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An Analytical Study of the O(N) Non-linear σ Models on Lattice*

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Abstract

In the variational cumulant expansion approach. we calculate the internal energy for the d-dimensional O(N) non-linear σ models on lattice. Specially, we give the results for two-dimensional O(3),O(4) and O(5) models. A comparison with Monte Carlo (MC) data is also presented. Our formalism can give the results for O(N) non-linear σ models (for any N and any dimensionality d) straightforward.

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1. Introduction

O(N) non-linear σ models in d=2 dimensions are of special interest to particle physicists due to their similarities with QCD. More than ten years ago, Wegner and Migdal [1] have conjectured that there are important similarities between two-dimensional O(N) non-liner σ models and four-dimensional SU(N) gauge theories. So far,this conjecture has been successful.

In both systems, renormalization group equation have the same structure[2]. Both theories($N \geq 3$) are asymptotically free, and for N=3 there exist instanton solutions [3]. Thus, it is an ideal laboratory for studying SU(N) gauge theory in O(N) σ models. Especially, these models are used as the laboratories to study the methods and problems to probe continuum behavior in asymptotically free theories on the lattice.

MC studies of the d-dimensional O(N) non-linear σ models on lattice have been for about ten years. It is the main goal to seek, to test and hopefully to demonstrate asymptotic scaling.

Many authors consider O(N) σ models on the lattice to investigate its nonperturbative effect, such as internal energy, mass gap, magnetic susceptibility and the continuum behavior,etc. There are many discussions [4-6] for the lattice O(N) models by numerical simulations. In this paper, we study the lattice d-dimensional O(N) non-linear σ models in the variational cumulant expansion approach. The formulation is given in Sec.2. In Sec.3, we give our results for two-dimensional O(3),O(4) and O(5) models for comparison with MC data.

2. The Formulation

Let us consider the d-dimensional O(N) non-linear σ models on lattice. The partition function of the system is defined as

$$Z = \int [d\sigma] e^{-S} \quad , \tag{1}$$

with the action(Wilson action)

$$S = -\beta \sum_{<i,j>} \vec{\sigma}_i \cdot \vec{\sigma}_j \quad , \tag{2}$$

where $\vec{\sigma}_i$ is a real unit N-dimensional vector at the i-th lattice site and $< i, j >$ are nearest-neighbor sites on a square lattice. $\beta = \frac{1}{T}$ is the inverse temperature and $[d\sigma]$ denotes the measure

$$[d\sigma] = \prod_i \prod_{\alpha}^{N} d\sigma_i^{\alpha} \quad , \tag{3}$$

and

$$\vec{\sigma}_i = \begin{pmatrix} \sigma_i^1 \\ \sigma_i^2 \\ \cdot \\ \\ \\ \sigma_i^N \end{pmatrix} \tag{4}$$

The σ_i^{α} satisfy the constraint

$$\sum_{\alpha=1}^{N} \sigma_i^{\alpha} \sigma_i^{\alpha} = 1 \quad . \tag{5}$$

The free energy F per site is obtained by

$$F = -\frac{1}{N_t} \ln Z \quad , \tag{6}$$

where N_t is the total number of sites. The average internal energy per site can be calculated. It is

$$E = \frac{\partial F}{\partial \beta} \quad . \tag{7}$$

Now let us use the variational cumulant expansion method to study these models. It is very difficult to compute the partition function analytically. In order to calculate it, let us introduce the effective action $S_0(\vec{J}, \vec{\sigma})$ which should be as "close" to the S as possible and easy to calculate. We choose

$$S_0 = -\sum_i \vec{J}_i \cdot \vec{\sigma}_i \quad , \tag{8}$$

where variational parameter $\vec{J}_i$ is defined for each site. $\vec{J}_i$ is a N-dimensional vector

$$\vec{J}_i = \begin{pmatrix} J_i^1 \\ J_i^2 \\ \cdot \\ \cdot \\ \cdot \\ J_i^N \end{pmatrix} \tag{9}$$

Due to the symmetry we put $\vec{J}_i = \vec{J}$ for each site. For simplicity for calculating. let us rotate the coordinate scheme to make the 1-axis parallel to $\vec{J}$. Thus, $J^1 = |\vec{J}| \equiv J$ and

$$J^2 = J^3 = ... = J^N = 0 \ . \tag{10}$$

Thus, the calculation could be proceeded in S_0 system instead of in S system. The effective action S_0 becomes

$$S_0 = -\sum_i J \ \sigma_i^1 \ . \tag{11}$$

The variational parameter J in S_0 is determined by minimizing free energy in cumulant expansion. (In this paper, we expand it up to K_1 order,see below.) According to cumulant expansion, the partition function is

$$Z = Z_0 \langle e^{S_0 - S} \rangle_0 = Z_0 \exp \left(\sum_{n=1}^{\infty} \frac{1}{n!} K_n \right) \ , \tag{12}$$

where

$$Z_0 \equiv \int [d\sigma] \ e^{-S_0(J, \sigma_i^1)} = (FI)^{N_i} \tag{13}$$

The single site integral is

$$FI \equiv \int d\vec{\sigma}_i \ e^{-J \sigma_i^1} \tag{14}$$

$$
\begin{aligned}
K_1 &= \langle S_0 - S \rangle_0 \ , \\
K_2 &= \langle (S_0 - S)^2 \rangle_0 - \langle S_0 - S \rangle_0^2 \ , \\
K_3 &= \langle (S_0 - S)^3 \rangle_0 - 3\langle S_0 - S \rangle_0 \langle (S_0 - S)^2 \rangle_0 + 2\langle S_0 - S \rangle_0^3 \ , \\
&\ldots \ .
\end{aligned} \tag{15}
$$

Because the modulus of $\vec{\sigma}_i$ is 1, so FI is an integral in the sphere surface

$$
FI = \int_0^{2\pi} d\phi_{N-1} \int_0^{\pi} d\phi_{N-2} \sin \phi_{N-2} \cdots \int_0^{\pi} d\phi_2 \sin^{N-3} \phi_2 \int_0^{\pi} d\phi_1 \sin^{N-2} \phi_1 e^{-J \cos \phi_1} \tag{16}
$$

We get

$$
FI = (2\pi)^{\frac{N}{2}} \frac{I_{\frac{N}{2}-1}(J)}{J^{\frac{N-2}{2}}} \ . \tag{17}
$$

The average in S_0 system is

$$
\langle \cdot \rangle_0 \equiv \frac{1}{Z_0} \int [d\sigma] e^{-S_0(J,\sigma^1)} (\cdot) \ , \tag{18}
$$

where $I_N(J)$ is N-th modified Bessel function.

For the convenience for calculating $K_1, K_2, \ldots$, we introduce some diagrammatic notations

$$
\begin{aligned}
\langle \mid \rangle_0 &\rightarrow \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle_0 \ , \\
\langle \cdot \rangle_0 &\rightarrow \langle \vec{J} \cdot \vec{\sigma}_i \rangle_0 = J \ \langle \ \sigma_i^1 \ \rangle_0 \ .
\end{aligned} \tag{19}
$$

Then $\langle \ S \ \rangle_0$, $\langle \ S_0 \ \rangle_0$, can be expressed as

$$
\begin{aligned}
\langle \ S \ \rangle_0 &= -\beta \ N_l d \ \langle \mid \rangle_0 \ , \\
\langle \ S_0 \ \rangle_0 &= -N_l \ \langle \cdot \rangle_0 \ .
\end{aligned} \tag{20}
$$

where d is the dimensionality. Now we can calculate K_n. (In this paper, we expand it up to K_3 order to calculate the internal energy.) For example,

$$
K_1 = \beta \ N_l d \ \langle \mid \rangle_0 - N_l \ \langle \cdot \rangle_0 \ . \tag{21}
$$

Similarly, we have

$$
\begin{aligned}
K_2 &= \beta^2 N_i d \, [\, \langle \, \| \, \rangle_0 - \langle \, | \, \rangle_0^2 \,] \\
&+\beta^2 N_i \, 2d \, (2d-1) \, [\, \langle \, |\_ \, \rangle_0 - \langle \, | \, \rangle_0^2 \,] \\
&-4\beta \, N_i d \, [\, \langle \, |. \, \rangle_0 - \langle \, | \, \rangle_0 \langle \cdot \rangle_0 \,] \\
&+N_i \, [\, \langle \cdot\cdot \rangle_0 - \langle \cdot \rangle_0^2 \,] \; , \\
&\cdots \; .
\end{aligned} \tag{22}
$$

Obviously only connected diagrams give non-zero contribution to K_n [7]. For example,the contributions of some of them are

$$
\begin{aligned}
\langle \, | \, \rangle_0 &= \langle \sigma^1 \rangle_0^2 \; , \\
\langle \cdot \rangle_0 &= J \langle \sigma^1 \rangle_0 \; , \\
\langle \, \| \, \rangle_0 &= \langle (\sigma^1)^2 \rangle_0^2 + \langle (\sigma^2)^2 \rangle_0^2 + \langle (\sigma^3)^2 \rangle_0^2 + \ldots \langle (\sigma^N)^2 \rangle_0^2 \; , \\
\langle \, \|\_ \, \rangle_0 &= \langle (\sigma^1)^3 \rangle_0 \langle (\sigma^1)^2 \rangle_0 \langle \sigma^1 \rangle_0 + \langle \sigma^1 \rangle_0^2 (\langle (\sigma^2)^2 \rangle_0^2 + \langle (\sigma^3)^2 \rangle_0^2 + \ldots + \langle (\sigma^N)^2 \rangle_0^2)
\end{aligned} \tag{23}
$$

The averages of all connected diagrams in Eq. (15) can be analytically calculated

$$
\begin{aligned}
\langle \sigma^1 \rangle_0 &= \frac{I_{\frac{N}{2}}(J)}{I_{\frac{N-2}{2}}(J)} \; , \\
\langle (\sigma^1)^2 \rangle_0 &= \frac{1}{J} \langle \sigma^1 \rangle_0 + \frac{I_{\frac{N+2}{2}}(J)}{I_{\frac{N-2}{2}}(J)} \; , \\
\langle (\sigma^2)^2 \rangle_0 &= \langle (\sigma^3)^2 \rangle_0 = \ldots = \langle (\sigma^N)^2 \rangle_0 = \frac{1}{J} \langle \sigma^1 \rangle_0 \; . \\
\langle (\sigma^1)^3 \rangle_0 &= \langle \sigma^1 \rangle_0 - \frac{N-1}{J} \frac{I_{\frac{N+2}{2}}(J)}{I_{\frac{N-2}{2}}(J)} \; .
\end{aligned} \tag{24}
$$

Substituting the variational parameter J determined by the minimizing free energy into (7), we get the internal energy E.

3. Results and Conclusion

We give the internal energy E for two-dimensional N=3, 4, and 5 models (shown in Fig. 1-3). MC data is from [4]. We see that our results are in good agreement with MC data in the strong and weak coupling region. There are a little discrepancy in the intermediate region.

Let's observe the weak expansion of internal energy of O(N) σ model, it is [8]

$$
\begin{aligned}
E &= 1 - \frac{N-1}{4\beta} - \frac{N-1}{32\beta^2} \\
&\quad - \frac{0.075(N-1) + 0.006(N-1)^2}{\beta^3} + O(\beta^{-4}) \ . \qquad (25)
\end{aligned}
$$

in d=2 dimensions.

Our calculation gives

$$
\begin{aligned}
E &= 1 - \frac{N-1}{2d\beta} - \frac{(N-1)(1+2N-3N^2+N^3)}{8d^2\beta^3} \\
&\quad + \frac{(N-1)(69+401N-250N^2-594N^3+821N^4-383N^5+64N^6)}{2048d^3\beta^3} \\
&\quad + O(\beta^{-4}) \ . \qquad (26)
\end{aligned}
$$

to the K_2 order, and

$$
\begin{aligned}
E &= 1 - \frac{N-1}{d\beta} + \frac{5(N-1)}{4d^2\beta} \\
&\quad - \frac{(N-1)(485+465N-858N^2+302N^3-11N^4+N^5)}{1024d^2\beta^2} \\
&\quad + \frac{(N-1)(357+593N-1050N^2+366N^3-11N^4+N^5)}{512d^3\beta^2} + O(\beta^{-3}) \qquad (27)
\end{aligned}
$$

to the K_3 order.(see Eqs.(7),(6) and(12))

Comparing Eqs.(25),(26)and(27),one can find again, if one determine free energy of the system to the first two order in the variational cumulant expansion, the best results of internal energy is also given by the first two order in this expansion.

Our results show that the variational cumulant expansion is also a promising theoretical method for the investigation of the lattice O(N) σ models.

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Figure Captions

Fig. 1: The internal energy for O(3) model(solid line), scaling approximation from Eq.(25) (dotted line), MC date taken from[4](crosses).

Fig. 2: The internal energy for O(4) model(solid line), scaling approximation from Eq.(25) (dotted line), MC date taken from[4](crosses).

Fig. 3: The internal energy for O(5) model(solid line), scaling approximation from Eq.(25) (dotted line), MC date taken from[4](crosses).

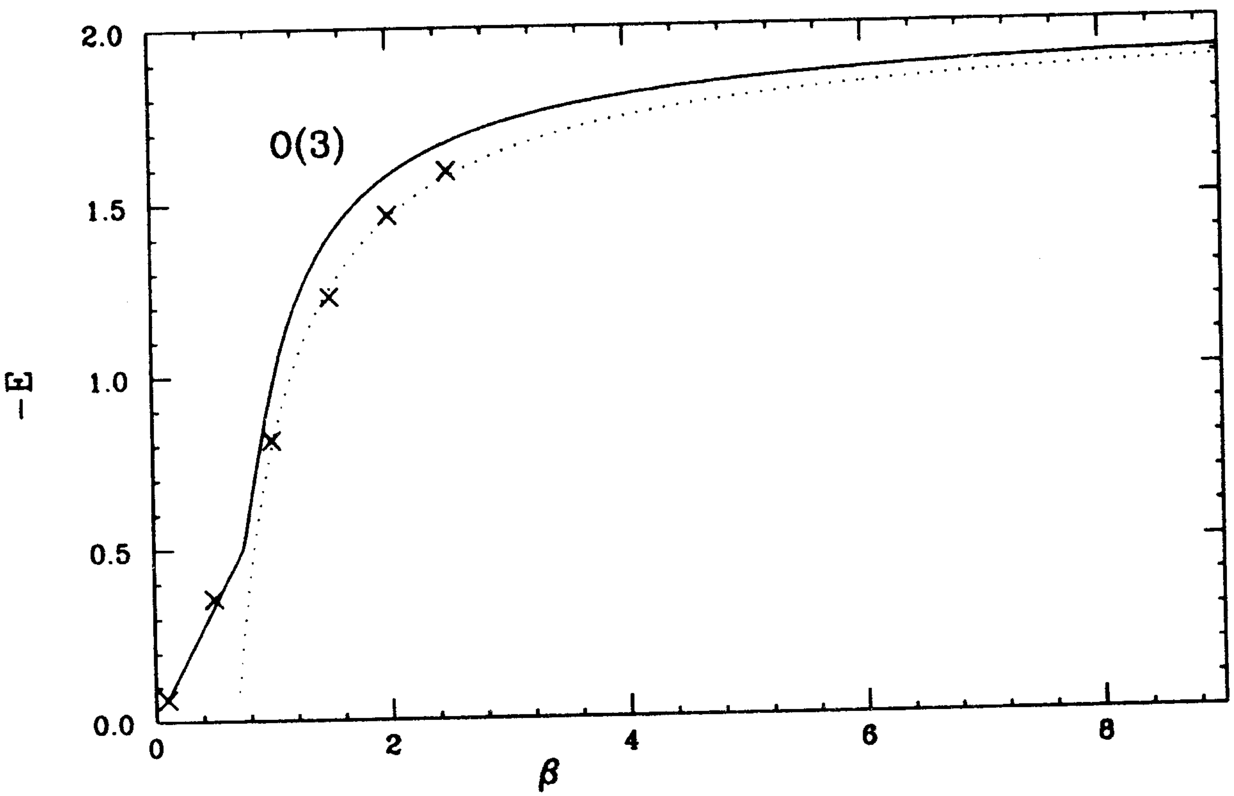

Fig.1

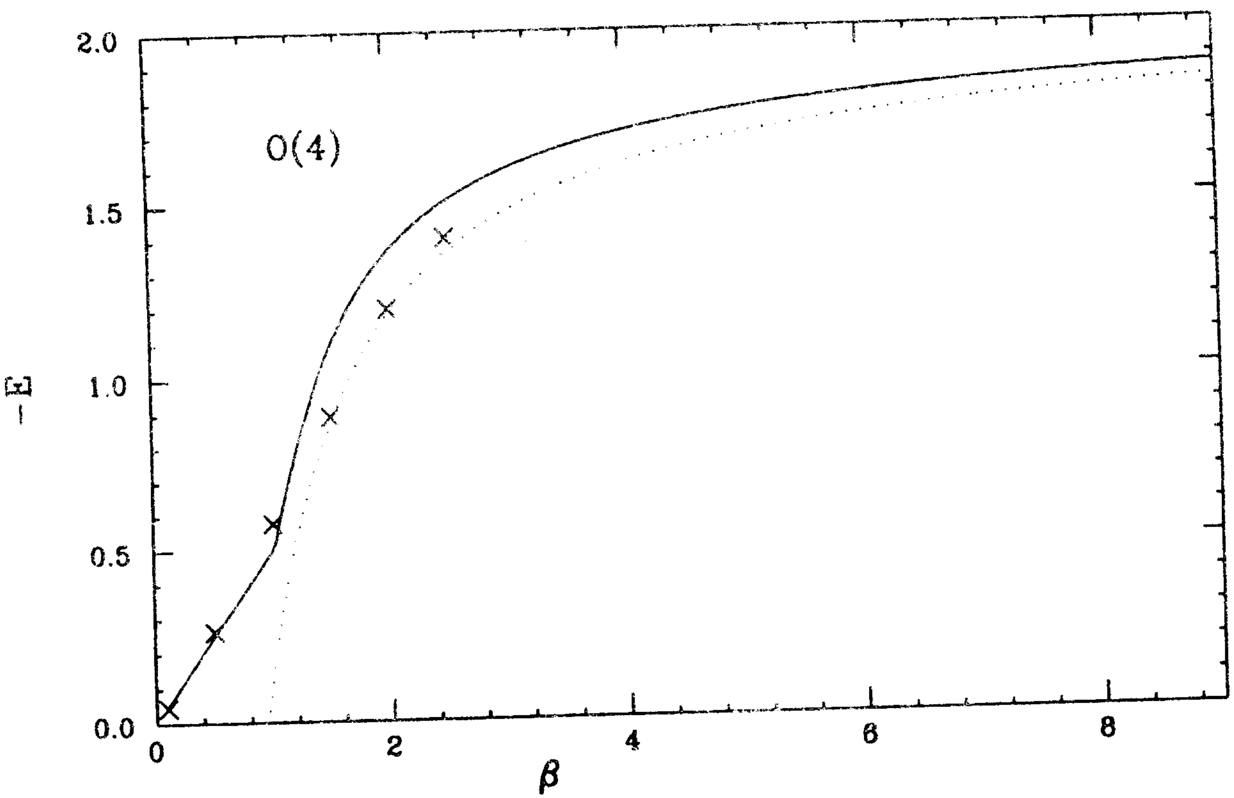

Fig.2

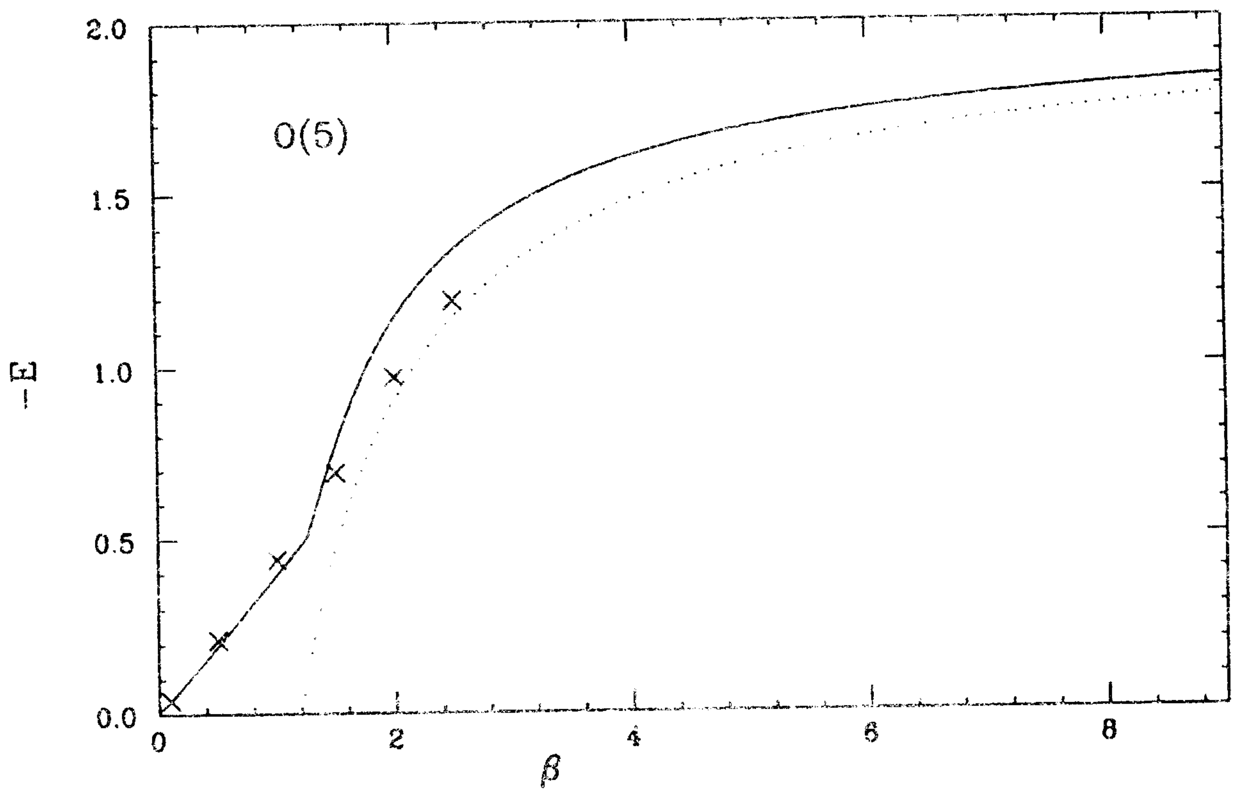

Fig.3