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AN EXACT SOLUTION OF EVOLUTION OPERATOR AND ITS APPLICATION TO COHERENT DESTRUCTION OF TUNNELING IN TWO-LEVEL SYSTEMS

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An exact solution of evolution operator and its application to coherent destruction of tunneling in two-level systems

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Abstract

The problem of two-level systems described by arbitrary time-dependent Hamiltonians is studied analytically. A closed-form solution of the evolution operator is obtained explicitly as a Neumann-Liouville expansion with the help of $SU(2)$ symmetry. From this exact solution, the results for any special case can be deduced. The application of this solution to a periodically driven two-level system leads to the apperance of dynamic localization, which is demonstrated to coincide with the onset of coherent destruction of tunneling.

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A two-level system, as well known, is relevant to varieties of topics in physics such as multiphoton transitions and quantum chaos in the presence of strong fields [1]; dynamics of the two-state system in dissipative environments [2]; Landau-Zener transitions in atomic collisions [3], molecular physics [4], mesoscopic systems [5], and solid-state physics [6]; geometric phase for neutron interferometry [7], magnetic resonance [8], and two-level atoms [9]; the problem of the solar-neutrino puzzle [10]; the question of self-trapping in disordered semiconductors [11]; to name but a few. In these studies, particular attention is devoted to the evolution operator, which plays an important role in quantum theory. Very recently, the present author find that the problem can be transformed into the study of a first-order differential equation, and gives out a general closed-form solution for the evolution operator [12]. In this paper, we address our another novel approach for this question. We find that this problem can also be solved exactly and explicitly as a Neumann-Liouville expansion with the help of $SU(2)$ symmetry. As the result, the general solution of the evolution operator is obtained as a closed-form expression, from which results for any special case can be deduced. To illustrate the validity of our method, we present two examples, both of which are of importance in quantum physics.

The time-dependent Hamiltonian of a two-level system can be written as

$$H(t) = \begin{pmatrix} a(t) & b(t)e^{-i\omega(t)} \\ b(t)e^{i\omega(t)} & -a(t) \end{pmatrix} \tag{1}$$

where $a(t)$, $b(t)$, and $\omega(t)$ are real functions and an inessential diagonal term in the evolution Hamiltonian was omitted.

The evolution operator $U(t,t_0)$ of the system, defined through the evolution state $|\psi(t)>$ as $|\psi(t)>= U(t,t_0)|\psi(t_0)>$, satisfies the evolution equation

$$i\frac{\partial}{\partial t}U(t,t_0) = H(t)U(t,t_0) \tag{2}$$

with the initial condition $U(t_0,t_0) = 1$, here we put $\hbar = 1$ throughout this paper.

By introducing

$$U(t,t_0) = U_0(t)U_I(t,t_0)U_0^\dagger(t_0) \tag{3}$$

with

$$U_0(t) = e^{-i\sigma_z\theta(t)} , \qquad \theta(t) = \int_0^t a(t')dt' . \tag{4}$$

where σ_z (as well as σ_x and σ_y used below) is a Pauli matrix, we obtain

$$i\frac{\partial}{\partial t}U_I(t,t_0) = H_I(t)U_I(t,t_0) \tag{5}$$

with the initial condition $U_I(t_0,t_0)=1$, where

$$H_I(t) = U_0^\dagger(t)\big[H(t) - i\frac{\partial}{\partial t}\big]U_0(t) \equiv X(t)\sigma_x + Y(t)\sigma_y \tag{6}$$

with

$$X(t) = b(t)\cos[\omega(t) - 2\theta(t)]\ , \qquad Y(t) = b(t)\sin[\omega(t) - 2\theta(t)]\ . \tag{7}$$

The solution of the evolution operator $U_I(t,t_0)$ can be expressed as

$$U_I(t,t_0) = T\exp\Big(-i\int_{t_0}^{t} H_I(t')dt'\Big)\ , \tag{8}$$

where T denotes time ordering. The use of Neumann-Liouville expansion [13] yields explicit form of Eq. (8)

$$U_I(t,t_0) = \sum_{m=0}^{\infty} R^{(m)}(t,t_0)\ , \tag{9}$$

where

$$R^{(0)}(t,t_0) = 1\ , \tag{10}$$

$$\begin{aligned} R^{(m)}(t,t_0) = (-i)^m\Big(\prod_{l=1}^{m}\int_{t_0}^{t} dt_l\Big)\theta(t_1-t_2)\theta(t_2-t_3)\cdots\theta(t_{m-1}-t_m) \\ \cdot H_I(t_1)H_I(t_2)\cdots H_I(t_m)\ , \qquad (m\neq 0)\ , \end{aligned} \tag{11}$$

with $\theta(t)=1$ for $t>0$ and 0 otherwise. Direct calculations show that

$$\begin{aligned} H_I(t_1)H_I(t_2)\cdots H_I(t_m) &= X(t_1t_2\cdots t_m) + iY(t_1t_2\cdots t_m)\sigma_z\ , && \text{if } m=2l\ , \\ &= X(t_1t_2\cdots t_m)\sigma_x + Y(t_1t_2\cdots t_m)\sigma_y\ , && \text{if } m=2l+1\ , \end{aligned} \tag{12}$$

where $X(t_1t_2\cdots t_m)$ and $Y(t_1t_2\cdots t_m)$ satisfy the following recurrence formulae

$$X(t_1t_2\cdots t_m) = X(t_1t_2\cdots t_{m-1})X(t_m) + Y(t_1t_2\cdots t_{m-1})Y(t_m)\ , \qquad (m\geq 2)\ , \tag{13}$$

$$Y(t_1t_2\cdots t_m) = X(t_1t_2\cdots t_{m-1})Y(t_m) - Y(t_1t_2\cdots t_{m-1})X(t_m)\ , \qquad (m\geq 2)\ . \tag{14}$$

It should be noticed that the Pauli matrices are not involved in Eqs. (13) and (14).

By defining

$$R_x^{(2m)}(t,t_0) = (-1)^m\Big(\prod_{l=1}^{2m}\int_{t_0}^{t} dt_l\Big)\theta(t_1-t_2)\cdots\theta(t_{2m-1}-t_{2m})X(t_1t_2\cdots t_{2m})\ , \tag{15}$$

$$R_y^{(2m)}(t,t_0) = (-1)^m\Big(\prod_{l=1}^{2m}\int_{t_0}^{t} dt_l\Big)\theta(t_1-t_2)\cdots\theta(t_{2m-1}-t_{2m})Y(t_1t_2\cdots t_{2m})\ , \tag{16}$$

$$R_x^{(2m+1)}(t,t_0) = (-1)^{m+1}\Big(\prod_{l=1}^{2m+1}\int_{t_0}^{t} dt_l\Big)\theta(t_1-t_2)\cdots\theta(t_{2m}-t_{2m+1})X(t_1t_2\cdots t_{2m+1})\ , \tag{17}$$

$$R_y^{(2m+1)}(t,t_0) = (-1)^{m+1}\Big(\prod_{l=1}^{2m+1}\int_{t_0}^{t} dt_l\Big)\theta(t_1-t_2)\cdots\theta(t_{2m}-t_{2m+1})Y(t_1t_2\cdots t_{2m+1})\ , \tag{18}$$

$$R_x^{(0)}(t,t_0) = 1\ , \qquad R_y^{(0)}(t,t_0) = 0\ , \tag{19}$$

Eq. (9) can be transformed into the following form

$$U_I(t,t_0) = \sum_{m=0}^{\infty} R_x^{(2m)}(t,t_0) + i\sigma_z\sum_{m=0}^{\infty} R_y^{(2m)}(t,t_0) + i\sigma_x\sum_{m=0}^{\infty} R_x^{(2m+1)}(t,t_0) + i\sigma_y\sum_{m=0}^{\infty} R_y^{(2m+1)}(t,t_0)\ . \tag{20}$$

Obviously $R_x^{(m)}(t,t_0)$ and $R_y^{(m)}(t,t_0)$ are also real functions. Considering the unitarity of the evolution operator $U_I(t,t_0)$, Eq. (20) can be rewritten as more compact form

$$U_I(t,t_0) = e^{i\vec{\sigma}\cdot\vec{\phi}(t,t_0)} \tag{21}$$

with

$$\vec{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z)\ , \tag{22}$$

$$\vec{\phi}(t,t_0) = \frac{\phi(t,t_0)}{\sin[\phi(t,t_0)]}\left(\sum_{m=0}^{\infty} R_x^{(2m+1)}(t,t_0),\ \sum_{m=0}^{\infty} R_y^{(2m+1)}(t,t_0),\ \sum_{m=0}^{\infty} R_y^{(2m)}(t,t_0)\right)\ , \tag{23}$$

$$\phi(t,t_0) = \cos^{-1}\left(\sum_{m=0}^{\infty} R_x^{(2m)}(t,t_0)\right)\ . \tag{24}$$

Thus, we obtain the final result for $U(t,t_0)$ as

$$U(t,t_0) = e^{-i\sigma_z\theta(t)}e^{i\vec{\sigma}\cdot\vec{\phi}(t,t_0)}e^{i\sigma_z\theta(t_0)}\ . \tag{25}$$

In practical calculations, it might be convenient to use the formulae

$$X(t_1t_2\cdots t_m) = \Big(\prod_{l=1}^{m} b(t_l)\Big)\cos\Big(\sum_{l=1}^{m}(-1)^l[\omega(t_l) - 2\theta(t_l)]\Big)\ , \tag{26}$$

$$Y(t_1t_2\cdots t_m) = (-1)^m\Big(\prod_{l=1}^{m} b(t_l)\Big)\sin\Big(\sum_{l=1}^{m}(-1)^l[\omega(t_l) - 2\theta(t_l)]\Big)\ , \tag{27}$$

which can be derived from Eqs. (13) and (14) straightforwardly.

The above results are for the most general time-dependent Hamiltonian (1), and thus the solution of the evolution operator for any special two-level system can be deduced from them. Here we present two examples to illustrate the validity of our method. First of all, we consider a special and simple case $\omega(t) - 2\theta(t) = \alpha$, α a constant. Thus we have

$$R_x^{(2m)}(t,t_0) = \frac{(-1)^m}{(2m)!}\Big(\int_{t_0}^t dt' b(t')\Big)^{2m} , \qquad R_y^{(2m)}(t,t_0) = 0 , \tag{28}$$

$$R_x^{(2m+1)}(t,t_0) = \frac{(-1)^{m+1}}{(2m+1)!}\Big(\int_{t_0}^t dt' b(t')\Big)^{2m+1} \cos\alpha, \tag{29}$$

$$R_y^{(2m+1)}(t,t_0) = \frac{(-1)^{m+1}}{(2m+1)!}\Big(\int_{t_0}^t dt' b(t')\Big)^{2m+1} \sin\alpha . \tag{30}$$

These result in

$$\vec{\phi}(t,t_0) = -\phi(t,t_0)\Big(\cos\alpha,\ \sin\alpha,\ 0\Big) , \qquad \phi(t,t_0) = \int_{t_0}^t dt' b(t') . \tag{31}$$

Substituting this equation into Eq. (25) yields the explicit expression of the evolution operator $U(t,t_0)$

$$U(t,t_0) = e^{-i\sigma_z \int_0^t dt' a(t')} e^{-i(\sigma_x \cos\alpha + \sigma_y \sin\alpha)\int_{t_0}^t dt' b(t')} e^{i\sigma_z \int_0^{t_0} dt' a(t')} . \tag{32}$$

This solution describes the resonant transition between two states, and can be widely applied to different fields.

Another example we want to show here is that of a periodically driven two-level system which undergoes multiple crossings [14,15]. In this model, $a(t) = A\cos\Omega t$, $b(t) = W$, and $\omega(t) = 0$, where W is a coupling constant, A and Ω are, respectively, amplitude and frequency of driving fields. This two-level system can be realized by coupling two propagation or two polarization modes of an optical ring resonator as done by Spreeuw *et al.*. By means of this optical ring resonator, they observed Landau-Zener transitions, Rabi oscillation with non-rotating-wave approximation signature, and Autler-Townes doublets [14]. Therefore it is meaningful to provide a general theoretical analysis for this problem.

Without loss of generality, we put $t_0 = 0$, and let the initial state $|\psi(0)>$ be $|\psi(0)> = \begin{pmatrix}1\\0\end{pmatrix} \equiv |1>$. Then we find that the probability $P(t)$ for the system staying in $|2> \equiv \begin{pmatrix}0\\1\end{pmatrix}$ at time t is

$$P(t) = \Big(\sum_{m=0}^{\infty} R_x^{(2m+1)}(t,0)\Big)^2 + \Big(\sum_{m=0}^{\infty} R_y^{(2m+1)}(t,0)\Big)^2 . \tag{33}$$

In the following discussion, we focus on the situation of high-frequency driving fields, assuming that $W << \Omega$. In that case, as the first-order approximation of W/Ω, we get

$$\sum_{m=0}^{\infty} R_x^{(2m+1)}(t,0) \simeq R_x^{(1)}(t,0) = -WJ_0(2A/\Omega)t - \frac{W}{\Omega}\sum_{m=1}^{\infty} J_{2m}(2A/\Omega)\frac{\sin(2m\Omega t)}{m} , \tag{34}$$

$$\sum_{m=0}^{\infty} R_y^{(2m+1)}(t,0) \simeq R_y^{(1)}(t,0) = 2\frac{W}{\Omega}\sum_{m=1}^{\infty} J_{2m-1}(2A/\Omega)\frac{1-\cos[(2m-1)\Omega t]}{2m-1} , \tag{35}$$

where J_m is the ordinary Bessel function. These results are valid only in a small time region because the first term in Eq. (34) grows with time linearly up to infinity, unless the ratio $2A/\Omega$ is a root of J_0. This means that higher-oredr corrections of $(W/\Omega)^m$, $(m > 1)$ are of importance, and should be included in the calculation. By the use of Eqs. (17), (18), (26) and (27), and through a long but straightforward calculation we obtain

$$\sum_{m=0}^{\infty} R_x^{(2m+1)}(t,0) = -\sin[WJ_0(2A/\Omega)t] - B(t) , \tag{36}$$

$$\sum_{m=0}^{\infty} R_y^{(2m+1)}(t,0) = -\frac{W}{\Omega}\Big\{\cos[WJ_0(2A/\Omega)t]\sum_{m=1}^{\infty} J_{2m-1}(2A/\Omega)\ \frac{2\cos[(2m-1)\Omega t]}{2m-1} + D(t)\Big\} , \tag{37}$$

where it can be shown that at the initial time $t = 0$

$$B(0) = 0 , \qquad D(0) = -\sum_{m=1}^{\infty} J_{2m-1}(2A/\Omega)\frac{2}{2m-1} , \tag{38}$$

and at time $t \neq 0$ and $J_0(2A/\Omega) \neq 0$

$$|B(t)| < \Big|\sin[WJ_0(2A/\Omega)t]\Big| , \tag{39}$$

$$|D(t)| < \Big|\cos[WJ_0(2A/\Omega)t]\sum_{m=1}^{\infty} J_{2m-1}(2A/\Omega)\ \frac{2\cos[(2m-1)\Omega t]}{2m-1}\Big| . \tag{40}$$

Thus the probability $P(t)$ reads

$$P(t) = \Big\{\sin[WJ_0(2A/\Omega)t] + B(t)\Big\}^2 + \Big(\frac{W}{\Omega}\Big)^2\Big\{\cos[WJ_0(2A/\Omega)t]\sum_{m=1}^{\infty} J_{2m-1}(2A/\Omega)\ \frac{2\cos[(2m-1)\Omega t]}{2m-1} + D(t)\Big\}^2 . \tag{41}$$

Obviously, if $J_0(2A/\Omega) \neq 0$, the first term $\sin^2[WJ_0(2A/\Omega)t]$ in Eq. (41) will dominate the dynamics for $P(t)$ due to the facts $W << \Omega$ and (39). Therefore we can rewrite the probability $P(t)$ approximately as

$$P(t) \simeq \sin^2[WJ_0(2A/\Omega)t] \tag{42}$$

This implies that the condition $J_0(2A/\Omega) \neq 0$ is of significance for the observation of the particle staying in $|2>$ at time t. The inverse situation is more intriguing. If the $2A/\Omega$ is a root of J_0, the probability to find the particle in $|2>$ vanishes almost entirely. This feature shows that, in striking contrast to our intuition, the particle can not tunnel to $|2>$ from the initial state $|1>$. It will remain localized in $|1>$ at the whole time of driving process if $J_0(2A/\Omega) = 0$. This phenomenon of dynamic localization is believed to be relevant to the coherent destruction of tunneling found by Grossmann *et al.* [16], and coincides with recent analysis for a similar model given by Kayanuma [17].

In summary, we have investigated the evolution behavior of two-level systems described by arbitrary time-dependent Hamiltonians. By means of the technique of the Lie algebra $SU(2)$, we obtained the closed-form solution of the evolution operator. From this general and exact result, the solutions for any special case can always be deduced. The application of our approach to a periodically driven two-level system yields the apperance of dynamic localization, which coincides with the onset of coherent destruction of tunneling. Since the two-level problem has very strong background on physics, our method presented in this paper will have potential applications to different fields.

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