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Topological Portal to the Dark Sector

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We propose a unique topological portal between quantum chromodynamics (QCD) and a dark sector characterized by a global symmetry breaking, which connects three QCD to two dark pions. When gauged, it serves as the leading portal between the two sectors, providing an elegant, self-consistent scenario of light thermal inelastic dark matter. The inherent antisymmetrization leads to diminished annihilations at later times and suppressed direct detection. However, novel collider signatures offer tremendous prospects for discovery at Belle II.

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I. INTRODUCTION

The quest to unravel the mysteries of dark matter (DM) stands as a pivotal challenge in contemporary physics. That DM is composed of a hidden particle physics sector is a well-motivated hypothesis [1]. While DM has primarily revealed its presence through gravitational interactions, the existence of other *portals* may not only be feasible but perhaps essential in elucidating aspects of cosmological evolution and accounting for the observed relic abundance [2].

In this letter, we propose a novel portal that links quantum chromodynamics (QCD) and a dark sector characterized by a global symmetry breaking. Our starting point is to seek a (hitherto overlooked) non-trivial topological interaction between the two theories in their confined phases. After a comprehensive and rigorous topological analysis of the possible global symmetries characterizing QCD-like dark sectors, we discover that there is a *unique* coset structure featuring a non-trivial Wess–Zumino–Witten (WZW) term [3, 4] which connects three QCD pions to two dark pions.

Having established the existence and uniqueness of this topological portal operator, we delve into the resulting phenomenology. The topological portal enables dark number conservation through a $\mathbb{Z}_2$ symmetry, ensuring the stability of the lightest dark pion. Furthermore, after quantum electrodynamics (QED) is properly included, the portal operator yields 4-point interactions involving one photon, one neutral QCD pion (π^0 or η), and two dark pions. The chiral Lagrangian power counting identifies these (appropriately gauged) WZW terms as the *leading* portal between the two sectors, which thus defines the dominant phenomenology. Prompted by this, we investigate whether such an operator can establish the correct relic abundance. Intriguingly, we uncover a self-consistent scenario whereby DM achieves thermal freeze-out at temperatures below the QCD phase transition. This phenomenon is primarily governed by late-stage interactions, captured correctly by the topological operator, hinting at the presence of DM in the GeV range [5].

The topological operator, being a differential form and therefore completely antisymmetric under exchanging pairs of fields, necessarily couples two *different* dark pion species. After freeze-out governed by $\chi_1\chi_2$ coannihilations, the heavier dark pion rapidly decays, leaving the lighter as the residual DM. This elegantly leads to diminished DM annihilation into SM particles at later times for the topological portal, skirting otherwise stringent annihilation constraints from CMB anisotropies on elastic s-wave scattering [6, 7]. In addition, direct detection experiments lack sensitivity due to the kinematic suppression of inelastic scatterings. Thus, we stumble upon an elegant and natural realization of the light thermal inelastic DM scenario [8, 9].

The key to probing this new portal lies in collider experiments. Novel and previously unexplored collider signatures are anticipated in current e^+e^- flavor factories, calling for the design of new search strategies. Depending on the mass splitting, the heavy dark pion exhibits decays that are either displaced or detector-stable. The future at Belle II looks incredibly bright, offering the potential to explore much of the intriguing parameter space defined by the relic abundance.

II. QCD-DARK TOPOLOGICAL PORTAL EFT

We consider a dark sector characterized by a global symmetry-breaking transition $K \to H$, delivering dark pions as the DM. The relevant degrees of freedom are the QCD pions plus dark pions, which are collectively described by a 4-d non-linear sigma model (NLSM) on a product coset

$$X = \frac{SU(3)_L \times SU(3)_R \times K}{SU(3)_{L+R} \times H} \cong SU(3) \times \frac{K}{H}\,, \quad (1)$$

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with global symmetry $G = SU(3)_L \times SU(3)_R \times K$. In addition to the usual terms appearing in the EFT construction of Callan, Coleman, Wess, and Zumino [10], which require a G-invariant metric on X, the action also admits non-trivial G-invariant *topological interactions* that can be constructed without a metric. In this paper, we explore EFTs in which such a topological interaction provides a portal through which DM interacts with the SM.

Jumping ahead, we discover a unique topological portal interaction, reported in Eq. (7), for a large class of motivated coset spaces. This section aims to formally derive the uniqueness of this interaction.

Recall that in pure QCD there already appears a topological interaction, the WZW term [3, 4]. This term originates from the existence of an $SU(3)_L \times SU(3)_R$-invariant closed 5-form $\omega_5 = \frac{-i}{480\pi^3} N_c \mathrm{Tr}\left(g^{-1}dg\right)^5$ on $SU(3)$, where g is the $SU(3)$-valued pion field. The action, evaluated on 4-d spacetime Σ, can be defined by extending $g(x)$ to a 5-d bulk manifold X whose boundary is Σ and integrating ω_5 thereon. Requiring the path integral phase $e^{2\pi i \int_X \omega_5}$ be independent of the choice of the bulk X forces the coefficient N_c be an integer [11–13]. Famously, N_c is fixed to be the number of colors in QCD by anomaly matching.

In addition to the invariant closed 5-form, QCD also features an invariant closed 3-form, $\omega_3 \sim \mathrm{Tr}(g^{-1}dg)^3$. In fact, there are *no other* $SU(3)_L \times SU(3)_R$ *invariant forms* on $SU(3)$ with which one could construct topological terms involving QCD pions. Simply by virtue of its degree, ω_3 does not appear in the pure QCD action. It does, however, appear as the topologically-conserved current in QCD, which can be identified with baryon number [11]. By coupling to a dark sector coset, ω_3 can be used to construct a second topological term. Such a *mixed* topological term requires there be a K-invariant closed 2-form Ω_2 on the dark coset K/H, which we can 'wedge' with ω_3 to form a G-invariant[1] closed 5-form. In our pursuit of a topological portal, we ignore 'theta-like' topological terms corresponding to closed 4-forms because these are locally total derivatives that cannot give rise to new local interactions with dark matter.)

p	1	Portal 2	3	4	SIMP 5
$H^p(SU(2))$	0	0	$\mathbb{R}$	–	–
$H^p(SU(n))$, $n \geq 3$	0	0	$\mathbb{R}$	0	$\mathbb{R}$
$H^p(SU(2)/SO(2))$	0	$\mathbb{R}$	–	–	–
$H^p(SU(3)/SO(3))$	0	0	0	0	$\mathbb{R}$
$H^p(SU(4)/SO(4))$	0	0	0	$\mathbb{R}$	$\mathbb{R}$
$H^p(SU(n)/SO(n))$, $n \geq 5$	0	0	0	0	$\mathbb{R}$
$H^p(SU(2n)/Sp(2n))$, $n \geq 2$	0	0	0	0	$\mathbb{R}$

TABLE I. de Rham cohomology in low degrees for all homogeneous spaces that are expected to arise as pNGB manifolds in QCD-like theories [19]. Entries shaded blue (of which there is only one) feature a topological portal interaction with the pions of QCD; entries shaded orange feature a 5-point dark pion WZW term, as employed in the strongly-interacting-massive-particle (SIMP) mechanism [20, 21]. No coset features both portal and SIMP interactions.

Which QCD-like dark sectors K/H feature such an invariant 2-form? If this transition is due to chiral symmetry breaking in the dark sector, the viable coset patterns take the form $SU(N)$, $SU(N)/SO(N)$, or $SU(2N)/Sp(2N)$. (One could, in principle, consider other options such as a complex projective space, $\mathbb{C}P^n$.) Because these K/H are not just homogeneous spaces but, moreover, symmetric spaces, one can show that invariant forms are in 1-to-1 with de Rham cohomology classes [17, 18]. Therefore, a dark coset K/H features a topological portal interaction iff $H^2_{\mathrm{dR}}(K/H) \neq 0$. From Tab. I, we find there is a *unique choice* of dark coset that features such a topological portal term, which is

$$K/H = SU(2)/SO(2) \cong S^2\,. \tag{2}$$

The closed, K-invariant 2-form in question is simply the unit-normalized volume form Ω_2 on S^2.

The topological term is then constructed from the G-invariant closed, integral, 5-form

$$\omega_{\mathrm{portal}} = \frac{n}{24\pi^2}\mathrm{Tr}\left(g^{-1}dg\right)^3 \wedge \Omega_2\,, \quad n \in \mathbb{Z}\,. \tag{3}$$

Like the WZW term, this interaction has an integer-quantized coefficient, with n determined by the details of the UV completion (which we do not specify). We now expand this 5-form in terms of the pion and dark pion fields. First, consider the QCD part. Expanding $g = \exp(2i\pi_a(x)t_a/f_\pi)$ where $t_a = \frac{1}{2}\lambda_a$ with λ_a being the Gell-Mann matrices, $g^{-1}dg = \frac{2i}{f_\pi}t_a d\pi_a + \mathcal{O}(\pi^2)$ implies

$$\mathrm{Tr}(g^{-1}dg)^3 = (2/f_\pi^3) f_{abc}\, d\pi_a \wedge d\pi_b \wedge d\pi_c + \mathcal{O}(\pi^4)\,, \tag{4}$$

where f_{abc} are the $SU(3)$ structure constants defined such that $[t_a, t_b] = i f_{abc} t_c$, and where we use antisymmetry of the wedge product to write $\mathrm{Tr}(t_a t_b t_c) \mapsto \mathrm{Tr}(t_a[t_b, t_c])/2$, which is proportional to f_{abc}. Our convention is such that the QCD pion kinetic term is $f_\pi^2 \mathrm{Tr}(\partial_\mu g\, \partial^\mu g^{-1})/4$, with $f_\pi = 92$ MeV.

Now consider the 'dark part'. Taking f_D to be the dark pion decay constant, let χ_1/f_D and χ_2/f_D be local coordinates on S^2 corresponding to canonically normalised fields in the vicinity of the vacuum point ($\chi_i = 0$). Then $\Omega_2 = \cos(\chi_1/f_D) d\chi_1 \wedge d\chi_2/(4\pi f_D^2) = d\chi_1 \wedge d\chi_2/(4\pi f_D^2) + \mathcal{O}(\chi^3)$. Combining with the QCD part, Eq. (3) gives

$$\omega_{\mathrm{portal}} = \frac{n}{96\pi^3 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij} d\pi_a d\pi_b d\pi_c d\chi_i d\chi_j\,, \tag{5}$$

now suppressing the wedges. Finally, by invoking the Poincaré lemma (and multiplying by $2\pi i$) we convert this

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[1] Because G is semi-simple, G-invariance of the closed 5-form is enough to guarantee G-invariance of the WZW action [14–16]. Because of the product structure of X, one can show that closedness, integrality, and invariance properties can be separately required of ω_3 and Ω_2.

to a Lagrangian, which is a locally-defined 4-form that we can integrate directly over spacetime:

$$\mathcal{L}^{e=0}_{\text{portal}} = \frac{in\epsilon^{\mu\nu\rho\sigma}}{48\pi^2 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij}\pi_a\partial_\mu\pi_b\partial_\nu\pi_c\partial_\rho\chi_i\partial_\sigma\chi_j \,. \tag{6}$$

Just as is the case for the familiar WZW term of QCD, it is crucial to gauge the electromagnetic subgroup $U(1)_Q \subset SU(3)_{L+R}$, generated by $Q = t_3 + t_8/\sqrt{3}$, in order to derive the leading phenomenological consequences of this Lagrangian. While the gauging of WZW terms is in general complicated [11, 22–25], here we wish to gauge only an abelian subgroup in an effective 2d WZW term — the QCD part. The procedure here simplifies to shifting ω_3 by $\frac{-e}{4\pi^2}F\wedge\text{Tr}\left(Qg^{-1}dg\right)$ where $F = dA$ is the QED field strength (see *e.g.* [26]). Evaluating the trace picks up contributions only from the π^0 and η. Wedging with Ω_2 to form the 'gauged portal interaction' 5-form, the full leading order Lagrangian after gauging QED is

$$\tilde{\mathcal{L}}_{\text{portal}} = \mathcal{L}^{e=0}_{\text{portal}} + \frac{ne\,\epsilon^{\mu\nu\rho\sigma}}{16\pi^2 f_\pi f_D^2}\left(\pi^0 + \frac{\eta}{\sqrt{3}}\right)F_{\mu\nu}\partial_\rho\chi_1\partial_\sigma\chi_2\,, \tag{7}$$

having again used the Poincaré lemma and multiplied by a factor $2\pi i$.

The gauged part of this operator, which couples $\gamma\pi^0(\eta)$ to the pair of dark pions, is an effective dimension-7 operator in our EFT. In terms of EFT power-counting, this is the lowest-dimension interaction involving both QCD and dark pions. The next such interaction occurs at dimension-8, namely $\frac{1}{f_\pi^2 f_D^2}(D_\mu\pi_a D^\mu\pi^a)(\partial_\nu\chi_i\partial^\nu\chi^i)$, where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. If the relevant energy scales are low with respect to f_π and f_D, one might expect the topological portal to be the leading portal in this EFT.

III. RELIC ABUNDANCE FROM THE TOPOLOGICAL PORTAL

We are now ready to investigate the phenomenology of the topological portal in cosmology and at colliders. For simplicity, we study the Boltzmann equations [2] in the approximation of negligible mass splitting, commenting on the mass splitting at the end of this section. We describe the evolution of $\chi_1 + \chi_2$ yield, Y_χ, defined as the ratio of the combined dark pion number density n_χ to the entropy density s,

$$\frac{dY_\chi}{dx} = -\sqrt{\frac{\pi g_*}{45}}\frac{M_{\text{P}} m_\chi}{x^2}\langle\sigma v\rangle\left(Y_\chi^2 - Y_{\text{eq}}^2\right)\,. \tag{8}$$

The evolution parameter is $x = m_\chi/T$ where m_χ is the dark pion mass, M_{P} is the Planck mass, and g_* is the effective number of relativistic degrees of freedom. The thermally averaged cross section times the Møller velocity, $v = \sqrt{|\mathbf{v}_1 - \mathbf{v}_2|^2 - |\mathbf{v}_1\times\mathbf{v}_2|^2}$, can be expressed as a single integral [2]

$$\langle\sigma v\rangle = \frac{\int_{4m_\chi^2}^\infty \sigma\sqrt{s}(s-4m_\chi^2)\,\text{K}_1(\sqrt{s}/T)\,\text{d}s}{8m_\chi^4 T\,\text{K}_2^2(m_\chi/T)}\,, \tag{9}$$

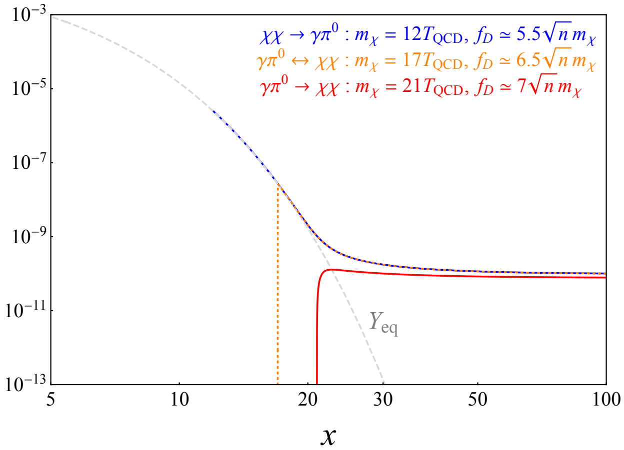

FIG. 1. The dark pion yield as a function of $x = m_\chi/T$. The different lines show qualitatively different possibilities for setting the relic abundance, depending on the cosmological history of dark pions. See the main text for details.

with σ being the unpolarised cross section for the process $\chi_1\chi_2 \to \pi^0\gamma$, $s = (p_{\chi_1} + p_{\chi_2})^2$, and K_i are the modified Bessel functions of the i-th order. Finally, the equilibrium yield Y_{eq} is a function of the evolution parameter x, namely $Y_{\text{eq}} = 45x^2\text{K}_2(x)/(4\pi^4 g_*)$.

For our topological portal interaction, we find the following unpolarised cross section

$$\sigma = \frac{n^2}{1536\,\pi^4}\,\frac{\alpha_{\text{Q}}}{f_\pi^2 f_D^4}\,s^{3/2}\sqrt{s - 4m_\chi^2}\,, \tag{10}$$

where $\alpha_{\text{Q}} = e^2/(4\pi)$ is the QED coupling constant, and we summed over the photon polarisations. After the integration in Eq. (9), we obtain

$$\langle\sigma v\rangle = \frac{n^2}{64\,\pi^{7/2}}\,\frac{\alpha_{\text{Q}} x_f^4}{f_\pi^2}\,\frac{x\,\text{G}^{3,0}_{1,3}\left(\begin{smallmatrix} & -2 & \\ -\frac{9}{2} & -\frac{1}{2} & \frac{1}{2}\end{smallmatrix}\,\middle|\,x^2\right)}{\text{K}_2^2(x)}\,, \tag{11}$$

where $x_f = m_\chi/f_D$, and $\text{G}(x^2)$ is the Meijer G-function [27].

Depending on the cosmological history of the dark pions, Eq. (8) can describe different scenarios setting the dark pion relic abundance. On the one hand, if the dark pions establish thermal equilibrium with the bath before the topological operator turns on at the QCD phase transition, then standard DM freeze-out occurs via $\chi\chi \to \gamma\pi^0$ at $x \sim 23$, after the topological portal is activated. This is shown by the blue line in Fig. 1, for which the topological operator turns on at $x_{\text{QCD}} := m_\chi/T_{\text{QCD}} = 12$. On the other hand, if the initial yield of the dark pions is negligible, and they are produced only after the QCD phase transition, they proceed with a quick thermalization followed by the freeze-out, and the correct relic abundance can still be attained. This is shown by an orange line in Fig. 1, for which the QCD phase transition occurs at $x = x_{\text{QCD}} = 17$.

Consequently, the correct yield today (Y_χ for $x \to \infty$)

Δm_χ	$\lesssim 1.7 m_{\pi^0}$	$\gtrsim 1.7 m_{\pi^0}$
Signature	$\pi^0 + \not{E}_T$	$\pi^0 + \not{E}_T + \mathrm{DV}(\pi^0\gamma\not{E}_T)$

TABLE II. Collider signatures. Here 'DV' indicates a displaced secondary vertex. The values of Δm_χ for which χ_2 lifetime is approximately 10^{-7} sec depend on the value of m_{χ_1}, and vary from $[1.3 - 2.1]\, m_{\pi^0}$ in the mass region $m_{\chi_1} \in [1 - 3.5]$ GeV.

can be achieved *irrespective* of the dark pion cosmological history. The topological portal thus robustly sets the correct relic abundance,

$$\Omega_\chi h^2 \approx \frac{2\, x_{\mathrm{QCD}}\, T_{\mathrm{QCD}}\, Y_\infty}{3.6 \cdot 10^{-9}\,\mathrm{GeV}} \approx 0.12\,, \tag{12}$$

through an interplay between the QCD phase transition onset relative to the dark pion mass, determined by x_{QCD}, and the $x_f = m_\chi / f_D$ ratio. In both scenarios, the QCD phase transition should happen no later than $x = x_{\max} = 23$; after this point, the dark pion yield drops below the value necessary to account for the DM relic abundance. This limiting case is shown by a red line in Fig. 1. The upper bound on x_{QCD} translates to an upper bound on m_χ. Specifically, to explain the relic abundance via the topological portal points towards *light thermal DM* with $m_\chi \lesssim 3.7$ GeV.

In the computation, we used the QCD phase transition temperature, $T_{\mathrm{QCD}} \sim 160$ MeV, and the number of effective degrees of freedom, $g_* \sim 18$, assuming the topological operator turns on only after the QCD phase transition [1]. While larger freeze-out temperatures are not inherently problematic, a correct description in this context would require full QCD rather than chiral perturbation theory. The fact that the correct relic abundance demands $f_D / m_\chi \sim \mathcal{O}(5.5 - 7) \times \sqrt{n}$ is consistent with the dark pions being the lightest dark states.

The non-zero mass splitting, $\Delta m_\chi = m_{\chi_2} - m_{\chi_1}$, leads to a suppression of the co-annihilation cross section. The dominant effect can be captured by introducing an exponential suppression factor $e^{-x\Delta}$ in Eq. (11), where $\Delta := \frac{\Delta m_\chi}{m_{\chi_1}}$ [28, 29]. The resulting f_D which fits the relic abundance is, accounting for the mass splitting,

$$f_D(\Delta) \approx f_D(0)\, e^{-\frac{x_{\max}\Delta}{4}}\,, \tag{13}$$

where the factor of 1/4 appearing in the exponent comes from the dependence $\langle \sigma v \rangle \propto x_f^4$ in Eq. (11). This is a numerically large suppression effect that we account for in understanding the viable parameter space; for example, for $\Delta = 5 \times \frac{m_{\pi^0}}{3.5\,\mathrm{GeV}}$ one finds $f_D(\Delta)/f_D(0) \approx 1/3$.

IV. NOVEL COLLIDER PHENOMENOLOGY

With a non-zero mass splitting, χ_2 decays to χ_1 shortly after freeze-out, leaving χ_1 as the sole dark matter component. As a result, DM annihilations at later times are suppressed, avoiding otherwise stringent indirect detection constraints from CMB anisotropies [6, 7]. Likewise, direct detection experiments are ineffective for the inelastic $\chi_1 \to \chi_2$ up-scattering due to kinematics, while the elastic scattering is a higher-order process.

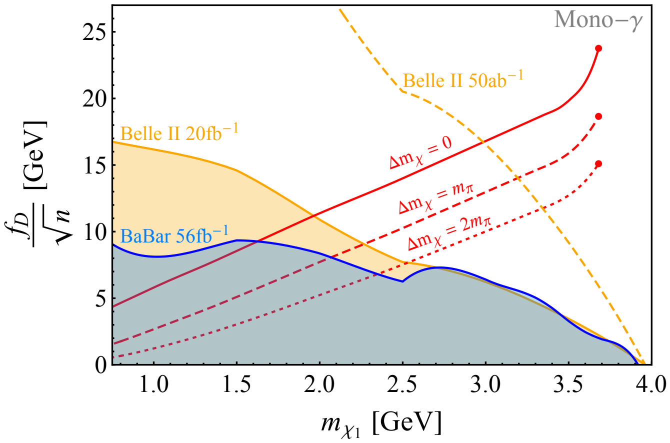

FIG. 2. Limits by the B-factories on the parameter space of dark pions with zero mass splitting. The signature corresponds to $\pi^0 + \not{E}_T$, where π^0 is identified as a photon such that mono-γ searches are used. The red lines delineate the parameter space points giving the correct DM relic abundance (solid: $\Delta m_\chi = 0$, dashed: $\Delta m_\chi = m_{\pi^0}$, dotted: $\Delta m_\chi = 2m_{\pi^0}$). Both Belle II exclusion lines correspond to the projections [30], while BaBar already performed a mono-γ search [31].

How, then, can we test this scenario? Interestingly, collider experiments offer a promising avenue through $e^+e^- \to \gamma^* \to \chi_1\chi_2\pi^0$ production. In particular, the required collider energy, high luminosity, and hermetic environment of Belle II result in exceptional sensitivity, potentially covering the full parameter space set by the DM relic abundance. The dark pion mass splitting Δm_χ suggests different search strategies, as summarised in Tab. II. For a small mass splitting, χ_2 is detector stable resulting in the final state with π^0 and missing energy $\not{E}_T$; otherwise, χ_2 decays to π^0, photon, and missing energy at a displaced vertex. (Prompt decays require a large mass splitting and are less motivated given Eq. (13).) Neither of these signatures has been explored in dedicated experimental analyses so far.

We observe that, when $\Delta m_\chi < m_{\pi^0}$, the decay $\chi_2 \to \chi_1\gamma\gamma\gamma$ through an off-shell π^0 gives a lifetime $\gtrsim \mathcal{O}(1)$ sec where the cosmological bounds kick in. Thus, the interesting mass range for colliders is $\Delta m_\chi > m_{\pi^0}$. We also found the decay rate for $\chi_2 \to \chi_1\gamma$, naïvely induced at 1-loop, to be zero.

To illustrate the potential of Belle II, we focus on the small Δm_χ scenario (detector-stable χ_2). We leave the other case for future work. The energy of π^0 originating from $e^+e^- \to \gamma^* \to \pi^0\chi_1\chi_2$ is such that the two photons from $\pi^0 \to \gamma\gamma$ cannot be sufficiently separated and are detected as a single photon [30]. Therefore, we recast the existing (proposed) *dark photon* search at BaBar [31]

(Belle II [30]) in the *mono*-γ channel, keeping in mind that a dedicated analysis would outperform our study, see also [32]. We simulate the signal events using MadGraph5_aMC@NLO [33] and apply the same event selection as reported in [30]. The overall analysis efficiency is slightly less compared to the dark photon search, resulting in the limits shown in Fig 2.

The red lines in the figure produce the correct DM relic abundance for three values of the mass splitting, and all terminate at $m_{\chi_1} \lesssim 3.7$ GeV as discussed in Sec. III. Already with $20\,\mathrm{fb}^{-1}$, Belle II can test a significant portion of the parameter space, slightly more than the existing BaBar exclusion based on $56\,\mathrm{fb}^{-1}$. This is due to the improved single photon trigger efficiency at Belle II. The projections for the full luminosity of $50\,\mathrm{ab}^{-1}$ (dashed orange) are extremely promising, covering most of the parameter space of interest. Finally, should Belle II discover such a signal in $\pi^0\chi_1\chi_2$, then observing the production of $\eta\chi_1\chi_2$ would serve as the next target (and arguably a 'smoking gun' for this topological portal), since it is predicted to occur at a fixed rate relative to the pion channel – see Eq. (7).

Finally, it is important to note that the targeted χ_i mass range for Belle II exceeds the reliable scope of our EFT description of the topological portal. An analysis involving a concrete UV completion is necessary to address this limitation, which is left for future work.

V. OUTLOOK

In this letter, we postulate a *unique* topological portal operator between QCD and a dark QCD-like sector, which successfully realizes the light thermal inelastic DM scenario while offering exciting signatures at Belle II.

Future explorations will be on two fronts. Firstly, developing an ultraviolet completion for our topological operator is crucial. We anticipate the involvement of a new state that mediates interactions between quarks and the dark sector. This should result in correlated signatures observable in other experiments, such as those conducted at the LHC. Secondly, whether Nature has selected this unique and subtle portal for primary communication between the visible and the dark sector is a question that Belle II could resolve by designing specific searches for its distinctive signatures, a course of action we highly recommend for the collaboration.

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