CERN-ACC-NOTE-2020-0054

August 2020

Giovanni.Iadarola@cern.ch

A Vlasov approach to study transverse instabilities driven by electron clouds

G. Iadarola, N. Mounet, L. Mether, and L. Sabato

Keywords: electron-cloud, instabilities

Abstract

We present a linearized method to study transverse instabilities due to electron clouds. It is based on a compact characterization of the cloud dipolar and quadrupolar forces, that can be easily obtained from quick single-pass numerical simulations. The long-term stability properties of the bunch are predicted by solving the linearized Vlasov equation, taking into account the dipolar forces introduced by the e-cloud along the bunch as well as the betatron tune modulation with the longitudinal coordinate due to the e-cloud quadrupolar forces. The identification of the beam coherent eigenmodes is achieved by solving an eigenvalue problem. An expression for the tune shift of the rigid-bunch mode is also derived.

1 Introduction

In this note we describe in detail the derivation of the Vlasov method applied in Ref. [1] for the study of transverse instabilities driven by electron clouds. The method is derived by extending the classical approach used for impedance-driven instabilities, as described in Refs. [2, 3], in order to handle:

- quadrupolar forces dependent on the longitudinal position of the particle along the bunch;
- dipolar coherent forces characterized by a discrete set of response functions.

2 First order Vlasov equation for arbitrary detuning

We consider the linearized Vlasov equation including a detuning term depending on the longitudinal phase space coordinates $\Delta Q(r,\phi)$:

$$\frac{\partial \Delta\psi}{\partial t} - \omega_0 \left(Q_{x0} + \Delta Q_{\Phi}(r,\phi) + \Delta Q_R(r)\right) \frac{\partial \Delta\psi}{\partial \theta_x} + \omega_s \frac{\partial \Delta\psi}{\partial \phi} = - \frac{df_0}{dJ_x} G_0\left(r\right) \sqrt{\frac{2J_x R}{Q_{x0}}} \sin\theta_x \frac{F_x^{coh}(z,t)}{m_0 \gamma v}\,, \quad (1)$$

where we have used the following notation: $\Delta\psi$ is the perturbation to the bunch phase space distribution; polar coordinates in the longitudinal phase space (r, ϕ) are defined such that:

$$z = r\cos\phi\,, \quad (2)$$

$$\delta = \frac{\omega_s}{v\eta} r \sin\phi\,; \quad (3)$$

polar coordinates in the transverse phase space (J_x, θ_x) are defined such that:

$$x = \sqrt{\frac{2J_x R}{Q_{x0}}} \cos\theta_x\,, \quad (4)$$

$$x' = \sqrt{\frac{2J_x Q_{x0}}{R}} \sin\theta_x\,; \quad (5)$$

with J_x being the horizontal action; ω_0 is the revolution angular frequency; Q_{x0} is the unperturbed betatron tune; $\omega_s = \omega_0 Q_s$ is the synchrotron angular frequency; η is the slippage factor, m_0 is the particle mass, v is its velocity and γ the corresponding relativistic factor; $R = 2\pi v/\omega_0$ is the accelerator radius; the unperturbed bunch distribution has been factorized as:

$$\psi_0(J_x, r) = f_0(J_x) G_0(r)\,; \quad (6)$$

$F_x^{coh}(z,t)$ is the transverse dipolar force due to the e-cloud:

$$\frac{F_x^{coh}}{m_0\gamma v} = -\frac{\partial \Delta H}{\partial x} = \left(\frac{dx'}{dt}\right)^{coh} = \frac{1}{m_0 \gamma v}\left(\frac{dP_x}{dt}\right)^{coh}\,, \quad (7)$$

with ΔH being the corresponding perturbation to the Hamiltonian and P_x being the transverse momentum.
Generalizing the method discussed in [4], we search for solutions in the form:

$$\Delta\psi\left(J_x,\theta_x,r,\phi,t\right) = e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp(\theta_x-\Delta\Phi(r,\phi))}\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}\,, \tag{8}$$

where the complex frequency Ω, the phase shift $\Delta\Phi(r,\phi)$ and the distribution functions $f^p(J_x)$ and $R_l^p(r)$ are to be found.
We compute the derivatives appearing in Eq. (1):

$$\frac{\partial\Delta\psi}{\partial t} = j\Omega\, e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp\theta_x}\cdot e^{-jp\Delta\Phi(r,\phi)}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}\,, \tag{9}$$

$$\begin{aligned}&\omega_0\left(Q_{x0}+\Delta Q_\Phi+\Delta Q_R\right)\frac{\partial\Delta\psi}{\partial\theta_x} = \\ &\quad e^{j\Omega t}\sum_{p=-\infty}^{+\infty}\left(jp\omega_0\left(Q_{x0}+\Delta Q_\Phi+\Delta Q_R\right)\right) f^p\left(J_x\right) e^{jp\theta_x}\cdot e^{-jp\Delta\Phi(z,\delta)}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-jl\phi}\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\omega_s\frac{\partial\Delta\psi}{\partial\phi} &= e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp\theta_x}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)\frac{\partial}{\partial\phi}e^{-j(p\Delta\Phi(r,\phi)+l\phi)} \\ &= \omega_s e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp\theta_x}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)\frac{\partial}{\partial\phi}e^{-j(p\Delta\Phi(r,\phi)+l\phi)} \\ &= e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp\theta_x}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-j(p\Delta\Phi(z,\delta)+l\phi)}\left(-jp\omega_s\frac{\partial\Delta\Phi}{\partial\phi}-jl\omega_s\right)\,.\end{aligned} \tag{11}$$

Substituting Eqs. (9)-(11) into Eq. (1) we obtain:

$$\begin{aligned}&e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^p\left(J_x\right) e^{jp\theta_x}\cdot\sum_{l=-\infty}^{+\infty} R_l^p(r)e^{-j(p\Delta\Phi(z,\delta)+l\phi)} \\ &\qquad\times\left(j\Omega - jp\omega_s\frac{\partial\Delta\Phi}{\partial\phi} - jl\omega_s - jp\omega_0\left(Q_{x0}+\Delta Q_\Phi+\Delta Q_R\right)\right) \\ &\qquad\qquad = -\frac{df_0}{dJ_x}G_0\left(J_z\right)\sqrt{\frac{2J_xR}{Q_{x0}}}\sin\theta_x\frac{F_x^{coh}(z,t)}{m_0\gamma v}\,.\end{aligned} \tag{12}$$

We can choose the function $\Delta\Phi$ in order to match the phase shift introduced by the term $\Delta Q_\Phi(r,\phi)$, by imposing:

$$\frac{\partial\Delta\Phi}{\partial\phi} = -\frac{\omega_0}{\omega_s}\Delta Q_\Phi(r,\phi)\,. \tag{13}$$

Assuming that such a function can be found (a practical procedure for cases of interest will be illustrated in the following Sec. 3), we can substitute Eq. (13) into Eq. (12) obtaining:

$$e^{j\Omega t}\sum_{p=-\infty}^{+\infty} f^{p}\left(J_x\right) e^{jp\theta_x} e^{-jp\Delta\Phi(r,\phi)} \sum_{l=-\infty}^{+\infty} R_l^{p}(r) e^{-jl\phi}\left(j\Omega - jp\omega_0\left(Q_{x0}+\Delta Q_R\right) - jl\omega_s\right)$$
$$= -\frac{df_0}{dJ_x} G_0(r)\sqrt{\frac{2J_xR}{Q_{x0}}}\left(\frac{e^{j\theta_x}-e^{-j\theta_x}}{2j}\right)\frac{F_x^{coh}(z,t)}{m_0\gamma v}. \quad (14)$$

As discussed in Refs. [3] and [5], it is possible to identify term by term the harmonics in θ_x, showing that all terms with $|p| \neq 1$ vanish. Assuming that the transverse betatron tune is much larger than the synchrotron tune, we can neglect the fast-oscillation term $p = -1$, as discussed in Ref. [4]. This allows retaining only the term $p = 1$, and leads to:

$$f^1(J_x) \propto \frac{df_0}{dJ_x}\sqrt{\frac{2J_xR}{Q_{x0}}}. \quad (15)$$

Therefore Eq. (8) becomes

$$\Delta\psi(J_x,\theta_x,r,\phi,t) = e^{j\Omega t} e^{j\theta_x}\frac{df_0}{dJ_x}\sqrt{\frac{2J_xR}{Q_{x0}}}\cdot e^{-j\Delta\Phi(r,\phi)}\cdot\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi} \quad (16)$$

(where the proportionality constant in Eq. (15) is absorbed in the unknowns $R_l(r)$), and Eq .(14) simplifies into

$$\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}\left(\Omega - Q_{x0}\omega_0 - \omega_0\Delta Q_R - l\omega_s\right) = e^{-j\Omega t}e^{j\Delta\Phi(r,\phi)}G_0(r)\frac{F_x^{coh}(z,t)}{2m_0\gamma v}. \quad (17)$$

3 Description of detuning sources

We consider detuning sources in the form:

$$\Delta Q(z,\delta) = \sum_{n=0}^{N} A_n z^n + B_n\delta^n, \quad (18)$$

which can be written in polar coordinates as:

$$\Delta Q(r,\phi) = \sum_{n=0}^{N} A_n r^n\cos^n\phi + \left(\frac{\omega_s}{v\eta}\right)^n B_n r^n\sin^n\phi. \quad (19)$$

This expression is valid for several sources introducing a detuning along the bunch (e-cloud, detuning impedance, RF quadrupoles) as well as for non-linear chromaticity of any order.
The detunding given by Eq. (19) can be decomposed in two terms, one responsible for detuning with longitudinal amplitude, and one responsible for head-tail phase shift:

$$\Delta Q(r,\phi) = \Delta Q_R(r) + \Delta Q_\Phi(r,\phi), \quad (20)$$

where the longitudinal amplitude-detuning term is defined as:

$$\Delta Q_R(r) = \frac{1}{2\pi}\int_0^{2\pi} \Delta Q(r,\phi)d\phi \tag{21}$$

and therefore the phase shift term $\Delta Q_\Phi(r,\phi)$ averages out over a full synchrotron period:

$$\frac{1}{2\pi}\int_0^{2\pi} \Delta Q_\Phi(r,\phi)d\phi = 0\,. \tag{22}$$

From Eq. (20), the term responsible for the head-tail phase shift can be written as:

$$\Delta Q_\Phi(r,\phi) = \Delta Q(r,\phi) - \Delta Q_R(r) = \Delta Q(r,\phi) - \frac{1}{2\pi}\int_0^{2\pi} \Delta Q(r,\phi)d\phi\,. \tag{23}$$

Using Eq. (19) we can write:

$$\Delta Q_\Phi(r,\phi) = \sum_n A_n r^n \left(\cos^n\phi - \frac{\overline{C}_n}{2\pi}\right) + \left(\frac{\omega_s}{v\eta}\right)^n B_n r^n \left(\sin^n\phi - \frac{\overline{S}_n}{2\pi}\right) \tag{24}$$

where:

$$\overline{C}_n = \int_0^{2\pi} \cos^n\phi\, d\phi\,, \tag{25}$$

$$\overline{S}_n = \int_0^{2\pi} \sin^n\phi\, d\phi\,. \tag{26}$$

Under these conditions we will show that Eq. (13) can be satisfied by a function $\Delta\Phi$ in the form:

$$\Delta\Phi(r,\phi) = -\frac{\omega_0}{\omega_s}\sum_{n=1}^{N} r^n \left[A_n\left(C_n(\phi) - \overline{C}_n\frac{\phi}{2\pi}\right) + \left(\frac{\omega_s}{v\eta}\right)^n B_n\left(S_n(\phi) - \overline{S}_n\frac{\phi}{2\pi}\right)\right]\,. \tag{27}$$

This can be proven by substituting Eqs. (19) and (27) into Eq. (13), obtaining:

$$\sum_{n=1}^{N} A_n r^n \cos^n\phi + \left(\frac{\omega_s}{v\eta}\right)^n B_n r^n \sin^n\phi = \sum_{n=1}^{N} r^n \frac{d}{d\phi}\left(A_n\, C_n(\phi) + \left(\frac{\omega_s}{v\eta}\right)^n B_n\, S_n(\phi)\right)\,. \tag{28}$$

Such a condition is automatically verified if:

$$C_n(\phi) = \int \cos^n\phi'\, d\phi'\,, \tag{29}$$

$$S_n(\phi) = \int \sin^n\phi'\, d\phi'\,. \tag{30}$$

These integrals can be computed recursively using:

$$\int \cos^n\phi\, d\phi = \frac{\cos^{n-1}\phi\sin\phi}{n} + \frac{n-1}{n}\int \cos^{n-2}\phi\, d\phi\,, \tag{31}$$

$$\int \sin^n\phi\, d\phi = -\frac{\sin^{n-1}\phi\cos\phi}{n} + \frac{n-1}{n}\int \sin^{n-2}\phi\, d\phi\,, \tag{32}$$

which can be rewritten in a compact form as:

$$C_n(\phi) = \frac{\cos^{n-1}\phi \sin\phi}{n} + \frac{n-1}{n} C_{n-2}(\phi)\,, \tag{33}$$

$$S_n(\phi) = -\frac{\sin^{n-1}\phi \cos\phi}{n} + \frac{n-1}{n} S_{n-2}(\phi)\,. \tag{34}$$

The first terms of the sequence are:

$$\begin{aligned} C_0(\phi) = \phi\,, \quad C_1(\phi) = \sin\phi\,, \quad C_2(\phi) = \tfrac{\phi}{2} + \tfrac{1}{2}\sin\phi\cos\phi\,, \\ S_0(\phi) = \phi\,, \quad S_1(\phi) = -\cos\phi\,, \quad S_2(\phi) = \tfrac{\phi}{2} - \tfrac{1}{2}\sin\phi\cos\phi\,. \end{aligned} \tag{35}$$

The constants in Eqs. (25) - (26) can be written as:

$$\overline{C}_n = C_n(2\pi) - C_n(0)\,, \tag{36}$$

$$\overline{S}_n = S_n(2\pi) - S_n(0)\,, \tag{37}$$

which gives:

$$\begin{aligned} \overline{C}_0 = 2\pi\,, \quad \overline{C}_1 = 0\,, \quad \overline{C}_2 = \pi\,, \quad \overline{C}_n = \tfrac{n-1}{n} C_{n-2}\,, \\ \overline{S}_0 = 2\pi\,, \quad \overline{S}_1 = 0\,, \quad \overline{S}_2 = \pi\,, \quad \overline{S}_n = \tfrac{n-1}{n} C_{n-2}\,. \end{aligned} \tag{38}$$

Substituting the Eqs. (36) and (37) into Eq. (27) shows that $\Delta\Phi(r,\phi)$ is periodic in ϕ:

$$\Delta\Phi(r, 2\pi) = \Delta\Phi(r, 0)\,, \tag{39}$$

which means that the accumulated phase shift over a single synchrotron period is zero.
In the case of first-order chromaticity there is only one term that is non-vanishing ($B_1 = Q'$). This gives:

$$\Delta\Phi(r,\phi) = -\frac{\omega_0}{v\eta} r B_1\, S_1(\phi) = \frac{Q'}{\eta R} r\cos(\phi)\,, \tag{40}$$

which coincides with the expression used in Refs. [3] and [4].
For the implementation in PyHEADTAIL, it is useful to express ΔQ_Φ (Eq. (24)) in Cartesian coordinates:

$$\Delta Q_\Phi(z,\delta) = \sum_{n=1}^{N} A_n \left(z^n - \frac{\overline{C}_n}{2\pi} r^n \right) + B_n \left(\delta^n - \frac{\overline{S}_n}{2\pi} \left(\frac{\omega_s}{v\eta} \right)^n r^n \right) \tag{41}$$

where:

$$r = \sqrt{z^2 + \left(\frac{v\eta}{\omega_s} \delta \right)^2}\,. \tag{42}$$

4 Description of the coherent force

To describe the coherent force from the e-cloud, we choose a set of real functions $h_n(z)$, satisfying orthogonality condition:

$$\int h_n\left(z\right) h_{n'}\left(z\right) dz = H_n^2 \delta_{n,n'} \tag{43}$$

where H_n is the norm of the function $h_n\left(z\right)$.
We expand the average transverse position along the bunch as:

$$\bar{x}\left(z\right) = \sum_{n=0}^{N} a_n h_n\left(z\right) \ . \tag{44}$$

Using the orthogonality condition, the coefficient a_n can be written as:

$$a_n = \frac{1}{H_n^2} \int \bar{x}\left(z\right) h_n\left(z\right) dz \, . \tag{45}$$

Replacing Eq. (45) into Eq.],44, we can write the following identity:

$$\bar{x}\left(z\right) = \sum_{n=0}^{N} \frac{h_n\left(z\right)}{H_n^2} \int d\tilde{z}\, \bar{x}\left(\tilde{z}\right) h_n\left(\tilde{z}\right) \tag{46}$$

As discussed in Ref. [1], we call $k_n(z)$ the dipolar response resulting from all e-clouds along the ring to the test function $h_n(z)$. For sufficiently small amplitude of the transverse distortion, the response of the e-cloud can be assumed to be linear and the resulting transverse kick along the bunch can be written using the superposition principle:

$$\Delta x'\left(z\right) = \sum_{n=0}^{N} a_n k_n\left(z\right) = \sum_{n=0}^{N} k_n\left(z\right) \int \bar{x}\left(\tilde{z}\right) \frac{h_n\left(\tilde{z}\right)}{H_n^2} d\tilde{z} \, . \tag{47}$$

A suitable set of test functions is:

$$h_n\left(z\right) = \begin{cases} \mathcal{A}_n \cos\left(2\pi \frac{n}{2} \frac{z}{L_{\text{bkt}}}\right), & \text{if } n \text{ is even} \\ \mathcal{A}_n \sin\left(2\pi \frac{n-1}{2} \frac{z}{L_{\text{bkt}}}\right), & \text{if } n \text{ is odd} \, . \end{cases} \tag{48}$$

where $\mathcal{A}_n$ are arbitrary constants and L_{bkt} is the length of the RF bucket.
Assuming that $\Delta x'$ is the integrated effect over one turn (in the smooth machine approximation [3]), we can write:

$$F_x^{coh} = \frac{dP_x}{dt} = m_0 \gamma v \frac{dx'}{dt} = m_0 \gamma v \frac{\Delta x'}{\Delta t} = \frac{m_0 \gamma v^2}{2\pi R} \Delta x' \, . \tag{49}$$

Using Eq. (47) we can write:

$$F_x^{coh}(z,t) = \frac{m_0 \gamma v^2}{2\pi R} \Delta x' = \frac{m_0 \gamma v^2}{2\pi R} \sum_{n=0}^{N} k_n\left(z\right) \int \bar{x}\left(\tilde{z}, t\right) \frac{h_n\left(\tilde{z}\right)}{H_n^2} d\tilde{z} \, , \tag{50}$$

where $\bar{x}(z')$ is the average transverse position of the bunch at the longitudinal position z, which can be written as:

$$\bar{x}(z,t) = \frac{1}{\lambda_0(z)} \iint d\tilde{x}d\tilde{x}' \int d\tilde{\delta}\, \tilde{x}\Delta\psi\left(\tilde{x},\tilde{x}'z,\tilde{\delta},t\right) , \tag{51}$$

where $\lambda_0(z)$ is the longitudinal bunch profile.
Substituting Eq. (51) into Eq. (50), we obtain:

$$F_x^{coh}(z,t) = \frac{1}{\lambda_0(z)} \frac{m_0\gamma v^2}{2\pi R} \iint d\tilde{x}d\tilde{x}' \iint d\tilde{z}d\tilde{\delta}'\tilde{x}\Delta\psi\left(\tilde{x},\tilde{x}',\tilde{z},\tilde{\delta},t\right) \sum_{n=0}^{N} k_n(z) \frac{h_n(\tilde{z})}{H_n^2} . \tag{52}$$

To express it in polar coordinates we recall the following identities [3]:

$$\iint d\tilde{x}d\tilde{x}' = \iint d\tilde{J}_x d\tilde{\theta}_x , \tag{53}$$

$$\iint d\tilde{z}d\tilde{\delta} = \frac{\omega_s}{v\eta} \iint \tilde{r}d\tilde{r}d\tilde{\phi} , \tag{54}$$

and we define $\hat{h}_n(z) = h_n(z)/\lambda_0(z)$. This allows rewriting Eq. (52) as:

$$F_x^{coh}(r,\phi,t) = \frac{m_0\gamma v\omega_s}{2\pi\eta R} \iint d\tilde{J}_x d\tilde{\theta}_x \iint \tilde{r}d\tilde{r}d\tilde{\phi} \times \sqrt{\frac{2\tilde{J}_x R}{Q_{x0}}} \cos\tilde{\theta}_x \Delta\psi\left(\tilde{J}_x,\tilde{\theta}_x,\tilde{r},\tilde{\phi}\right) \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2} . \tag{55}$$

We substitute the expression of $\Delta\psi$ from Eq. (16):

$$F_x^{coh}(r,\phi,t) = \frac{m_0\gamma v\omega_s}{2\pi\eta R} \iint d\tilde{J}_x d\tilde{\theta}_x \iint \tilde{r}d\tilde{r}d\tilde{\phi} \sqrt{\frac{2\tilde{J}_x R}{Q_{x0}}} \cos\tilde{\theta}_x \, e^{j\Omega t} e^{j\tilde{\theta}_x} \frac{df_0}{dJ_x} \sqrt{\frac{2\tilde{J}_x R}{Q_{x0}}} \cdot e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2} \tag{56}$$

and we reorder:

$$F_x^{coh}(r,\phi,t) = \frac{m_0\gamma v\omega_s}{\pi\eta R Q_{x0}} e^{j\Omega t} \int d\tilde{J}_x \tilde{J}_x \frac{df_0}{dJ_x} \int d\tilde{\theta}_x\, e^{j\tilde{\theta}_x} \cos\tilde{\theta}_x \iint \tilde{r}d\tilde{r}d\tilde{\phi} \cdot e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi) \frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2} . \tag{57}$$

From [3] we recall:

$$\int_0^{2\pi} d\tilde{\theta}_x e^{j\tilde{\theta}_x} \cos\tilde{\theta}_x = \pi, \tag{58}$$

$$\int_0^{+\infty} d\tilde{J}_x\, \tilde{J}_x \frac{df_0}{d\tilde{J}_x} = \left[\tilde{J}_x f_0\left(\tilde{J}_x\right)\right]_0^{+\infty} - \int_0^{+\infty} d\tilde{J}_x f_0\left(\tilde{J}_x\right) = -\frac{N}{2\pi} \tag{59}$$

which allows writing the final expression for the coherent force:

$$F_x^{coh}(r,\phi,t) = -\frac{Nm_0\gamma v\omega_s}{2\pi\eta Q_{x0}}e^{j\Omega t}\iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \times \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi)\frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}. \quad (60)$$

5 Integral equation

We substitute the expression of the coherent force from Eq. (60) into Eq. (17) obtaining:

$$\sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}(\Omega - Q_{x0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = -\frac{N\omega_s}{4\pi\eta Q_{x0}}e^{j\Delta\Phi(r,\phi)}G_0(r) \times \iint \tilde{r}d\tilde{r}d\tilde{\phi}\cdot e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi)\frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}. \quad (61)$$

We multiply both sides by $e^{jl\phi}$ and integrate with respect to ϕ. Using the orthogonality condition

$$\int_0^{2\pi} d\phi\, \mathrm{e}^{jl\phi}\mathrm{e}^{-jl'\phi} = 2\pi\delta_{l,l'} \quad (62)$$

we obtain:

$$R_l(r)(\Omega - Q_{x0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = -\frac{N\omega_s}{8\pi^2\eta Q_{x0}}G_0(r)\int d\phi e^{jl\phi}e^{j\Delta\Phi(r,\phi)} \times \iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi)\frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}. \quad (63)$$

Using the re-normalization of $G_0(z)$ given by Eq. (110) in Appendix A:

$$g_0(r) = \frac{\omega_s}{v\eta}G_0(r), \quad (64)$$

we obtain the following integral equation:

$$R_l(r)(\Omega - Q_{x0}\omega_0 - \omega_0\Delta Q_R - l\omega_s) = -\frac{Nv}{8\pi^2 Q_{x0}}g_0(r)\int d\phi e^{jl\phi}e^{j\Delta\Phi(r,\phi)} \times \iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n(r\cos\phi)\frac{\hat{h}_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}. \quad (65)$$

6 Radial expansion

We expand the radial function $R_l(r)$ as follows:

$$R_l(r) = W_l(r)\sum_{m=0}^{+\infty} b_{lm}f_{lm}(r), \quad (66)$$

where $W_l(r)$ is an arbitrary regular function and the functions $f_{lm}(r)$ satisfy the orthogonality condition:

$$\int f_{lm}(r) f_{lm'}(r) w_l(r) dr = F_{lm} \delta_{m,m'} \tag{67}$$

with $w_l(r)$ being a suitable weight function.
Using the orthogonality condition we obtain:

$$\int dr\, w_l(r) f_{lm}(r) \frac{R_l(r)}{W_l(r)} = b_{lm} F_{lm} \,. \tag{68}$$

7 Eigenvalue problem

In this section we consider the special case in which $\Delta Q_R = 0$, while more general cases will be treated in Sec. 8.
Applying the integral

$$\int dr\, w_l(r) f_{lm}(r) \frac{(*)}{W_l(r)} \tag{69}$$

to both sides of Eq. (65) we obtain:

$$\begin{aligned} b_{lm} F_{lm} \left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) = &- \frac{Nv}{8\pi^2 Q_{x0}} \iint dr d\phi\, w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} e^{jl\phi} e^{j\Delta\Phi(r,\phi)} \\ &\times \iint \tilde{r} d\tilde{r} d\tilde{\phi}\, e^{-jl'\tilde{\phi}} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) \sum_{n=0}^{N} k_n \left(r\cos\phi\right) \frac{\hat{h}_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} \,. \end{aligned} \tag{70}$$

Using Eq. (66) we can write:

$$\begin{aligned} b_{lm} F_{lm} \left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) = &- \frac{Nv}{8\pi^2 Q_{x0}} \iint dr d\phi\, w_l(r) f_{lm}(r) e^{jl\phi} e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} \\ &\iint \tilde{r} d\tilde{r} d\tilde{\phi}\, e^{-jl'\tilde{\phi}} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} W_{l'}(\tilde{r}) \sum_{l'm'} b_{l'm'} f_{l'm'}(\tilde{r}) \sum_{n=0}^{N} k_n \left(r\cos\phi\right) \frac{\hat{h}_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} \,. \end{aligned} \tag{71}$$

Reorganizing, we obtain the following eigenvalue problem:

$$\begin{aligned} b_{lm} F_{lm} \left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) = &- \frac{Nv}{8\pi^2 Q_{x0}} \\ &\times \sum_{l'm'} b_{l'm'} \sum_{n=0}^{N} \iint dr\, d\phi\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} e^{jl\phi} k_n \left(r\cos\phi\right) \\ &\times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} \,, \end{aligned} \tag{72}$$

which can be rewritten in the compact form:

$$b_{lm} \left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) = \sum_{l'm'} M_{lm,l'm'} b_{l'm'} \,, \tag{73}$$

where the matrix elements are expressed as:

$$M_{lm,l'm'} = -\frac{Nv}{8\pi^2 Q_{x0} F_{lm}} \sum_{n=0}^{N} \iint dr\, d\phi\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} e^{jl\phi} k_n\left(r\cos\phi\right)$$

$$\times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} . \quad (74)$$

7.1 Laguerre polynomials

Following the approach implemented in the DELPHI code [3, 2], for each value of l, we expand $R_l(r)$ as:

$$R_l(r) = \left(\frac{r}{r_b}\right)^{\lambda|l|} e^{-ar^2} \sum_{m=0}^{+\infty} c_l^m L_m^{|l|}\left(ar^2\right) \quad (75)$$

where $L_n^{|l|}$ are the generalized Laguerre polynomials and a and λ are tunable parameters.
The orthogonality condition for the Laguerre polynomials is:

$$\int d(ar^2)\, e^{-ar^2} \left(ar^2\right)^{|l|} L_m^{|l|}\left(ar^2\right) L_{m'}^{|l|}\left(ar^2\right) = \frac{(|l|+m)!}{m!} \delta_{m,m'} , \quad (76)$$

which, by explicitly writing the differential becomes:

$$\int dr\, 2ar\, e^{-ar^2} \left(ar^2\right)^{|l|} L_m^{|l|}\left(ar^2\right) L_{m'}^{|l|}\left(ar^2\right) = \frac{(|l|+m)!}{m!} \delta_{m,m'} . \quad (77)$$

Equations (75) and (77) coincide with Eqs. (66) and (67) if we define:

$$W_l(r) = \left(\frac{r}{r_b}\right)^{\lambda|l|} e^{-ar^2} , \quad (78)$$

$$w_l(r) = 2ar\, e^{-ar^2} \left(ar^2\right)^{|l|} , \quad (79)$$

$$f_{lm}(r) = L_m^{|l|}\left(ar^2\right) , \quad (80)$$

$$F_{lm} = \frac{(|l|+m)!}{m!} . \quad (81)$$

We assume that the unperturbed distribution is Gaussian:

$$\lambda_0(z) = \frac{N}{\sqrt{2\pi}\sigma_b} e^{-\frac{z^2}{2\sigma_b^2}} , \quad (82)$$

$$g_0\left(r\right) = \frac{1}{2\pi\sigma_b^2} e^{-\frac{r^2}{2\sigma_b^2}} , \quad (83)$$

and we use Eqs. (78) - (83) to compute the integrals in Eq. (74):

$$\iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \frac{W_{l'}(\tilde{r})}{\lambda_0(\tilde{r}\cos\tilde{\phi})} f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} =$$
$$\frac{\sqrt{2\pi}\sigma_b}{N} \iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \left(\frac{\tilde{r}}{r_b}\right)^{\lambda|l'|} e^{-a\tilde{r}^2\left(1-\frac{\cos^2\tilde{\phi}}{2a\sigma_b^2}\right)} L_{m'}^{|l'|}\left(a\tilde{r}^2\right) e^{-jl'\tilde{\phi}} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} =$$
$$\frac{\sqrt{2\pi}\sigma_b}{N} \int \tilde{r}d\tilde{r} \left(\frac{\tilde{r}}{r_b}\right)^{\lambda|l'|} L_{m'}^{|l'|}\left(a\tilde{r}^2\right) \int d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\, e^{-a\tilde{r}^2\left(1-\frac{\cos^2\tilde{\phi}}{2a\sigma_b^2}\right)} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} e^{-jl'\tilde{\phi}}\,, \tag{84}$$

$$\iint dr\, d\phi\, w_l(r) e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} f_{lm}(r) e^{jl\phi} k_n\left(r\cos\phi\right) =$$
$$\frac{1}{\pi\sigma_b^2} \iint dr\, d\phi\, e^{j\Delta\Phi(r,\phi)}\, ar\, \left(ar^2\right)^{|l|} e^{-\frac{r^2}{2\sigma_b^2}} \left(\frac{r_b}{r}\right)^{\lambda|l|} L_m^{|l|}\left(ar^2\right) e^{jl\phi} k_n\left(r\cos\phi\right) =$$
$$\frac{a}{\pi\sigma_b^2} \int r\, dr\ a^{|l|} r_b^{\lambda|l|} r^{(2-\lambda)|l|} L_m^{|l|}\left(ar^2\right) e^{-\frac{r^2}{2\sigma_b^2}} \int d\phi\, e^{j\Delta\Phi(r,\phi)}\, k_n\left(r\cos\phi\right) e^{jl\phi}\,. \tag{85}$$

This allows obtaining an explicit expression for the matrix of the eigenvalue problem:

$$M_{lm,l'm'} = -\frac{va}{4\pi^2\sqrt{2\pi}Q_{x0}\sigma_b} \frac{m!}{(|l|+m)!}$$
$$\times \sum_{n=0}^{N} \int \tilde{r}d\tilde{r} \left(\frac{\tilde{r}}{r_b}\right)^{\lambda|l'|} L_{m'}^{|l'|}\left(a\tilde{r}^2\right) \int d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}\, e^{-a\tilde{r}^2\left(1-\frac{\cos^2\tilde{\phi}}{2a\sigma_b^2}\right)} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} e^{-jl'\tilde{\phi}}$$
$$\times \int r\, dr\ a^{|l|} r_b^{\lambda|l|} r^{(2-\lambda)|l|} L_m^{|l|}\left(ar^2\right) e^{-\frac{r^2}{2\sigma_b^2}} \int d\phi\, e^{j\Delta\Phi(r,\phi)}\, k_n\left(r\cos\phi\right) e^{jl\phi}\,, \tag{86}$$

which can be computed using numerical integration.

8 Effect of detuning with longitudinal amplitude

We now consider the more general case in which the term ΔQ_R in Eq. (65) does not vanish:

$$R_l(r)\left(\Omega - Q_{x0}\omega_0 - l\omega_s - \Delta Q_R(r)\omega_0\right) = -\frac{Nv}{8\pi^2 Q_{x0}} e^{j\Delta\Phi(r,\phi)} g_0(r)$$
$$\times \int d\phi e^{jl\phi} \iint \tilde{r}d\tilde{r}d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} k_n\left(r\cos\phi\right) \frac{\hat{h}_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2}\,. \tag{87}$$

We apply the integral $\frac{1}{F_{lm}}\int dr\, w_l(r) f_{lm}(r) \frac{(*)}{W_l(r)}$ to both sides of Eq. (87) obtaining

$$b_{lm}\left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) - \frac{\omega_0}{F_{lm}} \int dr\, w_l(r) f_{lm}(r) \frac{R_l(r)\Delta Q_R(r)}{W_l(r)} = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}\,, \tag{88}$$

where the matrix $M_{lm,l'm'}$ is given by Eq. (74)).
We substitute the expansion of the radial function from Eq. (66)

$$R_l(r) = W_l(r) \sum_{m'=0}^{+\infty} b_{lm'} f_{lm'}(r) \tag{89}$$

obtaining:

$$b_{lm}\left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) - \frac{\omega_0}{F_{lm}} \sum_{m'=0}^{+\infty} b_{lm'} \int dr\, w_l(r) \Delta Q_R(r) f_{lm}(r) f_{lm'}(r) = \sum_{l'm'} M_{lm,l'm'} b_{l'm'}\,. \tag{90}$$

By defining an auxiliary matrix:

$$\tilde{M}_{lm,l'm'} = \delta_{l,l'} \frac{\omega_0}{F_{lm}} \int dr\, w_l(r) \Delta Q_R(r) f_{lm}(r) f_{lm'}(r) \tag{91}$$

we can rewrite the eigenvalue problem from Eq. (90) as:

$$b_{lm}\left(\Omega - Q_{x0}\omega_0 - l\omega_s\right) = \sum_{l'm'} \left(M_{lm,l'm'} + \tilde{M}_{lm,l'm'}\right) b_{l'm'}\,. \tag{92}$$

Using the expansion in Laguerre polynomials defined in Eqs. (79) - (81), we can write the following expression for $\tilde{M}_{lm,l'm'}$:

$$\tilde{M}_{lm,l'm'} = \delta_{l,l'} \frac{\omega_0}{F_{lm}} \int dr\, 2ar\, e^{-ar^2} \left(ar^2\right)^{|l|} \Delta Q_R(r) L_m^{|l|}\left(ar^2\right) L_{m'}^{|l|}\left(ar^2\right)\,. \tag{93}$$

where the integral can be evaluated numerically.

9 Rigid bunch tune shift

The average transverse position along the bunch can be expressed using Eq. (51):

$$\bar{x}(z) = \frac{1}{\lambda_0(z)} \iint d\tilde{x} d\tilde{x}' \int d\tilde{\delta}\, \tilde{x} \Delta\psi\left(\tilde{x}, \tilde{x}' z, \tilde{\delta}\right)\,. \tag{94}$$

We substitute into Eq. (94) the expression of the perturbation from Eq. (16), obtaining:

$$\bar{x}(z) = \frac{1}{\lambda_0(z)} e^{j\Omega t} \frac{2R}{Q_{x0}} \int d\tilde{J}_x\, J_x \frac{df_0}{dJ_x} \int d\tilde{\theta}_x\, \cos\theta_x\, e^{j\theta_x} \int d\tilde{\delta} \cdot e^{-j\Delta\Phi(r,\phi)} \cdot \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}\,. \tag{95}$$

Using Eqs. (58) and (59), we obtain

$$\bar{x}(z) = -\frac{NR}{Q_{x0}} \frac{1}{\lambda_0(z)} e^{j\Omega t} \int d\tilde{\delta}\, e^{-j\Delta\Phi(r,\phi)} \sum_{l=-\infty}^{+\infty} R_l(r) e^{-jl\phi}\,. \tag{96}$$

We substitute the expansion of $R_l(r)$ from Eq. (75) into Eq. (96), obtaining:

$$\bar{x}(z) = -\frac{NR}{Q_{x0}} \frac{1}{\lambda_0(z)} e^{j\Omega t} \sum_{lm} b_{lm} \int d\tilde{\delta}\, e^{-j\Delta\Phi(r,\phi)} W_l(r) f_{lm}(r) e^{-jl\phi}\,. \tag{97}$$

We consider the special case in which we have an eigenvector having as only non-zero element b_{00}, and we assume that there is no head-tail phase shift $\Delta\Phi(r,\phi) = 0$. In that case:

$$\bar{x}(z) = -\frac{NR}{Q_{x0}} \frac{1}{\lambda_0(z)} e^{j\Omega t} b_{00} f_{00} \int d\tilde{\delta} W_0(r)\,. \tag{98}$$

If we choose:

$$a = \frac{8}{r_b^2} = \frac{1}{2\sigma_b^2} \tag{99}$$

we have that $W_0(r)$ is proportional to $g_0(r)$ and therefore:

$$\int d\tilde{\delta} W_0(r) = K \int d\tilde{\delta} g_0(r) = K\lambda_0(z)\,. \tag{100}$$

We substitute Eq. (100) in Eq. (98): obtaining:

$$\bar{x}(z) = -\frac{NR}{Q_{x0}} K e^{j\Omega t} b_{00} f_{00} = x_0\, e^{j\Omega t}\,. \tag{101}$$

This shows that, under the mentioned hypotheses (in particular $\Delta\Phi(r,\phi) = 0$), if an eigenmode corresponding to a rigid bunch oscillation exists, the corresponding eigenvector will be in the form:

$$b_{lm} = b_{00}\, \delta_l \delta_m\,. \tag{102}$$

The corresponding complex frequency can be found substituting Eq. (102) into Eq. (92) obtaining:

$$\overline{\Omega} - Q_{x0}\omega_0 = M_{00,00} + \tilde{M}_{00,00}\,. \tag{103}$$

Summary

We have derived a linearized approach for the study of transverse beam instabilities driven by e-clouds using the Vlasov method. For this purpose the quadrupolar forces introduced by the e-cloud along the bunch are described using a polynomial, while the dipolar forces are described using a discrete set of response functions. With these assumptions, the identification of the beam coherent eigenmodes is achieved by solving an eigenvalue problem. An expression for the tune shift of the rigid-bunch mode is also derived.

Acknowledgements

We are very grateful to Elias Métral for his strong encouragement and support to this research and for numerous suggestions and discussions on this subject. We would like to thank Gianluigi Arduini, Hannes Bartosik, Kevin Li, Giovanni Rumolo and Michael Schenk for providing valuable input and support.

Appendix A g_0 renormalization

The function G_0 is normalized such that [3]:

$$\int_0^\infty dJ_z G_0\left(J_z\right) = \frac{1}{2\pi}\,. \tag{104}$$

Taking into account that:

$$r = \sqrt{\frac{2J_z v\eta}{\omega_s}}\,, \tag{105}$$

$$J_z = r^2\frac{\omega_s}{2v\eta}\,, \tag{106}$$

$$dJ_z = \frac{\omega_s}{v\eta} r\,dr\,, \tag{107}$$

we obtain:

$$\int_0^\infty rdrG_0\left(r\right) = \frac{v\eta}{\omega_s}\frac{1}{2\pi}\,. \tag{108}$$

We can introduce a renormalized function:

$$g_0(r) = \frac{\omega_s}{v\eta}G_0(r)\,. \tag{109}$$

Using Eq. (108), we obtain:

$$\int rdr\, g_0(r) = \frac{\omega_s}{v\eta}\int rdr\, G_0(r) = \frac{1}{2\pi}\,, \tag{110}$$

which is the normalization condition used in DELPHI [2].

A.1 Gaussian beam

For a Gaussian beam, we can show that the distribution

$$g_0\left(r\right) = \frac{1}{2\pi\sigma_b^2}e^{-\frac{r^2}{2\sigma_b^2}} \tag{111}$$

satisfies Eq. (110):

$$\begin{aligned}\int_0^\infty rdrg_0\left(r\right) = \frac{1}{2\pi\,\sigma_b^2}\int_0^\infty re^{-\frac{r^2}{2\sigma_b^2}}dr = \frac{1}{2\pi}\int_0^\infty \frac{r}{\sigma_b^2}e^{-\frac{r^2}{2\sigma_b^2}}dr \\ = -\frac{1}{2\pi}\int_0^\infty d\left(e^{-\frac{r^2}{2\sigma_b^2}}\right) = \frac{1}{2\pi}\,.\end{aligned} \tag{112}$$

Appendix B Perevedentsev formalism as a special case

E. Perevedentsev in Ref. [6] describes the dipolar effect of the e-cloud using a generalized wake field:

$$\Delta x' = \frac{2\pi R}{m_0\gamma v^2}F_x^{coh} = \frac{e^2}{m_0\gamma v^2}\int d\tilde{z}\lambda_0(\tilde{z})\bar{x}(\tilde{z})W_x^{dip}(z,\tilde{z}) \tag{113}$$

or the corresponding generalized impedance defined so that:

$$W_x^{dip}(z,\tilde{z}) = -\frac{j}{4\pi}\iint d\omega d\tilde{\omega} Z_x^{dip}(\omega,\tilde{\omega})\, e^{i(\omega z-\tilde{\omega}\tilde{z})/v}\,. \tag{114}$$

We can show that such a description is mathematically equivalent to the one introduced in Eq. (47):

$$\Delta x'(z) = \sum_{n=0}^{N} a_n k_n(z) = \sum_{n=0}^{N} k_n(z)\int \bar{x}(\tilde{z})\frac{h_n(\tilde{z})}{H_n^2}d\tilde{z}\,. \tag{115}$$

In fact, Eq. (115) can be rewritten as:

$$\begin{aligned}\Delta x'(z) &= \int d\tilde{z}\,\bar{x}(\tilde{z})\sum_{n=0}^{N} k_n(z)\frac{h_n(\tilde{z})}{H_n^2}\\ &= \frac{e^2}{m_0\gamma v^2}\int d\tilde{z}\,\lambda_0(\tilde{z})\bar{x}(\tilde{z})\left(\frac{m_0\gamma v^2}{e^2\lambda_0(\tilde{z})}\sum_{n=0}^{N} k_n(z)\frac{h_n(\tilde{z})}{H_n^2}\right).\end{aligned} \tag{116}$$

Comparing Eq. 116 with Eq. (113) we obtain:

$$W_x^{dip}(z,\tilde{z}) = \frac{m_0\gamma v^2}{e^2\lambda_0(\tilde{z})}\sum_{n=0}^{N} k_n(z)\frac{h_n(\tilde{z})}{H_n^2}\,, \tag{117}$$

which allows to write Perevedentsev's generalized wakefield using our set of response functions.

Conversely, the set of response functions $k_n(z)$ can be written in terms of W_x^{dip} by simply substituting $\bar{x}(z) = h_n(z)$ in Eq. (113):

$$k_n(z) = \frac{e^2}{m_0\gamma v^2}\int d\tilde{z}\lambda_0(\tilde{z})h_n(\tilde{z})W_x^{dip}(z,\tilde{z})\,. \tag{118}$$

We want to show that Perevedentsev's coupling matrix is a special case of the one derived in this work. We recall our matrix from Eq. (74), which can be rewritten as:

$$\begin{aligned}M_{lm,l'm'} = &-\frac{Nv}{8\pi^2 Q_{x0}F_{lm}}\iint dr\, d\phi\, w_l(r)f_{lm}(r)e^{j\Delta\Phi(r,\phi)}\frac{g_0(r)}{W_l(r)}e^{jl\phi}\\ &\times\iint \tilde{r}d\tilde{r}d\tilde{\phi}e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})}W_{l'}(\tilde{r})f_{l'm'}(\tilde{r})e^{-jl'\tilde{\phi}}\left[\frac{1}{\lambda_0(\tilde{r}\cos\tilde{\phi})}\sum_{n=0}^{N} k_n(r\cos\phi)\frac{h_n(\tilde{r}\cos\tilde{\phi})}{H_n^2}\right].\end{aligned} \tag{119}$$

The expression in the square brackets can be rewritten using Eq. (117):

$$M_{lm,l'm'} = -\frac{Ne^2}{8\pi^2 m_0 \gamma v Q_{x0} F_{lm}} \iint dr\, d\phi\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} e^{jl\phi}$$
$$\times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} W_x^{dip}(r\cos\phi, \tilde{r}\cos\tilde{\phi}) \,. \quad (120)$$

Using Eq. (114) we obtain:

$$M_{lm,l'm'} = \frac{j}{4\pi} \frac{Ne^2}{8\pi^2 m_0 \gamma v Q_{x0} F_{lm}} \iint dr\, d\phi\, w_l(r) f_{lm}(r) e^{j\Delta\Phi(r,\phi)} \frac{g_0(r)}{W_l(r)} e^{jl\phi}$$
$$\times \iint \tilde{r} d\tilde{r} d\tilde{\phi} e^{-j\Delta\Phi(\tilde{r},\tilde{\phi})} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) e^{-jl'\tilde{\phi}} \iint d\omega d\tilde{\omega} Z_x^{dip}(\omega, \tilde{\omega})\, e^{i(\omega r\cos\phi - \tilde{\omega}\tilde{r}\cos\tilde{\phi})/c} \,. \quad (121)$$

We reorganize the expression as:

$$M_{lm,l'm'} = \frac{j}{4\pi} \frac{Ne^2}{8\pi^2 m_0 \gamma v Q_{x0} F_{lm}} \iint d\omega d\tilde{\omega} Z_x^{dip}(\omega, \tilde{\omega})$$
$$\times \int dr\, w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int d\phi\, e^{j\Delta\Phi(r,\phi) + j\omega r\cos\phi/c} e^{jl\phi}$$
$$\times \int \tilde{r} d\tilde{r}\, W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \int d\tilde{\phi}\, e^{-j\Delta\Phi(\tilde{r},\tilde{\phi}) - j\tilde{\omega}\tilde{r}\cos\tilde{\phi}/c} e^{-jl'\tilde{\phi}} \,. \quad (122)$$

To proceed, we need to make the same assumptions as in Ref. [6], limiting ourself to the head-tail phase shift from linear chromaticity and neglecting the quadrupolar effects of the e-cloud, obtaining:

$$M_{lm,l'm'} = \frac{j}{4\pi} \frac{Ne^2}{8\pi^2 m_0 \gamma v Q_{x0} F_{lm}} \iint d\omega d\tilde{\omega} Z_x^{dip}(\omega, \tilde{\omega})$$
$$\times \int dr\, w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int d\phi\, e^{j(\omega - \omega_\xi) r\cos\phi/c} e^{jl\phi}$$
$$\times \int \tilde{r} d\tilde{r}\, W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \int d\tilde{\phi}\, e^{-j(\tilde{\omega} - \omega_\xi)\tilde{r}\cos\tilde{\phi}/c} e^{-jl'\tilde{\phi}} \quad (123)$$

where ω_ξ is the chromatic frequency shift:

$$\omega_\xi = -\frac{Q'}{\eta R} \,. \quad (124)$$

By exploiting the following identity [3]:

$$\int_0^{2\pi} d\phi e^{-jl\phi} e^{jY\cos\phi} = 2\pi j^l J_l(Y) \quad (125)$$

we can write:

$$\int_0^{2\pi} d\tilde{\phi} e^{-jl'\tilde{\phi}} e^{j(\tilde{\omega} - \omega_\xi)\tilde{r}\cos\tilde{\phi}/v} = 2\pi j^{l'} J_{l'}\left(\frac{(\tilde{\omega} - \omega_\xi)\tilde{r}}{v}\right)$$
$$\int_0^{2\pi} d\phi\, e^{jl\phi} e^{-j(\omega - \omega_x i)\frac{r\cos\phi}{v}} = \mathrm{conj}\left(\int_0^{2\pi} d\phi\, e^{-jl\phi} e^{j(\omega - \omega_\xi)\frac{r\cos\phi}{v}}\right) = 2\pi j^{-l} J_l\left(\frac{(\omega - \omega_\xi) r}{v}\right) \quad (126)$$

Substituting these results into Eq. (123) we obtain:

$$M_{lm,l'm'} = j\frac{Ne^2\pi}{8\pi^2 m_0\gamma v Q_{x0}F_{lm}} j^{l'-l} \iint d\omega d\tilde{\omega} Z_x^{dip}(\omega,\tilde{\omega})$$

$$\times \int dr\, w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} J_{l'}\left(\frac{(\tilde{\omega}-\omega_\xi)\tilde{r}}{v}\right) \int \tilde{r} d\tilde{r}\, W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) J_l\left(\frac{(\omega-\omega_\xi)r}{v}\right). \quad (127)$$

Specializing for Gaussian beams and generalized Laguerre polynomials (Eqs. (78) - (83)), with the choice of constants made in DELPHI:

$$\lambda = 1, \quad a = \frac{1}{2\sigma_b}, \quad r_b = 4\sigma_b\,, \quad (128)$$

Eq. (127) coincides with the coupling matrix obtained by Perevedentsev, i.e. Eq. (26) in Ref. [6].

Appendix C Impedance case

For the case of an impedance (assuming zero chromaticity) the dipolar coherent force can be written as:

$$F_x^{coh}(z,t) = \frac{e^2}{2\pi R}\iint d\tilde{z} d\tilde{\delta} \iint d\tilde{x} d\tilde{x}' \psi\left(\tilde{x},\tilde{x}',\tilde{z},\tilde{\delta},t\right) \tilde{x} W_x^{dip}(\tilde{z}-z) \quad (129)$$

$$= \frac{e^2}{2\pi R}\int d\tilde{z} W_x^{dip}(\tilde{z}-z) \int d\tilde{\delta} \iint d\tilde{x} d\tilde{x'} \tilde{x} \psi\left(\tilde{x},\tilde{x}',\tilde{z},\tilde{\delta},t\right) \quad (130)$$

$$= \frac{e^2}{2\pi R}\int d\tilde{z} \lambda_0(\tilde{z}) \bar{x}(z,t) W_x^{dip}(\tilde{z}-z)\,. \quad (131)$$

Using Eq. (49):

$$\Delta x' = \frac{2\pi R}{m_0\gamma v^2} F_x^{coh} = \frac{e^2}{m_0\gamma v^2}\int d\tilde{z}\lambda_0(\tilde{z})\bar{x}(\tilde{z}) W_x^{dip}(\tilde{z}-z) \quad (132)$$

the response to a generic test function $h_n(z)$ can be written as:

$$k_n(z) = \frac{e^2}{m_0\gamma v^2}\int d\tilde{z}\lambda_0(\tilde{z}) h_n(\tilde{z}) W_x^{dip}(\tilde{z}-z)\,. \quad (133)$$

From the definition of impedance:

$$W_x(z) = -\frac{j}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega\frac{z}{v}} Z_x(\omega) \quad (134)$$

we can write:

$$k_n(z) = \frac{e^2}{m_0\gamma v^2}\int d\hat{z}\lambda_0(\hat{z}) h_n(\hat{z}) \frac{-j}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega\frac{\hat{z}-z}{v}} Z_x(\omega)$$

$$= \frac{-je^2}{2\pi m_0\gamma v^2}\int d\omega Z_x(\omega) e^{-j\omega\frac{z}{v}} \int d\hat{z}\lambda_0(\hat{z}) h_n(\hat{z}) e^{j\omega\frac{\hat{z}}{v}}\,. \quad (135)$$

This can be substituted in Eq. (74), obtaining:

$$
\begin{aligned}
M_{lm,l'm'} &= \frac{jNe^2}{16\pi^3 m_0 \gamma v Q_{x0} F_{lm}} \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \\
&\times \int d\phi\, e^{jl\phi} \int d\tilde{\phi} \frac{1}{\lambda_0(\tilde{r}\cos\tilde{\phi})} e^{-jl'\tilde{\phi}} \sum_{n=0}^{N} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} \int d\omega Z_x(\omega) e^{-j\omega\frac{r\cos\phi}{v}} \int d\hat{z} \lambda_0(\hat{z}) h_n(\hat{z}) e^{j\omega\frac{\hat{z}}{v}} \\
&= \frac{jNe^2}{16\pi^3 m_0 \gamma v Q_{x0} F_{lm}} \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \int d\phi\, e^{jl\phi} \int d\tilde{\phi} \frac{1}{\lambda_0(\tilde{r}\cos\tilde{\phi})} e^{-jl'\tilde{\phi}} \\
&\times \int d\omega Z_x(\omega) e^{-j\omega\frac{r\cos\phi}{v}} \sum_{n=0}^{N} \frac{h_n\left(\tilde{r}\cos\tilde{\phi}\right)}{H_n^2} \int d\hat{z} \lambda_0(\hat{z}) h_n(\hat{z}) e^{j\omega\frac{\hat{z}}{v}} .
\end{aligned} \tag{136}
$$

Using Eq. (46), we can write

$$
\lambda_0(\zeta) e^{j\omega\frac{\zeta}{v}} = \sum_{n=0}^{N} \frac{h_n(\zeta)}{H_n^2} \int d\tilde{z}\, \lambda_0(z) e^{j\omega\frac{z}{v}} h_n(\tilde{z}) . \tag{137}
$$

Substituting into Eq. (136) for $\zeta = \tilde{r}\cos\tilde{\phi}$ we obtain:

$$
\begin{aligned}
M_{lm,l'm'} &= \frac{jNe^2}{16\pi^3 m_0 \gamma v Q_{x0} F_{lm}} \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \\
&\times \int d\phi\, e^{jl\phi} \int d\tilde{\phi} \frac{1}{\lambda_0(\tilde{r}\cos\tilde{\phi})} e^{-jl'\tilde{\phi}} \int d\omega Z_x(\omega) e^{-j\omega\frac{r\cos\phi}{v}} \lambda_0(\tilde{r}\cos\tilde{\phi}) e^{j\omega\frac{\tilde{r}\cos\tilde{\phi}}{v}} \\
&= \frac{jNe^2}{16\pi^3 m_0 \gamma v Q_{x0} F_{lm}} \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \\
&\times \int d\phi\, e^{jl\phi} \int d\tilde{\phi} e^{-jl'\tilde{\phi}} \int d\omega Z_x(\omega) e^{-j\omega\frac{r\cos\phi}{v}} e^{j\omega\frac{\tilde{r}\cos\tilde{\phi}}{v}} .
\end{aligned} \tag{138}
$$

Exploiting the following identity [3]:

$$
\int_0^{2\pi} d\tilde{\phi} e^{-jl\tilde{\phi}} e^{j\xi\cos\tilde{\phi}} = 2\pi j^l J_l(\xi) \tag{139}
$$

we can write:

$$
\begin{aligned}
\int_0^{2\pi} d\tilde{\phi} e^{-jl'\tilde{\phi}} e^{j\omega\frac{\tilde{r}\cos\tilde{\phi}}{v}} &= 2\pi j^{l'} J_{l'}\left(\frac{\omega\tilde{r}}{v}\right) \\
\int_0^{2\pi} d\phi\, e^{jl\phi} e^{-j\omega\frac{r\cos\phi}{v}} &= 2\pi j^{-l} J_{-l}\left(-\frac{\omega r}{v}\right) = 2\pi j^{-l}(-1)^l J_l\left(-\frac{\omega r}{v}\right) \\
&= 2\pi j^{-l} J_l\left(\frac{\omega r}{v}\right)
\end{aligned} \tag{140}
$$

and therefore:

$$
\begin{aligned}
&\int d\phi\, e^{jl\phi} \int d\tilde{\phi} e^{-jl'\tilde{\phi}} \int d\omega Z_x(\omega) e^{-j\omega\frac{r\cos\phi}{v}} e^{j\omega\frac{\tilde{r}\cos\tilde{\phi}}{v}} = \\
&\qquad \int d\omega Z_x(\omega) \int d\phi\, e^{jl\phi} e^{-j\omega\frac{r\cos\phi}{v}} \int d\tilde{\phi} e^{-jl'\tilde{\phi}} e^{j\omega\frac{\tilde{r}\cos\tilde{\phi}}{v}} \\
&\qquad = 4\pi^2 j^{l'-l} \int d\omega Z_x(\omega) J_{l'}\left(\frac{\omega\tilde{r}}{v}\right) J_l\left(\frac{\omega r}{v}\right) .
\end{aligned} \tag{141}
$$

Equation (141) can be substituted into Eq. (138), obtaining:

$$M_{lm,l'm'} = \frac{jNe^2}{4\pi m_0 \gamma v Q_{x0} F_{lm}} j^{l'-l} \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) \times \int d\omega Z_x(\omega) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right) J_l \left(\frac{\omega r}{v}\right), \quad (142)$$

which can be reorganized as:

$$M_{lm,l'm'} = \frac{jNe^2}{4\pi m_0 \gamma v Q_{x0} F_{lm}} j^{l'-l} \int d\omega Z_x(\omega) \int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} J_l \left(\frac{\omega r}{v}\right) \int \tilde{r}\, d\tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right). \quad (143)$$

With the assumptions described in Sec. 7.1, we can write:

$$\int dr\ w_l(r) f_{lm}(r) \frac{g_0(r)}{W_l(r)} J_l \left(\frac{\omega r}{v}\right) = \int dr\, 2ar\, e^{-ar^2} \left(ar^2\right)^{|l|} L_m^{|l|} \left(ar^2\right) g_0(r) \left(\frac{r}{r_b}\right)^{-|l|} e^{ar^2} J_l \left(\frac{\omega r}{v}\right) = 2a^{|l|+1} r_b^{|l|} \int dr\, r^{|l|+1} L_m^{|l|} \left(ar^2\right) g_0(r) J_l \left(\frac{\omega r}{v}\right), \quad (144)$$

$$\int d\tilde{r}\, \tilde{r} W_{l'}(\tilde{r}) f_{l'm'}(\tilde{r}) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right) = r_b^{-|l'|} \int d\tilde{r}\, \tilde{r}^{|l'|+1} e^{-a\tilde{r}^2} L_{m'}^{|l'|} \left(a\tilde{r}^2\right) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right). \quad (145)$$

Substituting into Eq. (143) we obtain:

$$M_{lm,l'm'} = \frac{jNe^2}{4\pi m_0 \gamma v Q_{x0} F_{lm}} j^{l'-l} \int d\omega Z_x(\omega) 2a^{|l|+1} r_b^{|l|} \int dr\, r^{|l|+1} L_m^{|l|} \left(ar^2\right) g_0(r) J_l \left(\frac{\omega r}{v}\right) \times r_b^{-|l'|} \int d\tilde{r}\, \tilde{r}^{|l'|+1} e^{-a\tilde{r}^2} L_{m'}^{|l'|} \left(a\tilde{r}^2\right) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right), \quad (146)$$

which can be rewritten as:

$$M_{lm,l'm'} = \frac{jNe^2}{2\pi m_0 \gamma v Q_{x0}} \frac{m!}{(|l|+m)!} j^{l'-l} a^{|l|+1} r_b^{|l|-|l'|} \int d\omega Z_x(\omega) \int dr\, r^{|l|+1} L_m^{|l|} \left(ar^2\right) g_0(r) J_l \left(\frac{\omega r}{v}\right) \int d\tilde{r}\, \tilde{r}^{|l'|+1} e^{-a\tilde{r}^2} L_{m'}^{|l'|} \left(a\tilde{r}^2\right) J_{l'} \left(\frac{\omega \tilde{r}}{v}\right). \quad (147)$$

C.1 Check against DELPHI

We want to compare Eq. 147 against the matrix used in the DELPHI code, which is [2]:

$$M_{ln,l'n'} = \frac{-j^{l'-l} n! \kappa \tau_b^{|l|-|l'|}}{2^{|l|}(n+|l|)!} \sum_{p=-\infty}^{+\infty} Z_x\left(\omega_p\right) G_{ln}\left(\omega_p\right) I_{l'n'}\left(\omega_p, a\right), \quad (148)$$

with:

$$\kappa = -j\frac{N\omega_0 e^2}{4\pi\gamma m_0 c Q_{x0}}\,, \tag{149}$$

$$G_{ln}(\omega,a) = (2a)^{|l|+1}\int_0^{+\infty}\tau^{1+|l|}L_n^{|l|}\left(a\tau^2\right)g_0(\tau)J_l(\omega\tau)\mathrm{d}\tau\,, \tag{150}$$

$$I_{ln}(\omega,a) = \int_0^{+\infty}\tau^{1+|l|}L_n^{|l|}\left(a\tau^2\right)e^{-a\tau^2}J_l(\omega\tau)\mathrm{d}\tau\,. \tag{151}$$

In the single-bunch approximation

$$\omega_0\sum_{p=-\infty}^{+\infty} = \int d\omega \tag{152}$$

Eq. (148) can be rewritten as:

$$\begin{aligned}
M_{ln,l'n'} &= \frac{j^{l'-l}n!\tau_b^{|l|-|l'|}}{2^{|l|}(n+|l|)!}j\frac{Ne^2}{4\pi\gamma m_0 c Q_{x0}}\int d\omega Z_x\left(\omega\right)G_{ln}\left(\omega\right)I_{l'n'}\left(\omega,a\right)\\
&= \frac{j^{l'-l}n!\tau_b^{|l|-|l'|}}{2^{|l|}(n+|l|)!}j\frac{Ne^2}{4\pi\gamma m_0 c Q_{x0}}(2a)^{|l|+1}\int d\omega Z_x\left(\omega\right)\\
&\times\int_0^{+\infty}\mathrm{d}\tau\,\tau^{1+|l|}L_n^{|l|}\left(a\tau^2\right)g_0(\tau)J_l(\omega\tau)\int_0^{+\infty}\tau^{1+|l'|}L_{n'}^{|l'|}\left(a\tau^2\right)e^{-a\tau^2}J_l(\omega\tau)\mathrm{d}\tau\\
&= \frac{jNe^2}{2\pi\gamma m_0 c Q_{x0}}\frac{n!}{(n+|l|)!}j^{l'-l}a^{|l|+1}\tau_b^{|l|-|l'|}\int d\omega Z_x\left(\omega\right)\\
&\times\int_0^{+\infty}\mathrm{d}\tau\,\tau^{1+|l|}L_n^{|l|}\left(a\tau^2\right)g_0(\tau)J_l(\omega\tau)\int_0^{+\infty}\mathrm{d}\tau\,\tau^{1+|l'|}e^{-a\tau^2}L_{n'}^{|l'|}\left(a\tau^2\right)J_l(\omega\tau)\,,
\end{aligned} \tag{153}$$

which indeed coincides with Eq. (147).

Bibliography

[1] G. Iadarola, L. Mether, N. Mounet, and L. Sabato, "Linearized method for the study of transverse instabilities driven by electron clouds," *Phys. Rev. Accel. Beams*, vol. 23, p. 081002, Aug 2020.

[2] N. Mounet, "Vlasov Solvers and Macroparticle Simulations," in *ICFA Mini-Workshop on Impedances and Beam Instabilities in Particle Accelerators, 18 - 22 Sep 2017, Benevento, Italy: Proceedings* (V. Brancolini, G. Rumolo, M. Masullo, and S. Petracca, eds.), pp. 77–85, 2018. CERN-2018-003-CP.

[3] N. Mounet, "Direct Vlasov solvers," in *Proc. of the CAS-CERN Accelerator School on Numerical Methods for Analysis, Design and Modelling of Particle Accelerators, Thessaloniki, Greece, 11–23 November 2018* (H. Schmickler, ed.), CERN Accelerator School Proceedings, CERN, Geneva, 2020. CERN-ACC-2020-0012.

[4] A. Chao, *Physics of collective beam instabilities in high-energy accelerators*. Wiley, New York, 1993, ISBN 978- 0-471-55184-3.

[5] Y.-H. Chin, "Hamiltonian formulation for transverse bunched beam instabilities in the presence of betatron tunespread," 1985. CERN-SPS-85-9-DI-MST.

[6] E. Perevedentsev, "Head-tail instability caused by electron cloud," in *ECLOUD'02: Mini-workshop on electron-cloud simulations for proton and positron beams, CERN, Geneva, Switzerland, 15-18 Apr 2002: Proceedings*, pp. 171–194, 4 2002. CERN-2002-001.