the above equation does not depend on j_2 and admits the automorphism symmetry

$$j_1 \longrightarrow (k+2)/2 - j_1, \quad j_{12} \longrightarrow (k+3)/2 - j_{12} \; , \tag{6.9}$$

which is not manifested in Eqs.(6.3-4). Equating Eqs.(6.8) and (6.3-4) one gets the following constraint relation

$$\frac{1}{3} j_2(j_2+1) + \delta_{j_2,0}\delta_{j_1,1}\delta_{j_{12},0} = \Delta'_{j_1,j_{12}}(k) + \frac{1}{(k+3)} j_{12}(j_{12}+1) - \frac{1}{(k+2)} j_1(j_1+1) \tag{6.10}$$

whose solution gives the duality transformation linking the conformal weights $\Delta_{r,s}$ and $\Delta_{j_1,j_{12}}$.

Another interesting consequence of our analysis is that it allows us to work out field realizations of the relevant perturbations of minimal models and more generally the deformations of $G_{k_1} \times G_{k_2}/G_{k_1+k_2}$ GWZW models. For example, writing the $\phi_{j_1,j_{12}}(k_1,k_2)$ perturbed action S as

$$S = S_0 + \lambda \int d^2z \; \phi_{j_1,j_{12}} \; , \tag{6.11}$$

where λ is a coupling parameter and S_0 is the critical action of the $SU_{k_1}(2)\otimes SU_{k_2}(2)/SU_{k_1+k_2}(2)$ GWZW theory given by

$$\begin{aligned} S_0[k_1, k_2; g_1, g_2, h] &= k_1 I[g_1] + k_2 I[g_2] - (k_1 + k_2 + 2) I[h] \\ &\quad + \int d^2z \ \mathrm{tr}(b\bar{\partial}c + \bar{b}\partial\bar{c}) \ , \end{aligned} \tag{6.12}$$

one can build up the WZW fields realizations of the $\phi_{j_1,j_{12}}$'s. We have for the first few terms:

$$\begin{array}{ll} \phi_{21} & \sim \mathrm{tr}(g_1^{-1}g_2 + g_2^{-1}g_1) \\ \phi_{12} & \sim \mathrm{tr}(g_2^{-1}h + h^{-1}g_2) \\ \phi_{22} & \sim \mathrm{tr}(g_1^{-1}h + h^{-1}g_1) \ . \end{array} \tag{6.13}$$

A complete classification list will be considered in a future occasion [28].

The last interesting consequence deals with topological WZW field theories. The study made in this paper shows that as k_1 goes to zero, the action S_{top}:

$$S_{top} = kI[g] - (k + c_v)I[h] + \int d^2z \ \mathrm{tr}(b\bar{\partial}c + \bar{b}\partial\bar{c}) \tag{6.14}$$

describes a G_k/G_k topological field theory. Vacua of this quantum field theory are given by the primary fields $\phi_{r,r}$, $1 \leq r \leq k+1$ having conformal weights $\Delta_{rr} = (r+1)(r-1)/[4(k+2)(k+3)]$. An interesting question which can be studied within the framework of GWZW theories is the $\phi_{r,r}$ relevant deformations of topological coset theories. Progress in this direction will be reported elsewhere.

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