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Katholieke Universiteit Nijmegen

Nijmegen preprint
HEN-381
February 1995

Density fluctuations in multiparticle production at high energies*

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*Invited review given at the Fourth Annual Seminar on Nonlinear Phenomena in Complex Systems, Minsk (Belarus), February 6-9, 1995

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Density fluctuations in multiparticle production at high energies*

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Abstract. Besides being important to describe high energy processes themselves, the dynamics of multiparticle production is part of the general field of non-linear phenomena and complex systems. Multiparticle dynamics is one of the rare fields of physics where *higher* order correlations are directly accessible in their full multi-dimensional characteristics under well controlled experimental conditions. This allows it to serve as an ideal testing ground for the development of advanced statistical methods.

Higher order correlations have indeed been observed as particle-density fluctuations. Approximate scaling with improving resolution gives evidence for a self-similar correlation effect. Quantum-Chromodynamics branching is a good candidate for a dynamical explanation of these correlations in e^+e^- collisions at CERN/LEP and, expectedly, also of those in pp collisions at future CERN/LHC energies, but also other sources such as identical particle Bose-Einstein interference effects contribute.

A particular question at the moment is the smooth transition from the QCD branching domain (gluon interference before hadronization) to the Bose-Einstein correlation domain (identical pion interference after hadronization). Both mechanisms have clearly been observed in e^+e^- collisions at CERN/LEP energies. The large amount of high resolution data being collected at LEP in the near future will allow to study the *genuine* (i.e. non-trivial) higher order correlations in both domains.

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1. Introduction

Recent years have witnessed a remarkably intense experimental and theoretical activity in search of scale-invariance and fractality in multihadron production processes. Besides being an important part of high energy physics itself, the dynamics of multiparticle production in collisions of elementary particles at high energies (multiparticle dynamics) is part of the general field of non-linear phenomena and complex systems. Studies of classical and quantum chaos, non-equilibrium dissipative processes, random media, growth phenomena and many more have all contributed to reveal the pervasive importance of self-similarity, power-laws and fractals in nature. Research in these fields is still in full swing and continues to uncover intriguingly simple and often surprisingly universal behaviour in complex, non-linear systems.

While considerable experience already exists in many fields for the study of two-component correlation, it is often in *higher*-order (i.e. multi-component) correlations that the most interesting properties manifest themselves, in the simultaneous interplay of a large number of components. The special significance of multiparticle dynamics for the development of advanced statistical methods lies in the fact that it is one of the rare fields of physics where higher order correlations are directly accessible in their full multi-dimensional characteristics under well controlled experimental conditions.

Higher order correlations have recently been observed as particle-density fluctuations in cosmic ray, nucleus-nucleus, hadron-hadron, e^+e^- and lepton-hadron experiments. To study these fluctuations in detail, normalized correlation integrals are being analyzed in phase-space domains of ever decreasing size. Approximate scaling with decreasing domain size is now observed in all types of collision, giving evidence for a correlation effect self-similar over a large range of the resolution (called intermittency, in analogy to a statistically similar problem in spatio-temporal turbulence).

Parton branching of Quantum-Chromodynamics predicts the type of correlations observed in e^+e^- collisions at CERN/LEP and, expectedly also in pp collisions at CERN/LHC energies. However, also other sources such as Bose-Einstein interference of identical particles contribute. Fast development of the applied technology has taken place over the last years, in particular in the extension from originally one-dimensional to full three-dimensional phase space analysis.

2. Method and Technology

2.1 Particle densities

A collision between particles a and b is assumed to yield exactly n particles in a sub-volume Ω of the total phase space Ω_{tot}. The single symbol y represents the kinematical variables needed to specify the position of each particle in this space (for example, y can be the the full four-momentum of a particle and Ω a cell in invariant phase space or simply the c.m. rapidity† of a particle and Ω an interval of length δy). The distribution of points in Ω can then be characterised by continuous probability densities $P_n(y_1, \ldots, y_n)$; $n = 1, 2, \ldots$. For simplicity we assume all final-state particles to be of the same type. In this case, the **exclusive** distributions $P_n(y_1, \ldots, y_n)$ can be taken fully symmetric in $y_1, \ldots, y_n$; they describe the distribution in Ω when the multiplicity is exactly n.

The corresponding **inclusive** distributions are given for $n = 1, 2, \ldots$ by:

$$\begin{aligned} \rho_n(y_1, \ldots, y_n) &= P_n(y_1, \ldots, y_n) \\ &+ \sum_{m=1}^{\infty} \frac{1}{m!} \int_\Omega P_{n+m}(y_1, \ldots, y_n, y'_1, \ldots, y'_m) \prod_{i=1}^{m} dy'_i \,. \end{aligned} \tag{1}$$

The inverse formula is

$$\begin{aligned} P_n(y_1, \ldots, y_n) &= \rho_n(y_1, \ldots, y_n) \\ &+ \sum_{m=1}^{\infty} (-1)^m \frac{1}{m!} \int_\Omega \rho_{n+m}(y_1, \ldots, y_n, y'_1, \ldots, y'_m) \prod_{i=1}^{m} dy'_i \,. \end{aligned} \tag{2}$$

Here, $\rho_n(y_1, \ldots, y_n)$ is the probability density for n points to be at $y_1, \ldots, y_n$, irrespective of the presence and location of any further points. Integration over an interval Ω in y yields

$$\begin{aligned} &\int_\Omega \rho_1(y) dy = \langle n \rangle \\ &\int_\Omega \int_\Omega \rho_2(y_1, y_2) dy_1 dy_2 = \langle n(n-1) \rangle \\ &\int_\Omega dy_1 \ldots \int_\Omega dy_q \rho_q(y_1, \ldots, y_q) = \langle n(n-1) \ldots (n-q+1) \rangle \,, \end{aligned} \tag{3}$$

where the angular brackets imply the average over the event ensemble.

2.2 Cumulant correlation functions

Besides the interparticle *correlations* we are looking for, the inclusive q-particle densities $\rho_q(y_1, \ldots, y_q)$ in general contain "trivial" contributions from lower-order densities. It is, therefore, advantageous to consider a new sequence of functions $C_q(y_1, \ldots, y_q)$ as those statistical quantities which vanish whenever one of their arguments becomes statistically independent of the others. Deviations of these functions from zero shall be addressed as *genuine* correlations.

The quantities with the desired properties are the correlation functions–also called (factorial) cumulant functions–or, in integrated form, Thiele's semi-invariants [1]. A formal proof of this property was given by Kubo [2]. The cumulant correlation

† Rapidity y is defined as $y = \frac{1}{2} \ln[(E + p_L)/(E - p_L)]$, with E the energy and p_L the longitudinal component of momentum vector $\vec{p}$ along a given direction (beam-particles, jet-axis, etc.); pseudorapidity is defined as $\eta = \frac{1}{2} \ln[(p + p_L)/(p - p_L)]$.

functions are defined as in the cluster expansion familiar from statistical mechanics via the sequence [3, 4, 5]:

$$\rho_1(1) = C_1(1), \tag{4}$$
$$\rho_2(1,2) = C_1(1)C_1(2) + C_2(1,2), \tag{5}$$
$$\begin{aligned}\rho_3(1,2,3) &= C_1(1)C_1(2)C_1(3) + C_1(1)C_2(2,3) + C_1(2)C_2(1,3) + \\ &\quad + C_1(3)C_2(1,2) + C_3(1,2,3);\end{aligned} \tag{6}$$

and, in general, by

$$\begin{aligned}\rho_m(1,\ldots,m) &= \sum_{\{l_i\}_m} \sum_{\text{perm.}} \underbrace{[C_1()\cdots C_1()]}_{l_1\,\text{factors}} \underbrace{[C_2(,)\cdots C_2(,)]}_{l_2\,\text{factors}} \cdots \\ &\quad \cdots \underbrace{[C_m(,\ldots,)\cdots C_m(,\ldots,)]}_{l_m\,\text{factors}}.\end{aligned} \tag{7}$$

Here, l_i is either zero or a positive integer and the sets of integers $\{l_i\}_m$ satisfy the condition

$$\sum_{i=1}^{m} i\, l_i = m. \tag{8}$$

The arguments in the C_i functions are to be filled by the m possible momenta in any order. The sum over permutations is a sum over all distinct ways of filling these arguments. For any given factor product there are precisely [4]

$$\frac{m!}{[(1!)^{l_1}(2!)^{l_2}\cdots(m!)^{l_m}]\, l_1! l_2! \cdots l_m!} \tag{9}$$

terms.

The relations (7) may be inverted with the result:

$$\begin{aligned}C_2(1,2) &= \rho_2(1,2) - \rho_1(1)\rho_1(2)\ , \\ C_3(1,2,3) &= \rho_3(1,2,3) - \sum_{(3)} \rho_1(1)\rho_2(2,3) + 2\rho_1(1)\rho_1(2)\rho_1(3)\ , \\ C_4(1,2,3,4) &= \rho_4(1,2,3,4) - \sum_{(4)} \rho_1(1)\rho_3(1,2,3) - \sum_{(3)} \rho_2(1,2)\rho_2(3,4) \\ &\quad + 2\sum_{(6)} \rho_1(1)\rho_1(2)\rho_2(3,4) - 6\rho_1(1)\rho_1(2)\rho_1(3)\rho_1(4).\end{aligned} \tag{10}$$

In the above relations we have abbreviated $C_q(y_1,\ldots,y_q)$ to $C_q(1,2,\ldots,q)$; the summations indicate that all possible permutations have to be taken (the number under the summation sign indicates the number of terms). Expressions for higher orders can be derived from the related formulae given in [6].

It is often convenient to divide the functions ρ_q and C_q by the product of one-particle densities. This leads to the definition of the normalized inclusive densities and correlations:

$$
\begin{aligned}
r_q(y_1, \dots, y_q) &= \rho_q(y_1, \dots, y_q)/\rho_1(y_1)\dots\rho_1(y_q), \qquad (11)\\
K_q(y_1, \dots, y_q) &= C_q(y_1, \dots, y_q)/\rho_1(y_1)\dots\rho_1(y_q). \qquad (12)
\end{aligned}
$$

In terms of these functions, correlations have been studied extensively for $q = 2$. Results also exist for $q = 3$, but usually statistics (i.e. number of events available for analysis) is too small to isolate genuine correlations. To be able to do that for $q \geq 3$, one has to apply moments defined via the integrals (3), but in limited phase space cells.

2.3 Cell-averaged factorial moments and cumulants

In practical work, with limited statistics, it is almost always necessary to perform averages over more than a single phase space cell. Let Ω_m be such a cell (e.g. a single rapidity interval of size δy) and divide the phase space volume into M non-overlapping cells Ω_m of size $\delta\Omega$ independent of m. Let n_m be the number of particles in cell Ω_m. Different cell-averaged moments may be considered, depending on the type of averaging.

Normalized cell-averaged factorial moments [7] are defined as

$$
\begin{aligned}
F_q(\delta y) &\equiv \frac{1}{M}\sum_{m=1}^{M}\frac{\langle n_m(n_m-1)\dots(n_m-q+1)\rangle}{\langle n_m\rangle^q} \qquad (13)\\
&\equiv \frac{1}{M}\sum_{m=1}^{M}\frac{\int_{\delta y}\rho_q(y_1,\dots,y_q)\prod_{i=1}^{q}dy_i}{\left(\int_{\delta y}\rho(y)dy\right)^q} \qquad (14)\\
&= \frac{1}{M(\delta y)^q}\sum_{m=1}^{M}\int_{\delta y}\frac{\rho_q(y_1,\dots,y_q)\prod_{i=1}^{q}dy_i}{(\bar{\rho}_m)^q}. \qquad (15)
\end{aligned}
$$

The whole rapidity interval ΔY is divided into M equal bins: $\Delta Y = M\delta y$; each y_i is within the δy-range and $\langle n_m\rangle \equiv \bar{\rho}_m\delta y \equiv \int_{\delta y}\rho_1(y)dy$. An example for $q = 2$ is given in Fig. 1a.

Likewise, cell-averaged normalized factorial cumulant moments may be defined as

$$
K_q(\delta y) = \frac{1}{M(\delta y)^q}\sum_{m=1}^{M}\int_{\delta y}\frac{C_q(y_1,\dots,y_q)\prod_{i=1}^{q}dy_i}{(\bar{\rho}_m)^q} \quad . \qquad (16)
$$

They are related to the factorial moments by‡

$$
\begin{aligned}
F_2 &= 1 + K_2 \quad ,\\
F_3 &= 1 + 3K_2 + K_3 \quad ,\\
F_4 &= 1 + 6K_2 + 3\overline{K_2^2} + 4K_3 + K_4 \,. \qquad (17)
\end{aligned}
$$

‡ The higher-order relations can be found in [8]

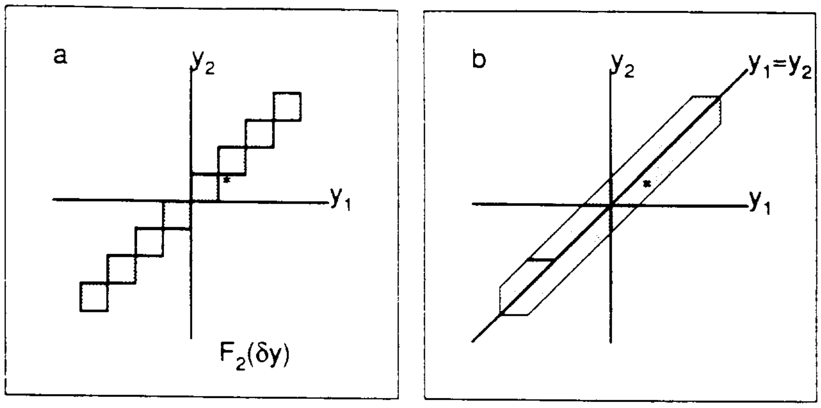

Figure 1. Integration domains for a) the second order factorial moment and b) the second order correlation integral. The small star in a) indicates the position of a particle pair with $|y_1 - y_2| < \delta y$ that is excluded from the F_2 calculation due to the binning. In b) the star is included.

In F_4 and higher-order moments, "bar averages" appear defined as $\overline{AB} \equiv \sum_m A_m B_m / M$.

To detect dynamical fluctuations in the density of particles produced in a high-energy collision, a way has to be devised to eliminate, or to reduce as much as possible, the statistical fluctuations (noise) due to the finiteness of the number of particles in the counting cell(s). This requirement can to a large extent be satisfied by studying factorial moments. It forms the basis of the factorial moment technique, known in optics, but rediscovered for multi-hadron physics in [7]. This crucial property does not apply to e.g. ordinary moments $\langle n^q \rangle / \langle n \rangle^q$.

The property of Poisson-noise suppression has made measurement of factorial moments a standard technique, e.g. in quantum optics, to study the statistical properties of arbitrary electromagnetic fields from photon-counting distributions. Their utility was first explicitly recognised, for the single time-interval case, in [9] and later generalised to the multivariate case in [10].

2.4 Correlation integrals

A fruitful recent development in the study of density fluctuations is the correlation strip integral method [11]. By means of integrals of the inclusive density over a strip domain of Fig. 1b, rather than a sum of the box domains of Fig. 1a, one not only avoids unwanted side-effects such as splitting of density spikes, but also drastically increases the integration volume (and therefore the statistical significance) at a given resolution.

Let us consider first the factorial moments F_q defined according to (14). As shown in Fig. 1a for $q = 2$, the integration domain $\Omega_B = \sum_{m=1}^{M} \Omega_m$ thus consists of M q-dimensional boxes Ω_m of edge length δy. A point in the m-th box corresponds to a pair (y_1, y_2) of distance $|y_1 - y_2| < \delta y$ and both particles in the same bin m. Points with $|y_1 - y_2| < \delta y$ which happen *not* to lie in the same but in adjacent bins (e.g. the asterix in Fig. 1a) are left out. The statistics can be approximately doubled by a change of the integration volume Ω_B to the strip-domain of Fig. 1b. For $q > 2$, the increase of integration volume (and reduction of squared statistical error) is in fact roughly proportional to the order of the correlation. The gain is even larger when working in two- or three-dimensional phase-space variables.

In terms of the strips (or hyper-tubes for $q > 2$), the correlation integrals become

$$F_q^S(\delta y) \equiv \frac{\displaystyle\int_{\Omega_s} \rho_q(y_1 \dots y_q) \Pi_i dy_i}{\displaystyle\int_{\Omega_s} \rho_1(y_1) \dots \rho_1(y_q) \Pi_i dy_i} \quad . \tag{18}$$

These integrals can be evaluated directly from the data after selection of a proper distance measure ($|y_i - y_j|, [(y_i - y_j)^2 + (\phi_i - \phi_j)^2]^{1/2}$, or better the four-momentum difference $Q_{ij}^2 = -(p_i - p_j)^2$) and after definition of a proper multiparticle topology (GHP integral [11], snake integral [12], star integral [13]).

The numerator of the factorial moments F_q can be determined by counting, for each event, the number of q-tuples that have a pairwise Q_{ij}^2 smaller than a given value Q^2 and then averaging over all events. Using the Heaviside unit step function Θ, this can mathematically be expressed as

$$F_q^S(Q^2) = \frac{1}{\text{norm}} \left\langle q! \sum_{i_1 < \dots < i_q} \prod_{\substack{all\ pairs \\ k_1, k_2}} \Theta(Q^2 - Q_{i_{k_1}, i_{k_2}}^2) \right\rangle \quad , \tag{19}$$

where the factor $q!$ takes into account the number of permutations within a q-tuple.

The normalization is obtained from "mixed" events constructed by random selection of tracks from different events in a track pool. The multiplicity of a mixed event is taken to be a Poissonian random variable, thereby ensuring that no extra correlations are introduced. A correction factor is applied for the difference in average multiplicity of the Poissonian and the experimental distribution. The mixed events are treated in the same way as real events.

2.5 Power-law scaling

The technique proposed in [7] consists in measuring the dependence of the normalized factorial moments (or correlation integrals) $F_q(\delta y)$ as a function of the resolution δy. For definiteness, δy is supposed to be an interval in rapidity, but the method generalises to arbitrary phase-space dimension, so also to the use of Q^2.

As pointed out above, the scaled factorial moments enjoy the property of "noise-suppression". High-order moments further act as a filter and resolve the large n_m tail of the multiplicity distribution. They are thus particularly sensitive to large density fluctuations at the various scales δy used in the analysis.

As proven in [7], a "smooth" (rapidity) distribution, which does not show any fluctuations except for the statistical ones, has the property that $F_q(\delta y)$ is independent of the resolution δy in the limit $\delta y \to 0$. On the other hand, if dynamical fluctuations exist and P_ρ is "intermittent", the F_q obey the power law

$$F_q(\delta y) \propto (\delta y)^{-\phi_q} \ , \quad (\delta y \to 0). \tag{20}$$

The powers ϕ_q (slopes in a double-log plot) are related [14] to the anomalous dimensions

$$d_q = \phi_q/(q-1) \ , \tag{21}$$

a measure for the deviation from an integer dimension. Equation (20) is a scaling law since the ratio of the factorial moments at resolutions L and ℓ

$$R = F_q(\ell)/F_q(L) = (L/\ell)^{\phi_q} \tag{22}$$

only depends on the ratio L/ℓ, but not on L and ℓ, themselves.

As pointed out above, the experimental study of correlations is difficult already for three particles. The close connection between correlations and factorial moments offers a possibility to measure higher-order correlations with the factorial moment technique at smaller distances than previously feasible. Via (21), the method further relates possible scaling behaviour of such correlations to the physics of fractal objects.

One further has to stress the advantages of factorial cumulants compared to factorial moments, since the former measure *genuine* correlation patterns, whereas the latter contain additional large combinatorial terms which may mask the underlying dynamical correlations.

The definition of "intermittency" given in (20), has its origin in other disciplines[§]. It rests on a loose parallel between the high non-uniformity of the distribution of energy dissipation, for example, in turbulent intermittency and the occurrence of large "spikes" in hadronic multiparticle final states. In the following we use the term "intermittency" in a weaker sense, meaning the rise of factorial moments with increasing resolution not necessarily according to a strict power law.

3. The state of the art

The suggestion that normalized factorial moments of particle distributions might show power-law behaviour has spurred a vigorous experimental search for (more or less) linear dependence of $\ln F_q$ on $-\ln \delta y$. A review of the present situation is given in [16].

As an example, we give Fig. 2 where NA22 data [17] are plotted as a function of $-\ln Q^2$, with all two-particle combinations in an n-tuple having $Q^2_{ij} < Q^2$. The

[§] For a masterly exposé of this subject see [15].

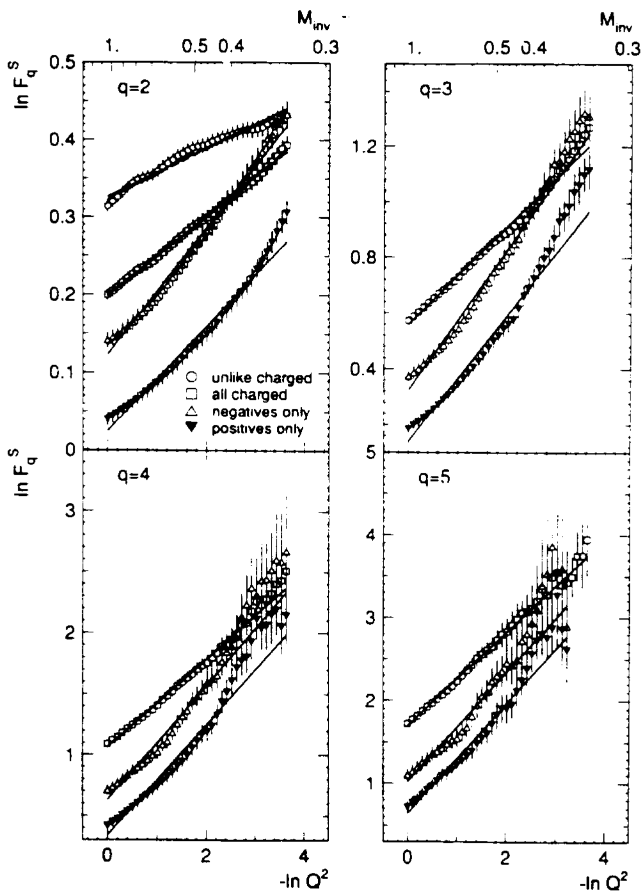

Figure 2. Correlation Integral Method applied to NA22 data in terms of Q^2.

following observations can be made:
i) with the (one-dimensional) distance measure Q^2, the moments show a steep rise with decreasing Q^2;
ii) negatives are much steeper than all-charged,
iii) F_2^S is flatter for $(+-)$ than for all charged or like-charge combinations.

The last two observations directly demonstrate the large influence of *identical* particle correlations on the factorial moments and their scaling behaviour. These results agree very well with results from the UA1 collaboration [18]. In [19] it has, furthermore, been shown in terms of the $K_q(\delta y)$ of (16) that genuine correlations exist at least up to order $q = 5$ in hadron-hadron collisions.

Of particular interest is a comparison of hadron-hadron to e^+e^- results in terms of same and opposite charges of the particles involved. This has been done for UA1 and DELPHI data in [20] and is shown in Fig. 3 for $q = 2$ (in fact, in this figure a differential form of (18) is presented). An important difference between UA1 and DELPHI can be observed on both sub-figures: For relatively large $Q^2 (> 0.03\ \mathrm{GeV}^2)$,

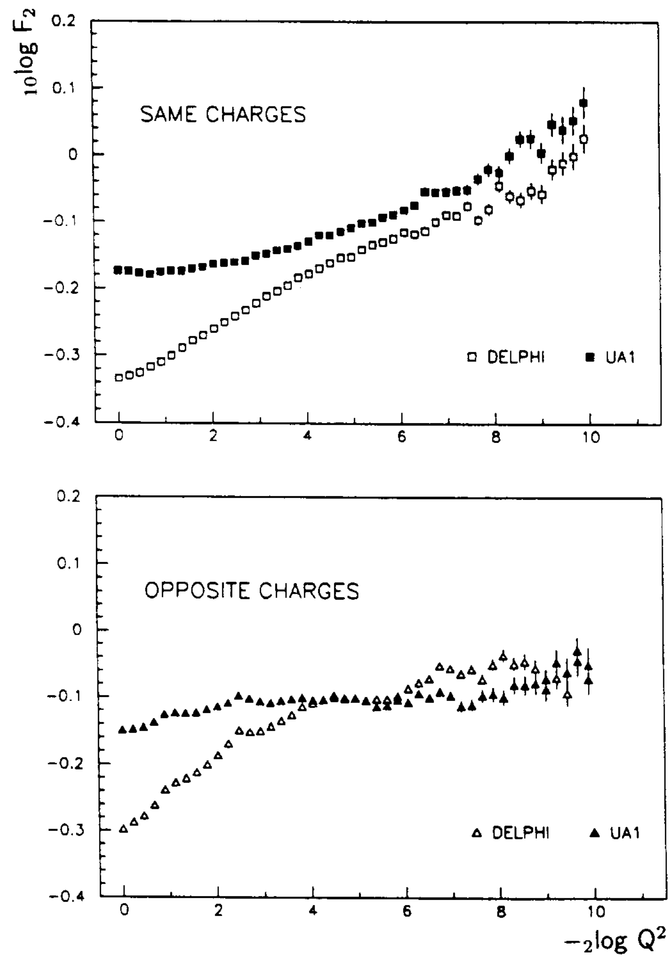

Figure 3. Comparison of correlation integrals for $q = 2$ in their differential form (in intervals $Q^2, Q^2 + dQ^2$) as a function of $_2\log(1/Q^2)$ for e^+e^- (DELPHI) and hadron-hadron collisions (UA1).

where Bose-Einstein effects do not play a major role, the e^+e^- data increase much faster with increasing $-_2\log Q^2$ than the hadron-hadron results. For e^+e^-, the increase in this Q^2 region is very similar for same and for opposite-sign charges. At small Q^2, however, the e^+e^- results approach the hadron-hadron results. The authors conclude that for e^+e^- at least two processes are responsible for the power-law behaviour: Bose-Einstein correlation following the evolution of jets. In hadron-hadron collisions at present collider energies mainly Bose-Einstein effects seem relevant.

The exact functional form of F_2^S is derived from the data of UA1 and NA22, again in its differential form, in Fig.4. Clearly, the data favour a power law in Q over a

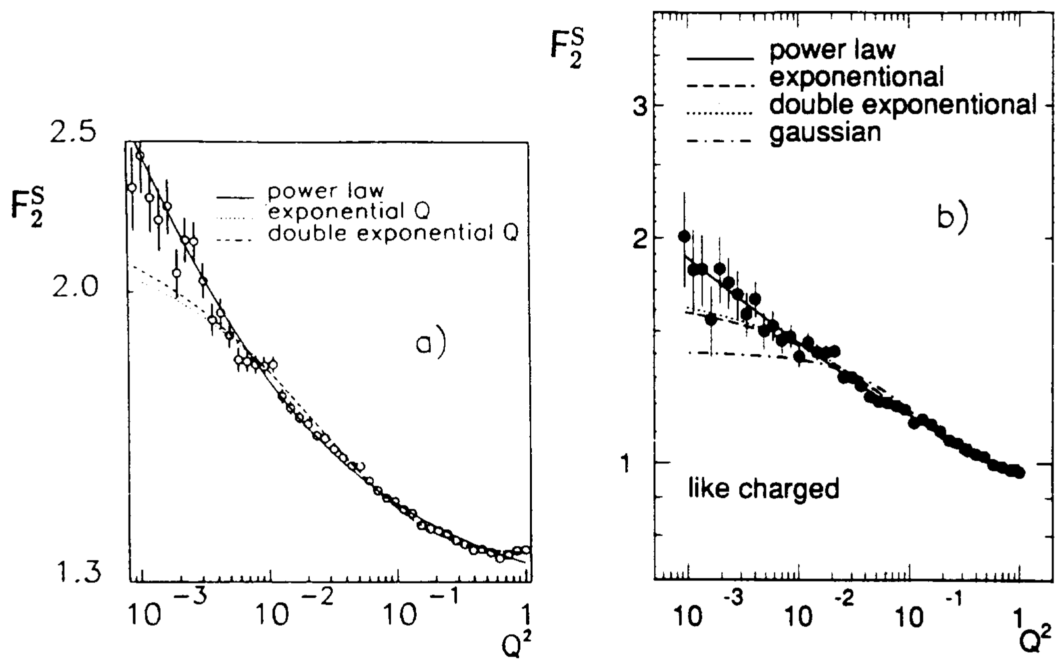

Figure 4. Correlation integrals F_2 (in their differential form) as a function of Q^2 for like-charged pairs in UA1 (preliminary) and NA22, compared to power-law, exponential, double-exponential and Gaussian fits, as indicated.

(non-scaling) exponential, double-exponential or Gaussian law.

If the observed effect is real, it supports a view recently developed in [21]. There, intermittency is explained from Bose-Einstein correlations between (like-sign) pions. As such, Bose-Einstein correlations from a static source are not power-behaved. A power law is obtained if i) the size of the interaction region is allowed to fluctuate, and/or ii) the interaction region itself is assumed to be a self-similar object extending over a large volume. Condition ii) would be realised if parton avalanches were to arrange themselves into self-organised critical states [22]. Though quite speculative at this moment, it is an interesting new idea with possibly far-reaching implications. We should mention also that in such a scheme intermittency is viewed as a final-state interaction effect and is therefore not troubled by hadronisation effects.

In perturbative QCD, on the other hand, the intermittency indices ϕ_q are directly related to the anomalous multiplicity dimension $\gamma_0 = (6\alpha_s/\pi)^{1/2}$ [23, 24, 25, 26] and, therefore, to the running coupling constant α_s. In the same theoretical context, it has been argued [24, 25, 26] that the opening angle χ between particles is a suitable and sensitive variable to analyse and well suited for these first analytical QCD calculations of higher order correlations. It is, of course, closely related to Q^2.

A first analytical QCD calculation [24] is based on the so-called double-log-approximation with angular ordering [27] (for a recent experimental study of angular ordering see [28]) and on local parton-hadron-duality [29]. A preliminary comparison with DELPHI data [30] gives encouraging results, even including an estimate for the running of the strong coupling constant α_S.

4. Summary

Multiparticle production in high-energy collisions is an ideal field to study genuine higher-order correlations. Methods also used in other fields are being tested and extended here for general application. Indications for genuine, approximately self-similar higher-order correlations are indeed found in hadron-hadron collisions, but need to be establised in their genuine and self-similar character in e^+e^- collisions at high energies. At large four-momentum distance Q^2, they are not only expected to be an inherent property of perturbative QCD, but are directly related to the anomalous multiplicity dimension and, therefore, to the running coupling constant α_s. At small Q^2, the QCD effects are complemented by Bose-Einstein interference of identical mesons carrying information on the unknown space-time development of particle production during the collision. The interplay between these two mechanisms, particularly important for an understanding of the process of hadronization, is completely unknown, at the moment.

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