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Chiral Symmetry on a Lattice

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Abstract

We show that on a lattice, chiral symmetry is a part of a larger invariance algebra. In consequence, we argue, the realisations of chiral symmetry are tied to this algebra. A remarkable aspect of this invariance is that the Fermi-sea is destroyed, and is replaced by a real-space paired quark-antiquark state.

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Introduction

Chiral symmetry has played an important part in our understanding of the physics of the strongly interacting matter[1]. Since the strong interactions are not amenable to perturbative methods in continuum field theories, these systems are conveniently handled on lattices. The chiral symmetry and its remnants on a lattice, have in the process been studied a great deal[2].

For light quarks, such as the u or the d, the Hamiltonian consists of a kinetic energy term, plus the interactions. The mass term, i.e. self-energy, is absent in the starting Hamiltonian. The theory has an exact chiral symmetry. A mass term, i.e. self-energy, breaks chiral symmetry explicitly. It is however believed, though not understood, that even in the absence of an explicit mass term, chiral symmetry breaks spontaneously, and a mass term is generated dynamically in the system.

On a lattice kinetic energy leads to hopping terms that allow quarks to hop from site to nearest neighbor site. The self-energy, i.e. the mass is the onsite bilinear term ; then of course, we have the interactions. If we concentrate for now on hopping and self-energy, clearly there is some symmetry, akin to chiral symmetry, that the hopping term does not share with self-energy. It is the purpose of this work to explore these symmetries of the hopping term.

There has been, in the recent past, an exploration[3] of the symmetries of hopping in the context of the Hubbard model. It has been shown that the kinetic energy term has a large symmetry on a lattice, and that symmetry generators naturally provide a framework for construction of multiparticle stable states. Indeed, if the interactions are included, an interesting subset of these multiparticle states, namely the singlets, continue to be solutions of this complete Hamiltonian. These singlet configurations, in the instance of the Hubbard model, lie in the middle of the spin density-wave (SDW) gap[3]. Thus, if particles or holes are introduced on to a half-filled state, they would preferably go into these coherent, singlet configurations. The other non-singlet coherent states are not solutions of the complete Hubbard Hamiltonian, including interactions.

There are three reasons for pursuing these ideas into relativistic quark dynamics. First, the Hubbard model, in its essence, and the theory of quarks are based on notions of gauge theory; both have particle interactions spatially limited due to screening. In the case of the Hubbard model, the coupling U, even if large, it is the scale of fluctuations about the singlet configuration that is of relevance. For quark-dynamics, the theory has strong interactions, but is asymptotically free. Both have metal-insulator, i.e., deconfinement transitions.

Second, both the theories have real space pairing of fermions that show remarkable properties. For the Hubbard model these are the coherent states that lead to real space superconductivity. For quarks the property of confinment is an established article of faith.

And third, that the Hubbard model has spin-singlet, real-space paired coherent solutions, and similar real space pairs have been shown to be solutions for the theory of Dirac quarks on a lattice[4]. There is, however, an extra symmetry of massless Dirac quarks that has not been discussed in this context. It is the chiral symmetry. That is the subject of this work.

In Section I we discuss the symmetry basis in the context of the Hubbard model. The algebra of the generators of this symmetry is explored and the reason the singlet state is a solution, in a mean-field sense of the complete Hamiltonian is discussed.

In Section II we rediscover this algebra in the context of Dirac equation on a lattice. It is shown that the chiral symmetry generator is part of this algebra. While both the Hubbard model and the Dirac equation are discussed on a 1+1 D lattice, the results are good for arbitary dimensional systems.

In Section III we discuss some consequences of this symmetry.

Section-I : The Symmetry Basis.

Our aim is to explore the symmetries that belong to hopping, but not to self-energy. To that purpose we digress briefly to the Hubbard model.

This model, proposed to elucidate the electron correlations and the consequent metal insulator transitions, has two parts. The Hamiltonian is :

$$H = -t \sum_{\substack{(ij)\\ \sigma}} (c^{+}_{i\sigma} c_{j\sigma} + h.c.) + U \sum_{i} n_{i\uparrow} n_{i\downarrow} \quad \ldots\ldots (1)$$

Where $c_i^{\dagger}$ creates an electron at site i, (ij) are the nearest neighbor sites, σ , the spin index; n_i is the number operator at site i.

The first term allows electrons to move from site to site; the second denotes the repulsive energy, U, that it takes for two electrons to be at the same site; the electrons that are further apart do not interact due to screening.

To analyse the symmetries of hopping, we concentrate on the first term of (1). It has been shown elsewhere[3] that if we construct a generator, e_1, as follows :

$$e_1 = \sum T_{ij}(\eta). \qquad \ldots\ldots (2)$$

where,

$$T_{ij}(\eta) = T_{ij}(+) = \pm T_{ij} \ \ldots (3)$$

$$T_{ij} = c^{\dagger}_{i\uparrow} c^{\dagger}_{j\downarrow} + c^{\dagger}_{j\uparrow} c^{\dagger}_{i\downarrow} \ \ldots\ldots (4)$$

If η is now chosen to alternate on links in this 1+1 dimensional lattice, e_1, then commutes with the hopping Hamiltonian. Note that e_1 creates a spin-singlet pair on the lattice, with identical amplitude for being on any link. The meaning of e_1 becomes clearer when we look at it in the Bloch basis. Essentially e_1 is a singlet pair that moves through the lattice with a center of mass momentum of $\pm\pi$; therefore has phases that alternate on links.

From e_1 we can construct e_{-1}, defined as

$$e_{-1} = e_1^{\dagger}, \ \ldots\ldots\ldots \quad (5)$$

which also commutes with hopping. From e_1 and e_{-1} we can generate the complete global symmetry basis of the hopping term as follows. Considering the commutator of e_1 and e_{-1} we obtain further commuting generators :

$$h_1 = \sum c^{+}_{i\sigma} c_{i\sigma}, \;\ldots\ldots \quad (6),$$ the global number and

$$h_2 = \sum c^{+}_{i\sigma} c_{(i+2)\sigma} \;\ldots\ldots \quad (7),$$ i.e., a translation by two lattice units.

Now, commutator of h_2 with $e_{\pm 1}$ gives us $e_{\pm 2}$, where

$$e_{+2} = \sum T_{i,i+3}(\eta) \;\ldots\ldots (8)$$

By repeating this procedure we get a closed algebra of global generators under periodic boundary conditions.

In (2) we created a spin singlet pair, but in as far as the hopping Hamiltonian is concerned there is nothing special about a singlet. We argue shortly, that when the interactions are present, such as in (1), only the singlet state continues to be a solution, in a mean-field sense, of the complete Hamiltonian. Thus, for hopping, we have shown[3] that generators, such as,

$$\hat{e}_1 = \sum_{(ij)} \hat{T}_{ij}(\eta); \quad \hat{T}_{ij} = c^{+}_{i\uparrow} c^{+}_{j\uparrow} \;\ldots\ldots \quad (9)$$

$$\check{e}_1 = \sum_{(ij)} \check{T}_{ij}(\eta); \quad \check{T}_{ij} = c^{+}_{i\downarrow} c^{+}_{j\downarrow} \;\ldots\ldots \quad (10)$$

$$\bar{e}_1 = \sum_{(ij)} \bar{T}_{ij}(\eta); \quad \bar{T}_{ij}(\eta) = c^{+}_{i\uparrow} c^{+}_{i\downarrow} + c^{+}_{i\downarrow} c^{+}_{j\uparrow} \;\ldots\ldots \quad (11)$$

all lead to closed global algebras.

The generators e_{+n} in (2;8) create a singlet pair that moves through the lattice at a center of mass momentum of $\pm\pi$. For e_{+1} the pair is made of two electrons separated by lattice spacing a; For e_{+2} the separation is 3a and so on. The translation generators and the global number together form a commuting set. Note that the vacuum state $|0\rangle$ is an eigenstate of the global number with eigenvalue zero. The state $|0\rangle$ of the medium is degenerate with the zero of band energy, that is the middle of the band.

At this point it is instructive to write the generators explicity in Bloch basis. The transformation that changes Wannier into Bloch state is :

$$c_k^+ = \frac{1}{N} \sum (\exp(ikx_i))c_{x_i}^+ \quad .... \quad (12)$$

therefore,

$$e_{+1} = \sum_k (\exp(ik))(c_{k\uparrow}^+ c_{\pi-k\downarrow}^+ - c_{k\downarrow}^+ c_{\pi-k\uparrow}^+) \quad .... \quad (13)$$

Here, we have set the lattice spacing to unity. The first exponential factor in the sum on the right hand side of equation (13) ensures that the electrons separated by a single lattice spacing pair into a singlet in e_{+1}. For e_{+2}, for example, this exponential factor changes to expi[3ka]. Thus, if e_{+1} is operated on $|0\rangle$, from (13), we obtain a state that has electrons of momentum k, pairing into a singlet with electrons of momentum π-k. The dispersions show that the pair has zero energy.

The pairs move with a center of mass momentum of π . The overall energy of this state, as expected, is the same as $|0\rangle$.

A state, such as $e_1|0\rangle$, is an eigenstate of global number with eigenvalue of two. That means a single pair is being created on the average. Further, since the commutator of global number, h_1, with $e_{\pm n}$ is:

$$[h_1, e_{\pm n}] = \pm 2\, e_{\pm n}; \quad \ldots\ldots\ldots \quad (14)$$

therefore, e_+ and e_- and global number have identical algebraic structure as c^+, c and local number. It is, therefore, tempting to construct a general multipair coherent state, denoted $|z\rangle$, of the form :

$$|z\rangle = (\exp(\sum_\beta z_\beta e_\beta - h.c.))|0\rangle, \quad \ldots\ldots \quad (15)$$

where z are a set of complex numbers.

Following, from equation (13), such a coherent state is represented in Bloch basis as :

$$|z\rangle = (\exp(\sum_\beta z_\beta \sum_k \exp(i(2\beta-1)k)(c^+_{k\uparrow} c^+_{\pi-k\downarrow} - c^+_{k\downarrow} c^+_{\pi-k\uparrow})$$

$$-h.c.))|0\rangle \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

There are two aspects of these $|z\rangle$ states that are remarkable. First, we know that for a purely hopping model the state that we usually construct is the fermi-sea $|FS\rangle$. The generators of symmetry of hopping, namely $e_{\pm n}$, displayed for example in eqn. (13), are such that :

$$e_n\, |FS\rangle = 0 \quad \ldots\ldots\ldots \quad (17)$$

Further, a completely filled fermi-sea, i.e., a filled band of electrons, has the same energy as the state $|0\rangle$ (since half the electrons lie below and the other half above $|0\rangle$. However, $|z\rangle$ is also degenerate with $|0\rangle$.

We, therefore, have a situation analogous to what is usually encountered in spontaneous symmetry breaking in that :

(1) we have degenerate ground states and

(2) the usual ground state, in this instance the $|FS\rangle$, is destroyed by some of the generators of symmetry.

Presumably the $|FS\rangle$ and the $|z\rangle$, even though degenerate in energy, lie in different sectors of the theory. In the sector of the fermi sea we have unpaired particles with its characteristic single particle excitation spectrum. On $|z\rangle$ we have a paired ground state with a completely different spectrum of states.[4] The $|z\rangle$ states have real-space singlet pairs. To the hopping term we can add, if we wish, a self-energy, i.e. a mass term. A mass term is directly proportional to the global number h_1, which as we know, is an element of the algebra. The other translation generators h commute with h_1. However, the pairing generators do not; their commutators with h_1 are given in eqn. (14). Therefore, the following scenario is obtained. The complete algebra is no longer a symmetry of the Hamiltonian. Interestingly, however, the following points need to be noted.

A. Even though $e_{\pm n}$ do not commute with h_1, a state formed by operating $e_{\pm n}$ on $|0\rangle$ is an eigenstate of h_1. This is obtained from eqn. (14).

B. In consequence, therefore, even in the absence of an explicit self-energy term in the Hamiltonian, self-energy has a non-zero expectation value in the $|z\rangle$ states. Thus, a mass term is spontaneously generated in this $|z\rangle$ phase.

C. Since h_1 is an element of an algebra represented by $|z\rangle$, it has a constant average value in the $|z\rangle$ states.

Let us investigate what happens when interactions of the type specified in (1) are included. We show now that in a mean field sense only the singlet $|z\rangle$ state survives : the other non-singlet $|z\rangle$ states made of generators listed in (9-11) do not survive the gauge interactions.

The spin-operator at site i, $\vec{S}_i$, may be represented as :

$$\vec{S}_i = \tfrac{1}{2} c^+_{i\sigma} \vec{\tau}_{\sigma\sigma'} c_{i\sigma'} \quad \dots\dots\dots\dots\dots \quad (18)$$

Where τ are the Pauli-matrices. Note that $|z\rangle$ states are representations of an algebra whose elements include the global number h_1. Thus, the expectation value of h_1 is a constant over the $|z\rangle$ states. Thus, the Hamiltonian (1) may be recast[3] in terms of the spin-variables as :

$$H = -t \sum (c^+_{i\sigma} c_{j\sigma} + h.c.) - (\tfrac{2}{3}) U \sum S_i^2 \quad \dots.. \quad (19)$$

Now, if we assume that the fluctuation around an average spin-state $\langle S_i \rangle$ is small, the mean-field Hamiltonian becomes :

$$H_{MF} = -t \sum (c^{\dagger}_{i\sigma} c_{j\sigma} + H.C.) + \tfrac{2}{3} U \sum \langle S_i \rangle^2$$

$$- \tfrac{4}{3} U \sum_i \langle S_i \rangle . S_i \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

For an uniform $\langle S_i \rangle = \langle S \rangle$, the last term is proportional to the total spin operator. For the singlet $|z\rangle$ state, $\langle S_i \rangle = 0$; thus the interaction terms add up to zero. For the other non-singlet $|Z\rangle$ states, they are not eigenstates of the total spin operator; therefore, are not solutions of the mean-field Hamiltonian.

In presence of interactions it is known that the usual band splits at the middle due to spin density waves in the system. At the middle of this gap we therefore have this singlet $|z\rangle$ state . On a half-filled band if charge carriers are introduced, they could go to these $|z\rangle$ states.

For a variety of reasons a symmetry that has played an important part in our understanding of the low energy hadronic spectrum is chiral symmetry, which, somewhat analogous to what we have discussed so far, is a symmetry of the kinetic energy (hopping), but not of mass (self-energy) terms. We now investigate the Dirac Hamiltonian on a 1+1 D lattice and rediscover the algebra that we have explored. We show further that the chiral symmetry generator is an element of this algebra.

We know that a realisation of chiral symmetry leads to a ground state with paired fermions. It thus appears that the quarks such as the u and the d, when put together do not make a fermi sea, unlike most fermions that we know. Instead quark-antiquark pairs populate the ground state of the system .

We remark in passing that this scenario is to be distinguished from that of neutrino-matter, where pairing occurs between fermions and holes in the ground state[5,6]. Similar particle-hole paired ground state also occurs for electron gas[7].

Section-II : Chiral Symmetry on a Lattice.

The theory of strong interactions, quantum chromodynamics (QCD), because of chiral symmetry in light quark sector, has a ground state that has pairs of fermions. It has been argued that this indicates a spontaneous breakdown of chiral invariance in the up and the down sector of the theory. There have been numerous evidence that this paired fermion basis is a consequence of Goldstone realisation of chiral symmetry that results then in modes such as the pions. Since we do have a paired fermion basis, it is interesting for us to explore to what extent chiral symmetry is linked to the invariance generators that we have explored thus far.

The Dirac matrices of interest[2] for this case are chosen as :

$$\gamma_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \alpha = \gamma_5 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \ldots\ldots\ldots (21)$$

A two component Dirac fermion ψ may be decomposed, using the γ_5 matrix into two states that are eigenstates of the γ_5 matrix. These two states denoted ψ_+ and ψ_- are defined as :

$$\Psi = \binom{\Psi_1}{\Psi_2} = \tfrac{1}{2}(1+\gamma_5)\,\psi + \tfrac{1}{2}(1-\gamma_5)\,\psi = \psi_+ + \psi_- \quad \ldots\ldots\ldots \quad (22)$$

Thus,

$$\Psi_+ = \tfrac{1}{2}\binom{\Psi_1+\Psi_2}{\Psi_1+\Psi_2} \quad \ldots\ldots\ldots \quad (23)$$

and

$$\Psi_- = \tfrac{1}{2}\binom{\Psi_1-\Psi_2}{-\Psi_1+\Psi_2} \quad \ldots\ldots\ldots \quad (24)$$

The Hamiltonian for this system of massless Dirac fermions is :

$$H = i\,\psi^+\alpha\,.\,\partial\Psi \quad \ldots\ldots\ldots \quad (25)$$

In terms of Ψ_+ and Ψ_-, the Hamiltonian is expressed in the form :

$$H = i\psi_+^+\,\partial\Psi_+ - i\,\psi_-^+\,\partial\Psi_- \quad \ldots\ldots\ldots \quad (26)$$

On a lattice, therefore, we can write the Hamiltonian in the form :

$$H = \frac{i}{2}\,\Psi_+^+(n)\,[\psi_+(n+1) - \psi_+(n-1)] \quad \ldots\ldots\ldots \quad (27)$$

$$- \frac{i}{2}\,\psi_-^+(n)\,[\psi_-(n+1) - \psi_-(n-1)]$$

We have, therefore, two sets of generators of symmetry, as follows .

In respect of the $\Psi_\pm$ objects we have

$$e^{\pm}_{+1} = \sum \psi_\pm^+(n)\,\psi_\pm^+(n+1) \quad \ldots\ldots\ldots \quad (28)$$

$$h_1^{\pm} = \sum_n \Psi_{\pm}^{+}(n)\, \Psi_{\pm}(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

$$h_2^{\pm} = \sum_n \Psi_{\pm}^{+}(n)\, \Psi_{\pm}(n+2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

$$e_{+2}^{\pm} = \sum_n \Psi_{\pm}^{+}(n)\, \Psi_{\pm}^{+}(n+3) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

and so on. We, therefore, have recovered exactly the algebra we had before in the context of the Hamiltonian (1). There are, however, two concurrent identical algebras in respect of the right-handed and the left-handed objects. Note, further, that factoring an i out in the Hamiltonian (25) has resulted in the phase η, that alternated in (2), becoming uniform through the links. The dispersions of $\Psi_{\pm}$ need to be taken note of[2]. We show now that the chiral symmetry generator is an element of the algebra we discussed in Section-I.

Let us look at the elements $h_1^{\pm}$. From these two generators we can, by linear combination, obtain

$$\sum_n (\psi_+^{+}(n)\, \psi_+(n) - \psi_-^{+}(n)\, \psi_-(n) \quad \ldots\ldots\ldots\ldots \quad (32)$$

$$= \sum \psi^{+}(n)\, \gamma_5\, \psi(n)$$

which is the generator of chiral rotations on a lattice.

It is instructive at this point to pause to take a careful look at the generators (28-31) of the algebra. First, as we have already noted the paired basis does not have the necessity of an η phase as in (2). That effectively means that the pairs, when formed, are static. In other words, the center of mass momentum of the pair is zero. Since the pairs have the same energy as $|0\rangle$, a fermion of momentum k couples to another of momentum -k, giving rise to an overall zero energy pair with no center of mass momentum. These static pairs contrast with the pairs of the Hubbard model that travel with c.m momenta of $\pm\pi$ through the lattice. Both the pairings, be it quarks or the Hubbard electrons, are in real space. It is interesting that in nature the pairings that we find for quarks are in real space; and holds also for high-temperature superconductivity.

Second, the generators of the h type that we encountered in the context of the Hubbard model, even though are translation generators, they in them have the possibility of creating electron-hole pairs. In the context of the relativistic Dirac Hamiltonian, these generators could create in addition particle-antiparticle pairs in the ground state. Since these have no overall phases, such as η, these are all static pairs with no center of mass momentum. Thus, for example, an up quark of momentum k is to pair with an antiup of momentum -k. However, to create a quark-hole or an electron-hole[6] pair it will be necessary to operate the h generators onto a fermi sea $|FS\rangle$; not on $|0\rangle$. Since in this work we are exploring alternatives to $|FS\rangle$, the particle-hole pairs need not worry us.

Section-III : Conclusions.

Chiral symmetry on a lattice is more than the mere on-site chiral rotation generator (32) that we constructed. Indeed for any pair of generators $h_i^{\pm}$ $(i\neq 1)$ we can construct a chiral translation through an even number of sites; and all these are symmetry transformations of the Hamiltonian. This may be done by creating the linear combination $h_i^{+}-h_i^{-}$. In a lattice, therefore, discrete translations, discrete chiral rotations and discrete chiral translations are symmetries. These $h_i^{\pm}$ operations in association with the $e_{\pm i}$ generators form a closed algebra.

What sort of a ground state do we envisage for this system? Well, we can always construct the $|FS\rangle$ of the up and the down quarks, but we have discussed already that we are not interested in this sort of fermi liquid states. The symmetries effectively provide us a way out of these fermi-liquids into states of paired basis.

The states of h_1 are in a way special. First, $|0\rangle$ is an eigenstate of this operator. Further, though h_1 can create an on-site quark-antiquark pair, but such a pair is going to simply annihilate, because they are on the same site. However, in $|0\rangle$ we can have quark-antiquark pairs generated by the translation generation h_i $(i\neq 1)$. Since they pair quarks and antiquarks separated by an even number of links, and the pairs are static, such an wavefunction, $|\bar{0}\rangle$, in terms of quark and antiquark creation operators a^+ and b^+ is :

$$|\bar{0}\rangle = (\exp(\sum_{\beta'} z_{\beta} \sum_{k} \exp\ (2i\beta k)(a_k^+ b_{-k}^+ \ - \text{h.c.})))\ |0\rangle \ldots. \quad (33)$$

Such a state $|\bar{0}\rangle$ is an eigenstate of global fermion number with eigenvalue zero, because there are exactly equal number of quarks and antiquarks. This is our fiducial state. On this we can operate with $e_{\pm i}$ arbitrary number of times to construct our ground state wave functions. Thus, the final wave function of the base state is of the type :

$$|z\rangle = (\exp(\sum_{\alpha} z_{\alpha} \sum_{k} \exp\ (i(2\alpha-1)k)(\Psi^{+}_{k}\Psi^{+}_{-k} - h.c.)))\ |\bar{0}\rangle \quad \ldots\ldots (34)$$

However, one might argue that this Z-liquid could not perhaps exist, because, first it is a color nonsinglet, and further, interactions have not been taken into account.

Well, the interactions contain anomalies, but in these asymptotically free theories it is possible to conceive of scenarios where the interactions have practically disappeared. So when the bag tears open, the liquid that oozes out could well be this Z.

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