UT-Komaba 94-9

Collective Field Applied to the Fractional Quantum Hall Effect in Double-Layer Electron Systems

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Abstract

Collective field theory is applied to the fractional quantum Hall effect in double-layer electron systems. Collective variables are electron density of each layer and its conjugate phase variable. In order to study the inter-layer electron hopping, we reconstruct collective field theory to make it applicable to the system without particle number conservation. It is shown that in the absence of the inter-layer hopping, the (mmm) Halperin state has a gapless mode which appears as a result of spontaneous breakdown of the SU(2)-diagonal symmetry. It dispersion relation changes from $\omega \sim |\vec{k}|$ to $|\vec{k}|^{3/2}$, because of shielding of the Coulomb repulsion. It is also shown that the Josephson(-type) phenomena occur in the presence of the inter-layer hopping.

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1 Introduction

There are various approaches for theoretical study of the fractional quantum Hall (FQH) effect[1]. The well-established concept in this phenomenon is that the ground state of the FQH system has the properties of an incompressible quantum liquid. Laughlin formulated the system as a 2-D quantum-mechanical model of N spinless fermions with charge $-e$ in perpendicular magnetic flux density B, interacting with each other by the Coulomb repulsion. The essence of Laughlin's theory[2] consists in the following variational wave function of this system;

$$\Psi_m(\vec{r}_1,\cdots,\vec{r}_N)=\left[\prod_{i<j}^{N}(\overline{z}_i-\overline{z}_j)^m\right]\cdot e^{-\sum_i\frac{|z_i|^2}{4\ell^2}}, \tag{1.1}$$

where m is an odd positive integer due to the Fermi statistics of electrons. Here we use complex coordinates, $z,\overline{z}=x\pm iy$, and $\ell=\sqrt{\frac{\hbar}{eB}}$ is the magnetic length. This state may be a unique ground state of the system which exhibits the above-mentioned properties, if the filling factor of the Landau level $\nu=2\pi\ell^2\frac{N}{A}$ ($A=$ the area of the 2-D system) takes the magic number; $\nu_0=\frac{1}{m}$. This particular number is associated with the quantized value of the Hall coefficient in the unit of $\frac{e^2}{h}$.

After the discovery of the peculiar type of the off-diagonal long range order (ODLRO) existing in the state (1.1)[3], it is found that the concept of the bosonization, which is possible only in one or two space dimensions[4], plays an important role in the FQH effect. In this context, the FQH effect is a kind of condensation of bosonized electrons[3, 5]. Now it is well-known that a 2-D system of N bosons coupled to a $U(1)$ Chern-Simons (C-S) gauge field with the coefficient $\frac{1}{4\pi\hbar m}$ gives a model describing the FQH effect, and it has been studied from various points of view[6].

One of the recent interests in this field is the (F)QH effect in the double-layer (DL) electron systems[7]. In order to assign the state in the FQH system, only one parameter, ν, is required in the SL case, while in the DL system we have more parameters, such as d and Δ_{SAS} other than the two filling factors in each layer. These new parameters parameterize the nature of interactions in the DL system. The parameter

d is the interlayer distance, which is of the same order as ℓ in recent experiments. On the other hand, Δ_{SAS} represents the tunneling amplitude between layers. More concrete definition will be given later. Due to these parameters, the DLFQH systems have much richer structure than the SL systems, and many interesting phenomena appear, such as the occurrence of even-denominator filling factor, the instability of the Hall state[8, 9], the superfluid-like properties, i.e., the appearance of a gapless mode[10, 11] and the Josephson(-type) effect[12, 11, 13, 14], etc.

Wave functions which describe Hall states in the DL systems have been proposed by Halperin[1][15], and their explicit form is

$$\begin{aligned}&\Psi_{lmn}(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}) \\ &=\left[\prod_{u<v}^{N^\uparrow}\left(\overline{z}_u^{\uparrow}-\overline{z}_v^{\uparrow}\right)^l\right]\cdot\left[\prod_{d<f}^{N^\downarrow}\left(\overline{z}_d^{\downarrow}-\overline{z}_f^{\downarrow}\right)^m\right]\cdot\left[\prod_{u=1}^{N^\uparrow}\prod_{d=1}^{N^\downarrow}\left(\overline{z}_u^{\uparrow}-\overline{z}_d^{\downarrow}\right)^n\right]\cdot e^{-\sum_u\frac{|z_u^{\uparrow}|^2}{4\ell^2}-\sum_d\frac{|z_d^{\downarrow}|^2}{4\ell^2}}.\end{aligned} \tag{1.2}$$

In Eq.(1.2), the superscripts $\uparrow$ and $\downarrow$ represent the upper- and lower-layer indices, respectively. Since the inter-electron interaction depends on the layer index because of the interlayer distance, the Halperin state with rather wide values of (lmn) can be realized as a unique ground state of the system[8]. The value of three integers l, m and n (l and m are odd) will be determined, depending on the parameters d and Δ_{SAS}, by the filling factor of each layer; $\nu^a=2\pi\ell^2\frac{N^a}{A}$. A is the area of each layer.[2] The total filling factor ν of the DL system is defined by $\nu=\nu^{\uparrow}+\nu^{\downarrow}$. The Halperin state Ψ_{lmn} is incompressible if the following magic fillings are realized;

$$\nu^{\uparrow}=\frac{m-n}{lm-n^2}\equiv\nu_0^{\uparrow},\quad \nu^{\downarrow}=\frac{l-n}{lm-n^2}\equiv\nu_0^{\downarrow}\quad\Longrightarrow\quad \nu=\frac{l+m-2n}{lm-n^2}\equiv\nu_0. \tag{1.3}$$

In the above expression, we assume $lm\neq n^2$ for simplicity, but the case with $lm=n^2$ will be more interesting to study. Detailed arguments will be developed in the main text.

The picture of the condensation of bosonized electrons still holds in the DLFQH effect. Throughout this paper, we will take this viewpoint to analyze the DL electron

[1] Originally, this wave function was proposed to incorporate real spin degrees of freedom.

[2] We assume that each layer has the same area.

system. Strictly speaking, this point of view is essentially based on a certain perturbative assumption. It is expected that the DL electron systems in strong magnetic field have rich phase structure besides the FQH state[16]. But the bosonization procedure chooses ("by hand") some particular type of ground states in the DL system, such as the Halperin state. However, in spite of this rather strong postulate on the ground states of the model, many interesting phenomena appear in this approach, and we will analyze the DL system in this framework.

This paper is organized as follows. A microscopic model of the DLFQH system will be given in Section 2. In order to represent the DL systems, we introduce a new quantum number called pseudospin. Then we consider a system of the bosonized electrons which carry pseudospin and couple to C-S gauge fields. Starting from this microscopic model, we will derive an effective theory of the DLFQH system in the present paper. Our strategy is using the collective variables[17], which are introduced in Section 3. Among these collective variables, the phase variable plays an essentially important role in this framework of the FQHE, just as in the case of superconductivity. Compared with the previously proposed formulation of effective theory of the (DL)FQH system, such as a Ginzburg-Landau (G-L) type description[3, 5] or a topological gauge theoretic approach[18], the present approach will give a plain description of the FQH effect, because the collective variables are observable quantities and they give clear physical picture. Especially, the collective-field approach will be systematically constructed from the microscopic model[19], and then in this formalism the relationship between the microscopic theory and the effective theory is quite clear. However, there is some difficulty in direct application of the collective-field approach to the DL system with the tunneling effect, which stems from the non-conservation of the electron number in each layer. This will be explained in Section 3.

In Section 4, study of the no-tunneling DLFQH systems will be given[20]. An effective theory is derived in terms of the collective variables[21, 22]. Ground states and excitations are explicitly obtained by solving the effective theory. Special attention is paid to the $l = m = n$ state, for there appears the Nambu-Goldstone boson as a result of the spontaneous breakdown of the $SU(2)$-diagonal symmetry[10, 11]. Shielding of the Coulomb repulsion changes its dispersion relation from $\omega(k) \sim |\vec{k}|$ to $|\vec{k}|^{3/2}$.

Section 5 will be devoted to the discussion on the DL system with the tunneling effect. Collective-field formalism is reconstructed in order to take account of the non-conservation of the particle number. The difficulty, which is explained in Section 3, is resolved by extending the space of states. Collective-field realization of the tunneling Hamiltonian is explicitly given.

In Section 6, in the framework of the effective theory obtained in Section 5, Josephson effect in DLFQH systems will be examined. Possibility of Josephson(-type) phenomenon was first suggested by Wen and Zee[12], and later it was studied by semi-classical analysis of a field-theoretical model of DLFQH system by Ezawa and Iwazaki[11, 13]. At present, however, there is controversy about it and it is not settled if it realy occurs. Discussion in Section 6 shows that Josephson(-type) effect occurs.

Section 7 will be devoted to discusssion and conclusion. Some technical calculations are summerized in Appendices A and B.

2 Model

In this section, we will give a model of the DLFQH system[23, 11] in both the quantum-mechanical and the second-quantized languages, which is the starting point for the later discussions.

We will represent electrons (effective mass M, charge $-e$) in the DLFQH system by (spinless) bosonized particles[3] in a two-dimensional ("single"-layer) surface, carrying an additional internal quantum number which corresponds to the layer index, $a = \uparrow, \downarrow$. Hereafter we shall call this pseudospin, $s^a = \pm 1$. Electrons with up (down) pseudospin lie in the upper (lower) layer. If there is no process with pseudospin-flip in the system, then the total pseudospin is conserved. We will take $SU(2)$-diagonal ($SU(2)_d$) as a symmetry group associated with this conservation (the conservation of the number of $\uparrow$ electrons *minus* that of $\downarrow$ electrons). This symmetry is realized in the case that the distance between the upper and lower layers is large and the superposition of up- and down-pseudospin states is negligible. Then the symmetry of the DLFQH without interlayer hopping systems is $U(1) \otimes SU(2)_d$ associated with the conservation of the total number and the total pseudospin, which means the conservation of the number of electrons in each layer.

The C-S gauge fields required on the bosonization of electrons in the DL system are introduced by gauging the above $U(1) \otimes SU(2)_d$ symmetry. In the following, we will take the basis $U(1)_\uparrow \otimes U(1)_\downarrow$ rather than $U(1) \otimes SU(2)_d$ and denote the associated C-S gauge fields as C_μ^a ($a = \uparrow, \downarrow$). Suppose that the Aharonov-Bohm (A-B) phase, $e^{im^{ab}\pi}$, which appears through the exchange of electrons with pseudospins s^a and s^b, is given ("by hand") as

$$(m^{ab}) = \begin{pmatrix} l & n \\ n & m \end{pmatrix}. \tag{2.1}$$

Because of the single-valuedness, m^{ab} must be an integer, and especially l and m are

[3]In this paper we will neglect the degrees of freedom of (real) spin of electrons in DL systems.

both odd due to the Fermi statistics. We take a sign convention that they are all positive. Note that there are *a priori* no grounds that l, m and n are all the same integer. When $lm \neq n^2$, the C-S action, which corresponds to the above A-B phase, is

$$\begin{aligned} S_{CS} &= \int dt d^2z L_{CS} \\ &= -\sum_{a,b} \frac{(m^{-1})^{ab}}{4\pi\hbar} \int dt d^2z\, \epsilon^{\mu\nu\lambda} C_\mu^a \partial_\nu C_\lambda^b. \end{aligned} \tag{2.2}$$

The case with $lm = n^2$ will be discussed later.

Then the quantum-mechanical Lagrangian of non-interacting [4] (bosonized) electrons $\{\vec{r}_i^a | a = \uparrow, \downarrow, i = 1, \cdots, N^a\}$ is

$$\begin{aligned} L_0 &= \sum_{a,i} \frac{M}{4} \left(\dot{\bar{z}}_i^a \dot{z}_i^a + \dot{z}_i^a \dot{\bar{z}}_i^a \right) + \sum_{a,i} e \left\{ \dot{z}_i^a A_z(z_i^a, \bar{z}_i^a) + \dot{\bar{z}}_i^a A_{\bar{z}}(z_i^a, \bar{z}_i^a) \right\} \\ &\quad + \sum_{a,i} \left\{ \dot{z}_i^a C_z^a(z_i^a, \bar{z}_i^a) + \dot{\bar{z}}_i^a C_{\bar{z}}^a(z_i^a, \bar{z}_i^a) + C_0^a(z_i^a, \bar{z}_i^a) \right\} + L_{CS} \\ &\quad - \sum_{(a,i)\neq(b,j)} \frac{\pi\hbar^2}{M} m^{ab} \delta^{(2)}(z_i^a - z_j^b), \end{aligned} \tag{2.3}$$

where $\sum_{a,i} \equiv \sum_{a=\uparrow,\downarrow} \sum_{i=1}^{N^a}$ and $\sum_{(a,i)\neq(b,j)} \equiv \sum_{a,i} \sum_{b,j} (1 - \delta^{ab}\delta_{ij})$. A_k ($k = x, y$) is the electromagnetic vector potential representing external uniform magnetic field B perpendicular to the plane and we adopt the symmetric gauge; $A_z(z, \bar{z}) = \frac{i}{4} B \bar{z}$. Statistical charge of (bosonized) electrons is taken to be -1 independent of pseudospin, and the C-S flux attached on each (bosonized) electron is antiparallel (the "$-$z" direction) to the external magnetic field (the "+z" direction). The second line of L_0 (2.3) is a topological term which corresponds to the winding angle and provides statistical properties for (bosonized) electrons. The last term of L_0 (2.3) represents the hard-core repulsion between (bosonized) electrons. The C-S gauge fields can be integrated out, and this model is expressed solely in terms of the electron coordinates $\{\vec{r}_i^a\}$. To this end, we explicitly solve the Coulomb gauge conditions $\partial C_{\bar{z}}^a + \bar{\partial} C_z^a = 0$ and the

[4]The only exception is the hard-core repulsion originating from the requirement of statistics[24]. See the last term of Eq.(2.3).

Gauss law constraints, which appear from the variation of L_0 (2.3) with respect to $C_0^a(z,\overline{z})$. The result is

$$C_z^a(z,\overline{z}) = -i\hbar \sum_{b,j} \frac{m^{ab}}{2} \frac{1}{z - z_j^b}. \tag{2.4}$$

Therefore we obtain the following quantum-mechanical Hamiltonian;

$$H_0^{QM} = -\sum_{a=\uparrow,\downarrow} \sum_{i=1}^{N^a} \frac{2\hbar^2}{M} \overline{D}_i^a D_i^a + \frac{N}{2}\hbar\omega_c, \tag{2.5}$$

where $N = N^\uparrow + N^\downarrow$ is the total number of electrons and $\omega_c = \frac{eB}{M}$ is the cyclotron frequency, and covariant derivatives are defined by

$$\begin{aligned} D_i^a &= \partial_i^a - \sum_{b,j}^{(a,i)} \frac{m^{ab}}{2} \frac{1}{z_i^a - z_j^b} + \frac{\overline{z}_i^a}{4\ell^2}, \\ \overline{D}_i^a &= \overline{\partial}_i^a + \sum_{b,j}^{(a,i)} \frac{m^{ab}}{2} \frac{1}{\overline{z}_i^a - \overline{z}_j^b} - \frac{z_i^a}{4\ell^2}, \end{aligned} \tag{2.6}$$

where $\partial_i^a = \frac{\partial}{\partial z_i^a}$ and $\sum_{b,j}^{(a,i)} = \sum_{b,j}(1 - \delta^{ab}\delta_{ij})$, and we have regularized the singularity which appears when coordinates of two electrons coincide.

This model (or its modified version) is exactly soluble[25], and the ground state is given by the following (bosonized) Halperin-type wave function;

$$\begin{aligned} &\Phi_{lmn}(\vec{r}_1^{\uparrow}, \cdots, \vec{r}_{N^\uparrow}^{\uparrow}; \vec{r}_1^{\downarrow}, \cdots, \vec{r}_{N^\downarrow}^{\downarrow}) \\ &= f(\overline{z}) \left[\prod_{u<v}^{N^\uparrow} \left| z_u^\uparrow - z_v^\uparrow \right|^l \right] \cdot \left[\prod_{d<f}^{N^\downarrow} \left| z_d^\downarrow - z_f^\downarrow \right|^m \right] \cdot \left[\prod_{u=1}^{N^\uparrow} \prod_{d=1}^{N^\downarrow} \left| z_u^\uparrow - z_d^\downarrow \right|^n \right] \cdot e^{-\sum_u \frac{|z_u^\uparrow|^2}{4\ell^2} - \sum_d \frac{|z_d^\downarrow|^2}{4\ell^2}}, \end{aligned} \tag{2.7}$$

where $f(\overline{z}) = f(\{\overline{z}_i^a\})$ is an arbitrary anti-holomorphic function. The filling factor ν of the DL system is given by the sum of that of each layer $\nu^a = 2\pi\ell^2 \frac{N^a}{A}$; $\nu = \nu^\uparrow + \nu^\downarrow$. In the presence of the inter-electron interaction such as the Coulomb repulsion, the Halperin state Φ_{lmn} with $f(\overline{z}) = 1$ describes an incompressible state if ν has the following magic filling;

$$\nu^\uparrow = \frac{m-n}{lm-n^2} \equiv \nu_0^\uparrow, \quad \nu^\downarrow = \frac{l-n}{lm-n^2} \equiv \nu_0^\downarrow \quad \Longrightarrow \quad \nu = \frac{l+m-2n}{lm-n^2} \equiv \nu_0. \tag{2.8}$$

In most of cases, however, the above microscopic model is not well suited for analytic studies of the FQHE other than numerical simulations of finite number of

electron system. Therefore we will construct an effective theory for further analysis of the DLFQHE in this paper.

First, let us translate the quantum-mechanical model L_0 (2.3) into the second-quantized form,

$$\begin{aligned} \mathcal{L}_0 &= \sum_a \left\{ \phi^{a\dagger}(i\hbar\partial_0 + C_0^a)\phi^a + \frac{\hbar^2}{2M} \sum_k (\mathcal{D}_k^a \phi^a)^\dagger (\mathcal{D}^{ak}\phi^a) \right\} \\ &\quad - \sum_{a,b} \frac{\pi\hbar^2}{M} m^{ab} \phi^{a\dagger}\phi^{b\dagger}\phi^b\phi^a + L_{CS}, \end{aligned} \tag{2.9}$$

where ϕ^a is a non-relativistic bosonized electron field operator with a pseudospin s^a and $\phi^{a\dagger}$ is its hermitian conjugate. The nontrivial equal-time commutation relations (ETCR's) among them are

$$\left[\phi^a(\vec{x},t), \phi^{b\dagger}(\vec{y},t) \right] = \delta^{ab} \cdot \delta^{(2)}(\vec{x} - \vec{y}). \tag{2.10}$$

Covariant derivatives are defined by $\mathcal{D}_k^a = \partial_k - \frac{i}{\hbar} C_k^a - \frac{i}{\hbar} e A_k$ $(k = x, y)$. This model is equivalent to the following Bogomol'nyi-type Lagrangian density[26, 24, 23, 11], which we denote $\mathcal{L}_0$ again;

$$\mathcal{L}_0 = \sum_a \left[\phi^{a\dagger}(i\hbar\partial_0 + C_0^a)\phi^a - \frac{2\hbar^2}{M} (\mathcal{D}^a\phi^a)^\dagger (\mathcal{D}^a\phi^a) - \frac{\hbar\omega_c}{2} \phi^{a\dagger}\phi^a \right] + L_{CS}, \tag{2.11}$$

where covariant derivatives are given in the complex notation; $\mathcal{D}^a\phi^a = (\partial - \frac{i}{\hbar} C_z^a + \frac{\overline{z}}{4\ell^2})\phi^a$. Solving the equations of the C-S gauge fields as in the above quantum-mechanical model, $C_z^a(z,\overline{z}) = -i\hbar \sum_b \frac{m^{ab}}{2} \int d^2w \frac{|\phi^b(w,\overline{w})|^2}{z-w}$, we obtain the following second-quantized Hamiltonian density

$$\mathcal{H}_0 = \sum_a \left[\frac{2\hbar^2}{M} (D^a\phi^a)^\dagger (D^a\phi^a) + \frac{\hbar\omega_c}{2} \phi^{a\dagger}\phi^a \right], \tag{2.12}$$

where

$$D^a = \partial - \sum_b \frac{m^{ab}}{2} \int d^2w \frac{|\phi^b(w,\overline{w})|^2}{z-w} + \frac{\overline{z}}{4\ell^2}. \tag{2.13}$$

This Hamiltonian is the starting point of our discussion. We will rewrite it in terms of the collective variables in subsequent sections, and study the ground state and excitations.

Before finishing this section, let us consider the case $lm = n^2$. Most interesting case is $l = m = n$, for the Halperin wave function describes a Hall state at specific filling factor in this case. Then, the A-B phase takes the following form;

$$(m^{ab}) = \begin{pmatrix} m & m \\ m & m \end{pmatrix}, \tag{2.14}$$

with m a positive odd integer. Therefore only one C-S gauge field $C_\mu^\uparrow + C_\mu^\downarrow$ with the C-S coefficient $\frac{1}{4\pi\hbar m}$ couples with electrons. In this case the quantum-mechanical Hamiltonian or the second-quantized Hamiltonian density is given by the formal replacement of m^{ab} with m in Eq.(2.6) or (2.13), respectively.

When $l = m = n$, the condition that the Halperin state Φ_{mmm} is a Hall state determines only the total filling;

$$\nu = \frac{1}{m}. \tag{2.15}$$

State with any value of $(\nu^\uparrow, \nu^\downarrow)$ with $\nu^\uparrow + \nu^\downarrow = \frac{1}{m}$ is a Hall state. Correspondingly to this situation, the $SU(2)_d$ symmetry is spontaneously broken. Then some interesting phenomena, such as the Josephson(-type) effect, are expected to be observed in this case, especially in the presence of the interlayer tunneling. More detailed discussion will be given later.

3 Collective-Field Approach

Ginzburg-Landau (G-L) type theory for the FQHE has been proposed, and it essentially corresponds to the (semi-)classical analysis of $\mathcal{L}_0$ given in Section 2 or its counterpart of single-layer cases[3, 5, 23, 11]. The G-L theory gives a fairly good phenomenological description of the FQHE, but it has some unsatisfactory aspects. Especially, its relationship to the microscopic theory is not so clear. We will not take this approach here. Alternatively, we will use the collective-field variables to describe the microscopic model of DLFQHE, and construct an effective theory in terms of them[19]. They are observable quantities and then the physical interpretation will

be very clear. This is one of the reason for develping the collective-field approach to the DLFQHE. In this section, we will give a brief review of the usual collective-field method[19], and enumerate some problems in application of this method to the above-setting model of the DLFQHE.

When one has a quantum mechanical system with N bosons and wants to construct an effective theory of it, it is sometimes very useful to rewrite the microscopic theory using the following collective variables[17]; the density variable

$$\rho(\vec{r}) = \sum_{i=1}^{N} \delta^{(d)}(\vec{r} - \vec{r}_i). \tag{3.1}$$

Formally, this rewriting is a point canonical transformation and, at least in principle, it can be performed without any ambiguity. In high-density thermodynamic limit, the quantity $\frac{1}{N}\rho(\vec{r})$ is expected to be smooth and well-defined. Since the system is bosonic, the original N-body wave function $\Phi(\vec{r}_1, \cdots, \vec{r}_N)$ of the particle coordinates $\{\vec{r}_i\}$ can be translated into the wave functional $\Phi[\rho]$ of the collective coordinates $\rho(\vec{r})$. Then the microscopic Schrödinger equation is replaced by a functional differential equation with respect to the density variable $\rho(\vec{r})$. Especially, if the system has fluid-like properties, this collective-field approach is very useful. As the (DL)FQH system has fluid-like properties, this method is expected to be promising for study of the FQHE.

As we have set up the model of the DLFQHE in the bosonized form, the collective-field approach is applicable[20, 27]. Collective variables are the density variables

$$\rho^a(z, \overline{z}) = \sum_{i=1}^{N^a} \delta^{(2)}(z - z_i^a) \tag{3.2}$$

and the phase variables $\vartheta^a(z, \overline{z}) = -i\frac{\delta}{\delta\rho^a(z,\overline{z})}$. No extra variables are needed (at least up to this stage), because our model Hamiltonian H_0^{QM} (2.5) is expressed completely in terms of only the electron coordinates $\{\vec{r}_i^a\}$. Then we could follow the usual steps for constructing the collective-field theory of the DLFQHE from the quantum mechanical model H_0^{QM} (2.5) by means of the above-mentioned point canonical transformation.

But this prospect is too optimistic. We have so far neglected interactions. While inter-electron interaction like the Coulomb repulsion can be handled in a straightforward manner, the interlayer tunneling effect leads to two difficulties coming from the fact that the number of electrons N^a in each layer is not conserved in the presence of this effect.

1. The particle-number-nonconserving system is untractable in the framework of the quantum mechanics. Then we do not have an adequate quantum-mechanical Hamiltonian including the interlayer tunneling.

2. Simple relation between the microscopic variables $\{\vec{r}_i^a\}$ and the collective variables $\rho^a(z,\overline{z})$ like Eq.(3.2) does not hold. Then we do not know how to introduce the collective variables in our model with the tunneling.

The first point is easily conquered by using the second-quantized language. In this lauguage, the interlayer tunneling effect is effectively described by the following hopping Hamiltonian density; [5]

$$\mathcal{H}_{hop} = -\frac{\triangle_{SAS}}{2}(\phi^{\uparrow\dagger}\phi^{\downarrow} + \phi^{\downarrow\dagger}\phi^{\uparrow}), \tag{3.3}$$

where $\triangle_{SAS}$ corresponds to the energy difference between the symmetric and anti-symmetric combination of the states $|\uparrow\rangle$ and $|\downarrow\rangle$.

The collective-field approach starting from the second-quantized system has also been well established, as far as the space of states is restricted onto the one with a fixed particle-number[22]. It is known to be equivalent to the above-mentioned approach starting from the quantum-mechanical system[27]. In this case collective variables are the following density and current operators;

$$\hat{\rho}^a(z,\overline{z}) = \phi^{a\dagger}(z,\overline{z})\cdot\phi^a(z,\overline{z}),$$

[5]Strictly speaking, this hopping term should be given in terms of the original electron field operator before bosonization. If so, the hopping Hamiltonian may be accompanied with the line integral of the C-S gauge field. This point will be discussed later and here we will take naively Eq.(3.3) as a hopping Hamiltonian.

$$\begin{aligned}
\hat{j}_z^a(z,\overline{z}) &= -i\phi^{a\dagger}(z,\overline{z})\cdot\partial\phi^a(z,\overline{z}), \\
\hat{j}_{\overline{z}}^a(z,\overline{z}) &= -i\phi^{a\dagger}(z,\overline{z})\cdot\overline{\partial}\phi^a(z,\overline{z}).
\end{aligned} \tag{3.4}$$

From now on, the hat symbol is attached to the collective-field operators to distinguish them from their eigenvalues. The relation between the current operators and the phase variables $\hat{\vartheta}^a(z,\overline{z})$ is determined by the commutation relations, which will be discussed later.

To describe the system with pseudospin flip (3.3), we will introduce additional collective variables as follows;

$$\begin{aligned}
\hat{\rho}^{ab}(z,\overline{z}) &\equiv \phi^{a\dagger}(z,\overline{z})\cdot\phi^b(z,\overline{z}), \\
\hat{j}_z^{ab}(z,\overline{z}) &\equiv -i\phi^{a\dagger}(z,\overline{z})\cdot\partial\phi^b(z,\overline{z}), \\
\hat{j}_{\overline{z}}^{ab}(z,\overline{z}) &\equiv -i\phi^{a\dagger}(z,\overline{z})\cdot\overline{\partial}\phi^b(z,\overline{z}).
\end{aligned} \tag{3.5}$$

These operators form closed commutation relations, which are listed in Appendix A. As discussed later, we can find the ρ-realization of them. In fact, if the inverse operators of the density $\hat{\rho}^{a-1}$ are well-defined, then these operators (3.5) are not independent of each other; i.e., $\hat{j}_{z,\overline{z}}^{ab}$'s are given by

$$\begin{aligned}
\hat{j}_z^{ab}(z,\overline{z}) &= \hat{\rho}^{ab}(z,\overline{z})\frac{1}{\hat{\rho}^b(z,\overline{z})}\hat{j}_z^b(z,\overline{z}), \\
\hat{j}_{\overline{z}}^{ab}(z,\overline{z}) &= \hat{\rho}^{ab}(z,\overline{z})\frac{1}{\hat{\rho}^b(z,\overline{z})}\hat{j}_{\overline{z}}^b(z,\overline{z}).
\end{aligned} \tag{3.6}$$

The existence of the inverse operators $\hat{\rho}^{a-1}$ is physically plausible, since eigenvalue of the density operators is positive, except for the particular configuration $\rho^a(z,\overline{z}) \equiv 0$. Then we regard only $\hat{\rho}^{ab}$ and $\hat{j}_{z,\overline{z}}^a$ as independent collective field operators.

Hint for solving the second subtlety consists in how to construct the space of states with variable particle number. Then we will first summarize the construction of the space of states with fixed particle number in the following section. Detailed discussion on the case with pseudospin flip will be given in Section 5.

4 Analysis for Non-interacting DLFQH Systems

In this section, we will write down an effective theory for the non-interacting DLFQH system with Hamiltonian density (2.5) or (2.12), in terms of the collective fields. This model corresponds to the strong-magnetic-field limit or low-mass limit of the DLFQH system without interlayer tunneling. In this case, there are two equivalent approaches as discussed in Section 3. One starts from the quantum-mechanical system (2.5), the other from the second-quantized system (2.12). We will take the latter one because it gives bases for the later discussions. This is an extension of the case of the single-layer FQHE, which has been previously studied[20].

4.1 Collective Fields and Hamiltonian

Let us rewrite the eigenvalue problem of $\mathcal{H}_0$ (2.12) in terms of the collective fields, $\hat{\rho}^a, \hat{j}_z^a$, and $\hat{j}_{\overline{z}}^a$ in Eq.(3.4). These variables satisfy the following equal-time commutation relations, which are closed and well-defined,

$$\begin{aligned}
\left[\hat{\rho}^a(z,\overline{z}), \hat{\rho}^b(w,\overline{w})\right] &= 0, \\
\left[\hat{j}_z^a(z,\overline{z}), \hat{j}_z^b(w,\overline{w})\right] &= -\delta^{ab}\partial_z\left(i\hat{j}_z^a(z,\overline{z})\,\delta^{(2)}(z-w)\right) - \delta^{ab} i\hat{j}_z^a(z,\overline{z})\,\partial_z\delta^{(2)}(z-w), \\
\left[\hat{j}_{\overline{z}}^a(z,\overline{z}), \hat{j}_{\overline{z}}^b(w,\overline{w})\right] &= -\delta^{ab}\overline{\partial}_z\left(i\hat{j}_{\overline{z}}^a(z,\overline{z})\,\delta^{(2)}(z-w)\right) - \delta^{ab} i\hat{j}_{\overline{z}}^a(z,\overline{z})\,\partial_z\delta^{(2)}(z-w), \\
\left[\hat{j}_z^a(z,\overline{z}), \hat{\rho}^b(w,\overline{w})\right] &= -\delta^{ab} i\hat{\rho}^a(z,\overline{z})\,\partial_z\delta^{(2)}(z-w), \\
\left[\hat{j}_{\overline{z}}^a(z,\overline{z}), \hat{\rho}^b(w,\overline{w})\right] &= -\delta^{ab} i\hat{\rho}^a(z,\overline{z})\,\overline{\partial}_z\delta^{(2)}(z-w). \qquad (4.1)
\end{aligned}$$

From the above relations, it is obvious that only the density variables can be simultaneously diagonalized. If the model is formulated only in terms of the density variables and derivatives with respect to them, we say that the model has ρ-realization. The above commutation relations are a basis for determining ρ-realization of the operators which appear in the Hamiltonian.

The non-interacting Hamiltonian (2.12) with $lm \neq n^2$ is easily rewritten with the above collective fields;

$$\begin{aligned}
H_0[\hat{j}_z^a, \hat{j}_{\overline{z}}^a; \hat{\rho}^a] &= \int d^2z\, \mathcal{H}_0(z,\overline{z}) \\
&= \sum_a \frac{2\hbar^2}{M}\int d^2z\, \left\{ \overline{\partial}\hat{\rho}^a(z,\overline{z}) - i\hat{j}_{\overline{z}}^a(z,\overline{z}) - \sum_c \frac{m^{ac}}{2}\int d^2\xi\, \frac{\hat{\rho}^a(z,\overline{z})\hat{\rho}^c(\xi,\overline{\xi})}{\overline{z}-\overline{\xi}} + \frac{z}{4\ell^2}\hat{\rho}^a(z,\overline{z}) \right\} \\
&\qquad \times \frac{1}{\hat{\rho}^a(z,\overline{z})} \cdot \left\{ i\hat{j}_z^a(z,\overline{z}) - \sum_b \frac{m^{ab}}{2}\int d^2w\, \frac{\hat{\rho}^a(z,\overline{z})\hat{\rho}^b(w,\overline{w})}{z-w} + \frac{\overline{z}}{4\ell^2}\hat{\rho}^a(z,\overline{z}) \right\} \\
&\quad + \sum_a \frac{\hbar\omega_c}{2}\int d^2z\, \hat{\rho}^a(z,\overline{z}). \qquad (4.2)
\end{aligned}$$

Hereafter we neglect, as a regularization, the divergent terms originating from the ordering of the operator products at the same spatial point. Of course, the non-interacting Hamiltonian with $l = m = n$ is given by the replacement of m^{ab} with m in the above equation. We must obtain ρ-realization of Eq.(4.2) as the next step. Before doing so, we will consider the space of states in the ρ-realization.

4.2 Construction of the Space of States

In this subsection we will set up the space of states realized in ρ. When the system is not linked to the external reservior and there is no interlayer hopping, the numbers of electrons $N^a = \int d^2z\, \rho^a(z,\overline{z})$ in each layer a are separately kept constant.[6] Then the relation between the particle coodinates $\{\vec{r}_i^a\}$ and the collective coordinates $\rho^a(z,\overline{z})$ is given by Eq.(3.2); $\rho^a(z,\overline{z}) = \sum_{i=1}^{N^a}\delta^{(2)}(z-z_i^a)$, and it is easy to find the ρ-realization of the state. Actually it relates to the quantum-mechanical wave function of N bosonized

[6] In order for the system to realize the (bosonized) Halperin state Φ_{lmn} with $lm \neq n^2$ as an incompressible ground state, we should choose each value N^a by fine tuning. Theoretically this value is determined by the consistency condition that the resulting expectation value of the density $\hat{\rho}^a$ with respect to Φ_{lmn} is equal to the average density $\rho_{\rm av}^a = \frac{N^a}{A}$; $\langle\hat{\rho}^a\rangle = \rho_{\rm av}^a$. The result is $N^a = \nu_0^a \frac{A}{2\pi\ell^2}$. When $l = m = n$, this fine tuning is unnecessary except for the tuning of the total number; $N = N^1 + N^l = \frac{1}{m}\frac{A}{2\pi\ell^2}$. In this case each N^a is fixed arbitrarily by hand.

electrons in the DL system;

$$\Phi(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}) \Longleftrightarrow \Phi[\rho^\uparrow,\rho^\downarrow]\Big|_{\rho^a(z,\overline{z})=\sum_{i=1}^{N^a}\delta^{(2)}(z-z_i^a)}. \tag{4.3}$$

But there is one more thing we have to consider. We must investigate the measure of the space of states. The inner product must be invariant under the change of variables: $\{\vec{r}_i^{\,a}\} \mapsto \{\rho^a(z,\overline{z})\}$, and then the measure of the functional space of $\Phi[\rho^\uparrow,\rho^\downarrow]$ is nontrivial;

$$\begin{aligned}\langle\Psi|\Phi\rangle &= \int(\prod_{u=1}^{N^\uparrow}d^2z_u^\uparrow)(\prod_{d=1}^{N^\downarrow}d^2z_d^\downarrow)\,\overline{\Psi(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow})}\cdot\Phi(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow})\\ &= \int\mathcal{D}\rho^\uparrow\mathcal{D}\rho^\downarrow\,J^{(0)}_{N^\uparrow N^\downarrow}[\rho^\uparrow,\rho^\downarrow]\cdot\overline{\Psi[\rho^\uparrow,\rho^\downarrow]}\cdot\Phi[\rho^\uparrow,\rho^\downarrow],\end{aligned} \tag{4.4}$$

where the additional weight factor $J^{(0)}_{N^\uparrow N^\downarrow}[\rho^\uparrow,\rho^\downarrow]$ corresponds to the Jacobian associated with the above change of variables. The physical meaning of this factor is the entropy of the system. In order to find the explicit expression of $J^{(0)}_{N^\uparrow N^\downarrow}[\rho^\uparrow,\rho^\downarrow]$, we will review the above argument in the second quantized formalism.

The state $|\Phi\rangle$ corresponding to the N-boson wave function $\Phi(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow})$ is given in the second-quantized language by

$$|\Phi\rangle = \int(\prod_{u=1}^{N^\uparrow}d^2z_u^\uparrow)(\prod_{d=1}^{N^\downarrow}d^2z_d^\downarrow)\,\Phi(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow})\cdot\Big|\,\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}\Big\rangle. \tag{4.5}$$

Here, a totally symmetric basis of the Hilbert space of $(N^\uparrow,N^\downarrow)$-boson states is constructed by $\phi^{a\dagger}$ as

$$\Big|\,\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}\Big\rangle = \prod_a\frac{1}{\sqrt{N^a!}}\left[\prod_{i=1}^{N^a}\phi^{a\dagger}(z_i^a,\overline{z}_i^a)\right]|0\rangle, \tag{4.6}$$

where the normalized vacuum $|0\rangle$ is defined by $\phi^a(z,\overline{z})\,|0\rangle = 0$. This basis state is an eigenstate of the density operator $\hat{\rho}^a(z,\overline{z})$ with eigenvalue $\sum_{i=1}^{N^a}\delta^{(2)}(z-z_i^a)$;

$$\hat{\rho}^a(z,\overline{z})\cdot\Big|\,\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}\Big\rangle = \sum_{i=1}^{N^a}\delta^{(2)}(z-z_i^a)\cdot\Big|\,\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}\Big\rangle. \tag{4.7}$$

Then, from Eq.(4.3), the relation between the state $|\Phi\rangle$ in (4.5) and the wave functional $\Phi[\rho^\uparrow,\rho^\downarrow]$ of ρ^a is obtained as follows;

$$\begin{aligned}|\Phi\rangle &= \int(\prod_{u=1}^{N^\uparrow}d^2z_u^\uparrow)(\prod_{d=1}^{N^\downarrow}d^2z_d^\downarrow)\,\Phi[\rho^\uparrow,\rho^\downarrow]\cdot\Big|\,\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^\uparrow}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^\downarrow}^{\downarrow}\Big\rangle\\ &= \Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\cdot\|N^\uparrow;N^\downarrow\rangle,\end{aligned} \tag{4.8}$$

where

$$\|N^\uparrow;N^\downarrow\rangle \equiv \prod_a\frac{1}{\sqrt{N^a!}}\left[\int d^2z\,\phi^{a\dagger}(z,\overline{z})\right]^{N^a}|0\rangle. \tag{4.9}$$

Note that $\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]$ is the operator-valued functional acting on $\|N^\uparrow;N^\downarrow\rangle$, and is of the same form with respect to its arguments as the wave functional $\Phi[\rho^\uparrow,\rho^\downarrow]$. The operand of $\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]$ consists of only $\|N^\uparrow;N^\downarrow\rangle$, which is unique up to the normalization because $N^\uparrow$, $N^\downarrow$ are fixed.

The calculation of the Jacobian factor is performed as follows. The inner product of the states $|\Phi\rangle$ and $|\Psi\rangle$ is

$$\begin{aligned}\langle\Psi|\Phi\rangle &= \langle N^\uparrow;N^\downarrow\|\,\Psi^\dagger[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\cdot\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\,\|N^\uparrow;N^\downarrow\rangle\\ &= \int\mathcal{D}\rho^\uparrow\mathcal{D}\rho^\downarrow\,\overline{\Psi[\rho^\uparrow,\rho^\downarrow]}\cdot\Phi[\rho^\uparrow,\rho^\downarrow] \qquad (4.10)\\ &\quad\times\langle N^\uparrow;N^\downarrow\|\prod_a\prod_z\delta\Big(\rho^a(z,\overline{z})-\hat{\rho}^a(z,\overline{z})\Big)\,\|N^\uparrow;N^\downarrow\rangle, \qquad (4.11)\end{aligned}$$

where the functional measure is $\mathcal{D}\rho^a = \prod_z d\rho^a(z,\overline{z})$, and the delta-function in the integrand is defined with respect to this measure; $\int\mathcal{D}\rho^a\prod_z\delta\Big(\rho^a(z,\overline{z})-\hat{\rho}^a(z,\overline{z})\Big)=1$. Compareing Eq.(4.4) with Eq.(4.11), the Jacobian is given by

$$\begin{aligned}J^{(0)}_{N^\uparrow N^\downarrow}[\rho^\uparrow,\rho^\downarrow] &= \langle N^\uparrow;N^\downarrow\|\prod_a\prod_z\delta\Big(\rho^a(z,\overline{z})-\hat{\rho}^a(z,\overline{z})\Big)\,\|N^\uparrow;N^\downarrow\rangle\\ &= \prod_a\int(\prod_{i=1}^{N^a}d^2z_i^a)\left[\prod_z\delta\left(\rho^a(z,\overline{z})-\sum_{i=1}^{N^a}\delta^{(2)}(z-z_i^a)\right)\right].\end{aligned} \tag{4.12}$$

This is calculated in the leading order of large-N^a as;

$$J^{(0)}_{N^\uparrow N^\downarrow}[\rho^\uparrow,\rho^\downarrow] \simeq \left[\prod_a\delta\left(\int d^2z\,\rho^a(z,\overline{z})-N^a\right)\right]\cdot e^{-\sum_a\int d^2z\,\rho^a(z,\overline{z})\cdot\log\frac{\rho^a(z,\overline{z})}{\rho^a_{\rm av}}}, \tag{4.13}$$

where $\rho^a_{\rm av}=\frac{N^a}{A}$ is the average density in the layer a[22].

Then we will determine the inner product of the functional space of $\Phi[\rho^\uparrow, \rho^\downarrow]$ as

$$\langle \Psi | \Phi \rangle = \int \langle \mathcal{D}\rho \rangle_0 \, e^{\mathcal{S}_0[\rho^\uparrow, \rho^\downarrow]} \cdot \overline{\Psi[\rho^\uparrow, \rho^\downarrow]} \cdot \Phi[\rho^\uparrow, \rho^\downarrow], \tag{4.14}$$

where the measure is defined by

$$\langle \mathcal{D}\rho \rangle_0 = \prod_a \prod_z d\rho^a(z, \overline{z}) \cdot \left[\prod_a \delta \left(\int d^2 z \, \rho^a(z, \overline{z}) - N^a \right) \right], \tag{4.15}$$

and the weight factor $e^{\mathcal{S}_0}$ is given by

$$\mathcal{S}_0[\rho^\uparrow, \rho^\downarrow] = -\sum_a \int d^2 z \left\{ \rho^a(z, \overline{z}) \log \rho^a(z, \overline{z}) - \rho^a_{\rm av} \log \rho^a_{\rm av} \right\}, \tag{4.16}$$

which corresponds to the entropy of the system.

4.3 ρ-Realization of the Current Operators

As we have finished constructing the space of states, the remaining problem is to find ρ-realization of the current operators and the Hamiltonian (4.2), which acts on the wave fuctional $\Phi[\rho^\uparrow, \rho^\downarrow]$. Since

$$\hat{j}^a_{z,\overline{z}}(z, \overline{z}) \, \| N^\uparrow; N^\downarrow \rangle = 0, \tag{4.17}$$

$\hat{j}^a_{z,\overline{z}}(z, \overline{z}) | \Phi \rangle = \left[\hat{j}^a_{z,\overline{z}}(z, \overline{z}), \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow] \right] \| N^\uparrow; N^\downarrow \rangle$. Then we only have to calculate the commutator $\left[\hat{j}^a_{z,\overline{z}}(z, \overline{z}), \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow] \right]$ and this is easily performed by using the commutation relations (4.1) (see Appendix A). The result is

$$\left[\hat{j}^a_{z(\overline{z})}(z, \overline{z}), \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow] \right] = -i \hat{\rho}^a(z, \overline{z}) \overset{(-)}{\partial} \frac{\delta}{\delta \hat{\rho}^a(z, \overline{z})} \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow]. \tag{4.18}$$

This is the operator relation acting on $\| N^\uparrow; N^\downarrow \rangle$ and it gives ρ-realization of the current operators acting on the wave functional $\Phi[\rho^\uparrow, \rho^\downarrow]$;

$$\hat{j}^a_{z(\overline{z})}(z, \overline{z}) \to -i \rho^a(z, \overline{z}) \overset{(-)}{\partial} \frac{\delta}{\delta \rho^a(z, \overline{z})} = \rho^a(z, \overline{z}) \overset{(-)}{\partial} \vartheta^a(z, \overline{z}). \tag{4.19}$$

Consequently the original eigenvalue problem $\int d^2 z \, \mathcal{H}_0(z, \overline{z}) | \Phi \rangle = E_0 | \Phi \rangle$ is rewritten to that of the wave functional $\Phi[\rho^\uparrow, \rho^\downarrow]$;

$$\mathbf{H}_0[\vartheta^a, \rho^a] \cdot \Phi[\rho^\uparrow, \rho^\downarrow] = E_0 \cdot \Phi[\rho^\uparrow, \rho^\downarrow], \tag{4.20}$$

where $\mathbf{H}_0$ is defined by $\mathbf{H}_0[\vartheta^a, \rho^a] = H_0[\rho^a \partial \vartheta^a, \rho^a \overline{\partial} \vartheta^a; \rho^a]$. H_0 is the same functional with respect to its arguments with Eq.(4.2). This equation (4.20) can be regarded as the Schrödinger equation of the wave fuctional $\Phi[\rho^\uparrow, \rho^\downarrow]$ as well as the operator relation acting on $\| N^\uparrow; N^\downarrow \rangle$.

4.4 Effective Theory of Non-interacting DLFQH Systems

Now the set up of the problem has been completed and all we have to do is to solve the eigenvalue problem (4.20). There are two directions for further strategy. For simplicity, let us discuss only the case $lm \neq n^2$. The similar arguments can be applied for the case $l = m = n$ through the replacement of m^{ab} with m.

The first one is of course the direct analysis of the above eigenvalue problem. We will call this Approach 1. Actually we can obtain the exact ground state with energy eigenvalue $E_0 = \frac{N}{2} \hbar \omega_c$, by solving the following equations[20];

$$\left(\partial \frac{\delta}{\delta \rho^a(z, \overline{z})} - \sum_b \frac{m^{ab}}{2} \int d^2 w \, \frac{\rho^b(w, \overline{w})}{z - w} + \frac{\overline{z}}{4\ell^2} \right) \Phi[\rho^\uparrow, \rho^\downarrow] = 0, \tag{4.21}$$

which correspond to the condition that all electrons in each layer lie in the lowest Laudau level (LLL). The general solution to this condition can be easily obtained as $\Phi[\rho^\uparrow, \rho^\downarrow] = \varphi[\rho^\uparrow, \rho^\downarrow] \, \Phi_{lmn}[\rho^\uparrow, \rho^\downarrow]$, where

$$\Phi_{lmn}[\rho^\uparrow, \rho^\downarrow] = e^{\sum_{a,b} \frac{m^{ab}}{2} \int d^2 z d^2 w \, \rho^a(z, \overline{z}) \cdot \log |z - w| \cdot \rho^b(w, \overline{w})} \cdot e^{-\frac{1}{4\ell^2} \int d^2 z \, |z|^2 \{ \rho^\uparrow(z, \overline{z}) + \rho^\downarrow(z, \overline{z}) \}}, \tag{4.22}$$

and $\varphi[\rho^\uparrow, \rho^\downarrow]$ is an arbitrary functional satisfying $\partial \frac{\delta}{\delta \rho^a(z, \overline{z})} \varphi[\rho^\uparrow, \rho^\downarrow] = 0$. It is easily verified that $\Phi_{lmn}[\rho^\uparrow, \rho^\downarrow]$ corresponds to the (bosonized) Halperin state (2.7), by substituting the relation (3.2). The expectation value of the density $\rho^a(z, \overline{z})$ in this state can be calculated and the result shows a uniform distribution of electron with density given by the magic filling $\frac{\nu^a_0}{2\pi\ell^2}$[28]. The degrees of freedom of $\varphi[\rho^\uparrow, \rho^\downarrow]$ represent vortex excitations. For example, the state with N^a_V vortices of vorticity m^a_{Vi} at ξ^a_i ($a = \uparrow, \downarrow, i = 1, \cdots, N^a_V$) above the Halperin state is given by

$$\varphi[\rho^\uparrow, \rho^\downarrow] = e^{\sum_a \sum_{i=1}^{N^a_V} m^a_{Vi} \int d^2 z \, \rho^a(z, \overline{z}) \cdot \log(\overline{z} - \overline{\xi}^a_i)}, \tag{4.23}$$

which turns out (generalized) Laughlin's quasi-hole wave function after substituting the relation (3.2)[28].

Then we can say that this Approach 1 is essentially equivalent to the microscopic quantum-mechanical theory through the relation (3.2); $\rho^a(z,\overline{z}) = \sum_{i=1}^{N^a} \delta^{(2)}(z - z_i^a)$.

The second prescription which we will call Approach 2 is developed as a (semi-)classical field theory in terms of the expectation value of the collective variables. To obtain the theory, we will reformulate Approach 1 as follows. Since the functional space in Approach 1 has a nontrivial inner product (4.14) with (4.15) and (4.16), then we will define the new functional space as

$$\tilde{\Phi}[\rho^\uparrow, \rho^\downarrow] \equiv e^{\frac{1}{2}\mathcal{S}_0[\rho^\uparrow,\rho^\downarrow]}\, \Phi[\rho^\uparrow, \rho^\downarrow], \tag{4.24}$$

so that the inner product is given by the usual one;

$$\langle \Phi | \Phi' \rangle = \int \langle \mathcal{D}\rho \rangle_0\, \overline{\tilde{\Phi}[\rho^\uparrow, \rho^\downarrow]} \cdot \tilde{\Phi}'[\rho^\uparrow, \rho^\downarrow], \tag{4.25}$$

where the measure $\langle \mathcal{D}\rho \rangle_0$ is defined by Eq.(4.15) again. According to this unitary transformation, the ρ-realized operator $\mathbf{O}[\vartheta^a, \rho^a]$ is also transformed as

$$\tilde{\mathbf{O}}[\vartheta^a, \rho^a] \equiv e^{\frac{1}{2}\mathcal{S}_0[\rho^\uparrow,\rho^\downarrow]}\, \mathbf{O}[\vartheta^a, \rho^a]\, e^{-\frac{1}{2}\mathcal{S}_0[\rho^\uparrow,\rho^\downarrow]} = \mathbf{O}\left[\vartheta^a - \frac{i}{2}(\log \rho^a + 1), \rho^a\right]. \tag{4.26}$$

In this space the hermiticity of the operator $\tilde{\mathbf{O}}$ is the usual one. Hereafter we will denote the expectation value with respect to the inner product (4.25) as $\langle \cdot \rangle$, while the one with respect to the inner product (4.14) will be denoted as $\langle\langle \cdot \rangle\rangle$. Note that the expectation value $\langle \mathbf{O} \rangle$ is real, while $\langle \tilde{\mathbf{O}} \rangle = \langle\langle \mathbf{O} \rangle\rangle$ is not always so. $\vartheta^a(z,\overline{z}) = -i\frac{\delta}{\delta \rho^a(z,\overline{z})}$ is such an example. In this space of ρ-realized states, the eigenvalue problem (4.20) is rewritten as follows;

$$\tilde{\mathbf{H}}_0[\vartheta^a, \rho^a] \cdot \tilde{\Phi}[\rho^\uparrow, \rho^\downarrow] = E_0 \cdot \tilde{\Phi}[\rho^\uparrow, \rho^\downarrow]. \tag{4.27}$$

Of course the direct analysis of this equation is completely equivalent to Approach 1.

Approach 2 is developed by taking the expectation value of Eq.(4.27). In addition we will apply the mean-field type approximation such that a n-point function of the

collective variables is replaced by the corresponding product of 1-point function of them. This prescription will be valid as a zeroth-order approximation when the system under study has fluid-like properties, and plays an essential role in our formalism. Then $\langle \tilde{\mathbf{H}}_0[\vartheta^a, \rho^a] \rangle$ is replaced by $\tilde{\mathbf{H}}_0[\langle \vartheta^a \rangle, \langle \rho^a \rangle]$ in this formalism. In the following, when there is no possibility of confusion, we will also denote $\langle \rho^a \rangle$ and $\langle \vartheta^a \rangle$ as ρ^a and ϑ^a, respectively. Then we will regard $\langle \tilde{\mathbf{H}}_0 \rangle$ as a Hamiltonian of "classical fields" ρ^a and ϑ^a, and we will treat it variationally, i.e., search for the stationary point with respect to $\rho^a(z,\overline{z})$ and $\vartheta^a(z,\overline{z})$ to calculate the (variational) ground state. Since the measure (4.15) of the functional space involves a electron-number constraint in each layer, then the resulting *effective* Hamiltonian is defined as follows;

$$\begin{aligned} H_0^{eff}[\vartheta^a, \rho^a; \lambda^a] &= \langle \tilde{\mathbf{H}}_0[\vartheta^a, \rho^a] \rangle + \sum_a \lambda^a \left(\int d^2z\, \rho^a(z,\overline{z}) - N^a \right) \\ &= \frac{2\hbar^2}{M} \sum_a \int d^2z\, \rho^a(z,\overline{z}) \overline{L}^a(z,\overline{z}) L^a(z,\overline{z}) + \sum_a \left(\frac{\hbar\omega_c}{2} + \lambda^a \right) \int d^2z\, \rho^a(z,\overline{z}) - \sum_a \lambda^a N^a, \end{aligned} \tag{4.28}$$

where the LL operators L^a's are

$$L^a(z,\overline{z}) = \frac{1}{2}\frac{\partial \rho^a(z,\overline{z})}{\rho^a(z,\overline{z})} + i\partial\vartheta^a(z,\overline{z}) - \sum_b \frac{m^{ab}}{2} \int d^2w\, \frac{\rho^b(w,\overline{w})}{z-w} + \frac{\overline{z}}{4\ell^2}, \tag{4.29}$$

and λ^a's are Lagrange multipliers which impose the constraints of the electron number in each layer, which are involved in the measure (4.15). We will regard Eq.(4.28) as an effective Hamiltonian for non-interacting DLFQH systems (with "classical fields" $\rho^a(z,\overline{z})$ and $\vartheta(z,\overline{z})$). By searching for a stationary point of this effective Hamiltonian, we can obtain the variational ground state. In this paper, we call this (type of) prescription the effective theory for the DLFQHE.

The effective theory of DLFQHE has been derived from the microscopic quantum-mechanical Hamiltonian. Also we can show, by practical calculation, that the expectation values of the collective-field operators calculated in Approach 1 are equal to those of the classical solution obtained in Approach 2. Then it is quite clear how the microscopic theory and the macroscopic effective theory are related.

As an example, let us examine the ground state of H_0^{eff}. Since H_0^{eff} is positive

definite, the ground state satisfies

$$L^a(z,\overline{z}) = 0. \tag{4.30}$$

These equations correspond to the LLL constraints (4.21). Let us find a solution with a constant density; $\rho^a(z,\overline{z}) = \rho^a_{\rm av} = \frac{N^a}{A}$. In this case, $L^a(z,\overline{z}) = i\partial\vartheta^a(z,\overline{z}) - \frac{\pi}{2}Q^a\overline{z}$, where $Q^a = \sum_b m^{ab}\rho^b_{\rm av} - \frac{1}{2\pi\ell^2}$, and Eq.(4.30) leads to the equation of $\vartheta^a(z,\overline{z})$, $(\overline{\partial}\partial + \partial\overline{\partial})\vartheta^a(z,\overline{z}) = 0$.

When $\vartheta^a(z,\overline{z})$'s are non-singular, the solution is given by $Q^a = 0$, i.e.,

$$\rho^a(z,\overline{z}) = \frac{\nu^a_0}{2\pi\ell^2}, \quad \text{and} \quad \vartheta^a(z,\overline{z}) = \text{constant}, \tag{4.31}$$

and the number of electrons in each layer must be fixed as $N^a = \nu^a_0\frac{A}{2\pi\ell^2}$. (See footnote 6.) This is just the same result as obtained in Approach 1, then this state certainly corresponds to the Halperin state.

When $\vartheta^a(z,\overline{z})$'s are singular, they take the following form;

$$\vartheta^a(z,\overline{z}) = \int d^2\xi\,\rho^a_V(\xi,\overline{\xi})\,\Im\log(\overline{z}-\overline{\xi}). \tag{4.32}$$

The quantities $\rho^a_V(z,\overline{z})$'s correspond to the density of vortices in the layer a. The LLL constraint (4.30) requires these to be spatially uniform, a constant-density solution;

$$\rho^a_V(z,\overline{z}) = \overline{\rho^a_V} = -\sum_b m^{ab}\rho^b_{\rm av} + \frac{1}{2\pi\ell^2}. \tag{4.33}$$

Then vortices are condensed in this state, and this state clearly corresponds to the first hierarchy state above the Halperin state[29]. Noting that in order for this state to be realized, the number of electrons in each layer is fixed according to Eq.(4.33); $N^a = \nu^a_0\frac{A}{2\pi\ell^2} - \sum_b\sum_{j=1}^{N^b_V}(m^{-1})^{ab}m^b_{Vj}$, where m^a_{Vi}' is the vorticity in the layer a as in the previous discussion. [7]

[7]The state with a single vortex appears as a solution with non-uniform density. But since we have taken the large-N limit in our formalism, we cannot see the effect of order $\frac{1}{N}$ correctly. Then solutions to Eq.(4.30) cannot well describe the state with a single vortex.

4.5 Effect of the Coulomb Interaction and Excitations

In the rest of this section, we will calculate the dispersion relation of the density fluctuations above the Halperin state. In the discussion given so far, we have neglected the inter-electron interaction. But since this calculation becomes more meaningful in the presence of such an interaction, we will take it into account below. As the inter-electron potential depends only on the positions of electrons, the corresponding interaction Hamiltonian in terms of the collective fields is given simply as $V[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]$, and the formalism developed in this section can apply without any modification.

We will consider the Coulomb-type interaction, whose explicit form is

$$V_{Coulomb}[\hat{\rho}^a] = \sum_{a,b}\frac{1}{2}\int d^2z d^2w\left(\hat{\rho}^a(z,\overline{z}) - \overline{\rho}^a\right)\cdot U^{ab}(|z-w|)\cdot\left(\hat{\rho}^b(w,\overline{w}) - \overline{\rho}^b\right),$$

$$\begin{aligned} U^{\uparrow\uparrow}(|z-w|) = U^{\downarrow\downarrow}(|z-w|) &= \frac{e^2}{4\pi\epsilon}\cdot\frac{1}{|z-w|}, \\ U^{\uparrow\downarrow}(|z-w|) = U^{\downarrow\uparrow}(|z-w|) &= \frac{e^2}{4\pi\epsilon}\cdot\frac{1}{\sqrt{|z-w|^2+d^2}}, \end{aligned} \tag{4.34}$$

where d is the interlayer distance. $\overline{\rho}^a$'s are the density of uniform background of neutralizing positive charge in the layer a. In the case without interlayer tunneling, they are equal to the average density $\rho^a_{\rm av}$ of electrons in each layer; $\overline{\rho}^a = \rho^a_{\rm av}$. This interaction stabilizes the Halperin state and makes it a unique ground state. The fluctuations above the Halperin state are described by

$$\begin{aligned} \rho^a(z,\overline{z}) &= \rho^a_{\rm av} + \delta\rho^a(z,\overline{z}), \quad \text{with} \quad \rho^a_{\rm av} = \frac{N^a}{A} = \frac{\nu^a_0}{2\pi\ell^2}, \\ \vartheta^a(z,\overline{z}) &= \vartheta^a_0 + \delta\vartheta^a(z,\overline{z}), \end{aligned} \tag{4.35}$$

where

$$\int d^2z\,\delta\rho^a(z,\overline{z}) = \int d^2z\,\delta\vartheta^a(z,\overline{z}) = 0, \tag{4.36}$$

and ϑ^a_0 is an arbitrary constant. Our strategy is based on a semi-classical treatment, i.e., we "re-quantize" the classical field theory (4.28), regarding $\delta\rho^a$ and $\delta\vartheta^a$

as mutually conjugate dynamical variables. By diagonalizing the quadratic part of the Hamiltonian, we obtain the excitation spectrum[27]. To this end, we rescale the fluctuation variables in order to adjust the order of N;

$$\begin{aligned} \delta\rho^a(z,\overline{z}) &= \sqrt{N^a}\,\eta^a(z,\overline{z}), \\ \delta\vartheta^a(z,\overline{z}) &= \frac{1}{\sqrt{N^a}}\,\xi^a(z,\overline{z}). \end{aligned} \tag{4.37}$$

The commutation relations are

$$\left[\eta^a(z,\overline{z}),\, \xi^b(w,\overline{w})\right] = i\delta^{ab}\,\delta^{(2)}(z-w), \tag{4.38}$$

and otherwise commute. Their Fourier transforms are defined by

$$\begin{aligned} \eta^a(z,\overline{z}) &= \frac{1}{A}\sum_{k\neq 0} |\vec{k}|\, e^{-ikz-i\vec{k}\overline{z}}\, q^a_k, \\ \xi^a(z,\overline{z}) &= \sum_{k\neq 0} \frac{1}{\hbar|\vec{k}|}\, e^{ikz+i\vec{k}\overline{z}}\, p^a_k, \\ \left[q^a_k,\, p^b_{k'}\right] &= i\hbar\delta^{ab}\,\delta_{kk'}. \end{aligned} \tag{4.39}$$

Rewriting the Hamiltonian,

$$H^{eff}_{no-hopping}[\vartheta^a,\rho^a;\lambda^a] = H^{eff}_0[\vartheta^a,\rho^a;\lambda^a] + V_{Coulomb}[\rho^a], \tag{4.40}$$

in terms of these variables, we obtain the following result;

$$\begin{aligned} H^{eff}_{no-hopping}[\vartheta^a,\rho^a;\lambda^a] &= \text{const.} + \sum_a \frac{\hbar\omega_c}{2} N^a \\ &+ \sum_a\sum_{k\neq 0}\frac{1}{2M}p^a_k p^a_{-k} + \sum_{a,b}\sum_{k\neq 0}\frac{1}{2}K^{ab}(k)q^a_k q^b_{-k} + \cdots, \end{aligned} \tag{4.41}$$

where the constant term originates in the ordering of operators, and the "spring constants" $K^{ab}(k)$ are given by

$$K^{ab}(k) = \delta^{ab}\frac{\hbar^2(\vec{k}^2)^2}{4M} + n^{ab}\sqrt{\nu^a_0\nu^b_0}\frac{\hbar^2}{M\ell^4} + m^{ab}\sqrt{\nu^a_0\nu^b_0}\frac{\hbar^2\vec{k}^2}{M\ell^2} + \frac{\sqrt{\nu^a_0\nu^b_0}}{2\pi\ell^2}\vec{k}^2U^{ab}(|\vec{k}|), \tag{4.42}$$

where

$$\begin{aligned} n^{ab} &= \sum_c \nu^c_0\, m^{ac}m^{bc} \\ &= \frac{1}{lm-n^2}\begin{pmatrix} l^2(m-n)+n^2(l-n) & 2lmn-(l+m)n^2 \\ 2lmn-(l+m)n^2 & m^2(l-n)+n^2(m-n) \end{pmatrix}. \end{aligned} \tag{4.43}$$

The first term of K^{ab} is a kinetic energy and the second is a "mass term". The third term corresponds to the effect of the hard-core repulsion and the fourth does to the inter-electron interaction. In the case of Eq.(4.34),

$$U^{ab}(|\vec{k}|) = \frac{e^2}{4\pi\epsilon}\frac{2\pi}{|\vec{k}|}e^{-(1-\delta^{ab})|\vec{k}|d}. \tag{4.44}$$

By diagonalizing K^{ab}, the eigenvalues $M\omega^2_\pm(k)$ are obtained as follows;

$$\begin{aligned} \omega^2_\pm(k) &= \left(\frac{\hbar\vec{k}^2}{2M}\right)^2 + \frac{2lm-(l+m)n}{lm-n^2}\omega_c\frac{\hbar\vec{k}^2}{2M} + \frac{l+m-2n}{lm-n^2}\frac{\omega_c}{4\pi\hbar}\vec{k}^2U^{\uparrow\uparrow}(|\vec{k}|) \\ &+ \frac{l^2(m-n)^2+m^2(l-n)^2+2n^2(l-n)(m-n)}{2(lm-n^2)^2}\omega_c^2 \\ &\pm \left[\frac{(l-m)^2}{(lm-n^2)^2}\left\{n\omega_c\frac{\hbar\vec{k}^2}{2M} + \frac{n(2lm-(l+m)n)}{2(lm-n^2)}\omega_c^2 + \frac{\omega_c}{4\pi\hbar}\vec{k}^2U^{\uparrow\uparrow}(|\vec{k}|)\right\}^2\right. \\ &\left. + \frac{4(l-n)(m-n)}{(lm-n^2)^2}\left\{n\omega_c\frac{\hbar\vec{k}^2}{2M} + \frac{n(2lm-(l+m)n)}{2(lm-n^2)}\omega_c^2 + \frac{\omega_c}{4\pi\hbar}\vec{k}^2U^{\uparrow\uparrow}(|\vec{k}|)\right\}^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.45}$$

The + mode is essentially associated with the $U(1)$ symmetry and the − mode with the $SU(2)_d$. In the case $l=m$ where the symmetry between the upper- and lower-layer ("inversion symmetry") is explict because of vanishing the total pseudospin $N^\uparrow = N^\downarrow$, the ± mode corresponds to $(q^\pm_{\vec{k}}, p^\pm_{\vec{k}})$, defined by $q^\pm_{\vec{k}} = \frac{1}{\sqrt{2}}(q^\uparrow_{\vec{k}} \pm q^\downarrow_{\vec{k}})$ and $p^\pm_{\vec{k}} = \frac{1}{\sqrt{2}}(p^\uparrow_{\vec{k}} \pm p^\downarrow_{\vec{k}})$, respectively. For $|\vec{k}|\,\ell \ll 1$,

$$\omega^2_+(k) \sim \left(\frac{\hbar\vec{k}^2}{2M}+\omega_c\right)^2 + \frac{l+m-2n}{lm-n^2}\frac{\omega_c}{4\pi\hbar}\vec{k}^2\left(U^{\uparrow\uparrow}(|\vec{k}|)+U^{\uparrow\downarrow}(|\vec{k}|)\right), \tag{4.46}$$

$$\omega^2_-(k) \sim \left(\frac{\hbar\vec{k}^2}{2M}+\frac{(l-n)(m-n)}{lm-n^2}\omega_c\right)^2 + \frac{l+m-2n}{lm-n^2}\frac{\omega_c}{4\pi\hbar}\vec{k}^2\left(U^{\uparrow\uparrow}(|\vec{k}|)-U^{\uparrow\downarrow}(|\vec{k}|)\right).$$

Please notice that when $d\to 0$, the effect of the inter-electron interaction to the − mode vanishes. Especially let us consider the long-wavelength limit, $|\vec{k}|\to 0$. When $\lim_{k\to 0}\vec{k}^2U^{ab}(|\vec{k}|) = 0$, which is satisfied by Eq.(4.44), we obtain

$$\begin{aligned} \omega_+(k\sim 0) &= \omega_c + \frac{l+m-2n}{lm-n^2}\frac{1}{2\hbar}\frac{e^2}{4\pi\epsilon}|\vec{k}| + \cdots, \\ \omega_-(k\sim 0) &= \frac{(l-n)(m-n)}{lm-n^2}\omega_c + \left(1+\frac{2}{l+m-2n}\frac{\frac{e^2}{4\pi\epsilon\ell}}{\hbar\omega_c}\frac{d}{\ell}\right)\omega_k + \cdots, \end{aligned} \tag{4.47}$$

where $\omega_k = \frac{\hbar^2\vec{k}^2}{2M}$. Except for the case $l = n$ or $m = n$, $\omega_\pm(0) > 0$. This means the rigidity of the ground state in the DLFQH system against the long-wavelength density fluctuations, that is, the incompressibility. The resulting perturbed ground state $|G\rangle$ is defined by $\frac{1}{\sqrt{2\hbar}}\left\{\sqrt{M\omega_\pm(k)}q^\pm_{\vec{k}} + \frac{i}{\sqrt{M\omega_\pm(k)}}p^\pm_{\vec{k}}\right\}|G\rangle = 0$, with the energy $\frac{N}{2}\hbar\omega_c$. This state gives $\langle G|\delta\rho^a|G\rangle = \langle G|\delta\vartheta^a|G\rangle = 0$. This means that within the above perturbative consideration the unperturbed uniform ground state (the Halperin state), $\rho^a(z,\overline{z}) = \rho^a_{\rm av}$, is still stable under such density fluctuations. That is, in our theory, any phase transition will not occur by the Coulomb repulsion.

In the case $l = m = n$, the same calculation as above leads to the vanishing of $\omega_-(0)$. This corresponds to the fact that the pseudospin $SU(2)_d$ symmetry is spontaneously broken. Corresponding Goldstone mode in this case is $\delta\vartheta^\uparrow - \delta\vartheta^\downarrow$. The $-$ mode has a linear dispersion;

$$\omega_-(k \sim 0) = \sqrt{\frac{\pi d}{M}\frac{e^2}{4\pi\epsilon}\frac{1}{2\pi\ell^2 m}}|\vec{k}| + \cdots. \tag{4.48}$$

Actually the above behavior might not be observed. The reason for this is as follows. The linear dispersion in the long wavelength limit has its origin in the difference between the inter- and intra-layer Coulomb repulsions. The tree-level Coulomb repulsion is often shielded in realistic systems due to the density fluctuation, and then the dispersion relation may be modified. For example, let us consider the quantum correction to the exchange of scalar photons in the RPA-type approximation (summing bubble diagrams) and suppose the contribution of a bubble diagram to be independent of pseudospin, which we denote as Π in the momentum space. $\Pi \to$ finite constant as $\vec{k} \to 0$. Then the screened Coulomb repulsion is given by

$$\tilde{U}^{ab}(|\vec{k}|) = \frac{e^2}{4\pi\epsilon}\frac{2\pi}{|\vec{k}| + \Pi}e^{-(1-\delta^{ab})|\vec{k}|d}. \tag{4.49}$$

In this case the dispersion relation of the $-$ mode in the infrared limit deviates from the linear behavior, and is given as follows;

$$\tilde{\omega}_-(k \sim 0) = \sqrt{\frac{\pi d}{M}\frac{e^2}{4\pi\epsilon}\frac{1}{\Pi}\frac{1}{2\pi\ell^2 m}}|\vec{k}|^{\frac{3}{2}} + \cdots. \tag{4.50}$$

By the existence of the Nambu-Goldstone mode, it is expected that at low energy this mode will govern the dynamics of the DLFQH system, and will generate some interesting phenomena like superfluids, especially when the tunneling between layers is present. This point will be discussed in Section 6.

5 Collective-Field Approach to DLFQH Systems with Interlayer Tunneling

In Section 4, we studied DLFQH systems without interlayer tunneling and determined the effective Hamiltonian in terms of the collective fields. Now we have reached the stage for considering effect of tunneling, and construct an effective theory of the DLFQH system.

5.1 Interaction Hamiltonian

As discussed in Introduction, there are two types of interactions in the DL electron systems. One is the inter-electron interaction and the other is the interlayer hopping due to the tunneling effect. The inter-electron Coulomb repulsion (4.34) respects the $SU(2)_d$-pseudospin symmetry and is expressed only in terms of $\hat{\rho}^a$ as discussed in Section 4. As the number of electrons N^a in each layer a is conserved, the formalism developed in the preceding sections works without any modification.

However, when the interlayer tunneling is present, the situation is drastically changed, as discussed briefly in Section 3. In this case N^a cannot be conserved any longer.[8] In the following we will develop the perturbative analysis around the Halperin-type ground state Φ_{lmn}, regarding the interlayer tunneling effect as a perturbation. Therefore we should take $\overline{\rho}^a$ to be the magic density of Φ_{lmn}; $\overline{\rho}^a = \frac{\nu^a_0}{2\pi\ell^2}$ in the case with $lm \neq n^2$, and then if the resulting (perturbative) ground state in the

[8] Of course, the fine tuning of the total number N of electrons is still required; $N = \nu_0\frac{A}{2\pi\ell^2}$ with $lm \neq n^2$ and $N = \frac{1}{m}\frac{A}{2\pi\ell^2}$ with $l = m = n$.

presence of tunneling has also the magic density, the analysis is consistent. Actually only the Halperin state with $l = m$ ($\nu^{\uparrow} = \nu^{\downarrow} = \frac{1}{m+n}$ in this case) satisfies the consistency condition and the state with $l \neq m$ becomes unstable due to the tunneling effect. In the case $l = m = n$, we will also take the above self-consistency requirement and expect that the following choice is correct; $\overline{\rho}^a = \frac{1}{2}\frac{N}{A} = \frac{1}{2m}\frac{1}{2\pi\ell^2}$ for each layer.

We will describe the interlayer tunneling effect by the interlayer hopping Hamiltonian (3.3). Then the new collective fields are required to express it; $\hat{\rho}^{ab}$, $\hat{j}^a_z$ and $\hat{j}^a_{\overline{z}}$ which are given in Section 3. Then the explicit form of the hopping Hamiltonian is

$$H_{hop}[\hat{\rho}^{ab}] = -\frac{\triangle_{SAS}}{2}\int d^2z\,\left\{\hat{\rho}^{\uparrow\downarrow}(z,\overline{z}) + \hat{\rho}^{\downarrow\uparrow}(z,\overline{z})\right\}, \tag{5.1}$$

and the total Hamiltonian is given by Eqs.(4.2), (4.34) and (5.1);

$$H_{total}[\hat{j}^a_z, \hat{j}^a_{\overline{z}}; \hat{\rho}^{ab}] = \mathbf{H}_0[\hat{j}^a_z, \hat{j}^a_{\overline{z}}; \hat{\rho}^a] + V_{Coulomb}[\hat{\rho}^a] + H_{hop}[\hat{\rho}^{ab}]. \tag{5.2}$$

ETCR's among $\hat{\rho}^{ab}$ and $\hat{j}^a_{z,\overline{z}}$ are given in Appendix A.

In the next step, we must express the total Hamiltonian (5.2) in terms of the collective fields ρ^a. To this end, we must first reconstruct the space of states which contains all the (N+1) spaces of electron numbers $= (N^{\uparrow}, N^{\downarrow})$ with $N = N^{\uparrow} + N^{\downarrow}$ fixed. It will be discussed in the following subsection.

5.2 Construction of the Space of States

When there is no tunneling effect in the DL system, we can treat each layer separately and have only to take their (tensor-)product, as discussed in Subsection 4.2. But in the presence of tunneling, we must consider the whole state with the only total number of electrons fixed. In this case pseudospin s^a_i becomes dynamical, and the quantum-mechanical wave function has the following form; $\Phi(\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N)$, which is totally symmetric under an exchange of coordinates $(\vec{r}_i, s^a_i)$. We want to find the corresponding wave functional of the collective variables. The collective fields (3.5) satisfy nontrivial commutation relations (see Appendix A) and then the state is still

expressible only by the (eigenvalue of) density variables $\hat{\rho}^a$, $\Phi[\rho^{\uparrow}, \rho^{\downarrow}]$. In order to find this ρ-realization of the state, we write down the state in the second-quantized formalism as in the discussion in Subsection 4.2;

$$|\Phi\rangle = \sum_{\{s^a_i\}}\int(\prod_{i=1}^{N} d^2z_i)\,\Phi(\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N)\cdot\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle, \tag{5.3}$$

where $\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle$ is a totally symmetric basis, defined by

$$\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle = \frac{1}{\sqrt{N!}}\left[\prod_{i=1}^{N}\phi^{a_i\dagger}(z_i,\overline{z}_i)\right]|0\rangle. \tag{5.4}$$

Then noticing that this basis is the eigenstate of $\hat{\rho}^a$;

$$\hat{\rho}^a(z,\overline{z})\cdot\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle = \sum_{i=1}^{N}\delta^{aa_i}\cdot\delta^{(2)}(z-z_i)\cdot\left|\vec{r}_1 s^a_1; \cdots, \vec{r}_N s^a_N\right\rangle, \tag{5.5}$$

we should set the relation between the quantum-mechanical wave function $\Phi(\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N)$ and the wave functional $\Phi[\rho^{\uparrow}, \rho^{\downarrow}]$ as

$$\Phi(\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N) \Longleftrightarrow \Phi[\rho^{\uparrow}, \rho^{\downarrow}]\Big|_{\rho^a(z,\overline{z})=\sum_{i=1}^{N}\delta^{aa_i}\delta^{(2)}(z-z_i)}. \tag{5.6}$$

Using this relation, the state $|\Phi\rangle$ (5.3) is translated into the wave functional $\Phi[\rho^{\uparrow}, \rho^{\downarrow}]$ as follows;

$$\begin{aligned}|\Phi\rangle &= \sum_{\{s^a_i\}}\int(\prod_{i=1}^{N} d^2z_i)\,\Phi[\rho^{\uparrow}, \rho^{\downarrow}]\cdot\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle \\ &= \Phi[\hat{\rho}^{\uparrow}, \hat{\rho}^{\downarrow}]\cdot\|N),\end{aligned} \tag{5.7}$$

where

$$\|N) = \sum_{N^{\uparrow}=0}^{N}({}_NC_{N^{\uparrow}})^{\frac{1}{2}}\cdot\|N^{\uparrow}; N-N^{\uparrow}\rangle. \tag{5.8}$$

From the first to the second line in Eq.(5.7), we have used the relation

$$\sum_{\{s^a_i\}}\int(\prod_i d^2z_i)\left|\vec{r}_1 s^a_1; \cdots; \vec{r}_N s^a_N\right\rangle = \frac{1}{\sqrt{N!}}\left(\sum_a\int d^2z\,\phi^{a\dagger}(z,\overline{z})\right)^N|0\rangle,$$

and performed the binominal expansion to rewrite it by the symmetric basis (4.9).

Corresponding to the above construction, the functional space of $\Phi[\rho^\uparrow, \rho^\downarrow]$ is also modified, especially, Jacobian factor in the inner product,

$$\begin{aligned}\langle\Psi|\Phi\rangle &= \sum_{\{s_i^a\}} \int (\prod_{i=1}^{N} d^2 z_i)\, \overline{\Psi(\vec{r}_1 s_1^a; \cdots; \vec{r}_N s_N^a)} \cdot \Phi(\vec{r}_1 s_1^a; \cdots; \vec{r}_N s_N^a) \\ &= \int \mathcal{D}\rho^\uparrow \mathcal{D}\rho^\downarrow\, J_N[\rho^\uparrow, \rho^\downarrow] \cdot \overline{\Psi[\rho^\uparrow, \rho^\downarrow]} \cdot \Phi[\rho^\uparrow, \rho^\downarrow]. \end{aligned} \tag{5.9}$$

From Eq.(5.8), the Jacobian factor in this case is given as

$$J_N[\rho^\uparrow, \rho^\downarrow] = \sum_{N^\uparrow=0}^{N} {}_N C_{N^\uparrow} \cdot J^{(0)}_{N^\uparrow, N-N^\uparrow}[\rho^\uparrow, \rho^\downarrow]. \tag{5.10}$$

This means just the composition of the entropy of each layer with the Gibbs correction, which is a natural result. The explicit calculation leads to the following expression in the leading order in large-N;

$$\begin{aligned} J_N[\rho^\uparrow, \rho^\downarrow] &\simeq \sum_{N^\uparrow=0}^{N} {}_N C_{N^\uparrow} \cdot \left[\prod_a \delta\left(\int d^2 z\, \rho^a(z,\overline{z}) - N^a\right)\right] \cdot e^{-\sum_a \int d^2 z\, \rho^a(z,\overline{z}) \cdot \log \frac{\rho^a(z,\overline{z})}{\rho^a_{av}}} \\ &\simeq \delta\left(\int d^2 z \left\{\rho^\uparrow(z,\overline{z}) + \rho^\downarrow(z,\overline{z})\right\} - N\right) \cdot e^{-\sum_a \int d^2 z\, \rho^a(z,\overline{z}) \cdot \log \frac{\rho^a(z,\overline{z})}{N/A}} \\ &\quad \times \int_{-\infty}^{+\infty} \frac{d\kappa}{2\pi} e^{i\frac{\kappa}{2}\int d^2 z \left(\rho^\uparrow(z,\overline{z}) - \rho^\downarrow(z,\overline{z})\right)} \cdot \frac{\sin(N+1)\frac{\kappa}{2}}{\sin\frac{\kappa}{2}}. \end{aligned} \tag{5.11}$$

We have used the Stirling formula. Physical meaning of the κ-integral can be easily seen, i.e., from the identity

$$\frac{\sin(N+1)\frac{\kappa}{2}}{\sin\frac{\kappa}{2}} = \sum_{r=0}^{N-1} \cos\left((N-2r)\frac{\kappa}{2}\right),$$

the κ-integral gives the constraint

$$\begin{aligned} \int d^2 z(\rho^\uparrow - \rho^\downarrow) &= N^\uparrow - N^\downarrow \\ N^\uparrow + N^\downarrow &= N, \end{aligned} \tag{5.12}$$

where $N^\uparrow$ and $N^\downarrow$ are *arbitrary* positive integers.

We will set the inner product of the functional space of $\Phi[\rho^\uparrow, \rho^\downarrow]$ as

$$\langle\Psi|\Phi\rangle = \int \langle\mathcal{D}\rho\rangle\, e^{\mathcal{S}[\rho^\uparrow, \rho^\downarrow]} \cdot \overline{\Psi[\rho^\uparrow, \rho^\downarrow]} \cdot \Phi[\rho^\uparrow, \rho^\downarrow], \tag{5.13}$$

where the measure is defined by

$$\langle\mathcal{D}\rho\rangle = \prod_a \prod_z d\rho^a(z,\overline{z}) \cdot \delta\left(\int d^2 z \left\{\rho^\uparrow(z,\overline{z}) + \rho^\downarrow(z,\overline{z})\right\} - N\right), \tag{5.14}$$

and the entropy factor is

$$\mathcal{S}[\rho^\uparrow, \rho^\downarrow] = -\int d^2 z \left\{\sum_a \rho^a(z,\overline{z}) \log \rho^a(z,\overline{z}) - \frac{N}{A}\log\frac{N}{A}\right\}. \tag{5.15}$$

The constraint (5.12) will be also taken into account when it is necessary.

5.3 ρ-Realization of the Hopping Hamiltonian

In this subsection, we will seek for the ρ-realization of $H_{hop}[\hat{\rho}^{ab}]$ (5.1). To this end, we have to consider the ρ-realization of $\int d^2 z\, \hat{\rho}^{ab}(z,\overline{z})$ with $(ab) = (\uparrow\downarrow)\,\text{or}\,(\downarrow\uparrow)$. It is expected that they are determined by the commutation relations (CR's) among the collective fields, $\hat{\rho}^{ab}$'s and $\hat{j}^a_{z,\overline{z}}$'s. (See Appendix A.) The CR's among $\hat{\rho}^a$'s and $\hat{j}^a_{z,\overline{z}}$'s, Eq.(4.1) and,

$$\left[\hat{\rho}^{ab}(z,\overline{z}),\, \hat{\rho}^{cd}(w,\overline{w})\right] = \left(\delta^{bc}\hat{\rho}^{ad}(z,\overline{z}) - \delta^{ad}\hat{\rho}^{cb}(z,\overline{z})\right) \delta^{(2)}(z-w), \tag{5.16}$$

form a closed, well-defined algebra. Using this fact, we will first find the ρ-realization of $\hat{j}^a_{z,\overline{z}}$ and $\hat{\rho}^{\uparrow\downarrow}$, $\hat{\rho}^{\downarrow\uparrow}$ only from Eqs.(4.1) and (5.16). In fact, the former has already been given in Eq.(4.18). The CR's between $\hat{\rho}^{ab}$'s and $\hat{j}^a_{z,\overline{z}}$ will be used to check the consistency of the resulting ρ-realizations.

Let us decomposing the operation of $\int d^2 z\, \hat{\rho}^{ab}(z,\overline{z})$ on the state $|\Phi\rangle$ (5.7) into two parts,

$$\begin{aligned} &\int d^2 z\, \hat{\rho}^{ab}(z,\overline{z})\, |\Phi\rangle \\ &= \int d^2 z \left[\hat{\rho}^{ab}(z,\overline{z}), \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow]\right] \|N) + \Phi[\hat{\rho}^\uparrow, \hat{\rho}^\downarrow] \cdot \int d^2 z\, \hat{\rho}^{ab}(z,\overline{z})\, \|N). \end{aligned} \tag{5.17}$$

Using the relations

$$\int d^2 z\, \hat{\rho}^{ab}(z,\overline{z})\, \|N) = \int d^2 z\, \hat{\rho}^a(z,\overline{z})\, \|N), \tag{5.18}$$

the second term on the right hand side of (5.17) is rewritten as

$$\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]\cdot\int d^2z\,\hat{\rho}^{ab}(z,\overline{z})\,\|N\rangle=\int d^2z\,\hat{\rho}^{a}(z,\overline{z})\,|\Phi\rangle. \tag{5.19}$$

The former commutator will be calculated using the commutation relations (5.16) with $(ab)=(\uparrow\downarrow)\,\mathrm{or}\,(\downarrow\uparrow)$ and $c=d$. The result is

$$\begin{aligned}\left[\hat{\rho}^{\uparrow\downarrow}(z,\overline{z}),\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]\right] &= F[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]\cdot e^{-\frac{\delta}{\delta\hat{\rho}^{\uparrow}(z,\overline{z})}+\frac{\delta}{\delta\hat{\rho}^{\downarrow}(z,\overline{z})}}\,\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}],\\ \left[\hat{\rho}^{\downarrow\uparrow}(z,\overline{z}),\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]\right] &= G[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]\cdot e^{+\frac{\delta}{\delta\hat{\rho}^{\uparrow}(z,\overline{z})}-\frac{\delta}{\delta\hat{\rho}^{\downarrow}(z,\overline{z})}}\,\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}].\end{aligned} \tag{5.20}$$

The functionals $F[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]$, $G[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]$ are undetermined at this stage. They are determined as follows by the full CR's and dimensional analysis, which still containes one free real parameter α;

$$\begin{aligned}F[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}] &= \alpha\hat{\rho}^{\uparrow}(z,\overline{z}),\\ G[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}] &= \frac{1}{\alpha}\hat{\rho}^{\downarrow}(z,\overline{z}).\end{aligned} \tag{5.21}$$

The value of α can be determined by another requirement that $\hat{\rho}^{\uparrow\downarrow}$ and $\hat{\rho}^{\downarrow\uparrow}$ should be mutually hermitian conjugate. This requirement is rather subtle due to the nontrivial measure of the space of states $\Phi[\rho^{\uparrow},\rho^{\downarrow}]$, and after some calculation it is found that $\alpha=\pm1$. (Some detailed calculation is given in Appendix B.) Finally, we will take the sign of α to be $+1$, as explicit operation supports. (See the following discussion)

From the above relations, we obtain the operator relation of $\int d^2z\,\hat{\rho}^{ab}(z,\overline{z})$ acting on $\|N\rangle$ as follows;

$$\int d^2z\,\hat{\rho}^{ab}(z,\overline{z})\cdot\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]=\int d^2z\,\hat{\rho}^{a}(z,\overline{z})\left\{1+e^{-\frac{\delta}{\delta\hat{\rho}^{a}(z,\overline{z})}+\frac{\delta}{\delta\hat{\rho}^{b}(z,\overline{z})}}\right\}\cdot\Phi[\hat{\rho}^{\uparrow},\hat{\rho}^{\downarrow}]. \tag{5.22}$$

This leads to the ρ-realization of $\int d^2z\,\hat{\rho}^{ab}(z,\overline{z})$'s which acts on the wave functional $\Phi[\rho^{\uparrow},\rho^{\downarrow}]$;

$$\begin{aligned}\int d^2z\,\hat{\rho}^{ab}(z,\overline{z}) &\rightarrow \int d^2z\,\rho^{a}(z,\overline{z})\cdot\left\{1+e^{-\frac{\delta}{\delta\rho^{a}(z,\overline{z})}+\frac{\delta}{\delta\rho^{b}(z,\overline{z})}}\right\}\\ &= \int d^2z\,\rho^{a}(z,\overline{z})\cdot\left\{1+e^{-i\left(\vartheta^{a}(z,\overline{z})-\vartheta^{b}(z,\overline{z})\right)}\right\}.\end{aligned} \tag{5.23}$$

Thus we obtain the ρ-realization $\tilde{H}_{hop}[\vartheta^{a},\rho^{a}]$ of the interlayer-hopping Hamiltonian (5.1);

$$\tilde{H}_{hop}[\vartheta^{a},\rho^{a}]=-\sum_{a}\frac{\triangle_{SAS}}{2}\int d^2z\,\rho^{a}(z,\overline{z})\cdot\left\{1+e^{-is^{a}\left(\vartheta^{\uparrow}(z,\overline{z})-\vartheta^{\downarrow}(z,\overline{z})\right)}\right\}. \tag{5.24}$$

Before finishing this subsection, we will examine the properties of $\tilde{H}_{hop}$ (5.24) by operating it to some explicit wave functionals $\Phi[\rho^{\uparrow},\rho^{\downarrow}]$ in order to verify our proposition for $\tilde{H}_{hop}$. Let us consider its operation on the Halperin wave functional $\Phi_{lmn}[\rho^{\uparrow},\rho^{\downarrow}]$ (4.22). Explicit calculation shows

$$\begin{aligned}&\tilde{H}_{hop}\cdot\Phi_{lmn}[\rho^{\uparrow},\rho^{\downarrow}]\\ &\sim -\frac{\triangle_{SAS}}{2}\int d^2z\,\rho^{\uparrow}(z,\overline{z})e^{-(l-n)\int d^2w\,\log|z-w|\cdot\rho^{\uparrow}(w,\overline{w})+(m-n)\int d^2w\,\log|z-w|\cdot\rho^{\downarrow}(w,\overline{w})}\,\Phi_{lmn}[\rho^{\uparrow},\rho^{\downarrow}]\\ &\quad -\frac{\triangle_{SAS}}{2}\int d^2z\,\rho^{\downarrow}(z,\overline{z})e^{+(l-n)\int d^2w\,\log|z-w|\cdot\rho^{\uparrow}(w,\overline{w})-(m-n)\int d^2w\,\log|z-w|\cdot\rho^{\downarrow}(w,\overline{w})}\,\Phi_{lmn}[\rho^{\uparrow},\rho^{\downarrow}].\end{aligned} \tag{5.25}$$

This result is easily translated into the microscopic description through the replacement (3.2) and we obtain the following expression;

$$\begin{aligned}&\tilde{H}_{hop}\cdot\Phi_{lmn}(\vec{r}_1^{\uparrow},\cdots,\vec{r}_{N^{\uparrow}}^{\uparrow};\vec{r}_1^{\downarrow},\cdots,\vec{r}_{N^{\downarrow}}^{\downarrow})\\ &\sim -\frac{\triangle_{SAS}}{2}\left\{\sum_{u=1}^{N^{\uparrow}}\frac{\prod_{d=1}^{N^{\downarrow}}|z_u^{\uparrow}-z_d^{\downarrow}|^{m-n}}{\prod_{v=1}^{N^{\uparrow}}|z_u^{\uparrow}-z_v^{\uparrow}|^{l-n}}+\sum_{d=1}^{N^{\downarrow}}\frac{\prod_{u=1}^{N^{\uparrow}}|z_d^{\downarrow}-z_u^{\downarrow}|^{l-n}}{\prod_{f=1}^{N^{\downarrow}}|z_d^{\downarrow}-z_f^{\downarrow}|^{m-n}}\right\}\cdot\Phi_{lmn}(z_1^{\uparrow},\cdots,z_{N^{\uparrow}}^{\uparrow};z_1^{\downarrow},\cdots,z_{N^{\downarrow}}^{\downarrow}).\end{aligned} \tag{5.26}$$

Therefore the operation of $\tilde{H}_{hop}$ to the Halperin state Φ_{lmn} is to exchange the layer index (i.e., attached statistical flux and magnitude of associated zeros) of electrons, which means the hopping of electrons. Then $\tilde{H}_{hop}$ gives the desirable result for the Halperin state. Similar calculation for the excited states above the Halperin state also supports Eq.(5.24).

As shown above, Eq.(5.24) has desirable properties describing the interlayer tunneling of electrons in the DLFQH system. In the following section, we construct an effective theory of the DLFQHE based on the above arguments.

5.4 Effective Hamiltonian of DLFQH Systems

In this subsection, we will summarize the arguments given so far, and write down the effective Hamiltonian of DLFQH systems with the interlayer tunneling in the collective-field language.

The total Hamiltonian is given in terms of the collective coordinate ρ^a as follows;

$$\mathbf{H}_{total}[\vartheta^a, \rho^a] = \mathbf{H}_0[\vartheta^a, \rho^a] + V_{Coulomb}[\rho^a] + \tilde{H}_{hop}[\vartheta^a, \rho^a]. \tag{5.27}$$

The explicit forms of $\mathbf{H}_0$, $V_{Coulomb}$ and $\tilde{H}_{hop}$ are given by Eqs.(4.2), (4.20), (4.34) and (5.24), respectively. The Schrödinger equation of the wave functional $\Phi[\rho^\uparrow, \rho^\downarrow]$ is

$$\mathbf{H}_{total}[\vartheta^a, \rho^a] \cdot \Phi[\rho^\uparrow, \rho^\downarrow] = E \cdot \Phi[\rho^\uparrow, \rho^\downarrow]. \tag{5.28}$$

The effective Hamiltonian in the same sense as in Subsection 4.4 (Approach 2) takes the following form;

$$\begin{aligned} H^{eff}_{total}[\vartheta^a, \rho^a; \lambda] &= e^{\frac{1}{2}\mathcal{S}[\rho^\uparrow,\rho^\downarrow]}\, \mathbf{H}_{total}[\vartheta^a, \rho^a]\, e^{-\frac{1}{2}\mathcal{S}[\rho^\uparrow,\rho^\downarrow]} + \lambda\Big(\sum_a \int d^2z \rho^a(z,\overline{z}) - N\Big) \\ &= \mathbf{H}_{total}\left[\vartheta^a + \frac{1}{2i}(\log\rho^a + 1), \rho^a\right] + \lambda\Big(\sum_a \int d^2z \rho^a(z,\overline{z}) - N\Big), \end{aligned}$$

where λ is a Lagrange multiplier giving the total-number constraint appearing in the measure (5.14). Using this effective Hamiltonian, we can perform a (semi-)classical analysis of the DLFQH systems. Perturbatively stable ground state is the Halperin state with $l = m$, Φ_{mmn}, and $\triangle_{SAS}$ term changes the dispersion relation of $-$ mode.

These are results for the case $lm \neq n^2$. It is expected, however, that more interesting phenomena will appear in the case $l = m = n$. Then we will focus on that case in the following section.

6 Josephson Effect in DLFQH Systems

In this section, we will consider the case $l = m = n$. As explained in Sections 2 and 4, as the interlayer tunneling effect is switched off, there exists only one C-S gauge

field $C^\uparrow_\mu + C^\downarrow_\mu$ associated with the total number of electrons, and the Nambu-Goldstone mode associated with the spontaneously broken $SU(2)_d$-pseudospin symmetry, $\vartheta^\uparrow - \vartheta^\downarrow$, appears. Any Φ_{mmm} with arbitrary $(N^\uparrow, N^\downarrow)$ is energetically-degenerate ground state, as long as $N^\uparrow + N^\downarrow = \frac{1}{m}\frac{A}{2\pi\ell^2}$. The actual state will be a linear combination of all $(N^\uparrow, N^\downarrow)$ states with an equal weight. As the interlayer tunneling effect is switched on, the density $< \rho^a >$ and the phase $< \vartheta^a >$ of each layer tend to be equal. That is, the coherent state will be generated. Since this Nambu-Goldstone mode is a phase of electrons and can of course couple to the external electromagnetic field, then this state may support a Josephson(-type) effect just as in the case of the superconducting state[30]. In fact, as the interlayer tunneling effect breaks the $SU(2)_d$-pseudospin symmetry explicitly, the Nambu-Goldstone mode acquires a gap of order $\triangle_{SAS}$. But in our perturbative treatment, $\triangle_{SAS}$ is considered to be small (compared to other typical energy scales, such as the Coulomb energy and the cyclotron energy). Then due to the low energy theorem, this mode may remember some properties of the Nambu-Goldstone boson and dominate the low energy dynamics of the system. Therefore we may still expect a Josephson(-type) effect by this mode[12, 11, 13]. We will examine this phenomenon in detail in the collective-field formalism constructed in the previous sections.

Now we introduce the following basis for later discussion;

$$\rho^\pm(z) = \rho^\uparrow(z) \pm \rho^\downarrow(z), \quad \vartheta^\pm(z) = \frac{1}{2}\left(\vartheta^\uparrow(z) \pm \vartheta^\downarrow(z)\right). \tag{6.1}$$

In the above expressions and hereafter, we will often abbreviate the arguments of variables. The ρ-realized Hamiltonian acting on the wave functional $\Phi[\rho^\uparrow, \rho^\downarrow]$ in this case is given by

$$\tilde{H}_{total}[\vartheta^a, \rho^a] = \mathbf{H}_0[\vartheta^a, \rho^a] + V_{Coulomb}[\rho^a] + \tilde{H}_{hop}[\vartheta^a, \rho^a] + \mu \int d^2z\, \rho^-(z), \tag{6.2}$$

and the effective Hamiltonian is

$$H^{eff}_{total}[\vartheta^a, \rho^a; \lambda] = \frac{2\hbar^2}{M}\sum_a \int d^2z\, \rho^a(z)\overline{L}^a(z)L^a(z)$$

rewritten (in the mean-field type approximation) as follows;

$$\dot{\rho}^a(z;t) = \frac{i}{\hbar}\left[\mathbf{H}_{total}[\vartheta^b,\,\rho^b],\rho^a(z;t)\right]\Big|_{\vartheta^a\to\vartheta^a-\frac{i}{2}(\log\rho^a+1)}, \tag{6.25}$$
$$\dot{\vartheta}^a(z;t) - \frac{i}{2}\frac{\dot{\rho}^a(z;t)}{\rho^a(z;t)} = \frac{i}{\hbar}\left[\mathbf{H}_{total}[\vartheta^b,\,\rho^b],\vartheta^a(z;t)\right]\Big|_{\vartheta^a\to\vartheta^a-\frac{i}{2}(\log\rho^a+1)}.$$

Actually these equations are found to be in accordance with the following canonical equations of H^{eff}_{total};

$$\dot{\rho}^a(z;t) = \frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\hbar\delta\vartheta^a(z;t)}\Big|_{\lambda=0}, \quad \hbar\dot{\vartheta}^a(z;t) = -\frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\delta\rho^a(z;t)}\Big|_{\lambda=0}. \tag{6.26}$$

The right hand side of the above equations have been already calculated in Eqs.(6.6) and (6.7). Since the generic treatment is hard to be done, we will focus on spatially uniform configurations, $\rho^a(z;t) = \rho^a_0(t)$ and $\vartheta^a(z;t) = \vartheta^a_0(t)$, for simplicity. The Halperin state Φ_{mmn} holds the constant distribution of ρ^a's against the interlayer electron hopping due to the existence of the Nambu-Goldstone mode. Then this assumption will be reasonable in our perturbative treatment. The uniformity of ϑ^a's is consistent, because the total density ρ^+_0 is fixed at the magic value $\frac{1}{m}\frac{1}{2\pi\ell}$ in that case;

$$\dot{\rho}^+_0(t) = -\frac{2\hbar}{M}\left\{\rho^+_0(t)(\partial\overline{\partial}+\overline{\partial}\partial)\vartheta^+(z;t) + \rho^-_0(t)(\partial\overline{\partial}+\overline{\partial}\partial)\vartheta^-(z;t)\right\} \equiv 0. \tag{6.27}$$

Note that these configurations satisfy the LLL constraints. Therefore the resulting equations are given as

$$\begin{aligned}
\dot{\rho}^-_0(t) &= \frac{2\triangle_{SAS}}{\hbar}\sqrt{\rho^\uparrow_0(t)\rho^\downarrow_0(t)}\,\sin 2\vartheta^-_0(t),\\
\dot{\vartheta}^+_0(t) &= \frac{\triangle_{SAS}-\hbar\omega_c}{2\hbar} + \frac{\triangle_{SAS}}{4\hbar}\frac{\rho^+_0}{\sqrt{\rho^\uparrow_0(t)\rho^\downarrow_0(t)}}\cos 2\vartheta^-_0(t),\\
\dot{\vartheta}^-_0(t) &= -\frac{\mu}{\hbar} - \frac{1}{2\hbar}\rho^-_0(t)2\pi\ell^2U - \frac{\triangle_{SAS}}{4\hbar}\frac{\rho^-_0(t)}{\sqrt{\rho^\uparrow_0(t)\rho^\downarrow_0(t)}}\cos 2\vartheta^-_0(t).
\end{aligned} \tag{6.28}$$

Since the tunneling current between layers is defined by

$$J(z;t) = -e\,\dot{\rho}^\uparrow(z;t), \tag{6.29}$$

then the above equations[31, 11] show that under the condition of $\rho^\uparrow_0 \approx \rho^\downarrow_0 \approx \frac{1}{2m}\frac{1}{2\pi\ell^2} \equiv \rho_0$, the dc current $J = -J_c\sin 2\vartheta^-_0$ will be generated in the case without the external potential $\mu = 0$ (dc Josephson effect), and the alternating current with a very high frequency $\frac{eV_0}{\hbar}$, $J(t) = -J_c\sin\left(2\vartheta^-_0 + \frac{eV_0}{\hbar}t\right)$, will appear in the presence of the constant external potential $2\mu = -eV_0$ (ac Josephson effect). Herein we define the critical current as $J_c = \frac{e\triangle_{SAS}}{\hbar}\rho_0$.

As long as we start from the Hamiltonian $\mathcal{H}_{hop}$ (3.3) for treating the interlayer tunneling effect, we inevitably reach the above Josephson effect. In deriving this result, there has been little ambiguity and the coherence of the Hall fluid plays a crucial role. It is this characteristics of the state that makes our semi-classical treatment meaningful. From the microscopic point of view, the coherence of the FQH state will enlarge the value of $\triangle_{SAS}$ which is proportional to the tunneling amplitude between layers, and will support the observability of the effect.

7 Summary and Discussions

In this paper, we studied the system of bosonized electrons with pseudospin coupled to the C-S gauge fields as a model of the DLFQH system, by using the collective-field formalism. In this formalism, the correspondence to the microscopic theory is quite clear.

When the interlayer tunneling is absent, the usual collective-field approach is applicable in a straightforward manner. We calculated the dispersion relation of the (long-wavelength) density fluctuations and showed that the Hall state is incompressible in the case $lm \neq n^2$. When $l = m = n$, the Nambu-Goldstone mode explicitly appears and it is found that these fluctuations do not generate any phase transition against the Halperin state. Our results are consistent with those obtained by Ezawa and Iwazaki[11] who studied the same model with ours by the strightforward semi-classical approximation.

For simplicity, let us consider the case with $\rho_0^\uparrow = \rho_0^\downarrow$, i.e., $\mu = 0$, then the normal modes $(q_{\vec{k}}^\pm, p_{\vec{k}}^\pm)$ are determined by

$$q_{\vec{k}}^+ = \frac{1}{\sqrt{2}}(q_{\vec{k}}^\uparrow + q_{\vec{k}}^\downarrow), \quad p_{\vec{k}}^+ = \frac{1}{\sqrt{2}}(p_{\vec{k}}^\uparrow + p_{\vec{k}}^\downarrow), \tag{6.19}$$
$$q_{\vec{k}}^- = \frac{1}{\sqrt{2\left(1+\frac{2M\triangle_{SAS}}{\hbar^2\vec{k}^2}\right)}}(q_{\vec{k}}^\uparrow - q_{\vec{k}}^\downarrow), \quad p_{\vec{k}}^- = \sqrt{\frac{1}{2}\left(1+\frac{2M\triangle_{SAS}}{\hbar^2\vec{k}^2}\right)}(p_{\vec{k}}^\uparrow - p_{\vec{k}}^\downarrow),$$

and the corresponding frequencies are

$$\omega_+(k) = \left[(\omega_k+\omega_c)^2 + \frac{1}{2m\hbar}\omega_c\frac{e^2}{4\pi\epsilon}|\vec{k}|(1+e^{-|\vec{k}|d})\right]^{\frac{1}{2}}, \tag{6.20}$$
$$\omega_-(k) = \left[\left(\omega_k+\frac{\triangle_{SAS}}{\hbar}\right)^2 + \frac{1}{2m\hbar}\omega_c\frac{e^2}{4\pi\epsilon}\left(\vec{k}^2+\frac{2\triangle_{SAS}}{\hbar\omega_c}\frac{1}{\ell^2}\right)\frac{1-e^{-|\vec{k}|d}}{|\vec{k}|}\right]^{\frac{1}{2}}.$$

In the small wavelength limit, we obtain the following result;

$$\omega_+(k\sim 0) = \omega_c + \frac{1}{2m\hbar}\frac{e^2}{4\pi\epsilon}|\vec{k}| + \cdots, \tag{6.21}$$
$$\omega_-(k\sim 0) = \frac{\triangle_{SAS}}{\hbar}\sqrt{1+\frac{1}{m}\frac{\frac{e^2}{4\pi\epsilon\ell}}{\triangle_{SAS}}\frac{d}{\ell}} - \frac{\frac{1}{4m\hbar}\frac{e^2}{4\pi\epsilon}}{\sqrt{1+\frac{1}{m}\frac{\frac{e^2}{4\pi\epsilon\ell}}{\triangle_{SAS}}\frac{d}{\ell}}}\frac{d^2}{\ell^2}|\vec{k}| + \cdots.$$

This result shows a peculiar behavior of the $-$ mode, that is, there appears a minimum of ω_- at a finite wave vector k_{min}, whose value is estimated as

$$k_{min} \sim \frac{1}{d}\left(4m\frac{\hbar\omega_c}{\frac{e^2}{4\pi\epsilon d}} + 2\frac{\hbar\omega_c}{\triangle_{SAS}}\frac{d^2}{\ell^2} + \frac{2}{3}\right)^{-1}. \tag{6.22}$$

As long as $\triangle_{SAS} \neq 0$, the gap $\omega_-(k_{min})$ is no-vanishing. This behavior is considered to be a sign of the collapse of the Hall state consisting of pseudospin-symmetric state[9]. But more carefull analysis will be required, because we have neglected the LLL constraints in this calculation. In addition, it is generated by the difference between the inter- and intra-layer Coulomb repulsions at tree level in the infrared region ($|\vec{k}| \to 0$). Then it might not be observed in realistic systems due to the shielding effect, which makes the Coulomb interaction short-ranged, as discussed at the end of Section 4.

It is not so difficult to impose the LLL constraints. In the present case, they give constraints on the operators $q_{\vec{k}}^a$ and $p_{\vec{k}}^a$. It is straightforward to apply the Dirac quantization method of constraint system to the $(q_{\vec{k}}^a, p_{\vec{k}}^a)$ system. The results will be reported elsewhere.

6.2 Josephson Effect

In this subsection, we will discuss the Josephson(-type) effect in the DLFQH system[12, 11, 13]. As discussed previously, in the case $l = m = n$, the Nambu-Goldstone mode ϑ^- apparently survives and as long as the explicit breakdown of the $SU(2)_d$-pseudospin symmetry due to the interlayer tunneling effect is small enough, and this mode will dominate the low-energy dynamics of the DLFQH system. Also from the above arguments, the ground state is found to be a coherent Hall fluid in pseudospin-symmetric states, then the characteristic phenomena may be expected to occur. One of the typical phenomena is the Josephson(-type) effect discussed below. Although the effective theory (6.3) (Approach 2) is originally set up for the static situation (eigenvalue problem), it is straightforward to reformulate it for such dynamical phenomena. That is, we will study the following Schrödinger equation;

$$i\hbar\frac{\partial}{\partial t}\Phi[\rho^\uparrow, \rho^\downarrow; t] = \mathbf{H}_{total}[\vartheta^a, \rho^a]\cdot\Phi[\rho^\uparrow, \rho^\downarrow; t]. \tag{6.23}$$

Effective theory is obtained by taking the expectation value of the above equation. But it is more convenient to rewrite it in the Heisenberg picture. The (expectation value of the) Heisenberg equation is given by

$$\langle\!\langle \dot{\rho}^a(z;t)\rangle\!\rangle = \frac{i}{\hbar}\left\langle\!\!\left\langle\left[\mathbf{H}_{total}[\vartheta^b, \rho^b], \rho^a(z;t)\right]\right\rangle\!\!\right\rangle, \tag{6.24}$$
$$\langle\!\langle \dot{\vartheta}^a(z;t)\rangle\!\rangle = \frac{i}{\hbar}\left\langle\!\!\left\langle\left[\mathbf{H}_{total}[\vartheta^b, \rho^b], \vartheta^a(z;t)\right]\right\rangle\!\!\right\rangle.$$

As in Section 4, we will regard "classical" fields ρ^a and ϑ^a as expectation values of the collective variables (in the Heisenberg picture). Then the above equations are

and the second of (6.8) becomes the following sine-Gordon type equation;

$$2(\partial\overline{\partial}+\overline{\partial}\partial)\vartheta^a(z)-s^a\frac{M\triangle_{SAS}}{\hbar^2}\left(\frac{\rho_0^\downarrow}{\rho_0^\uparrow}\right)^{-\frac{s^a}{2}}\sin 2\vartheta^-(z)=0. \tag{6.10}$$

Here we define

$$U\equiv\frac{1}{2\pi\ell^2}\int d^2w\,\left(U^{11}(|z-w|)-U^{11}(|z-w|)\right)\simeq\frac{e^2d}{4\pi\epsilon\ell^2}(1+\mathcal{O}(\frac{d}{R})), \tag{6.11}$$

where R is an infrared cutoff corresponding to the radius of the system. If $\vartheta^a(z)$'s are harmonic, Eq.(6.10) shows that a stable solution will be realized at one of the bottoms of the cosine potential, e.g., $\vartheta^-(z)=0$. Moreover when $\vartheta^a(z)$'s take a nonsingular configuration, Eq.(6.10) requires that $Q=0$ which means the magic filling factor of the total density, and that $\vartheta^a(z)$'s are constant;

$$\rho^+(z)=\rho_0^+=\frac{1}{m}\frac{1}{2\pi\ell^2},\quad\text{and}\quad\vartheta^a(z)=\vartheta_0^a,\quad\text{with}\quad\vartheta_0^\uparrow=\vartheta_0^\downarrow. \tag{6.12}$$

Note that this configuration satisfies the LLL constraints. The physical picture of this state is as follows. Suppose that the inter-layer Coulomb repulsion is smaller than $\triangle_{SAS}$. Then the energy eigen states of one-electron states in the LLL in the DL system is splitted into two states with the gap $\triangle_{SAS}$; the pseudospin-symmetric and pseudospin-antisymmetric states. The ground state is the state in which all electrons are in the pseudospin-symmetric state. When the inter-electron Coulomb repulsion is turned on as a perturbation, this state will become a Hall fluid. The above state (6.12) corresponds to this. The distribution of electrons in each layer is determined by Eq.(6.10) as

$$2\mu+\rho_0^-\left(2\pi\ell^2U+\frac{\triangle_{SAS}}{2\sqrt{\rho_0^\uparrow\rho_0^\downarrow}}\right)=0. \tag{6.13}$$

When we want to set at $\rho_0^\uparrow=\rho_0^\downarrow=\frac{1}{2m}\frac{1}{2\pi\ell^2}$, the Lagrange multiplier μ must be equal to zero. Or when μ is the external potential,

$$\rho_0^-\simeq-\frac{2\mu}{\frac{1}{m}U+\triangle_{SAS}}\rho_0^+, \tag{6.14}$$

where we assume $|\rho_0^-|\ll\rho_0^+$.

State with vortices condensed corresponds to a kink-type solution of Eq.(6.10). The effect of such configurations will be discussed elsewhere.

The semi-classical study of the fluctuations around the above ground state can also be performed in the same way as in Subsection 4.5,

$$\rho^a(z)=\rho_0^a+\delta\rho^a(z),\quad\text{with}\quad\rho_0^a\simeq\left(1-s^a\frac{2\mu}{\frac{1}{m}U+\triangle_{SAS}}\right)\frac{1}{2m}\frac{1}{2\pi\ell^2},$$
$$\vartheta^a(z)=\vartheta_0^a+\delta\vartheta^a(z),\quad\text{with}\quad\vartheta_0^\uparrow=\vartheta_0^\downarrow, \tag{6.15}$$

where $\delta\rho^a(z)$'s and $\delta\vartheta^a(z)$'s satisfy Eq.(4.38). If we neglect the nonperturbative effects such as a tunneling between the valleys of cosine potential, i.e., kink-type configurations, then we obtain the following equation in the Gaussian approximation[9];

$$\begin{aligned}H^{eff}_{total}[\vartheta^a,\rho^a;\lambda] &= \text{const.}+\left(\frac{\hbar\omega_c}{2}-\frac{\triangle_{SAS}}{2}\right)N+\mu\cdot\triangle N\\ &+ (\rho_0^\uparrow-\overline{\rho}^\uparrow)^2A\cdot2\pi\ell^2U-\triangle_{SAS}\sqrt{\rho_0^\uparrow\rho_0^\downarrow}A\\ &+ \sum_{a,b}\sum_{\vec{k}\neq0}\frac{1}{2M}G^{ab}(k)p^a_{\vec{k}}p^b_{-\vec{k}}+\sum_{a,b}\sum_{\vec{k}\neq0}\frac{1}{2}K^{ab}(k)q^a_{\vec{k}}q^b_{-\vec{k}}+\cdots,\end{aligned} \tag{6.16}$$

where the term of "const." originates in the ordering of operators and cancels the zero-point energy of the density waves as in Subsection 4.5. Due to $\triangle_{SAS}$, the "metric" becomes nontrivial one;

$$G^{ab}(k)=\delta^{ab}+\frac{M\triangle_{SAS}}{\hbar^2\vec{k}^2}\cdot\begin{pmatrix}\sqrt{\frac{\rho_0^\downarrow}{\rho_0^\uparrow}} & -1\\ -1 & \sqrt{\frac{\rho_0^\uparrow}{\rho_0^\downarrow}}\end{pmatrix}, \tag{6.17}$$

and the "spring constants" are shifted as follows;

$$\begin{aligned}K^{ab}(k) &= \delta^{ab}M\omega_k^2+m2\pi\ell^2\sqrt{\rho_0^a\rho_0^b}M\omega_c^2\\ &+ \sqrt{\rho_0^a\rho_0^b}2\vec{k}^2\left(\frac{\pi\hbar^2}{M}m+\frac{1}{2}U^{ab}(|\vec{k}|)\right)+\frac{\triangle_{SAS}}{4}\vec{k}^2\cdot\begin{pmatrix}\sqrt{\frac{\rho_0^\downarrow}{\rho_0^\uparrow}} & -1\\ -1 & \sqrt{\frac{\rho_0^\uparrow}{\rho_0^\downarrow}}\end{pmatrix}.\end{aligned} \tag{6.18}$$

[9]Here we use the same notations as in Subsection 4.5 except for ρ_0^aA instead of N^a.

$$
\begin{aligned}
&+ \left(\frac{\hbar\omega_c}{2} - \frac{\triangle_{SAS}}{2} + \lambda\right)\int d^2z\,\rho^+(z) + \mu\int d^2z\,\rho^-(z) \\
&+ \frac{1}{2}\sum_{a,b}\int d^2z d^2w\left(\rho^a(z)-\overline{\rho}^a\right)\cdot U^{ab}(|z-w|)\cdot\left(\rho^b(w)-\overline{\rho}^b\right) \\
&- \triangle_{SAS}\int d^2z\,\sqrt{\rho^\uparrow(z)\rho^\downarrow(z)}\cdot\cos 2\vartheta^-(z) - \lambda N, \qquad (6.3)
\end{aligned}
$$

where the LL operators are defined by

$$
L^a(z) = \frac{1}{2}\frac{\partial\rho^a(z)}{\rho^a(z)} + i\partial\vartheta^a(z) - \frac{m}{2}\int d^2w\,\frac{\rho^+(w)}{z-w} + \frac{\overline{z}}{4\ell^2}. \qquad (6.4)
$$

Here we consider the chemical potential term explicitly corresponding to the potential difference 2μ between layers. This term will be treated in two different ways. In the first one it plays a role of another Lagrange multiplier (besides λ). That is, the system is not isolated and the reservior is linked to each layer to keep the particle numbers constant. In this case μ must be treated as a variable to fix the difference of the electron numbers in two layers $\triangle N = \int d^2z\,\rho^-(z)$, which is given by hand. In the second way it corresponds to the physical quantity, such as an external voltage-drop. In this case it is regarded as a parameter in the theory, given by an experimental condition, and $\triangle N$ will be determined according to μ. In the following we will call the μ term the external potential term in the second case.

The density of underlying positive charge $\overline{\rho}^a$ is unknown and we will determine it through the consistency condition that the ground state with uniform density at $\mu = 0$ is neutralized by $\overline{\rho}^a$. The result will be found to be $\overline{\rho}^a = \frac{1}{2m}\frac{1}{2\pi\ell^2}$.

For later convenience, let us calculate the variation of H^{eff}_{total}. The variation with respect to the density is

$$
\begin{aligned}
\frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\delta\rho^a(z)} &= \frac{2\hbar^2}{M}\overline{L}^a(z)L^a(z) - \frac{\hbar^2}{M}\left(\frac{\overline{\partial}(\rho^a(z)L^a(z))}{\rho^a(z)} + \frac{\partial(\rho^a(z)\overline{L}^a(z))}{\rho^a(z)}\right) \\
&+\frac{\hbar^2}{M}\sum_b\int d^2w\,m\rho^b(w)\left(\frac{L^b(w)}{\overline{z}-\overline{w}} + \frac{\overline{L}^b(w)}{z-w}\right) + \frac{\hbar\omega_c}{2} - \frac{\triangle_{SAS}}{2} + \lambda + s^a\mu \qquad (6.5) \\
&+\sum_b\int d^2w\,U^{ab}(|z-w|)\cdot\left(\rho^b(w)-\overline{\rho}^b\right) - \frac{\triangle_{SAS}}{2}\left(\frac{\rho^\uparrow(z)}{\rho^\downarrow(z)}\right)^{-\frac{s^a}{2}}\cos 2\vartheta^-(z),
\end{aligned}
$$

and the variation with repect to the phase is

$$
\begin{aligned}
\frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\delta\vartheta^a(z)} &= -\frac{i\hbar^2}{M}(\partial\overline{\partial}-\overline{\partial}\partial)\rho^a(z) + \frac{2i\hbar^2}{M}\left(\overline{\partial}\rho^a(z)\,L^a(z) - \partial\rho^a(z)\,\overline{L}^a(z)\right) \qquad (6.6) \\
&+\frac{2\hbar^2}{M}\rho^a(z)\,(\partial\overline{\partial}+\overline{\partial}\partial)\vartheta^a(z) + s^a\cdot\triangle_{SAS}\sqrt{\rho^\uparrow(z)\rho^\downarrow(z)}\cdot\sin 2\vartheta^-(z).
\end{aligned}
$$

It is worth while mentioning that in the above discussions the LLL constraints $L^a(z) \equiv 0$, which often play an important role in the FQHE, are not taken into account. If the dynamics of the system is restricted to the LLL, we should replace the kinetic term of the effective Hamiltonian by the following LLL constraints, i.e.,

$$
\begin{aligned}
&\frac{2\hbar^2}{M}\sum_a\int d^2z\,\rho^a(z)\overline{L}^a(z)L^a(z) \\
&\qquad\Longrightarrow\quad \sum_a\int d^2z\left(\chi^a(z)\overline{L}^a(z) + \overline{\chi}^a(z)L^a(z)\right), \qquad (6.7)
\end{aligned}
$$

where $\chi^a(z)$ and $\overline{\chi}^a(z)$ are Lagrange multipliers imposing the LLL constraints. In the following we will forget the LLL constraints for a while, and analyze H^{eff}_{total}.

6.1 Semi-classical Analysis

First, we will determine the (variational) ground state. The ground state is obtained by solving the equations;

$$
\frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\delta\rho^a(z)} = 0, \quad \text{and} \quad \frac{\delta H^{eff}_{total}[\vartheta^b,\rho^b;\lambda]}{\delta\vartheta^a(z)} = 0. \qquad (6.8)
$$

We will restrict ourselves to solutions with uniform density; $\rho^a(z) \equiv \rho^a_0$. In this case the LL operators are $L^a(z) = i\partial\vartheta^a(z) - \frac{\pi}{2}Q\overline{z}$, where $Q = m\rho^+_0 - \frac{1}{2\pi\ell^2}$ ($\rho^+_0 = \frac{N}{A}$). Then the first equation of (6.8) becomes

$$
\begin{aligned}
&\frac{2\hbar^2}{M}\overline{\partial}\vartheta^a(z)\partial\vartheta^a(z) - \frac{i\hbar^2}{M}(\overline{\partial}\partial - \partial\overline{\partial})\vartheta^a(z) - \frac{i\pi\hbar^2}{M}Q(z\partial - \overline{z}\overline{\partial})\vartheta^a(z) + \frac{\pi^2\hbar^2}{2M}Q^2|z|^2 \\
&+\frac{\hbar^2}{M}\sum_b m\rho^b_0\int d^2w\left(\frac{L^b(w)}{\overline{z}-\overline{w}} + \frac{\overline{L}^b(w)}{z-w}\right) - \frac{\triangle_{SAS}}{2}\left(\frac{\rho^\uparrow_0}{\rho^\downarrow_0}\right)^{-\frac{s^a}{2}}\cos 2\vartheta^-(z) \qquad (6.9) \\
&+s^a(\rho^\uparrow_0 - \overline{\rho}^\uparrow)\,2\pi\ell^2 U + \frac{\pi\hbar^2}{M}Q + \frac{\hbar\omega_c}{2} - \frac{\triangle_{SAS}}{2} + s^a\mu + \lambda = 0,
\end{aligned}
$$

In order to describe the interlayer tunneling effect, we use the hopping Hamiltonian (3.3) of bosonized electron ϕ^a. As mentioned in footnote 5, in the original sense, we should express this term by the fermionic field operators ψ^a. The relation between two operators ϕ^a and ψ^a is as follows;

$$\begin{aligned}\psi^a(x) &= e^{+\frac{i}{\hbar}\int_{\gamma(x)} C^a_\mu(y)dy^\mu}\cdot\phi^a(x) \\ &= e^{\sum_b \frac{m^{ab}}{2}\int d^2w\,\rho^b(w)\int_{\gamma(x)}\left(\frac{dz}{z-w}-\frac{d\bar{z}}{\bar{z}-\bar{w}}\right)}\cdot\phi^a(x),\end{aligned} \tag{7.1}$$

where $\gamma(x)$ represents a path starting from x to the base point (e.g. infinity). Then the hopping terms are accompanied by the line integral of the pseudospin-$SU(2)_d$ C-S gauge field;

$$\psi^{\uparrow\dagger}(x)\psi^{\downarrow}(x) = \phi^{\uparrow\dagger}(x)\,e^{-\frac{i}{\hbar}\int_{\gamma(x)}\left(C^\uparrow_\mu(y)-C^\downarrow_\mu(y)\right)dy^\mu}\phi^{\downarrow}(x), \quad \text{etc.} \tag{7.2}$$

However, this modification will not affect the case of our particular interest of $l = m = n$, because the pseudospin-$SU(2)_d$ gauge field is a decoupled pure gauge in that case, and then the difference between $\psi^{\uparrow\dagger}\psi^{\downarrow}$ and $\phi^{\uparrow\dagger}\phi^{\downarrow}$ is only a trivial c-number phase factor. For the case $lm \neq n^2$, it is also verified that the above line integral does not affect the discussion on the variational ground state and the dispersion relation of the excitations.

We succeeded in constructing the space of states in the presence of the interlayer tunneling effect, realizing $\mathcal{H}_{hop}$ (3.3) in ρ, and formulating the collective-field theory for the full DLFQH system. We believe that the expression $\tilde{H}_{hop}$ (5.24) describe correctly the properties of electron tunneling. On the other hand, in the state with some vortices like (4.32), there might occur the tunneling of these vortices through the naive considerations. But when operating $\tilde{H}_{hop}$ to $\Phi[\rho^\uparrow, \rho^\downarrow]$ with (4.23) explicitly, we will find that it is only the hopping of electrons, and not of vortices. This is because the distribution of vortices in the state (4.23), $\{\xi^a_i\}$, is inserted by hand, i.e., the degrees of freedom of vortices are not prepared as dynamical variables in the space of states with the location of $\{\xi^a_i\}$ fixed. In order to treat them dynamically, we have

to make a duality-type transformation[20];

$$\Phi[\rho^\uparrow, \rho^\downarrow] = \int \langle \mathcal{D}\vartheta_{reg}\,\mathcal{D}\rho_V\rangle\, e^{i\sum_a \int d^2z\,\rho^a(z)\left(\vartheta^a_{reg}(z)+\vartheta^a_{sing}[z;\rho^a_V]\right)}\cdot\Xi[\vartheta^\downarrow_{reg}, \vartheta^\downarrow_{reg}; \rho^\downarrow_V, \rho^\downarrow_V], \tag{7.3}$$

where $\vartheta^a_{reg}(z)$'s and $\vartheta^a_{sing}[z;\rho^a_V]$'s represent the regular and singular part of the phase variables of electrons; $\vartheta^a(z) = \vartheta^a_{reg}(z) + \vartheta^a_{sing}[z;\rho^a_V]$, respectively. $\rho^a_V(z)$'s represent the vorticity-density defined by $\rho^a_V(z) = -\frac{1}{2\pi}\epsilon^{ij}\partial_i\partial_j\vartheta^a_{sing}[z;\rho^a_V]$. Actually, even if we carry out the above transformation, the phenomenon of tunneling of vortices is *not* allowed, as pointed out by Wen and Zee[14], due to the consistency of the theory originating from the Dirac's quantization condition. More detailed discussions about this point will also be given elsewhere.

Based on the ρ-realization of the total Hamiltonian, we wrote down the effective theory for the full DLFQH system, and investigated some properties of the system with $l = m = n$. The dispersion relation of the long-wavelength density fluctuations is calculated within the perturbative assumption. The effect of the interlayer tunneling affects only the mode associated with the $SU(2)_d$-pseudospin symmetry, and the Nambu-Goldstone mode ϑ^- acquires a mass of order $\triangle_{SAS}$. The dispersion relation of this mode exhibits a local minimum at a finite wave vector k_{min}. It may be considered that this is a sign of the collapse of the $\frac{1}{m}$-Hall fluid of pseudospin-symmetric states, but the more careful treatment, in which the LLL constraints are taken into account, will be required as mentioned above. In the end we studied the Josephson(-type) effect. As long as based on our formalism, the interlayer tunneling effect generates the Josephson coupling energy, and inevitably leads to the phenomenon just like the Josephson effect when the phase difference between layers is present. In this phenomenon the coherence of the (mmm)-Hall state plays a key role just as in the superconducting state. The critical current is determined by $J_c = \frac{e\triangle_{SAS}}{\hbar}\rho_0$ where $\rho_0 = \frac{1}{2m}\frac{1}{2\pi\ell^2}$, and the frequency in the ac effect with a voltage drop V_0 is $\frac{eV_0}{\hbar}$.

Finally, we will give some comments about the bosonization procedure, which we supposed throughout this paper. As discussed in Introduction, it is considered that this prescription is based on some perturbative assumption. If we want to overcome

this point, a fermionic description of the (DL)FQH system will be required by all means. In this case the collective-field formalism may be difficult to apply to the full (DL)FQH system, but it is possible to write down the effective theory of low-lying bosonic excitations above the ground state, such as density waves, by using the collective variables[32]. It is considered that this is also a perturbative model but may involve a little more informations than the present model, because this may link the Hall state with different (lmn)[8, 11]. This arguments will be developed elsewhere.

Appendix

A ETCR's among the Collective Fields

In this appendix A, we will give the equal-time commutation relations (ETCR's) between the collective-field operators; $\hat{\rho}^{ab}$ and $\hat{j}^{ab}_{z,\overline{z}}$, defined by (3.5). The fundamental ETCR's of ϕ^a and $\phi^{a\dagger}$ are given in Eq.(2.10), then we obtain the following relations,

$$
\begin{aligned}
&\left[\hat{\rho}^{ab}(z,\overline{z}),\, \hat{\rho}^{cd}(w,\overline{w})\right] = \\
&\quad \left(\delta^{bc}\hat{\rho}^{ad}(z,\overline{z}) - \delta^{ad}\hat{\rho}^{cb}(z,\overline{z})\right)\delta^{(2)}(z-w), \\
&\left[\hat{j}^{ab}_{z(\overline{z})}(z,\overline{z}),\, \hat{j}^{cd}_{z(\overline{z})}(w,\overline{w})\right] = \\
&\quad -\delta^{ad}\, i\hat{j}^{cb}_{z(\overline{z})}(z,\overline{z})\overset{(-)}{\partial}_z\delta^{(2)}(z-w) - \delta^{bc}i\overset{(-)}{\partial}_z\left(\hat{j}^{ad}_{z(\overline{z})}(z,\overline{z})\,\delta^{(2)}(z-w)\right) \\
&\quad -\Big\{\delta^{ad}\Big(i\overset{(-)}{\partial}\hat{\rho}^{c}(z,\overline{z}) + \hat{j}^{c}_{z(\overline{z})}(z,\overline{z})\Big)\frac{1}{\hat{\rho}^{c}(z,\overline{z})}\hat{j}^{cb}_{z(\overline{z})}(z,\overline{z}) \\
&\qquad - \delta^{bc}\Big(i\overset{(-)}{\partial}\hat{\rho}^{a}(z,\overline{z}) + \hat{j}^{a}_{z(\overline{z})}(z,\overline{z})\Big)\frac{1}{\hat{\rho}^{a}(z,\overline{z})}\hat{j}^{ad}_{z(\overline{z})}(z,\overline{z})\Big\}\,\delta^{(2)}(z-w), \\
&\left[\hat{j}^{ab}_{z}(z,\overline{z}),\, \hat{j}^{cd}_{\overline{z}}(w,\overline{w})\right] = \\
&\quad -\delta^{ad}\, i\hat{j}^{cb}_{z}(z,\overline{z})\,\overline{\partial}_z\delta^{(2)}(z-w) - \delta^{bc}i\partial_z\left(\hat{j}^{ad}_{\overline{z}}(z,\overline{z})\,\delta^{(2)}(z-w)\right) \\
&\quad -\Big\{\delta^{ad}\left(i\overline{\partial}\hat{\rho}^{c}(z,\overline{z}) + \hat{j}^{c}_{\overline{z}}(z,\overline{z})\right)\frac{1}{\hat{\rho}^{c}(z,\overline{z})}\hat{j}^{cb}_{z}(z,\overline{z}) \\
&\qquad - \delta^{bc}\left(i\partial\hat{\rho}^{a}(z,\overline{z}) + \hat{j}^{a}_{z}(z,\overline{z})\right)\frac{1}{\hat{\rho}^{a}(z,\overline{z})}\hat{j}^{ad}_{\overline{z}}(z,\overline{z})\Big\}\,\delta^{(2)}(z-w), \\
&\left[\hat{j}^{ab}_{z(\overline{z})}(z,\overline{z}),\, \hat{\rho}^{cd}(w,\overline{w})\right] = \\
&\quad \left(\delta^{bc}\,\hat{j}^{ad}_{z(\overline{z})}(z,\overline{z}) - \delta^{ad}\,\hat{j}^{cb}_{z(\overline{z})}(z,\overline{z})\right)\delta^{(2)}(z-w) - \delta^{bc}\,i\hat{\rho}^{ad}(z,\overline{z})\overset{(-)}{\partial}_z\delta^{(2)}(z-w)
\end{aligned}
\tag{A.1}
$$

These relations are closed within the collective-field operators $\{\hat{\rho}^{ab}, \hat{j}^{ab}_{z,\overline{z}}\}$ and well-defined, because the existence of the inverse operators $(\hat{\rho}^{a})^{-1}$ is physically plausible. From the above relations, the maximal commutative set is only $\{\hat{\rho}^{a}\}$. Then we may realize in ρ the all ETCR composed of the collective-field operators.

The subset $\{\hat{\rho}^{a}, \hat{j}^{a}_{z,\overline{z}}\}$, which is required in the case without the interlayer tunneling

effect, also forms a closed, well-defined algebra;

$$
\begin{aligned}
&\left[\hat{\rho}^a(z,\overline{z}),\hat{\rho}^b(w,\overline{w})\right]=0,\\
&\left[\hat{j}^a_{z(\overline{z})}(z,\overline{z}),\hat{j}^b_{z(\overline{z})}(w,\overline{w})\right]=-\delta^{ab}\overset{(-)}{\partial}_z\left(i\hat{j}^a_{z(\overline{z})}(z,\overline{z})\,\delta^{(2)}(z-w)\right)-\delta^{ab}i\hat{j}^a_{z(\overline{z})}(z,\overline{z})\overset{(-)}{\partial}_z\delta^{(2)}(z-w),\\
&\left[\hat{j}^a_{z}(z,\overline{z}),\hat{j}^b_{\overline{z}}(w,\overline{w})\right]=-\delta^{ab}\overline{\partial}_z\left(i\hat{j}^a_{z}(z,\overline{z})\,\delta^{(2)}(z-w)\right)-\delta^{ab}i\hat{j}^a_{\overline{z}}(z,\overline{z})\,\partial_z\delta^{(2)}(z-w),\\
&\left[\hat{j}^a_{z(\overline{z})}(z,\overline{z}),\hat{\rho}^b(w,\overline{w})\right]=-\delta^{ab}i\hat{\rho}^a(z,\overline{z})\overset{(-)}{\partial}_z\delta^{(2)}(z-w).
\end{aligned}
\tag{A.2}
$$

From these relations, we obtain the following ρ-realization of the commutator with respect to the current operators $\hat{j}^a_{z,\overline{z}}$;

$$
\left[\hat{j}^a_{z(\overline{z})}(z,\overline{z}),\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]=-i\hat{\rho}^a(z,\overline{z})\overset{(-)}{\partial}\frac{\delta}{\delta\hat{\rho}^a(z,\overline{z})}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow].
\tag{A.3}
$$

This operator relation will be used in Section 4.

In addition, since $\hat{j}^{ab}_{z,\overline{z}}$'s satifies the relations (3.6), the minimum information involved in (A.1) other than (A.2) consists of only the following;

$$
\begin{aligned}
&\left[\hat{\rho}^a(z,\overline{z}),\,\hat{\rho}^{bc}(w,\overline{w})\right]=(\delta^{ab}-\delta^{ac})\,\hat{\rho}^{bc}(z,\overline{z})\delta^{(2)}(z-w),\\
&\left[\hat{\rho}^{\uparrow\downarrow}(z,\overline{z}),\,\hat{\rho}^{\downarrow\uparrow}(w,\overline{w})\right]=\left(\hat{\rho}^\uparrow(z,\overline{z})-\hat{\rho}^\downarrow(z,\overline{z})\right)\delta^{(2)}(z-w),\\
&\left[\hat{j}^a_{z(\overline{z})}(z,\overline{z}),\,\hat{\rho}^{bc}(w,\overline{w})\right]=(\delta^{ab}-\delta^{ac})\,\hat{j}^{bc}_{z(\overline{z})}(z,\overline{z})\delta^{(2)}(z-w)-\delta^{ab}i\hat{\rho}^{bc}(z,\overline{z})\overset{(-)}{\partial}_z\delta^{(2)}(z-w),
\end{aligned}
\tag{A.4}
$$

where $(bc)=(\uparrow\downarrow)\,\text{or}\,(\downarrow\uparrow)$. From these relations, the commutator of $\hat{\rho}^{ab}$ or $\hat{j}^{ab}_{z,\overline{z}}$ will be realized in ρ as

$$
\left[\hat{\rho}^{ab}(z,\overline{z}),\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]=\hat{\rho}^a(z,\overline{z})\cdot e^{-\frac{\delta}{\delta\hat{\rho}^a(z,\overline{z})}+\frac{\delta}{\delta\hat{\rho}^b(z,\overline{z})}}\cdot\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow],
\tag{A.5}
$$

and

$$
\left[\hat{j}^{ab}_{z(\overline{z})}(z,\overline{z}),\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]=-i\hat{\rho}^a(z,\overline{z})\cdot e^{-\frac{\delta}{\delta\hat{\rho}^a(z,\overline{z})}+\frac{\delta}{\delta\hat{\rho}^b(z,\overline{z})}}\cdot\overset{(-)}{\partial}\frac{\delta}{\delta\hat{\rho}^b(z,\overline{z})}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow].
\tag{A.6}
$$

During the above calculations, we used the hermitian condition $(\hat{\rho}^{\uparrow\downarrow})^\dagger=\hat{\rho}^{\downarrow\uparrow}$, which seems to be nontrivial due to the unusual inner product of the space of states. This point will be discussed in Appendix B. These equations will give a basis for determining the ρ-realization of $\hat{\rho}^{\uparrow\downarrow}$ and $\hat{\rho}^{\downarrow\uparrow}$ in Section 5.

B ρ-Realization of $\hat{\rho}^{\uparrow\downarrow}$ and $\hat{\rho}^{\downarrow\uparrow}$

In this Appendix B, we will address the ρ-realization of the interlayer hopping term in the functional space constructed in Section 5. Especially we will focus on the hermitian condition of the hopping operators $\hat{\rho}^{\uparrow\downarrow}$ and $\hat{\rho}^{\downarrow\uparrow}$.

At first, we will prove the ρ-realization of the commutator with respect to $\hat{\rho}^{\uparrow\downarrow}$ and $\hat{\rho}^{\downarrow\uparrow}$, Eq.(A.5). We will find the following expressions through the first relation of (A.4);

$$
\begin{aligned}
\left[\hat{\rho}^{\uparrow\downarrow}(z,\overline{z}),\,\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]&=F[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\cdot e^{-\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})}+\frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow],\\
\left[\hat{\rho}^{\downarrow\uparrow}(z,\overline{z}),\,\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]&=G[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\cdot e^{+\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})}-\frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow].
\end{aligned}
\tag{B.1}
$$

The functionals $F[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]$ and $G[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]$ are undetermined at this stage. They will be determined by using dimensional analysis and the second relation of (A.4) up to two arbitrary parameters $\alpha,\beta\,(\alpha\neq\beta)$;

$$
F[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]=\alpha\hat{\rho}^\uparrow(z,\overline{z})+\beta\hat{\rho}^\downarrow(z,\overline{z}),\quad G[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]=\frac{1}{(\alpha-\beta)^2}\left\{\beta\hat{\rho}^\uparrow(z,\overline{z})+\alpha\hat{\rho}^\downarrow(z,\overline{z})\right\}.
\tag{B.2}
$$

Suppose that the ρ-realization of the commutator with respect to the current operators $\hat{j}^a_{z,\overline{z}}$ is taken to be Eq.(A.3) still in this full case. If we can consistently choose α and β so that the equations (A.3) and (B.1), (B.2) fulfill the last relation of (A.4), then the complete ρ-realization of the full ETCR's (A.1) will be determined. After some algera, we will find $\beta=0$, but $\alpha(\neq 0)$ still remains arbitrary;

$$
\begin{aligned}
\left[\hat{\rho}^{\uparrow\downarrow}(z,\overline{z}),\,\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]&=\alpha\,\hat{\rho}^\uparrow(z,\overline{z})\cdot e^{-\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})}+\frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow],\\
\left[\hat{\rho}^{\downarrow\uparrow}(z,\overline{z}),\,\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow]\right]&=\frac{1}{\alpha}\,\hat{\rho}^\downarrow(z,\overline{z})\cdot e^{+\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})}-\frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}}\Phi[\hat{\rho}^\uparrow,\hat{\rho}^\downarrow].
\end{aligned}
\tag{B.3}
$$

The value of α cannot be determined only by the ETCR's(A.1), and another requirement is needed. We will consider as such a requirement the hermitian condition; $\left(\hat{\rho}^{\uparrow\downarrow}\right)^\dagger=\hat{\rho}^{\downarrow\uparrow}$. This condition is evident in the second-quantized formalism, but becomes nontrivial in the functional space constructed in Section 5 due to the unusual

inner product (5.13). From this condition we will obtain $\alpha = \frac{1}{\alpha}$ i.e. $\alpha = \pm 1$ as shown below. The sign of α should be taken as $+1$, because the relation between $a = b$ and $a \neq b$ in the ρ-realization of $\hat{\rho}^{ab}$ is logically more natural in that choice. Thus we will obtain the ρ-realization of the ETCR with respect to $\hat{\rho}^{ab}$'s to be Eq.(A.5). From this, we will easily obtain Eq.(A.6), by using the relation (3.6).

The residual problem is to prove $\alpha = \frac{1}{\alpha}$ by the hermitian condition. In the functional space constructed in Section 5, this condition is express as follows;

$$\langle \Phi | \hat{\rho}^{\uparrow\downarrow}(z,\overline{z}) \Phi' \rangle = \langle \hat{\rho}^{\downarrow\uparrow}(z,\overline{z}) \Phi | \Phi' \rangle, \quad \text{for} \, \forall |\Phi\rangle, |\Phi'\rangle. \tag{B.4}$$

The bracket is taken with respect to the inner product in Eq.(5.13). Let us calculate the left-hand side of the above equation explicitly.

$$\begin{aligned} \langle \Phi | \hat{\rho}^{\uparrow\downarrow}(z,\overline{z}) \Phi' \rangle &= \int \langle \mathcal{D}\rho \rangle \, e^{\mathcal{S}[\rho^\uparrow,\rho^\downarrow]} \cdot \overline{\Phi[\rho^\uparrow,\rho^\downarrow]} \cdot \alpha \rho^\uparrow e^{-\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})} + \frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}} \, \Phi'[\rho^\uparrow,\rho^\downarrow] \\ &= \int \langle \mathcal{D}\rho \rangle \, \Gamma[\rho^\uparrow,\rho^\downarrow] \cdot \overline{\left\{ e^{+\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})} - \frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}} \, \Phi[\rho^\uparrow,\rho^\downarrow] \right\}} \cdot \Phi'[\rho^\uparrow,\rho^\downarrow]. \end{aligned} \tag{B.5}$$

Here we perform the integration by parts. Since the functional measure in this space is given by Eq.(5.14), then this seems to require a nontrivial calculation. But the total-number constraint involved in Eq.(5.14) implies the exclusion of the zero mode from the integration variable, then the behavior of the surface term may be rather mild as compared with the case without the constraint. Therefore this will make the naive partial integration well-defined. Γ is given as follows;

$$\begin{aligned} \Gamma[\rho^\uparrow,\rho^\downarrow] &= e^{+\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})} - \frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}} \left\{ \alpha \rho^\uparrow(z,\overline{z}) e^{\mathcal{S}[\rho^\uparrow,\rho^\downarrow]} \right\} e^{-\frac{\delta}{\delta\rho^\uparrow(z,\overline{z})} + \frac{\delta}{\delta\rho^\downarrow(z,\overline{z})}} \\ &\sim \alpha \rho^\uparrow(z,\overline{z}) \cdot \frac{\rho^\downarrow(z,\overline{z})}{\rho^\uparrow(z,\overline{z})} e^{\mathcal{S}[\rho^\uparrow,\rho^\downarrow]}. \end{aligned} \tag{B.6}$$

In this calculation, we regularize all $\delta^{(2)}(z=0)$ terms as in the main text. Then the right-hand side of Eq.(B.5) reduces to $\langle \hat{\rho}^{\downarrow\uparrow}(z,\overline{z}) \Phi | \Phi' \rangle$ when $\alpha = \frac{1}{\alpha}$.

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