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Yu-Fen LIU and Zhong-Yuan ZHU

P.O.Box 2735, Beijing 100080, The People's Republic of China

Telefax : (86)-1-2562587 Telephone : 2568348

Telex : 22040 BAOAS CN Cable : 6158

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Yu-Fen LIU and Zhong-Yuan ZHU
Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing 100080, China
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Abstract
Inhomogeneous nonlinear gauge field theory for fermions is studied in detail. We show that the Lagrangian of the standard model can be rewritten in terms of the nonlinear connection.

I. Introduction

In this paper we try to explain in an elementary way the basic notions and principles of the nonlinear gauge field theory for spinor fields. This theory opens up a new style and a new aim in the spirit of the great unification of physical fields as dreamed of by Einstein. Instead of the usual Kaluza–Klein type theory[1] in which only bosonic fields are metric fields, we construct the theory in which all fields are connection fields. Strictly speaking, in our theory apart from the conventional electromagnetic and Yang–Mills fields, fermionic fields as well as Higgs field (commonly regarded as matter fields) also become here gauge fields. In fact from the mathematical point of view, the gauge fields of the bosons introduced first by H. Weyl,[2] Yang and Mills[3] are the most simple kind of the linear homogeneous connections on a vector bundle, geometrization of fermion needs to introduce inhomogeneous nonlinear connection on a nonvector bundle.

Our theory postulates a total Lagrangian to be a square of the nonlinear curvature tensor (the Yang–Mills's type Lagrangian). No additional matter stress tensor is needed, because here the usual matter fields are gauge fields and already contained in the curvature tensor. In the above sense our theory is a self-sourced interacting.

The inhomogeneous nonlinear connections gave us the possibility to geometrize the fermion in the framework of the Kaluze–Klein type theory, so that fermions can be considered as components of the metric in higher-dimensional space.[9]

II. Nonlinear Connection

We begin with briefly reviewing the main mathematical aspects of the nonlinear connection theory. Let $P(M,G)$ be a principal fiber bundle over a manifold M with group G. For each $u \in P$, let $T_u(P)$ be the tangent space of P at u and the vertical subspace is G_u, the subspace of $T_u(P)$ consisting of vectors tangent to the fiber through u. A *connection* Γ in P is an assignment of a subspace Q_u of $T_u(P)$ to each $u \in P$ such that[4]

(a) $T_u(P) = G_u + Q_u$ (direct sum);

(b) $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in G$, where R_a is the transformation of P induced by $a \in G$, $R_a u = ua$;

(c) The horizontal subspace Q_u depends differentiably on u.

A vector $X \in T_u(p)$ is called vertical (resp. horizontal) if it lies in G_u (resp. Q_u). By (a), every vector $X \in T_u(P)$ can be uniquely written as $X = Y + Z$ where $Y \in G_u$ and $Z \in Q_u$.

We can formulate the theory of the above connection in terms of the local coordinates.[5] Let $E(M,F,G,P)$ be the fiber bundle associated with P, where F is a manifold on which G acts on the left. The connection Γ on $P(M,G)$ can induce a connection B on $E(M,F,G,P)$. Since $\sigma \in G$ induces an differential automorphism $\sigma : F \to F$, the local coordinate representation of this automorphism is denoted by $\tilde{y}^A = (\sigma y)^A = \varphi^A(\sigma, y)$, where $y \in F$ and $(A = 1, 2, \ldots, m_1)$. The infinitesimal transformations $\lambda_i^A(y) = \{[\partial\varphi^A(\sigma, y)]/(\partial\sigma i)\}_{\sigma=e}$ where e is the identity of

G and its coordinate is always assumed to be 0, form a base of linear algebra isomorphic to g; more precisely, the real linear space generated by the vector fields

$$Y_i = \lambda_i^A(y)\frac{\partial}{\partial y^A}, \qquad i = 1, 2, \ldots, m_2 . \tag{1}$$

is isomorphic to g if we define the bracket

$$[Y_i, Y_j] = \left(\lambda_i^B \frac{\partial \lambda_j^A}{\partial y^B} - \lambda_j^B \frac{\partial \lambda_i^A}{\partial y^B}\right)\frac{\partial}{\partial y^A} = C_{ij}^k Y_k , \tag{2}$$

where C_{ij}^k are structure constants of the Lie algebra. In conventional mathematical terminology the word "representation" is normally reserved for matrix (i.e. linear homogeneous) representations. But in our theory, it is inconvenient to make this restriction. In principle, the $\lambda_i^A(y)$ of Eq. (1) may be quite general. They may be independent of y^A or they may be complicated functionals of y^A. The only restriction on them is the identity (2). In the local coordinate neighborhood $E_u(x^N, y^A)$, where $x \in M$, let

$$Z_N = \frac{\partial}{\partial x^N} + \Gamma_N^i Y_i = \frac{\partial}{\partial x^N} + B_N^A(x,y)\frac{\partial}{\partial y^A} \tag{3}$$

be the horizontal vector. Then the functions $B_N^A(x,y) = \Gamma_N^i \lambda_i^A$ define a connection in a wider sense. A connection on a vector bundle is called here linear, otherwise we call it non-linear. If $y^A = y^A(x)$ is the local coordinate representation of a section $s : M \to E$, then the covariant derivatives of the section s can be defined by

$$y_{\|N}^A \equiv \frac{\partial y^A}{\partial x^N} - B_N^A(x,y) , \tag{4}$$

which is invariant in the sense that

$$\tilde{y}_{\|N}^A = y_{\|M}^B \frac{\partial \tilde{y}^A}{\partial y^B}\frac{\partial x^M}{\partial \tilde{x}^N} , \tag{5}$$

where

$$\frac{\partial \tilde{x}^M}{\partial x^N}\tilde{B}_M^A(\tilde{x},\tilde{y}) = \frac{\partial \tilde{y}^A}{\partial y^B}B_N^B(x,y) + \frac{\partial \tilde{y}^A}{\partial x^N} . \tag{6}$$

We can prove that

$$[Z_N, Z_M] = R_{NM}^i Y_i = R_{NM}^A(x,y)\frac{\partial}{\partial y^A} = (B_{M,N}^A - B_{N,M}^A + B_{M,D}^A B_N^D - B_{N,D}^A B_M^D)\partial_A \tag{7}$$

is the vertical tensor. Under the local coordinate transformation

$$\tilde{R}_{NM}^A(\tilde{x},\tilde{y}) = \frac{\partial \tilde{y}^A}{\partial y^B}R_{KL}^B \frac{\partial x^K}{\partial \tilde{x}^N}\frac{\partial x^L}{\partial \tilde{x}^M} . \tag{8}$$

The functions $R_{NM}^i(x,y)$ and $R_{NM}^A(x,y)$ define a curvature tensor.

The Yang-Mills-type action is defined as

$$\mathbf{A} = -k\int dV(R_{NM}^i R_{KL}^j G_{ij} G^{NK} G^{ML}) , \tag{9}$$

where G^{NK} is the metric of the base space M and $dV = \sqrt{|\det(G_{NM})|}\, d^n x$ is an invariant volume element on it. For simplicity, here we restrict ourselves only to the case with the base space being flat (without gravitational field). We want to point out that it is possible to construct the matrix G_{ij} by means of λ_i^A in the following way : $G_{ij} = a\lambda_i^A g_{AB}\lambda_j^B + bC_{il}^k C_{jk}^l$ where a and b are arbitrary constants and g_{AB} is invariant metric on F which transforms as $\tilde{g}_{AB} = (\partial y^C/\partial \tilde{y}^A)(\partial y^D/\partial \tilde{y}^B) g_{CD}$.

The concept of the above connection on $E(M, F, G, P)$ contains all the usual connections (gauge fields) used in physics. For example, for $U(1) \times SU(2)$ gauge theory, let σ^o and σ^i be corresponding group parameters. Then

$$Z_\mu = \partial_\mu + B_\mu^0 \partial_0 + A_\mu^i Y_i , \tag{10}$$

$$[Z_\mu, Z_\nu] = (B_{\nu,\mu}^0 - B_{\mu,\nu}^0)\partial_0 + (A_{\nu,\mu}^i - A_{\mu,\nu}^i + gC_{jk}^i A_\mu^j A_\nu^k)Y_i = F_{\mu\nu}^0 \partial_0 + F_{\mu\nu}^i Y_i \tag{11}$$

where

$$Y_iY_j - Y_jY_i = gC^k_{ij}Y_k\,, \qquad Y_i = \frac{g}{2}\lambda^A_i\partial_A = \frac{g}{2}T^A_{iB}y^B\partial_A\,, \tag{12}$$

and T^A_{iB} are generators of $SU(2)$ group. We see that $F^0_{\mu\nu}$ and $F^i_{\mu\nu}$ are usual electromagnetic field and Yang–Mills field respectively. We entitle the above connections linear homogeneous connections or linear gauge fields.

The simplest example of a nonlinear connection is the inhomogeneous affine connection which we will discuss in the next section.

III. Gauge Theory of the Translation Group

For the first time, gauge theory of the Poincare group was first brought into play by Kibble, Frolov, Sciama at the beginning of 60s in order to generalize R. Utiyama's gauge version of gravity which had left open the question on the gauge status of tetrad gravitational fields. At the same time, gauge potentials of spatial translations appeared to acquire satisfactory physical utilization to the gauge theory of dislocations in continuous media.[6]

The translation group T^4 is a subgroup of the Poincare group and the affine group $A(4,R)$. In the conventional gauge theory of the affine group, one faces the problem of physical interpretation for both gauge translation potentials and sections $y^a(x)$ of the affine tangent bundle TM. The gauge theory of dislocations is based on the fact that, in the presence of dislocations, displacement vectors of small deformations are determined only with accuracy to gauge translations $u^a \to u^a + \sigma^a(x)$. In this theory, the gauge translational potentials $N^a_\mu(x)$ describe a plastic distortion. The covariant derivative

$$\nabla_\mu u^a = D_\mu u^a - N^a_\mu(x) \tag{13}$$

coincides with the elastic distortion and the strength

$$F^a_{\mu\nu} = D_\mu N^a_\nu(x) - D_\nu N^a_\mu(x) \tag{14}$$

describes the dislocation density. Some authors[7] had studied the theory of dislocation and disclination continuum and found that the geometrical properties of plastic imperfection are closely related to that of a nonriemannian space, which can be described by metric, torsion and curvature tensor. They pointed out that the dislocation density $F^d_{\mu\nu}$ plays the role of Cartan torsion.

Many authors attempted to use the seeming identity of the tensor ranks of tetrad functions e^d_μ and gauge potentials N^d_μ of the translation subgroup of Poincare group. We are especially interested in this situation, in which the Poincare structure group of a bundle contracts to its Lorentz subgroup. In this case a global section $y^a(x)$ of the associated bundle exists in the quotient spaces. The connections of the localized Poincare group is defined as

$$Z_\mu = \partial_\mu + \Gamma^a_{\mu b}y^b\partial_a + N^a_\mu\partial_a\,, \tag{15}$$

where $\Gamma^a_{\mu b}$ and N^a_μ denote a homogeneous Lorentz connection and an inhomogeneous translation connection respectively. The covariant derivative of the section $y^a(x)$ can be defined by

$$\nabla_\mu y^a = \partial_\mu y^a - \Gamma^a_{\mu b}y^b - N^a_\mu = D^\Gamma_\mu y^a - N^a_\mu\,, \tag{16}$$

which is invariant in the sense that under the above two (homogeneous and inhomogeneous) gauge transformations

$$\tilde{y}^a = S^a_b(\sigma_1)y^b + \sigma^a_2\,; \qquad \widetilde{\nabla_\mu y^a} = \left(\frac{\partial\tilde{y}^a}{\partial y^b}\right)\nabla_\mu y^b = S^a_b(\sigma_1)\nabla_\mu y^b\,. \tag{17}$$

The transformation property of the (homogeneous and inhomogeneous) connections are

$$\tilde{\Gamma}^a_{\mu b} = S^a_d\Gamma^d_{\mu c}(S^{-1})^c_b + S^a_{d,\mu}(S^{-1})^d_b\,, \tag{18}$$

$$\tilde{N}^a_\mu = S^a_dN^d_\mu + \partial_\mu\sigma^a_2 - S^a_d\Gamma^d_{\mu c}(S^{-1})^c_b\sigma^b_2 - S^a_{d,\mu}(S^{-1})^d_b\sigma^b_2\,. \tag{19}$$

We decompose a translation connection in two parts

$$N^a_\mu(x) = D^\Gamma_\mu y^a + h^a_\mu = n^a_\mu(x) + h^a_\mu(x)\,. \tag{20}$$

One easily sees that just the components $n^a_\mu = D^\Gamma_\mu y^a$ are responsible for the inhomogeneous transformation law of the connection N^a_μ under gauge translations, while h^a_μ remains invariant under these transformations and satisfies the linear law of gauge-Lorentz transformations. The curvature tensor corresponding to the above connections is dependent only on h^a_μ. Moreover, there is always a certain translation gauge, where the inhomogeneous part n^a_μ of the translation connection N^a_μ equals zero, and N^a_μ coincides with the part h^a_μ. One sees at once the agreement of the tensor ranks of the translational gauge potentials h^a_μ and vierbein field e^a_μ. For a long time this superficial agreement stimulated repeated attempts to describe the tetrad gravitational fields in the framework of gauge gravitational theory as gauge potentials of the translation group. The current situation is not at all clear cut, however.[8].We shall not discuss this question here. We only want to point out that for some physical considerations[9] it is possible to choose a very special metric compatible gauge potentials of Poincare group such that the second part of translation connection $h^a_\mu = -e^a_\mu$. In other words, without losing invariant property of the theory, we can assume that $D^\Gamma_\mu g_{ab} = 0$ and

$$\nabla_\mu y^a = \partial_\mu y^a - \Gamma^a_{\mu b} y^b - N^a_\mu = D^\Gamma_\mu y^a - N^a_\mu = e^a_\mu\,. \tag{21}$$

In this case the curvature tensor corresponding to the localized translation group will play the role of the Cartan torsion.

IV. Inhomogeneous Spinor Connection.

In this section we will discuss another very special inhomogeneous connection in the spinor space. By using this kind of connection we will prove that Dirac field can be considered as an inhomogeneous nonlinear spinor connection which corresponds to the special inhomogeneous translational transformations in the spinor space. The most important results of our work are connections (36), (49) (in this section) and connection (63) (in the next section). In fact these connections were at the first obtained by means of the computer. So, in general one could directly define connection B^A_N as (36), (49) or (63), and omit the following discussions from the article. We feel, however, that it is worthwhile to take some time to understand certain ideas rising in the definition of these connections.

Dirac spinors are complex, so we must work on a complex spinor space F_S, where the coordinate system can usually be chosen as $\theta^A = (\theta^\alpha, \theta^{*\alpha})^T$, the symbol $*$ denotes complex conjugate. In physics the useful scalar is $\bar\psi\psi = (\psi^{*T}\gamma_0)\psi$, where $\bar\theta$ is a linear combination of θ^* i.e. $\bar\theta = (\theta^{*T}\gamma_0)$, denotes a Dirac conjugate of θ. If we consider the Dirac field with the presence of an electromagnetic field, then it is better to use another spinor coordinate $\theta^A = (\theta^\alpha, \theta^{\dot\alpha})^T$, where $\theta^{\dot\alpha}$ is an another linear combination of θ^*, i.e. $\theta^{\dot\alpha} = (\eta_c C\bar\theta^T)^{\dot\alpha}$ which denotes a charge conjugate of θ^α, where $C = C_{\dot\alpha\beta} = -C_{\beta\dot\alpha}$ is a charge conjugation matrix and η_c is an arbitrary unobservable phase generally taken as being to unity. Thus if ψ^α describes the motion of a particle of charge e, with magnetic moment μ, then $(\psi^c)^{\dot\alpha}$ describes an antiparticle of charge $-e$ with magnetic moment $-\mu$. In these coordinates

$$2\bar\theta\theta = (\theta^T, \bar\theta) G_1 \begin{pmatrix} \theta \\ \bar\theta^T \end{pmatrix} = (\theta^T, \theta^{*T}) G_2 \begin{pmatrix} \theta \\ \theta^* \end{pmatrix} = (\theta^{\alpha T}, \theta^{\dot\alpha} T) G_{AB} \begin{pmatrix} \theta^\beta \\ \theta^{\dot\beta} \end{pmatrix}, \tag{22}$$

where

$$G_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & \gamma_0^T \\ \gamma_0 & 0 \end{pmatrix}, \qquad G_{AB} = \begin{pmatrix} 0 & -C_{\alpha\dot\beta} \\ C_{\dot\alpha\beta} & 0 \end{pmatrix}$$

can be considered as metrics on the spinor space in different coordinates systems.

Consider a set of matrices $S_\pm$ with arbitrary parameters σ^d_1

$$S^\alpha_{\pm\beta} = I^\alpha_\beta + \sigma^d_1 i\gamma^\alpha_{d\rho}\Big(\frac{1\pm\gamma_5}{2}\Big)^\rho_\beta = \exp\Big[\sigma^d_1 i\gamma^\alpha_{d\rho}\Big(\frac{1\pm\gamma_5}{2}\Big)^\rho_\beta\Big]. \tag{23}$$

It is easy to prove that $S_\pm(\sigma_1)S_\pm(\sigma_1') = S_\pm(\sigma_1+\sigma_1')$ and $(S^{-1})_\pm(\sigma_1) = S_\pm(-\sigma_1)$, $S_\pm(0) = I$. It means that the set of matrices $S_\pm(\sigma_1) \in G_S$, with parameters σ_1^d constitute 4-dimensional Abelian group.[9] Now let us construct another reducible representations of this group with the following matrices

$$\Lambda_1 = \begin{pmatrix} S_- & \\ & S_-^{-1T} \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} S_- & \\ & S_-^* \end{pmatrix}, \qquad S_B^A = \begin{pmatrix} S_- & \\ & S_+ \end{pmatrix}$$

where $S_+ = (CS_-^{-1}C^{-1})^T$. We can prove that

$$\Lambda_1^T G_1 \Lambda_1 = G_1\,, \quad \Lambda_2^T G_2 \Lambda_2 = G_2\,, \quad S^T G S = G\,,$$

which means that these group transformations keep the metrics G_{AB} in Eq. (22) on the spinor space F_S unchanged. Without loss of generality, in this paper we will work only with the third form of the metric and corresponding matrix representation of the gauge group. We will construct a gauge field theory being invariant under the action of this group. For this, the coordinates of the spinor space will be chosen as (by using similarity transformation $y^A = \Omega_B^A \theta^B$ which keeps the metrics G_{AB} unchanged)

$$y^\alpha = \Big[I_\beta^\alpha + b^d i\gamma_{d\rho}^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\beta^\rho\Big]\theta^\beta\,, \qquad y^{\dot\alpha} = \Big[I_{\dot\beta}^{\dot\alpha} + b^d i\gamma_{d\dot\rho}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\beta}^{\dot\rho}\Big]\theta^{\dot\beta} \tag{24}$$

here b^d are arbitrary constants, and $\gamma_{d\dot\rho}^{\dot\alpha} = -C_{\dot\rho\beta}\gamma_{d\gamma}^{\beta}C^{\gamma\dot\alpha} \rightarrow (\gamma_d^T = -C\gamma_d C^{-1})$.

Notice that the matrices $\gamma_{d\rho}^\alpha[(1-\gamma_5)/2]_\beta^\rho$ are nilpotent, so we can formally rewrite the above spinors $y^A = (y^\alpha, y^{\dot\alpha})^T$ in another form

$$y^\alpha = \theta^\beta \exp\Big[b^d i\gamma_{d\rho}^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\beta^\rho\Big], \qquad y^{\dot\alpha} = \theta^{\dot\beta} \exp\Big[b^d i\gamma_{d\dot\rho}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\beta}^{\dot\rho}\Big]. \tag{25}$$

We see that b^d can be considered as arbitrary "phase angles". The transformation $\tilde y^A = S_B^A(\sigma_1)y^B$ can induce the translational transformation in the "phase angles" manifold, i.e. $\tilde b^a = b^a + \sigma_1^a$. Moreover, it is important to notice that although y^A include the arbitrary "phase angles" b^d formally, but in our case

$$y^A G_{AB} y^B = 2y^{\dot\alpha}C_{\dot\alpha\beta}y^\beta = 2\theta^{\dot\alpha}C_{\dot\alpha\beta}\theta^\beta = 2\bar\theta\theta\,, \tag{26}$$

so they can disappear, at least from a Lagrangian (scalar) for fermion[9] which was constructed by means of y^A. The situation is the same as the Lagrangian of the $U(1)$-invariant charged field $\Phi(x^\mu, t) = \Phi(x^\mu)\exp(-it)$ with an arbitrary phase angle t.

Next we introduce an idempotent projector operators $(P_L)_B^A$ and its complement $V_B^A = (I-P_L)_B^A$.

$$\begin{aligned}(P_L)_\beta^\alpha &= \Big[I + y^d i\gamma_d\Big(\frac{1-\gamma_5}{2}\Big)\Big]_\gamma^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\beta^\gamma,\\ (P_L)_{\dot\beta}^{\dot\alpha} &= \eta\Big[I + y^d i\gamma_d\Big(\frac{1+\gamma_5}{2}\Big)\Big]_{\dot\gamma}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\beta}^{\dot\gamma}\end{aligned} \tag{27}$$

where $y^d \in M^I$ and M^I is Minkowski space. We may verify that

$$(P_L)(P_L) = (P_L)\,, \qquad (P_L)V = 0\,, \qquad VV = V. \tag{28}$$

Two idempotents P_L and V are said to mutually annihilate. Any spinor $y^A \in F_S$ can be decomposed in two parts $y^A = (P_L)_B^A y^B + V_B^A y^B = y_L^A + y_R^A$ and one can verifies that $y_L^A G_{AB} y_L^B = y_R^A G_{AB} y_R^B = 0$. Under transformation $S_B^A(\sigma_1) \in G_{SL}$, the spinor $y^A \in F_S$ transforms as

$$\tilde y^A = S_B^A y^B = S_B^A y_L^B + y_R^A\,. \tag{29}$$

This means that in F_S the group G_{SL} acts only on its projection subspace F_{SL}! The explicit form of $y_L^A = (y_L^\alpha, y_L^{\dot\alpha})^T \in F_{SL}$ is

$$\begin{aligned} y_L^\alpha &= \Big[I_\beta^\alpha + y^d i\gamma_{d\rho}^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\beta^\rho\Big]\theta_{L_0}^\beta = \Big[I_\gamma^\alpha + y^d i\gamma_{d\rho}^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\gamma^\rho\Big]\Big(\frac{1-\gamma_5}{2}\Big)_\beta^\gamma\Big]\theta^\beta\,, \\ y_L^{\dot\alpha} &= \eta\Big[I_{\dot\beta}^{\dot\alpha} + y^d i\gamma_{d\dot\rho}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\beta}^{\dot\rho}\Big]\theta_{L_0}^{\dot\beta} = \eta\Big[I_{\dot\gamma}^{\dot\alpha} + y^d i\gamma_{d\dot\rho}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\gamma}^{\dot\rho}\Big]\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\beta}^{\dot\gamma}\theta^{\dot\beta}\,, \end{aligned} \tag{30}$$

where $\theta_{L_0}^\beta$ is the usual left-handed spinor. Generally, y_L^A is not conventional left-handed spinor, it includes right-handed "phase angles", these angles can be gauged away by gauge transformation. We entitle to call $y_L^A \in F_{SL}$ as "generalized left-handed" spinors. (Experimentalist assert that they have not observed fermions with $V+A$ weak interaction. So for a reason that will be clear shortly we will work only with generalized left-handed spinors.) It is interesting to notice that $y_L^A \in F_{SL}$ are closely related to the "null twistors" which was studied earlier by R. Penrose.[10,11] One may verify that a point in M^I, with standard Minkowski coordinates $y^d \in M^I$ (metric diag.(+1,-1,-1,-1)), is said to be "incident" with the null twistor, i.e. this kind of twistors has the property that its homogeneous co-ordinates y_L^α satisfy $y_L^{\dot\alpha} C_{\dot\alpha\beta} y_L^\beta = 0$. This is the so-called standard flat-space twistor correspondence. The geometry is clearest in terms of the projective twistor space. Considering y_L^α to be fixed and looking for real solutions $y^d \in M^I$ of Eq. (30), it turns out that a solution exists only if $y_L^{\dot\alpha} C_{\dot\alpha\beta} y_L^\beta = 0$. These solutions for y^d in the real Mincowski space M^I, constitute a null straight line (null geodesics) whenever $y_L^{\dot\alpha} C_{\dot\alpha\beta} y_L^\beta = 0$, $\theta_{L_0}^\beta \neq 0$ and every null straight line in M^I arises in this way. If $\theta_{L_0}^\beta = 0$, the point in the twistor space can be interpreted in the space M^I as an α–plane (or null geodesic) at infinity. To understand these one has to go to compactified space. We shall not enter into the details of this case.

We see that $\sigma_1^a \in G_{SL}$ induces an differential automorphism $\sigma_1 : F_S \to F_S$ or $\sigma_1 : F_{SL} \to F_{SL}$

$$\begin{aligned} \tilde y^A &= \varphi^A(\sigma_1, y) = S_B^A y^B \ \to\ \tilde b^d = b^d + \sigma_1^d\,, \\ \tilde y_L^A &= \varphi^A(\sigma_1, y_L) = S_B^A y_L^B \to \tilde y^d = y^d + \sigma_1^d\,, \\ \tilde y_L^A \tilde G_{AB} \tilde y_L^B &= y_L^A G_{AB} y_L^B = 2\eta\theta_{L_0}^{\dot\alpha} C_{\dot\alpha\beta}\theta_{L_0}^\beta = 0\,. \end{aligned} \tag{31}$$

As we mentioned in Ref. [9], for some physical considerations we can consider σ_1^d as the parameters of Poincare translation in the spinor space which induced by translation in the tangent bundle TM of space–time manifold and discussed in the previous section. Let y^d is coordinate of the tangent space TM. If e_μ^d is the orthogonal tetrad (vierbein), then for a particular choice of origin of coordinate sistem we can take $y^d = x^\mu e_\mu^d + h^d$, where h^d denotes an arbitrary "origin-point" in TM. We know that the law of physics is not dependent upon arbitrary choice of the origin of TM. The infinitesimal transformation of this group in the spinor space F_S is

$$\lambda_d^A(y) = \Big[\frac{\partial\varphi^A(\sigma_1, y)}{\partial\sigma_1^d}\Big]_{\sigma_1=0} = \begin{cases} \lambda_d^\alpha(y) = i\gamma_{d\beta}^\alpha\Big(\frac{1-\gamma_5}{2}\Big)_\delta^\beta y^\delta \\ \lambda_d^{\dot\alpha}(y) = i\gamma_{d\dot\beta}^{\dot\alpha}\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\delta}^{\dot\beta} y^{\dot\delta}\,. \end{cases} \tag{32}$$

Localization of this transformation leads to introducing the corresponding connections (gauge fields) in different representation spaces

$$Z_\mu = \partial_\mu + N_\mu^d(x)\lambda_d^A(y)\partial_A\,, \qquad Z_\mu = \partial_\mu + N_\mu^a(x)\partial_a\,. \tag{33}$$

For the reasons as we discussed in the previous section we choose the metric compatible connection satisfying $\partial_\mu y^a(x) - N_\mu^a(x) = e_\mu^a$ (we work in torsion free flat space where $e_\mu^a = \delta_\mu^a$). It is easy to prove that the corresponding curvature tensor dependent only on vierbein field e_μ^a, and in our case equal zero. We can choice suitable gauge such that $y^a = 0$ and $N_\mu^a = -\delta_\mu^a$, another very useful gauge is $N_\mu^a = 0$ and $y^a = x^\mu e_\mu^a$. From mathematical point of view this is the case with a flat connection on a trivial bundle.[5] Physicaly it means that in the flat space (without gravitational field) we have not observed the new type of massless particles, so N_μ^a must be only the pure gauge which can be gauged away by suitable choice of the origin of coordinate sistems.

Next we introduce another inhomogeneous group transformation $\sigma_2 \in G_{SP}$ which induces another differential automorphism $\sigma_2 : F_S \longrightarrow F_S$

$$\tilde{y}^A = \varphi^A(\sigma_2, y) = y^A + \sigma_2^A\,, \tag{34}$$

$$\lambda_B^A(y) = \left[\frac{\partial\varphi^A(\sigma_2, y)}{\partial\sigma_2^B}\right]_{\sigma_2=0} = \delta_B^A\,. \tag{35}$$

We may consider this transformation as a displacement in the spinor space, which means that the choice of an origin of the coordinate system does not change the property of the spinor space.

Localization of these transformations leads to the introduction of the corresponding connections (gauge fields). The explicit form of these connections is dependent on the choice of the representation space. For example

$$Z_\mu = \partial_\mu + N_\mu^a(x)\lambda_a^A(y)\partial_A + W_\mu^B(x)\lambda_B^A\partial_A = \partial_\mu + \Gamma_{\mu B}^A y^B\partial_A + W_\mu^A\partial_A\,. \tag{36}$$

If $y^A = y^A(x)$ is the local coordinate representation of a section $s : M \longrightarrow E_S$, then the covariant derivatives of a section s can be defined by

$$y_{\|\mu}^A = y^A{}_{,\mu} - \Gamma_{\mu B}^A y^B - W_\mu^A\,, \tag{37}$$

which is invariant in the sense that under the above two (homogeneous and inhomogeneous) gauge transformations

$$\tilde{y}^A = S_D^A(\sigma_1)y^D + \sigma_2^A\,, \qquad \tilde{y}_{\|\mu}^A = (\frac{\partial\tilde{y}^A}{\partial y^B})y_{\|\mu}^B = S_B^A(\sigma_1)y_{\|\mu}^B\,. \tag{38}$$

The transformation property of the above connections are·

$$\tilde{N}_\mu^d = N_\mu^d + \sigma_{1,\mu}^d\,, \tag{39}$$

$$\begin{aligned}\widetilde{W}_\mu^\alpha &= S_\beta^\alpha(\sigma_1)W_\mu^\beta + \left[\sigma_2^\alpha{}_{,\mu} - (N_\mu^d + \sigma_1^d{}_{,\mu})i\gamma_{d\delta}^\alpha\left(\frac{1-\gamma_5}{2}\right)_\beta^\delta\sigma_2^\beta\right]\,,\\ \widetilde{W}_\mu^{\dot\alpha} &= S_{\dot\beta}^{\dot\alpha}(\sigma_1)W_\mu^{\dot\beta} + \left[\sigma_2^{\dot\alpha}{}_{,\mu} - (N_\mu^d + \sigma_1^d{}_{,\mu})i\gamma_{d\dot\delta}^{\dot\alpha}\left(\frac{1+\gamma_5}{2}\right)_{\dot\beta}^{\dot\delta}\sigma_2^{\dot\beta})\right]\,.\end{aligned} \tag{40}$$

Let $n_\mu^A = D_\mu^\Gamma y^A = \partial_\mu y^A - \Gamma_{\mu B}^A y^B$. As we mentioned in the previous section one can expand a translation connection to two parts

$$W_\mu^A(x) = n_\mu^A(x) + h_\mu^A(x)\,. \tag{41}$$

We can prove that the curvature tensor which corresponding to these connections and determined by Eq. (7) is dependent only on covariant vector-spinors h_μ^A. In other words, it is possible to shift a zero point (origin of coordinate system) by dislocational transformation of $\sigma_2 \in G_{SP}$ such that $W_\mu^A = h_\mu^A$. By using projection operator introduced in Eq. (27), we decompose h_μ^A in two parts $h_\mu^A = (P_L)_B^A h_\mu^B + V_B^A h_\mu^B = L_\mu^A + R_\mu^A$. The transformation property of h_μ^A is

$$\tilde{h}_\mu^A = \left(\frac{\partial\tilde{y}^A}{\partial y^B}\right)h_\mu^B = S_B^A(\sigma_1)h_\mu^B = S_B^A(\sigma_1)[(P_L)_D^B h_\mu^D] + [V_B^A h_\mu^B]\,. \tag{42}$$

One easily sees that just the components $L_\mu^A = (P_L)_B^A h_\mu^B$ (null vector-spinors) are responsible for the transformation law of the gauge field, and as we mentioned in the above to be "incident" with the null geodesics in Minkowski space-time. So, without losing invariant property of the theory, we can assume that $V_B^A h_\mu^B = R_\mu^A = 0$. In this case, the covariant part of the connection h^A_μ must be of the form of generalized left-handed null vector-spinor

$$h^A_\mu = \begin{cases} L^\alpha_\mu = \Big[I^\alpha_\beta + y^d i\gamma^\alpha_{d\rho}\Big(\frac{1-\gamma_5}{2}\Big)^\rho_\beta\Big] L^\beta_{o\mu} = \Big[I^\alpha_\gamma + y^d i\gamma^\alpha_{d\rho}\Big(\frac{1-\gamma_5}{2}\Big)^\rho_\gamma\Big]\Big(\frac{1-\gamma_5}{2}\Big)^\gamma_\beta \theta^\beta_\mu \\ L^{\dot\alpha}_\mu = \eta\Big[I^{\dot\alpha}_{\dot\beta} + y^d i\gamma^{\dot\alpha}_{d\dot\rho}\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\rho}_{\dot\beta}\Big] L^{\dot\beta}_{o\mu} = \eta\Big[I^{\dot\alpha}_{\dot\gamma} + y^d i\gamma^{\dot\alpha}_{d\dot\rho}\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\rho}_{\dot\gamma}\Big]\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\gamma}_{\dot\beta}\theta^{\dot\beta}_\mu . \end{cases} \tag{43}$$

Next, we introduce an invariant tensor

$$\tau^A_{\mu\nu B} = (P_L)^A_C \begin{pmatrix} \frac{1}{4}\gamma_\mu\gamma_\nu & 0 \\ 0 & \frac{1}{4}\gamma_\mu\gamma_\nu \end{pmatrix}^C_B , \tag{44}$$

which is invariant in the sense that

$$S^A_B(\sigma^d_1)[\tau^B_{\mu\nu C}(y^d)]S^{-1}{}^C_D(\sigma^d_1) = \tau^A_{\mu\nu D}(y^d + \sigma^d_1) . \tag{45}$$

Notice $\tau^A_{\mu\lambda D}\tau^D_{\lambda\nu B} = \tau^A_{\mu\nu B}$. With the help of $\tau^A_{\mu\nu B}$, one can further decompose L^α_μ in two parts

$$\begin{aligned} L^\alpha_\mu &= L^T{}^\alpha_\mu + \tau^\alpha_{\mu\nu\beta}L^{\nu\beta} = L^T{}^\alpha_\mu + \Big[I^\alpha_\gamma + y^d i\gamma^\alpha_{d\rho}\Big(\frac{1-\gamma_5}{2}\Big)^\rho_\gamma\Big] i\gamma^\gamma_{\mu\beta}\Big[\frac{1}{4i}\gamma^\beta_{\nu\delta}\Big(\frac{1-\gamma_5}{2}\Big)^\delta_\epsilon L^{\nu\epsilon}\Big] \\ &= L^T{}^\alpha_\mu + \Big[I^\alpha_\gamma + y^d i\gamma^\alpha_{d\rho}\Big(\frac{1-\gamma_5}{2}\Big)^\rho_\gamma\Big] i\gamma^\gamma_{\mu\beta}R^\beta \\ L^{\dot\alpha}_\mu &= L^{T\dot\alpha}_\mu - \Big[I^{\dot\alpha}_{\dot\gamma} + y^d i\gamma^{\dot\alpha}_{d\dot\rho}\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\rho}_{\dot\gamma}\Big] i\gamma^{\dot\gamma}_{d\dot\beta}R^{\dot\beta}_\mu . \end{aligned} \tag{46}$$

It means that L^α_μ includes a generalized left-handed vector-spinor field $L^T{}^\alpha_\mu$ (with the constraints $i\gamma^\beta_{\mu\gamma}[(1-\gamma_5)/2]^\gamma_\alpha L^T{}^\alpha_\mu = 0$), and a spin $\frac{1}{2}$ right-handed spinor singlet

$$R^\beta = \frac{-i}{4}\gamma^\beta_{\mu\gamma}\Big(\frac{1-\gamma_5}{2}\Big)^\gamma_\alpha L^\alpha_\mu .$$

According to Eq. (7), the curvature tensor corresponding to these connections is defined by

$$\begin{aligned} [Z_\mu, Z_\nu] = F^d_{\mu\nu}Y_d + F^\beta_{\mu\nu}Y_\beta + F^{\dot\beta}_{\mu\nu}Y_{\dot\beta} &= (N^d_{\nu},_\mu - N^d_{\mu},_\nu)Y_d \\ &+ \Big[W^\alpha_\nu,_\mu - W^\alpha_\mu,_\nu + N^d_\nu i\gamma^\alpha_{d\beta}\Big(\frac{1-\gamma_5}{2}\Big)^\beta_\delta W^\delta_\mu - N^d_\mu i\gamma^\alpha_{d\beta}\Big(\frac{1-\gamma_5}{2}\Big)^\beta_\delta W^\delta_\nu\Big]\partial_\alpha \\ &+ \Big[W^{\dot\alpha}_\nu,_\mu - W^{\dot\alpha}_\mu,_\nu + N^d_\nu i\gamma^{\dot\alpha}_{d\dot\beta}\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\beta}_{\dot\delta} W^{\dot\delta}_\mu - N^d_\mu i\gamma^{\dot\alpha}_{d\dot\beta}\Big(\frac{1+\gamma_5}{2}\Big)^{\dot\beta}_{\dot\delta} W^{\dot\delta}_\nu\Big]\partial_{\dot\alpha} . \end{aligned} \tag{47}$$

The existence of the torsion field in Nature is an open question, so if one does not like it, then at the beginning without losing invariant property of the theory, one can assume that $T^d_{\mu\nu} = F^d_{\mu\nu} = 0$. And finally we get only the Lagrangian which describes the motion of the spin $\frac{1}{2}$ right-handed spinor singlet R^A and the motion of the left-handed vector-spinor field $L^{TA}_{0\,\mu}$.

$$\begin{aligned} \mathrm{L} = &-\frac{1}{2}F^{\dot\alpha}_{\mu\nu}C_{\dot\alpha\beta}F^\beta_{\mu\nu} = 6R^{\dot\alpha}C_{\dot\alpha\beta}(i\gamma^\beta_{\mu\alpha})R^\alpha,_\mu - 6R^{\dot\alpha},_\mu C_{\dot\alpha\beta}(i\gamma^\beta_{\mu\alpha})R^\alpha \\ &+ L^{T\dot\alpha}_{0\,\nu}C_{\dot\alpha\beta}(i\gamma^\beta_{\mu\alpha})L^{T\alpha}_{0\,\nu},_\mu - L^{T\dot\alpha}_{0\,\nu},_\mu C_{\dot\alpha\beta}(i\gamma^\beta_{\mu\alpha})L^{T\alpha}_{0\,\nu} + 2[L^{T\dot\alpha}_{0\,\mu}C_{\dot\alpha\beta}R^\beta - R^{\dot\alpha}C_{\dot\alpha\beta}L^{T\beta}_{0\,\mu}],_\mu . \end{aligned} \tag{48}$$

We may ignore the last divergence term. This term may be integrated up to the topological quantum number. Variation of Eq. (48) with respect to $L^{TA}_{0\mu}$, and using the method of Lagrange multipliers, we can prove that $L^{TA}_{0\mu} = L^{TA}_{01\mu} + \frac{1}{2}\partial_\mu L^{TA}_{02}$ includs a spin $\frac{1}{2}$ left-handed spinor singlet $L^{TA}_{02} \equiv L^{TA}_{0\mu,\mu}$ which satisfies massless Dirac equation and the spin $\frac{3}{2}$ left-handed vector spinor $L^{TA}_{01\mu}$ satisfies $\gamma_\mu\partial_\mu L^T_{01\nu} - \gamma_\nu L^T_{02} = 0$; $\gamma_\mu L^T_{01\mu} = 0$. It is important to notice that condition $L^{TA}_\mu = 0$ is gauge invariant, so it does not break invariant property of the theory.

We know that the charged field has an arbitrary phase angle which we denoted here by x^5 and subjected to the condition of cyclicity i.e. $\Psi(x^\mu, x^5) = \Psi(x^\mu)e^{-ig_1x^5}$. The Lagrangian does not depend on this phase angle. This condition reflects the independence of an observed phenomena on the coordinate x^5 and is called the condition of cylindricity. It implies the orthogonality of the space-time manifold with the phase manifold. If we regard that parameters of localized gauge group σ^o, σ^i, σ_2^α, $\sigma_2^{\dot\alpha}$ are functions not only of space-time coordinate x^μ but also are functions of phase angle x^5, then it leads to introduce an additional gauge fields (for simplicity we will work in $y^d = 0$, $N_\mu^d = -\delta_\mu^d$ gauge, and take $L^T{}_\mu^\alpha = 0$).

$$\begin{aligned} Z_\mu &= \partial_\mu + B_\mu^0\partial_0 + A_\mu^i Y_i + \Big[N_\mu^d(i\gamma_{d\beta}^\alpha)\Big(\frac{1-\gamma_5}{2}\Big)_\delta^\beta y^\delta + i\gamma_{\mu\beta}^\alpha R^\beta\Big]\partial_\alpha \\ &+ \Big[N_\mu^d(i\gamma_{d\dot\beta}^{\dot\alpha})\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\delta}^{\dot\beta} y^{\dot\delta} - i\gamma_{\mu\dot\beta}^{\dot\alpha} R^{\dot\beta}\Big]\partial_{\dot\alpha}\,, \\ Z_5 &= \partial_5 + a_1\sigma_5^0\partial_0 + a_2\pi_5^j Y_j + ia_3 L_5^\alpha\partial_\alpha + ia_4 L_5^{\dot\alpha}\partial_{\dot\alpha}\,. \end{aligned} \tag{49}$$

On the analogy of Kaluza–Klein theory, we introduce an ansatz of "dimentional reduction", taking the electromagnetic field B_μ^0 and the Yang–Mills field A_μ^i to be independent of the extra coordinate x^5; and assume that the left-handed fermion field $L^\alpha(x^\mu, x^5, y^0) = L^\alpha(x^\mu)$ $\exp(-ig_2y^0 - ig_1x^5)$, $L^{\dot\alpha}(x^\mu, x^5, y^0) = L^{\dot\alpha}(x^\mu)\exp(ig_2y^0 + ig_1x^5)$ the right-handed fermion field $R^\alpha(x^\mu, x^5, y^0) = R^\alpha(x^\mu)\exp(-ig_2y^0 - ig_1x^5)$, $R^{\dot\alpha}(x^\mu, x^5, y^0) = R^{\dot\alpha}(x^\mu)\exp(ig_2y^0 + ig_1x^5)$, $\sigma(x^\mu)$ is a scalar field, π^k is a triplet of pseudoscalars (pions). $\sigma(x)$ and $\pi^3(x)$ are neutral (i.e. $\partial_5\sigma = \partial_5\pi^3 = \partial_0\sigma = \partial_0\pi^3 = 0$). There are two charged fields $\pi^+(x, y^0) = \pi^+(x)\exp(ig_\pi y^0)$ and $\pi^-(x, y^0) = \pi^-(x)\exp(-ig_\pi y^0)$, so that $\pi^1 = \frac{1}{2}(\pi^+ + \pi^-)$ and $\pi^2 = -\frac{i}{2}(\pi^+ - \pi^-)$ are real.

According to Eq. (9), the corresponding Lagrangian density are:

$$\begin{aligned} \mathrm{L} &= F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4}F_{\mu\nu}^k F_{\mu\nu}^k - \frac{1}{2}F_{\mu\nu}^{\dot\alpha} C_{\dot\alpha\beta} F_{\mu\nu}^\beta \\ &- G^{55}\big(F_{\mu5}^{\dot\alpha} C_{\dot\alpha\beta} F_{\mu5}^\beta + \frac{1}{2}F_{\mu5}^0 F_{\mu5}^0 + \frac{1}{2}F_{\mu5}^k F_{\mu5}^k\big) \\ &= -\frac{1}{4}F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4}F_{\mu\nu}^k F_{\mu\nu}^k + 6R^{\dot\alpha} C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta)\nabla_\mu R^\alpha - 6\nabla_\mu R^{\dot\alpha} C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta) R^\alpha \\ &+ \lambda_2 a_3 a_4[L^{\dot\alpha} C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta)\nabla_\mu L^\alpha - \nabla_\mu L^{\dot\alpha} C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta) L^\alpha] \\ &- \lambda_2(g_2 a_1\sigma + g_1)(a_3 R^{\dot\alpha} C_{\dot\alpha\beta} L^\beta + a_4 L^{\dot\alpha} C_{\dot\alpha\beta} R^\beta) \\ &+ \frac{1}{2}\lambda_2[a_1^2(\partial_\mu\sigma)(\partial_\mu\sigma) + a_2^2(\nabla_\mu\pi^k)(\nabla_\mu\pi^k)] \end{aligned} \tag{50}$$

where $G^{55} = -\lambda_2$ and

$$\begin{aligned} \nabla_\mu\pi^k &= (\pi^k{}_{,\mu} + g_\pi B_\mu^0 C_{3i}^k\pi^i + \frac{g}{2}A_\mu^i C_{ij}^k\pi^j)\,, \\ \nabla_\mu R^\alpha &= (R^\alpha{}_{,\mu} - ig_2 B_\mu R^\alpha)\,, \\ \nabla_\mu R^{\dot\alpha} &= (R^{\dot\alpha}{}_{,\mu} + ig_2 B_\mu R^{\dot\alpha})\,. \end{aligned} \tag{51}$$

The scalar field σ and pions π^k can acquire the mass after further enlargement of dimension of space-time manifold. It is possible to take $x^5 = \delta_0^5 y^0 = y^0$, in this case we can get the similar result (except by constant factors).

V. Geometrization of Electro-Weak Interaction

Now, let us study the Lagrangian of electro-weak interaction[12] using the concept of inhomogeneous nonlinear gauge field we have just introduced. We will show that it is possible to construct this Lagrangian in terms of connections and curvature tensor. For practical purposes, it is certainly useful to rewrite the Weinberg–Salam Lagrangian in our own notations. For this, let

$$\Phi(x^\mu, x^5, y^0) = \Phi(x^\mu)\exp\Big(i\mu x^5 - \frac{ig'y^0}{2k_1}\Big) = \begin{pmatrix}\phi^I \\ \phi^{II}\end{pmatrix} \tag{52}$$

be a doublet of the Higgs–Goldston complex scalar field, and

$$L(x^\mu, y^0) = L(x^\mu)\exp\Big(\frac{ig'y^0}{2k_1}\Big) = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{53}$$

be a charged left-handed spinor doublet . We rewrite them in the following form

$$\begin{cases} H^1 = \frac{1}{2}(\phi^I + \phi^{*I}), \\ H^2 = \frac{i}{2}(-\phi^I + \phi^{*I}), \\ H^3 = \frac{1}{2}(\phi^{II} + \phi^{*II}), \\ H^4 = \frac{i}{2}(-\phi^{II} + \phi^{*II}), \end{cases} \quad \begin{cases} L^{\beta 1} = \nu_e, \\ L^{\beta 2} = -i\nu_e, \\ L^{\beta 3} = e_L, \\ L^{\beta 4} = -ie_L, \end{cases} \quad \begin{cases} L^{\dot\alpha 1} = \nu_e^c, \\ L^{\dot\alpha 2} = i\nu_e^c, \\ L^{\dot\alpha 3} = e_L^c, \\ L^{\dot\alpha 4} = ie_L^c, \end{cases} \tag{54}$$

where H^A are real and ν_e^c is charge conjugation of the left handed neutrino spinor ν_e. The right-handed spinor singlet is denoted by

$$\frac{1}{2}(1-\gamma_5)e = R^\alpha(x^\mu, x^5, y^0) = R^\alpha(x)\exp(ig'y^0k_1^{-1} - i\mu x^5)\,. \tag{55}$$

In these notations the Lagrangian of the Standard model can be rewritten as

$$\begin{aligned} \mathbf{L} = & \Big(\partial_\mu\Phi^\dagger + \frac{ig'}{2}B_\mu\Phi^\dagger + \frac{ig}{2}\Phi^\dagger\tau_i A_\mu^i\Big)\Big(\partial_\mu\Phi - \frac{ig'}{2}B_\mu\Phi - \frac{ig}{2}A_\mu^i\tau_i\Phi\Big) \\ & + \frac{1}{2}\bar{L}i\gamma^\mu\Big(\partial_\mu L + \frac{ig'}{2}B_\mu L - \frac{ig}{2}A_\mu^i\tau_i L\Big) - \frac{1}{2}(\partial_\mu\bar{L} - \frac{ig'}{2}B_\mu\bar{L} + \frac{ig}{2}\bar{L}\tau_i A_\mu^i)i\gamma^\mu L \\ & + \frac{1}{2}\bar{R}i\gamma^\mu(\partial_\mu R + ig'B_\mu R) - \frac{1}{2}(\partial_\mu\bar{R} - ig'B_\mu\bar{R})i\gamma^\mu R \\ & + |\mu^2|\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 - g_e(\bar{R}\Phi^\dagger L + \bar{L}\Phi R) - \frac{1}{4}F_{\mu\nu}^0F_{\mu\nu}^0 - \frac{1}{4}F_{\mu\nu}^iF_{\mu\nu}^i \\ = & (\nabla_\mu H^A)(\nabla_\mu H^A) + |\mu^2|HH - \lambda(HH)^2 \\ & + \frac{1}{4}(\nabla_\mu L^{\dot\alpha A})C_{\dot\alpha\beta}(\nabla_\mu L^{\beta A}) + \frac{1}{2}(\nabla_\mu R^{\dot\alpha})C_{\dot\alpha\beta}(\nabla_\mu R^\alpha) \\ & - g_e(R^{\dot\alpha}C_{\dot\alpha\beta}L^{\beta A}H_A + H_A L^{\dot\alpha A}C_{\dot\alpha\beta}R^\beta) - \frac{1}{4}F_{\mu\nu}^0F_{\mu\nu}^0 - \frac{1}{4}F_{\mu\nu}^iF_{\mu\nu}^i\,, \end{aligned} \tag{56}$$

where τ_i $(i = 1,2,3)$ are Pauli matrices, and

$$\begin{aligned} \nabla_\mu H^A &= (\partial_\mu H^A + k_1 B_\mu H^A{}_{,0} + \frac{g}{2}A_\mu^i T_{iB}^A H^B)\,, \\ \nabla_\mu L^{\alpha A} &= (\partial_\mu L^{\alpha A} + k_1 B_\mu L^{\alpha A}{}_{,0} + \frac{g}{2}A_\mu^i T_{iB}^A L^{\alpha B} - i\gamma_{\mu\gamma}^\alpha L^{\gamma A})\,, \\ \nabla_\mu R^\alpha &= (\partial\mu R^\alpha + k_1 B_\mu R^\alpha{}_{,0} - i\gamma_{\mu\gamma}^\alpha R^\gamma)\,. \end{aligned} \tag{57}$$

Notice that, in these notations the Lagrangian for fermions has the same form as that for bosons. Without loss of generality, the generators of the gauge group $SU(2)$ are chosen in such a way that

$$T_{iB}^A y^B = \lambda_i^A = \begin{pmatrix} -y^4 & y^3 & -y^2 & y^1 \\ y^3 & y^4 & -y^1 & -y^2 \\ -y^2 & y^1 & y^4 & -y^3 \end{pmatrix}. \tag{58}$$

As mentioned in the previous section, the base space can be enlarged. The coordinates of the base space M are $x^N = (x^\mu, x^5, x^a)$, where x^5 and x^a are coordinates of represantation space of groups $U(1)$ and $SU(2)$. We introduce a matrix

$$K_b^A = \frac{1}{y}\begin{pmatrix} y^4 & y^3 & -y^2 & -y^1 \\ y^3 & -y^4 & -y^1 & y^2 \\ -y^2 & y^1 & -y^4 & y^3 \\ y^1 & y^2 & y^3 & y^4 \end{pmatrix} \tag{59}$$

where $y = \sqrt{y^A y_A}$. We can consider it as an orthogonal frame, because $K_b^A \delta^{ab} K_a^B = \delta^{AB}$. Moreover it is important to notice that if $(J_a)_B^A = (yK_a^A)_{,B}$ then

$$J_a J_b - J_b J_a = 2C_{ij}^k J_k \delta_a^i \delta_b^j , \quad J_a T_i - T_i J_a = 0 , \quad T_j T_i - T_i T_j = 2C_{ji}^k T_k . \tag{60}$$

Now we are in a position to study the standard model using the concept of the nonlinear connection. If we demand that generators of gauge groups σ^0, σ^i, σ_2^α, $\sigma_2^{\dot\alpha}$ are not only functions of x^μ but also are functions of x^5 and x^a, then more gauge field components must be introduced. In general we must take $Z_N = \partial_N + B_N^A \partial_A$ as

$$\begin{aligned} Z_\mu &= \partial_\mu + B_\mu^0 \partial_0 + B_\mu^i Y_i + B_\mu^\alpha \partial_\alpha + B_\mu^{\dot\alpha} \partial_{\dot\alpha} \\ Z_5 &= \partial_5 + B_5^0 \partial_0 + B_5^i Y_i + B_5^\alpha \partial_\alpha + B_5^{\dot\alpha} \partial_{\dot\alpha} \\ Z_a &= \partial_a + B_a^0 \partial_0 + B_a^i Y_i + B_a^\alpha \partial_\alpha + B_a^{\dot\alpha} \partial_{\dot\alpha} \end{aligned} \tag{61}$$

Especially, if we restrict ourselves to the standard model of electroweak interaction only, then for simplicity we can take $B_5^0 = B_5^i = B_5^\alpha = B_5^{\dot\alpha} = B_a^i = 0$. In the $N_\mu^d = -\delta_\mu^d$ gauge

$$\begin{aligned} Z_\mu &= \partial_\mu + k_1 B_\mu^0 \partial_0 + A_\mu^i Y_i + \Big[-(i\gamma_{\mu\beta}^\alpha)\Big(\frac{1-\gamma_5}{2}\Big)_\delta^\beta y^\delta + (i\gamma_{\mu\beta}^\alpha) R^\beta\Big]\partial_\alpha \\ &\quad - \Big[(i\gamma_{\mu\dot\beta}^{\dot\alpha})\Big(\frac{1+\gamma_5}{2}\Big)_{\dot\delta}^{\dot\beta} y^{\dot\delta} + (i\gamma_{\mu\dot\beta}^{\dot\alpha}) R^{\dot\beta}\Big]\partial_{\dot\alpha} \\ Z_5 &= \partial_5 \\ Z_a &= \partial_a + ik_0 K_a^B H_B^0 \partial_0 + K_a^B L_B^\alpha \partial_\alpha + K_a^B L_B^{\dot\alpha} \partial_{\dot\alpha} \end{aligned} \tag{62}$$

where k_n are constants. The metrics on the horizontal and vertical spaces are taken as

$$G_{AB} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \delta_{ij} & 0 & 0 \\ 0 & 0 & 0 & -C_{\alpha\dot\beta} \\ 0 & 0 & C_{\dot\alpha\beta} & 0 \end{pmatrix}, \qquad G^{NM} = \begin{pmatrix} \delta^{\mu\nu} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \delta^{ab} \end{pmatrix}. \tag{63}$$

According to Eq. (9), the total Yang–Mills type Lagrangian is defined as

$$\begin{aligned} \mathbf{A} &\equiv -\frac{k}{4}\int_M \sqrt{|G|}dV \, (R_{NM}^A R_{KL}^B G_{AB} G^{NK} G^{ML}) \\ &= k\int_M \sqrt{|G|}dV \Big[-\frac{\lambda_1}{4} F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i - \frac{1}{2} F_{\mu\nu}^{\dot\alpha} C_{\dot\alpha\beta} F_{\mu\nu}^\beta \\ &\quad - \lambda_3 F_{\mu b}^{\dot\alpha} C_{\dot\alpha\beta} F_{\mu b}^\beta - \frac{\lambda_1\lambda_3}{2} F_{\mu b}^0 F_{\mu b}^0 - \frac{\lambda_1\lambda_3^2}{4} F_{ab}^0 F_{ab}^0 - \frac{\lambda_1\lambda_2\lambda_3}{2} F_{a5}^0 F_{a5}^0\Big]. \end{aligned} \tag{64}$$

The base space is a product of three spaces, $M = M_1^4 \times M_2^1 \times M_3^4$, where M_1^4 is a 4-dimensional Minkowski space and an internal space $M_2^1 \times M_3^4$ is a 5-dimensional compact space of a size being necessarily much shorter than any length we have ever measured.

After redefining the constants k_n and λ_m we can get the Lagrangian density of the standard model in our notations. Here

$$\begin{aligned} -\frac{\lambda_1}{4} F_{\mu\nu}^0 F_{\mu\nu}^0 - \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i = &- \frac{\lambda_1 k_1^2}{4}(B_{\nu,\mu}^0 - B_{\mu,\nu}^0)(B_{\nu,\mu}^0 - B_{\mu,\nu}^0) \\ &- \frac{1}{4}(A_{\nu,\mu}^i - A_{\mu,\nu}^i + gC_{jk}^i A_\mu^j A_\nu^k)(A_{\nu,\mu}^i - A_{\mu,\nu}^i + gC_{jk}^i A_\mu^j A_\nu^k) \end{aligned} \tag{65}$$

are the Lagrangian terms of the electromagnetic field and the Yang–Mills field respectively;

$$-\frac{1}{2} F_{\mu\nu}^{\dot\alpha} C_{\dot\alpha\beta} F_{\mu\nu}^\beta = 6[R^{\dot\alpha} C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta)(R^\alpha{}_{,\mu} + ig' B_\mu R^\alpha) - (R^{\dot\alpha}{}_{,\mu} - ig' B_\mu R^{\dot\alpha}) C_{\dot\alpha\beta}(i\gamma_{\mu\alpha}^\beta) R^\alpha] \tag{66}$$

is the Lagrangian term which describes the free motion of the right-handed spinor singlet.

$$\begin{aligned}\lambda_3 F^{\dot\alpha}_{\mu b} C_{\dot\alpha\beta} F^{\beta}_{\mu b} = \lambda_3 \Big[& L^{\dot\alpha}_{A} C_{\dot\alpha\beta}(i\gamma^{\beta}_{\mu\alpha})\Big(\partial_\mu L^{\alpha A} + \frac{ig'}{2} B_\mu L^{\alpha A} + \frac{g}{2} A^{i}_{\mu} T^{A}_{iB} L^{\alpha B}\Big) \\ & - \Big(\partial_\mu L^{\dot\alpha A} - \frac{ig'}{2} B_\mu L^{\dot\alpha A} + \frac{g}{2} A^{i}_{\mu} T^{A}_{iB} L^{\dot\alpha B}\Big) C_{\dot\alpha\beta}(i\gamma^{\beta}_{\mu\alpha}) L^{\alpha}_{A}\Big] \\ & - 4g' k_1^{-1} k_0 \lambda_3 (R^{\dot\alpha} C_{\dot\alpha\beta} L^{\beta}_{A} H^{A} + H^{A} L^{\dot\alpha}_{A} C_{\dot\alpha\beta} R^{\beta})\end{aligned} \tag{67}$$

is the Lagrangian term which describes the motion of the left handed spinor doublet, it includes the mass term of the electron e.

$$\begin{aligned} & -\frac{\lambda_1\lambda_3}{2} F^{0}_{\mu b} F^{0}_{\mu b} - \frac{\lambda_1\lambda_3^2}{4} F^{0}_{ab} F^{0}_{ab} - \frac{\lambda_1\lambda_2\lambda_3}{2} F^{0}_{a5} F^{0}_{a5} \\ & = \frac{\lambda_1\lambda_3 k_0^2}{2} \nabla_\mu H^{A} \nabla_\mu H^{A} - \frac{\lambda_1\lambda_3^2 k_0^4 g'^2}{8k_1^2} (H^{A} H^{A})^2 + \frac{\lambda_1\lambda_2\lambda_3\mu_1^2 k_0^2}{2} (H^{A} H^{A})\end{aligned} \tag{68}$$

is the Lagrangian term of the Higgs–Goldston field with ϕ^4 potential.

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