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Coherent Synchrotron Radiation, Wake Field and Impedance

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Abstract

Coherent synchrotron radiation is considered here in terms of the wake field and the longitudinal impedance on a curved trajectory. Shielding by the vacuum chamber as well as bunching of the beam play an important role for this radiation. The physical restrictions on machine parameters arising from the considerations are presented. A possible explanation for the existing controversy in different formulae for the total power of radiation, along with a correct formulae for this quantity, are presented.

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1 Introduction

As a source of radiation with unique characteristics, synchrotron radiation is used in a wide variety of applications. The maximum of the power spectrum of the synchrotron radiation lies at very short wave lengths, and most of it is produced by each particle of the bunch independently, i.e. *incoherently.*

Nevertheless, the spectrum of the synchrotron radiation extends all the way up to wave lengths comparable to the length of the orbit. In this range of wave lengths, the EM fields of different particles in the bunch can interfere constructively, enhancing the radiated power. Such *coherent radiation* is in many respect similar to the radiation produced by a bunch due to its interaction with the environment. In particular, the power of both types of radiation is proportional to the square of the bunch current, and they both can be described in terms of an effective impedance.

There is one peculiarity of the wake function due to the synchrotron radiation. Since radiation travels on a chord it can catch up with a bunch even for an ultra-relativistic particle. Hence, the interaction of a particle with the radiated field is not restricted to the trailing particles only. In fact, the longitudinal wake function due to radiation is much stronger in front of the bunch than behind it.

The power lost by a bunch in the form of radiation can be viewed as one which is lost by its current flowing through an effective impedance. The calculation of this impedance and an estimate of the radiation power is the goal of this note.

2 Incoherent Synchrotron Radiation

We start with a list of the most important formulae for incoherent synchrotron radiation which we need as reference background. Figure 1 represents the geometry and the coordinate system.

Let us assume that a point charge e of the mass m moves in the horizontal plane $z = 0$ in a magnetic field with vertical component H. The radius ρ of its circular orbit, its velocity v, its revolution frequency ω_0 and its energy $\mathcal{E}$ are connected in the following way:

$$\omega_0 = \frac{v}{\rho} = \frac{eB}{m\gamma}, \quad \mathcal{E} = evB\rho \ . \tag{1}$$

The synchrotron radiation is emitted on angular frequencies $\omega = n\omega_0$, where n is a harmonic number of radiation. The angular and spectral distribution of the radiated power $P_n(\theta, \alpha)$ is defined by formula[1]:

$$dP_n(\theta, \phi) = \frac{n^2 r_0 \mathcal{E}_0 \omega_0}{2\pi\rho}[\tan^2\theta J_n^2(n\beta\cos\theta) + \beta^2 J_n'(n\beta\cos\theta)]d\Omega \ , \tag{2}$$

where θ and ϕ are the polar and the azimuthal angles of the unit vector in the direction of the radiation, $r_0 \equiv e^2/4\pi\epsilon_0 mc^2$ is the classical radius of a particle, and $\mathcal{E}_0 \equiv mc^2$ is the particle rest energy.

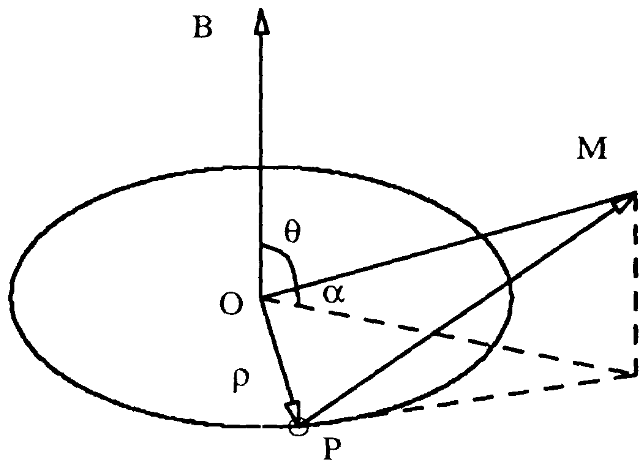

Figure 1: Geometry and coordinate system for the motion in magnetic field. Letter M marks an observation point of the radiation emitted by a charge from point P in the direction of vector PM.

Let us introduce the more convenient variables $\alpha \equiv \pi/2 - \theta$ and $\epsilon \equiv \alpha^2 + \gamma^{-2}$. When γ is large, both these quantities are very small. Using now the asymptotic expansions of the Bessel function of large index when its argument is of the order of magnitude of its index one gets:

$$dP_n(\theta,\phi) = \frac{n^2 r_0 \mathcal{E}_0 \omega_0}{6\pi^3 \rho}[\epsilon^2 K_{2/3}^2(\frac{n}{3}\epsilon^{3/2}) + \epsilon\alpha^2 K_{1/3}^2(\frac{n}{3}\epsilon^{3/2})]d\Omega \ . \tag{3}$$

The modified Bessel functions of the second kind of the orders 1/3 and 2/3 decay exponentially when their argument is larger then 1:

$$\begin{aligned} K_{1/3}^2(x) &\approx \frac{\pi}{2}\frac{e^{-2x}}{x} \ , \\ K_{2/3}^2(x) &\approx \frac{\pi}{2}(\frac{2}{3})^{2/3}\frac{e^{-2x}}{x^{1/3}} \quad \text{for} \quad x > 1 \ . \end{aligned} \tag{4}$$

From this expression follows that the angular distribution of the radiation of the ultra-relativistic particle with $\gamma \gg 1$ is concentrated in a narrow cone around the instantaneous particle velocity with a width $\delta\theta \cong \gamma^{-1}$. When $\alpha < \gamma^{-1}$, the harmonic number n is limited by the condition $n < n_c$, where $n_c \cong 3\gamma^3/2$, and the radiation goes into angles smaller then $\alpha \cong \gamma^{-1}$. The harmonic number n for the major part of the radiation is large. That explains the fact that classical dynamics is sufficient to describe the radiation in the most parts of the spectrum. When $\alpha > \gamma^{-1}$, the radiation of harmonics $n \ll n_c$ is emitted mainly into angles smaller then $\alpha \cong n^{-1/3}$.

The total radiated power is obtained by integration of Eqs. 2 or 3 over the angles, and by summing over the harmonic numbers n. The result is:

$$P_{tot} = \frac{2}{3}\frac{r_0 \mathcal{E}_0 \omega_0}{\rho}\gamma^4 \ . \tag{5}$$

A comprehensive but suscinct description of incoherent synchrotron radiation can be found in work[2]. It contains also some useful integrals of functions which are relevant to this problem.

3 Coherent Synchrotron Radiation and Shielding

Particles radiate coherently at a certain wave length λ when their relative distances are smaller than half their wavelength. The EM field of such particles interfere constructively, and the resulting field is the same as the field radiated by one particle with the total charge equal to the sum of all charges. If the characteristic length of the longitudinal charge distribution is σ_s, then the condition for coherence of the radiation in free space is:

$$\sigma_s \le \frac{\lambda}{2} \ . \tag{6}$$

Unlike the case of incoherent radiation, for which the radiation power is proportional to the particle number N in the bunch, the power of coherent radiation is proportional to N^2. To understand how this comes about, consider two particles moving in a magnetic field on a circular trajectory. Their positions have azimuthal angles ϕ_1 and ϕ_2 with respect to some arbitrary azimuth taken as zero. Correspondingly, their currents have the phase factors $\exp(-i\phi_1)$ and $\exp(-i\phi_2)$. The n-th harmonics of the fields excited by the particles have the phase factors $\exp(-in\phi_1)$ and $\exp(-in\phi_2)$.

To find the total field created by all N particles of a bunch, we need to sum their individual fields taking into account the corresponding phase factors. For the radiation power averaged over the longitudinal particle distribution $S(\phi)$ we find:

$$P_n|\Sigma_k^N e^{-in\phi_k}|^2 = NP_n + N(N-1)P_n f_n \ , \tag{7}$$

where P_n is the incoherent power emitted by each particle, and the form factor for coherent radiation f_n is:

$$f_n = \left(\int d\phi \cos(n\phi) S(\phi)\right)^2 \ . \tag{8}$$

For example, for a Gaussian distribution with the rms bunch length σ_s

$$S(\phi) = \frac{\rho}{\sqrt{2\pi}\sigma_s} \exp(-\frac{\phi^2\rho^2}{2\sigma_s^2}) \tag{9}$$

$$f_n = \exp(-\frac{n^2\sigma_s^2}{\rho^2}) \ . \tag{10}$$

Since $\lambda \approx \rho/n$, from these equations the coherence condition Eq. 6 follows once more in a more formal way.

The first term on the right hand side of Eq. 7 gives the power of incoherent radiation of the bunch, which is proportional to the number of particles in the bunch or to its current. The second term gives the power of coherent radiation, which for $N \gg 1$ is proportional to the square of the number of particles or the bunch current I. Such a signature of coherent radiation makes it suitable to describe it by an effective impedance Z_n which can be defined in accordance with Ohm's law $P_n = Z_n I_n^2$.

For all practical particle distributions, the condition Eq. 6 limits coherent synchrotron radiation to the microwave range of frequencies. Hence, the wave lengths of the radiated power are usually smaller or comparable to the transverse size of the vacuum chamber and/or the distance to poles of a magnet. Under these conditions the presence of walls and magnets can substantially change the radiation and its characteristics. When the walls are conductive, the induced charges tend to decrease the EM fields created by the charge itself. Poles of a magnet have similar effect due to the induced currents. This phenomenon is referred to as *shielding* and is the stronger, the closer the induced charges. From this consideration it is clear that the shielding must depend strongly on the size of the vacuum chamber or the distance to the poles.

4 Lengths of Longitudinal Coherence, Absorption and Shielding

To take into account the presence of conductive walls around a bunch, and to evaluate the shielding effect for coherent radiation, we need to modify the coherence condition Eq. 6.

Consider, for example, a case of two parallel conductive plates placed above and below the plane of motion at $z = \pm h/2$. The tangential components of the electric field should satisfy the proper boundary conditions on the plates. For perfectly conducting plates these components should be zero. That means that the EM field must contain factors which make appropriate components of the field vanish on the plates. Hence, the radiation field then has to be expanded in double series containing two integers: the radial harmonic number n, and the vertical harmonic number p. Accordingly, the wave vector $\mathbf{k}$, $|k| = \omega/c$ of the EM field acquires the vertical component k_z which cannot be smaller then π/h - the lowest value of the vertical wave number which is consistent with the boundary condition. As we discussed in the previous subsection, the opening angle α in which the radiation is emitted is smaller then $n^{-1/3}$, that means that $\pi/h < k\alpha = \omega\alpha/c = n\alpha/\rho < n^{2/3}/\rho$. Here ρ is the local radius of curvature of the trajectory due to the magnetic field. Hence, only harmonics with $n > (\pi\rho/h)^{3/2}$ can be emitted coherently. As we will show below, the more exact *longitudinal coherence condition* is:

$$n > n_{th} \equiv \frac{1}{\sqrt{3}}\left(\frac{\pi\rho}{h}\right)^{3/2} . \tag{11}$$

Usually ρ is much bigger then h. Then the radiation can be emitted coherently only harmonics with rather large numbers.

That means in turn that the condition of coherence in the presence of conductive plates is rather more stringent than the one in free space, Eq. 6:

$$\sigma_s < \frac{\lambda_{th}}{2} \approx \frac{\rho}{n_{th}} = \frac{h}{\pi}\sqrt{\frac{h}{\pi\rho}} . \tag{12}$$

The finite conductivity of the plates can be taken into account by introducing the *effective absorption length* L_{ab} defined as the length in the longitudinal direction in which the field decays by a factor e due to absorption in the walls. For radiation at the wave length λ, it can be estimated as $L_{ab} \approx Q\lambda/2\pi$[3]. The quality factor Q is the ratio of the field energy stored in a vacuum chamber with typical size h (proportional to its volume per unit length h^2) to the field absorbed in the wall within the skin depth of thickness δ (proportional to the volume of the absorption also per unit length $h\delta$):

$$L_{ab} \approx \frac{\rho h}{n\delta(\omega)} \approx \frac{\rho h}{n^{1/2}\delta(\omega_0)} . \tag{13}$$

The effect of the finite conductivity is usually small since the absorption length L_{ab} is much larger then the *effective longitudinal shielding length*

$$L_{sh} \approx \rho/n^{1/3} . \tag{14}$$

The latter is defined as the length in which the phase of the field slips with respect to the phase of the bunch current by π:

$$\omega t - k_{\|} s \equiv [\frac{\omega}{c} - k(1 - \alpha^2/2)] L_{sh} \approx \pi . \tag{15}$$

As we will show, the power spectrum of coherent radiation increases toward shorter wave lengths. This increase is limited by the bunch length since the radiation ceases to be coherent when the wave length becomes smaller then the bunch length.

5 Wake Field and Impedance due to Synchrotron Radiation

The EM field of a point charge e moving on a circular trajectory has been derived by Schott[1]. The ϕ component of the radiated electric field simplifies for an ultra-relativistic particle. For $\gamma \gg 1$ it can be written in the following form [4]:

$$E_{\phi}(\phi) = -\frac{2U_0}{2\pi e \rho} w(\phi) , \tag{16}$$

where $U_0 = 4\pi r_0 \mathcal{E}_0 / 3\rho$ is the energy loss per revolution, and

$$w(\phi) = -\begin{cases} 0 & \text{for} \quad \mu < 0 \ , \\ \frac{1}{2} & \text{for} \quad \mu = 0 \ , \\ W(\mu) & \text{for} \quad \mu > 0 \ . \end{cases} \tag{17}$$

Here the wake function $W(\mu)$ is expressed as function of the dimension-less quantity $\mu \equiv 3\gamma^3\phi/2$. The sign convention is such that *in front of the particle* $\phi > 0$ and $\mu > 0$:

$$W(\mu) = \frac{9}{4}\frac{d}{d\mu}\frac{\cosh(\frac{5}{3}\mathrm{Arsinh}\mu) - \cosh(\mathrm{Arsinh}\mu)}{\sinh(2\mathrm{Arsinh}\mu)} . \tag{18}$$

Directly in front of the particle $\phi = 0$ and $\mu = 0$, while directly behind the the particle $\phi = 2\pi$ and $\mu = 3\pi\gamma^3$. The total radiated power is $P_{tot} = ecE_{\phi}(0)$ and is given by Eq. 5.

Figure 2 presents function $W(\mu)$.

The longitudinal impedance Z_n due to synchrotron radiation at a harmonic number n is as usual the Fourier transform of the longitudinal electric field, normalized by a charge current: $Z_n \equiv 2\pi\rho\tilde{E}(n)/I_n$, where

$$\tilde{E}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi E_{\phi}(\phi) e^{-in\phi} , \tag{19}$$

and $I_n = e\omega_0/2\pi$. For $n \ll \gamma^3$ the impedance is:

$$Z_n = Z_0 \pi \beta n \Big(J'_{2n}(2n\beta) - i\mathbf{E}'_{2n}(2n\beta) \Big) . \tag{20}$$

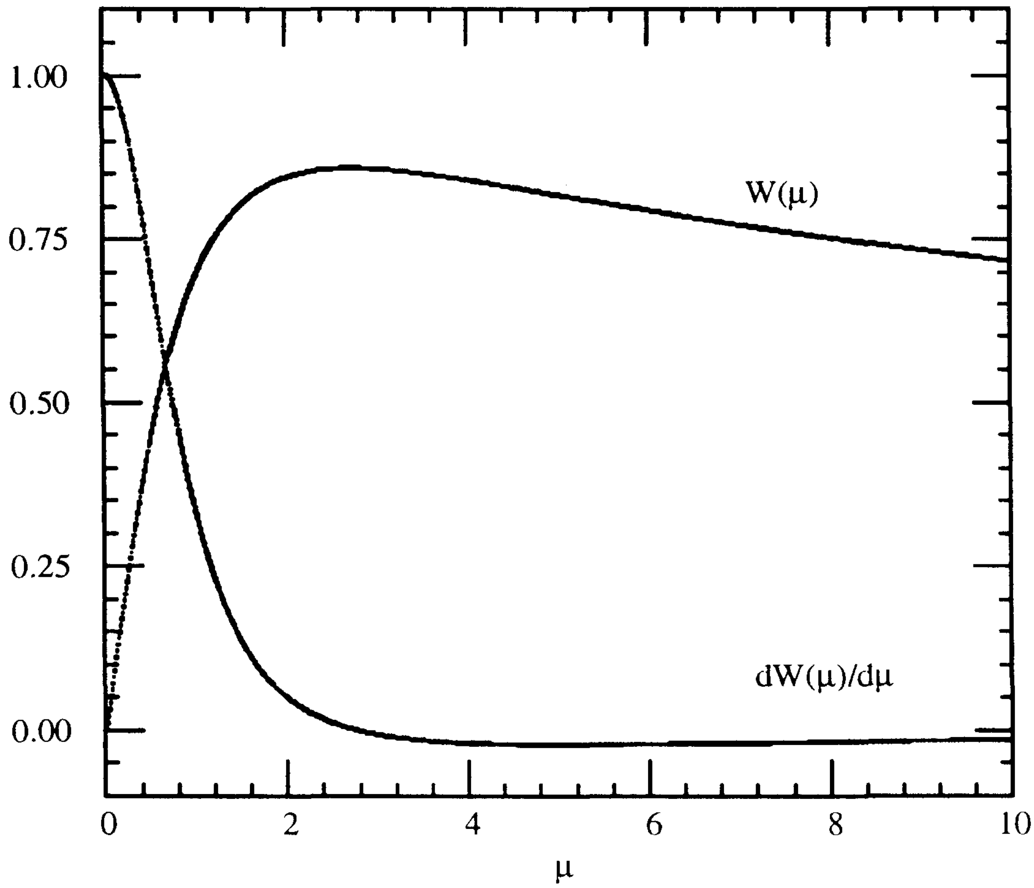

Figure 2: Longitudinal wake potential for synchrotron radiation

Here the Bessel and the Weber functions are defined in the usual way:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi dt \cos(nt - x\sin t)$$

and

$$\mathbf{E}_n(x) = \frac{1}{\pi}\int_0^\pi dt \sin(nt - x\sin t)$$

.

Using the asymptotic expansions of the Bessel and Weber functions for a large index, in the region where their arguments are of the order of the index, one obtains the result[5]:

$$Z_n = Z_0 \frac{\Gamma(2/3)}{3^{1/3}}(\frac{\sqrt{3}}{2} + i\frac{1}{2})n^{1/3} \ . \tag{21}$$

Coherent radiation of the ultra-relativistic particle $\gamma \gg 1$ is produced mostly at harmonic numbers in the range $n_{th} \ll n_c$. For such n, the significant contribution to the integrals defining the Bessel and the Weber functions comes from the range of t which satisfies the inequality $nt^3 \leq 1$. Hence, $t \ll 1$ and

$$\begin{aligned} J_n'(2n) &\approx \frac{1}{\pi}\int_0^\infty dt \ t\sin(nt^3/3) \ , \\ \mathbf{E}_n'(2n) &\approx -\frac{1}{\pi}\int_0^\infty dt \ t\cos(nt^3/3) \ . \end{aligned} \tag{22}$$

The upper limit of the integration can be extended to infinity without much error. From here the result Eq.21 is recovered if one takes into account the expression for the Γ function:

$$\begin{aligned} \Gamma(x) &= \frac{1}{\cos(\pi x/2)}\int_0^\infty dt \ t^{x-1}\cos t \ , \\ \Gamma(x) &= \frac{1}{\sin(\pi x/2)}\int_0^\infty dt \ t^{x-1}\sin t \ , \quad \text{for} \quad 0 < Re\ x < 1 \ , \end{aligned} \tag{23}$$

For a circular machines the result Eq. 21 can be written in a form which enters in most criteria for collective instabilities of the bunch: $ReZ_n/n \approx Z_0/n^{2/3}$. This ratio decrease with the increase of n. On the other hand for the values $n < n_{th}$, cf. Eq. 11, the radiation and, respectively, impedance decreases exponentially. That means that $Re\ Z_n/n$ has a maximum at $n \approx n_{th}$:

$$Re(\frac{Z_n}{n})_{max} \approx 300\frac{h}{2\rho} \quad \text{Ohm} \ . \tag{24}$$

This result was first obtained by Faltens and Laslett[5].

The result Eq. 21 was obtained for radiation in free space. Now we turn to the effect of shielding. To study it we will consider the case when a charge moves between two conducting plates positioned at $z = \pm h/2$ above and below the plane of the orbit[6],[7].

The n-th harmonics of the EM field of a point charge moving in the mid-plane between the plates can be found using Green's function which in this case should also vanish on at $z = \pm h/2$. Such a function is readily available[8]:

$$G_n(r,\phi,z;r',\phi',z') = \frac{i}{2h}\sum_{p=1}^{\infty} \sum_{m=-\infty}^{\infty} \sin p\pi(\frac{z}{h}+\frac{1}{2}) \sin p\pi(\frac{z'}{h}+\frac{1}{2}) e^{im(\phi-\phi')} G_r(r,r') \ , \tag{25}$$

where

$$G_r(r,r') = \begin{cases} J_m(\gamma_p r) H_m^{(1)}(\gamma_p r') & \text{for } r < r' \ , \\ J_m(\gamma_p r') H_m^{(1)}(\gamma_p r) & \text{for } r > r' \ . \end{cases} \tag{26}$$

Here $\gamma_p = \sqrt{(n\beta/\rho)^2 - (p\pi/h)^2}$.

The power of radiation at the nth harmonic is:

$$P_n = Re\left(4inr_0\mathcal{E}_0\omega_0 \int_{-\pi}^{\pi} d(\phi-\phi') G_n(\rho,\phi,0;\rho,\phi',0) [1-\beta^2\cos(\phi-\phi')] e^{-in(\phi-\phi')} \right) \ . \tag{27}$$

After performing the integration over the angles one gets:

$$P_n = \frac{4\pi n r_0 \mathcal{E}_0 \omega_0}{h} \sum_{p=1,3,\ldots}^{p<nh\beta/\pi\rho} \left(\beta^2 J_n'^2 + \frac{(p\pi\rho/h)^2}{(n\beta)^2 - (p\pi\rho/h)^2} J_n^2 \right) \ , \tag{28}$$

where the argument of the Bessel functions is

$$\gamma_p \rho = \sqrt{(n\beta)^2 - (p\pi\rho/h)^2} \ . \tag{29}$$

Only propagating modes with the numbers $p < nh\beta/\pi\rho$ contribute to the energy loss, since for evanescent modes the product $J_m H_m^{(1)}$ is purely imaginary. From now on we assume that $\beta = 1$.

The real part of the impedance due to synchrotron radiation can be found from expression Eq. 28 by applying Ohm's law: $Z_n = P_n/I_n^2$:

$$Z_n = Z_0 n \frac{\rho}{h\beta^2} \sum_{p=1,3,\ldots} \left(\beta^2 J_n'^2 + \frac{(p\pi\rho/h)^2}{n^2 - (p\pi\rho/h)^2} J_n^2 \right) \ . \tag{30}$$

The same result can be obtained by employing another approach. For example, one can use existing results for the EM fields excited by a point charge in a perfectly conductive pill-box cavity of the radius b and the gap h placed in a plane perpendicular to the magnetic field.

The EM fields which are excited in a closed cavity by a point charge and which satisfy all the boundary conditions on its surface can be easily written using expansions into cylindrical waves [9]. Such an approach was used by several authors [10],[11],[12],[13],[14].

Unlike an open structure where radiation has a way for an escape, the radiation in a closed cavity can only be absorbed in its walls. That situation brings to appearance in the real part of the impedance narrow peaks around the resonance frequencies of the cavity. In an idealized situation of a perfectly conducting walls the resonance peaks become δ-functions. That means that the equilibrium between the emission and absorption of the radiation (which is described by such solutions) can be reached only in an infinitely large time and, hence, have no practical significance. But for the purpose of evaluating the radiated EM field in an open structure, the desired result can be obtained from such a solution in the limit $b \to \infty$.

The double Fourier transform of the longitudinal component of the electric field E_ϕ, one in time at the frequency ω, the second one in the azimuthal angle ϕ at the harmonic number n, is proportional to a function $Z(n,\omega)$. The radiation field most effectively interacts with a particle when its phase velocity is equal to the particle velocity. In terms of the function $Z(n,\omega)$ that implies that the impedance is its value at the point $\omega = n\omega_0$.

The quantity $Z(n,\omega)$ for a pill-box cavity can be taken, for example, from the paper by Warnock and Morton[11].

$$Z(n,\omega) = in\pi^2 Z_0 \frac{\rho}{h} \sum_{p=1,3,5,\ldots} \left(\frac{\alpha_p^2}{\gamma_p^2} \frac{J_n(x)}{J_n(y)} q_n(x,y) + \frac{\omega\rho}{nc} \frac{J_n'(x)}{J_n'(y)} s_n(x,y) \right) , \qquad (31)$$

where

$$\alpha_p = \pi p/h, \ \gamma_p = \sqrt{(\omega/c)^2 - \alpha_p^2} , \qquad (32)$$

$$q_n(x,y) = J_n(x)Y_n(y) - J_n(y)Y_n(x) , \qquad (33)$$

$$s_n(x,y) = J_n'(x)Y_n'(y) - J_n'(y)Y_n'(x) , \qquad (34)$$

and $x = \gamma_p \rho, \quad y = \gamma_p b$.

To compare this expression with the previous result, we need to find its limit for $b \to \infty$. The main contribution comes from the vicinity of the zeros of the denominators, which are defined by: $\gamma_p b = \nu_{nm}$ and $\gamma_p b = \mu_{nm}$, where ν_{nm} and μ_{nm} are the roots of the Bessel function and of its derivative: $J_n(\nu_{nm}) = 0$ and $J_n'(\mu_{nm}) = 0$. These roots define the resonance frequencies:

$$\omega_{\nu_{nm}} = \sqrt{(\frac{\nu_{nm}}{b})^2 + (\frac{\pi p}{h})^2} , \quad \omega_{\mu_{nm}} = \sqrt{(\frac{\mu_{nm}}{b})^2 + (\frac{\pi p}{h})^2}. \qquad (35)$$

Expanding the Bessel functions in the vicinity of these points we get an expression for $Z(n,\omega)$ which is the double sum over n and m of two terms. One is the sum over the frequencies $\omega_{\nu_{nm}}$, another is similar sum over roots the frequencies $\omega_{\mu_{nm}}$:

$$Z_n = -i\pi Z_0 \frac{\rho}{h} n[\Sigma_\nu + \Sigma_\mu] \ , \tag{36}$$

where

$$\Sigma_\nu = \sum_{pm} \frac{1}{\gamma_p b - \nu_{nm}} (\frac{\alpha_p}{\gamma_p})^2 J_n^2(\frac{\rho}{b}\nu_{nm}) \frac{Y_n(\nu_{nm})}{J_n'(\nu_{nm})} \ , \tag{37}$$

$$\Sigma_\mu = \sum_{pm} \frac{1}{\gamma_p b - \mu_{nm}} J_n'^2(\frac{\rho}{b}\mu_{nm}) \frac{Y_n'(\mu_{nm})}{J_n''(\mu_{nm})} \ . \tag{38}$$

In the first sum, the main contribution comes from large values of m, the sum over m can be evaluated replacing it by integration. The real part of the integral arises from the residue of the pole at $\gamma_p b = \nu_{nm}$. The ratio $Y_n(b\gamma_p)/J_n'(b\gamma_p)$ is approximately 1 for $b \to \infty$. Hence, the sum over m in Σ_ν is approximately $iJ_n^2(\gamma_p\rho)/\pi$. Similarly, the sum over m in Σ_μ is $iJ_n'^2(\gamma_p\rho)/\pi$.

Calculating now function $Z(n,\omega)$ for $\omega = n\omega_0$ we obtain the expression for the impedance Eq. 30.

6 Spectrum of Coherent Synchrotron Radiation

The power of coherent synchrotron radiation P_n emitted at the n-th harmonic can be obtained by multiplying the impedance Z_n due to synchrotron radiation by the square of the current of the circulating charge $(Ne\omega_0/2\pi)^2$. We can use Eq. 30 for the impedance.

In the ultra-relativistic case, and when the condition $\pi\rho/h \gg 1$ is fulfilled, most of the radiation is produced at the sufficiently large harmonic numbers n. That makes the following asymptotic expansion of the Bessel functions valid:

$$\begin{aligned} J_n(\gamma_p\rho) &= \frac{1}{\sqrt{3}\pi}(\frac{p\pi\rho}{nh}) K_{1/3}[\frac{1}{3n^2}(\frac{p\pi\rho}{h})^3] \ , \\ J_n'(\gamma_p\rho) &= \frac{1}{\sqrt{3}\pi}(\frac{p\pi\rho}{nh})^2 K_{2/3}[\frac{1}{3n^2}(\frac{p\pi\rho}{h})^3] \ . \end{aligned} \tag{39}$$

Since the modified Bessel functions $K_\nu(x)$ are exponentially small, cf. Eq. 4, only harmonics with numbers $n > (p\pi\rho/h)^{3/2}$ contribute to the the sum in Eq. 30. Hence, $n \gg p\pi\rho/h$ and the second term in Eq. 30 can be simplified. The result is [6],[7]:

$$P_n = \frac{N^2 r_0 \mathcal{E}_0 \omega_0}{\rho} \frac{4\rho}{3\pi h} n \sum_{p=1,3,\ldots}^{p<nh/\pi\rho} (\frac{p\pi\rho}{nh})^4 \Big(K_{2/3}^2(x) + K_{1/3}^2(x) \Big) \ , \tag{40}$$

where the argument of the modified Bessel functions $K_\nu(x)$

$$x = \frac{1}{3n^2}(\frac{p\pi\rho}{h})^3 . \tag{41}$$

The main contribution to the sum in Eq. 40 is produced by the region in which argument of the modified Bessel functions is small. Outside this region, the functions K_ν exponentially small. The condition $x < 1$ for $p = 1$ is the same as the one given in Eq. 12. When $n_{th} \gg 1$ (see definition in Eq. 11), the sum in Eq. 40 can be replaced by integration over x. Hence,

$$P_n = \frac{2C}{3^{1/3}\pi^2}\frac{N^2 r_0 \mathcal{E}_0 \omega_0}{\rho} n^{1/3} \quad \text{for} \quad n_{th} < n < \gamma^3 , \tag{42}$$

where[2]

$$C = \int_0^\infty dx x^{2/3}\Big(K^2_{2/3}(x) + K^2_{1/3}(x)\Big) \approx 3.68 . \tag{43}$$

One can see that the spectrum of coherent radiation is essentially the same as that of incoherent radiation. The extra factor N increases the power by a large amount, making the bunch moving in a magnetic field a very powerful source of the electro-magnetic radiation. Formula Eq. 42 defines the radiation spectrum for $n_{th} < n$.

To define the spectrum of coherent synchrotron radiation for the range $n < n_{th}$, we need to evaluate the sum in Eq. 40 for $x > 1$. To do this we can use the asymptotic expansion of the functions K_ν for $\nu = 1/3$ and $2/3$, Eq. 3. Since these expressions fall exponentially with increasing x, it is enough to keep only the first term in the sum over p. Then we get:

$$P_n = \frac{2^{5/3}}{\pi}\frac{N^2 r_0 \mathcal{E}_0 \omega_0}{\rho} n^{1/3}(\frac{n^2_{th}}{2n^2})^2 \exp(-\frac{n^2_{th}}{2n^2}) \quad \text{for} \quad n < n_{th} , \tag{44}$$

which defines the shape of the spectrum below the threshold.

7 Total Power of Coherent Synchrotron Radiation

To obtain the total power of coherent synchrotron radiation, the power spectrum Eq. 42, should be multiplied by the spectrum of the bunch distribution and summed over all allowed harmonic numbers n.

For a particular case of the Gaussian distribution, the particle density spectrum is given by formulae in Eq.10:

$$P_{coh} \approx \frac{N^2 r_0 \mathcal{E}_0 \omega_0}{\rho} \sum_{n_{th}}^{n_c} n^{1/3} e^{-(\frac{n\sigma_\parallel}{\rho})^2} . \tag{45}$$

Since the harmonic numbers n which produce the main contribution to the sum in this equation are very large, the summation can be replaced by an integration over $x = n^2\sigma_s^2/\rho^2$:

$$P_{coh} \approx \frac{N^2 r_0 \mathcal{E}_0 \omega_0}{2\rho}(\frac{\rho}{\sigma_s})^{4/3} F(x_{th}) , \tag{46}$$

where the form factor is

$$F(x_{th}) = \int_{x_{th}}^{x_c} dx x^{-1/3} e^{-x} \tag{47}$$

and the limits of integration are

$$x_{th} \equiv \frac{2}{3}(\frac{\pi\rho}{h})^3(\frac{\sigma_s}{\rho})^2 , \quad x_c \equiv \frac{9}{4}\gamma^6(\frac{\sigma_s}{\rho})^2 \gg 1 . \tag{48}$$

Since the integrand in Eq. 47 is exponentially small for large x, the integration in it can be extended without introducing any error to ∞. Then the form factor $F(x)$ reduces to the incomplete Γ-function of the order 2/3: $F(x) \equiv \Gamma(2/3, x_{th})$.

For small and large values of its argument, estimates can be obtained by using appropriate expansions of this function.

1. In the first case when $x_{th} \ll 1$, $F(x_{th}) = \Gamma(2/3)$. The total power of the radiation is given by the formula Eq. 46 with this value of $F(x_{th})$. This expression was first derived by Schiff[15].

2. In the second, most usual case when $x_{th} \gg 1$, $F(x_{th}) \approx x_{th}^{-1/3} e^{-x_{th}}$. The total power of the radiation in this case is exponentially small.

Similar results could be obtained for other particle distributions, e.g., for the uniform distribution in the range $-\sigma_s < s < \sigma_s$. The density spectrum of such a distribution is $f_n = [\sin\ x/x]^2$, where $x = n\sigma_s/\rho$. Although it is different from the one for the Gaussian distribution, nevertheless, for $x \gg 1$ it is also a sharp function of its argument. In the limit $n \to \infty$, it tends to the δ-function: $f_n \to \pi\delta(\sigma_s/\rho)/n$. That probably was overlooked in the paper[7]. In the derivation of the total power of radiation the order of summation over the eigen numbers p and n was interchanged and then $sin^2\ x$ has been replaced by its average value 1/2. As the result, the total power of radiation given in this paper has no exponential term $\exp(-x_{th})$ Apart from this their expression agrees with Eq. 46 if $F(x_{th})$ is replaced by $x_{th}^{-1/3}$.

Figure 3 represents function $F(x_{th})$.

The total power of coherent synchrotron radiation is mainly determined by parameter x_{th} Eq. 48. It depends on two different physical entities: (a) *shielding*, which is characterized by the ratio $\rho/\ h$, and (b) *bunching*, which is characterized by the ratio σ/ρ. In all storage rings for a high particle energy, usually $x_{th} \gg 1$ and coherent radiation is very week. In order to be able to observe this type of radiation in a storage ring, it should be specially designed in such a way as to limit the magnitude of the parameter $x_{th} \approx 1$. This

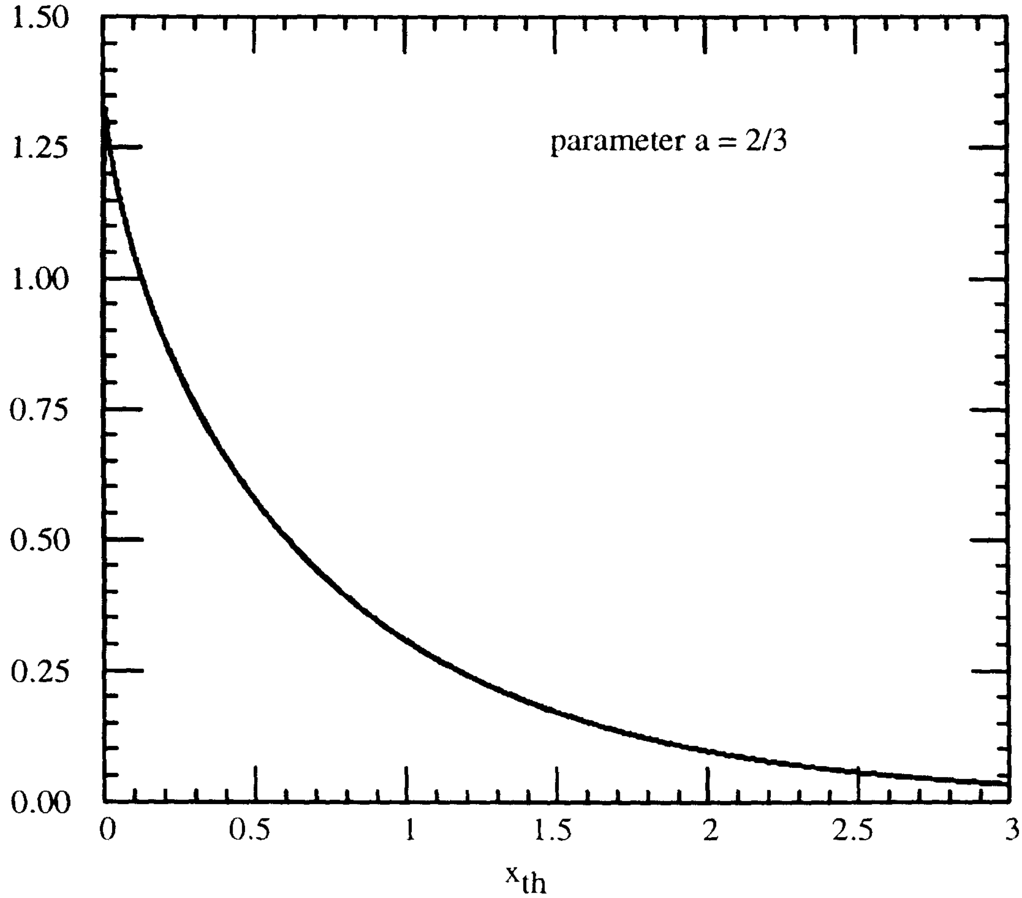

Figure 3: Function $F(x_{th})$, cf. Eq. 47

can be achieved when the ratio ρ/h is not too large and the ratio σ/ρ is not too small enough . In other words, the transverse size of the vacuum chamber should be large and the longitudinal effective size of the bunch should be small.

It is interesting to evaluate x_{th} for conditions in which an attempt was done to observe coherent synchrotron radiation[16] albeit with a linac beam. Using the parameters of the beam and apparatus ($\rho = 2.44$ m, $h = 0.2$ m, $\sigma \approx 1$ mm) given in the paper, the value of parameter $x_{th} \approx 6.3 \cdot 10^{-3}$. Such a small value of x_{th} seems to indicate that the experimental conditions are not optimized for the goal of the observation of coherent synchrotron radiation.

In the proposal[17] to build a special storage ring for that goal the shielding is not considered. The size of this ring vacuum chamber is not presented. That does not allow to estimate parameter x_{th} and viability of the proposal.

8 Conclusion

The problem of coherent synchrotron radiation is reviewed. Several important aspects of the radiation – coherence, bunching, shielding and absorption are considered and relevant parameters are presented. An existing discrepancy in the estimates of the shielding of radiation is explained and removed. The shape of the radiation spectrum for frequencies below its maximum is calculated. The total power of coherent synchrotron radiation is obtained and the parameter which governs its magnitude is introduced.

In any existing large storage ring practically no coherent radiation might be expected to influence its performance or to be observed under normal mode of its operation. With careful design it might be possible to build a storage ring as a source of very powerful radiation in the millimeter and sub-millimeter wave length region.

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