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THE NAVIER-STOKES LIMIT OF STATIONARY SOLUTIONS OF THE NONLINEAR BOLTZMANN EQUATION*

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Abstract. We consider the flow of a gas in a channel whose walls are kept at fixed (different) temperatures. There is a constant external force parallel to the boundaries which may themselves also be moving. The system is described by the stationary Boltzmann equation to which are added Maxwellian boundary conditions with unit accommodation coefficient. We prove that when the temperature gap, the relative velocity of the planes and the force are all sufficiently small, there is a solution which converges, in the hydrodynamic limit, to a local Maxwellian with parameters given by the stationary solution of the corresponding compressible Navier-Stokes equations with no-slip boundary conditions. Corrections to this Maxwellian are obtained in powers of the Knudsen number with a controlled remainder.

Keywords: Hydrodynamical limit, stationary Navier-Stokes equations, kinetic theory.

1. Introduction.

The behavior of macroscopic systems in steady nonequilibrium situations is a subject of great intrinsic and practical interest and one which was close to Onsager's heart [1]. The simplest cases are those which have some symmetries. These include the uni-directional flow between parallel plates or coaxial cylinders in which the steady nonequilibrium flow is maintained by an external body force or pressure gradient, and/or by translating the walls at some prescribed speed, as in the classical Poiseille and Couette flows. The hydrodynamic description of such systems has been much studied and the stability properties of the flow for small values of the control parameters are known. The appearance of instabilities, for some critical values of the parameters, is also proven, at least for the linearized equations [2].

While much less is known about these problems from the microscopic point of view. Onsager was able to use properties of the microscopic dynamics to derive exact results about the symmetry properties of the transport coefficients appearing

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in the linear hydrodynamic equations. To do this he had to make a very plausible assumption about the equivalence of transport of matter, heat, etc. resulting from the regression of spontaneous fluctuations in an equilibrium system and that induced by macroscopic gradients or forces which obey linear laws. The validity of these linear laws, such as Fourier's law of heat conduction, and of the hydrodynamic description itself was then as now based on experiments rather than derived from the more fundamental laws governing the motion of atoms or molecules. To actually derive the hydrodynamic equations in a mathematical, rigorous way from the underlying microscopic dynamics is a formidable task which is now in an active but still early stage of development [3,4].

The study of these problems at the kinetic level of the Boltzmann equation is an intermediate step in this program. It is useful from the conceptual point of view because, while many of the features of the microscopic description survive, the mathematical analysis is simpler than the fully microscopic one. In addition, it is also of practical interest in situations in which the fluid is sufficiently rarefied for the Boltzmann equation to give an accurate description of the microscopic state. The hydrodynamical behavior away from boundary layers or shocks is recovered by expanding in the Knudsen number, the ratio of the mean free path to the scale of macroscopic gradients. Such expansions have been extensively investigated, and we refer to [5] and references quoted therein. The validity of such an expansion, relative to the Euler behavior, in the time dependent case without boundaries, was proven in [6, 7, 8]. One of the difficulties in dealing with stationary problems is due to the fact that the boundary is essential and in a thin layer (of the size of the mean free path) near the boundary the space variations are not as slow as the hydrodynamical ones. Therefore one has to deal with a boundary layer expansion too. In [9] the two intertwined expansions are discussed in the case of the thermal layer.

In a recent paper [10], we considered the case of a gas between two walls subject to a force parallel to the walls. The walls were held at equal temperatures and there where no-slip boundary conditions. We proved there, for a sufficiently small force, the validity of a truncated expansion in the Knudsen number, whose lowest order is the local Maxwellian with parameters satisfying the hydrodynamical equations (the stationary compressible Navier-Stokes equations). The next orders involve boundary layer corrections as well as kinetic corrections in the bulk; the first order kinetic corrections are actually responsible of the dissipative effects and determine the form of the hydrodynamical equations. The proof in [10] used explicitly the symmetry between the two walls which prevents its direct application to more general situations. In this paper we extend the above results to the case in which the two planes are at different temperatures and can move with respect to each other, provided that the difference in temperature and the relative velocity, as well as the force, are small enough. In this more general situation we need to modify the proof to take into account boundary terms which were absent in [10] because of symmetry. This will be discussed in section 3 where we sketch the proof, pointing out the necessary modifications, while we refer to [10] for details. A complete formulation of the problem and a precise statement of the results is given in section 2.

We note here that the case of the fluid between two coaxial, non rotating cylinders, at different temperatures and subject to an external force parallel to the axis, can be reduced to a form very similar to the one discussed in this paper, so that our results apply also to this case. Not covered in this paper is the case of a channel with a force orthogonal to the walls or that of rotating cylinders, that will be presented in a forthcoming paper [11]. We note alas that the restriction of our result to the small values of the external driving parameters prevents its application to the most interesting situations in which instabilities arise. Finally we mention that, like for the hydrodynamical equations, explicit steady solutions [12–14] are available in some special cases for the BGK model and for the Boltzmann equation for Maxwell molecules. While such solutions are found for all Knudsen numbers and also for large values of the external parameters, they are only valid in the bulk. Since they do not match the boundary conditions, their range of applicability is effectively reduced to the case of small Knudsen numbers. The stability of such solutions is still an unexplored field.

2. Formulation and results.

We consider the stationary Boltzmann equation for the distribution function $f(r', v)$ on the space scale of the mean free path in the presence of an external force G:

$$v \cdot \nabla' f + G \cdot \nabla_v f = Q(f, f). \tag{2.1}$$

The velocity of the particles, $v = (v_x, v_y, v_z)$ is in $\mathbb{R}^3$, while the position r' is in a three dimensional slab $\Omega_\varepsilon = \{(x', y', z') \in \mathbb{R}^3 \text{ s.t. } |y'| < \varepsilon^{-1}\}$; ε^{-1} the size of the box in microscopic units, over which there are significant variations in temperature, velocity, etc., will be the scaling parameter. ∇' denotes the gradient with respect to (x', y', z'). $Q(f, g)$ is the usual Boltzmann collision operator for a hard spheres gas; we refer to [10] for all details. Since we are interested in the solutions of (2.1)

in the limit $\varepsilon \to 0$, it is convenient to rewrite it in rescaled (macroscopic) space coordinates $(x, y, z) = \varepsilon(x', y', z')$. In the new variables (2.1) becomes

$$v \cdot \nabla f + \frac{1}{\varepsilon} G \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f, f), \tag{2.2}$$

and the space domain becomes $\Omega = \{(x, y, z) \in \mathbb{R}^3 \text{ s.t. } |y| < 1\}$. The walls (i.e. the planes $y = \pm 1$) are assumed to be at fixed temperatures $T_\pm$, say $T_+ \geq T_-$, and to move parallel to the xz-plane at speeds $U_\pm$. We model collisions with the walls via Maxwell boundary conditions with unit accommodation coefficient, i.e. we assume that the distribution of "incoming" velocities after a collision with the walls is given by

$$f(-1, v) = \alpha_- \overline{M}_-(v), \quad v_y > 0, \tag{2.3}$$

$$f(1, v) = \alpha_+ \overline{M}_+(v), \quad v_y < 0, \tag{2.4}$$

with $f(\pm 1, v) \equiv f(x, \pm 1, z, v)$ and

$$\overline{M}_\pm(v) = \frac{1}{2\pi T_\pm^2} e^{-[v - U_\pm]^2 / 2T_\pm}, \tag{2.5}$$

normalized so that $\int_{v_y \gtrless 0} |v_y| \overline{M}_\pm(v) dv = 1$.

The $\alpha_\pm$ depend on the distribution of "outgoing" velocities in such a way that the net current to the walls vanishes,

$$\langle v_y f \rangle \equiv \int_{\mathbb{R}^3} v_y f(\pm 1, v) dv = 0 \quad \text{for } y = \pm 1, \tag{2.6}$$

Condition (2.6) and the normalization of $\overline{M}_\pm$ imply:

$$\alpha_\pm = \pm \int_{v_y \gtrless 0} v_y f(\pm 1, v) dv \tag{2.7}$$

Namely, $\alpha_\pm$ represent the outgoing (from the fluid to the walls) fluxes of mass in the direction y. More general boundary conditions could be allowed (see for example [5]), but we restrict ourselves to this case for the sake of concreteness.

In the following, as in [10], we assume that the force field is in the x-direction and has a strength of order ε^2, i.e. $G = (\varepsilon^2 F, 0, 0)$. We also assume that F is constant in space. The scaling factor ε^2 for the force is required, as discussed in [10], to get stationary solutions. Larger forces cannot in general be equilibrated by the boundary dissipative mechanisms of the Boltzmann fluid. We look for solutions of (2.2) depending only on the y space coordinate. The Boltzmann equation then becomes

$$v_y \frac{\partial f}{\partial y} + \varepsilon F \frac{\partial f}{\partial v_x} = \frac{1}{\varepsilon} Q(f, f) \tag{2.8}$$

with $f \geq 0$ and

$$\int_{-1}^{1} dy \langle f \rangle = m \tag{2.9}$$

for some positive constant m. We use the notation $\langle g \rangle = \int_{\mathbb{R}^3} g(v)dv$. Note that the space variables x and z can be restricted to a square with periodic boundary conditions, without any change in the equations.

All the result of this paper extend immediately to a case of a fluid between two coaxial cylinders of macroscopic radii $a_1 < a_2$. In this case we use cylindrical coordinates (r, ϕ, x) and substitute $v_r(\partial f/\partial r)$ for $v_y(\partial f/\partial y)$ in (2.8). The boundary conditions are now given for $r = a_1$ and $r = a_2$, for $v_r > 0$ and $v_r < 0$ respectively. Setting $a_1 = 0$, the condition for $r = a_1$ is replaced by the condition that the solution on the axis be even in v_r, a situation which resembles more the case discussed in [10].

The hydrodynamical regime

When ε is small the solution of the Boltzmann equation is expected to describe behavior close to the hydrodynamical one, in the sense that, to the lowest order, f is given by a local Maxwellian, with parameters determined by the solution of a set of hydrodynamical equations. At higher order in ε there are both bulk and boundary layer corrections. The proof of this assertion for the boundary value problem (2.3)-(2.8) is the main result of this paper.

In [10] we considered the situation $T_+ = T_-$, $U_\pm = 0$. This has the symmetry $(y, v_y) \to (-y, -v_y)$, which was used heavily in the proofs. In this paper we extend the proof in [10] to the case where there is no such symmetry. We prove that when the force, the difference of temperature and the relative velocity of the planes are sufficiently small then it is possible, for small ε, to construct a solution to (2.8) of the form

$$f = M + \sum_{n=1}^{6} \varepsilon^n f_n + \varepsilon^3 f_R. \tag{2.10}$$

Here $M \equiv M_{\rho,U,T}$ is the local Maxwellian with parameters $T = T(y)$, $\rho = \rho(y)$ and $U = (u(y), 0, w(y))$ given by the solution of the stationary hydrodynamical

equations

$$\frac{d}{dy}(\rho T) = 0 \tag{2.11}$$

$$\frac{d}{dy}(\eta(T)\frac{du}{dy}) + \rho F = 0 \tag{2.12}$$

$$\frac{d}{dy}(\eta(T)\frac{dw}{dy}) = 0 \tag{2.13}$$

$$\frac{d}{dy}(\kappa(T)\frac{dT}{dy}) + \eta(T)[(\frac{du}{dy})^2 + (\frac{dw}{dy})^2] = 0 \tag{2.14}$$

These equations are to be solved with no-slip boundary conditions $U(\pm 1) = U_\pm$ on the thermal walls at temperatures $T_\pm > 0$ and we fix $\int_{-1}^{1} \rho(y)dy = m$. The thermal conductivity $\kappa(T)$ and the viscosity coefficient $\eta(T)$ are strictly positive functions of the temperature, given by well known expressions for which we refer to [5].

We note that the transport coefficints are described by the term f_1 in (2.10), which contains the main contribution to the heat flow and momentum dissipation. Therefore they are of order ε at the microscopic level of the distribution function f, but they are of finite size on the Navier-Stokes time scale. They are responsible for the conversion of mechanical work into heat and of the transport of heat to the boundary. See Section 4 for more comments about this point.

The corrections f_n in (2.10) are the sum of three terms, $f_n = B_n + b_n^+ + b_n^-$, with B_n describing f in the bulk while $b_n^\pm$ give boundary layer corrections, sensibly different from 0 only near the boundary. The bulk terms B_n satisfy the following set of equations, which correspond to a sort of Hilbert expansion: for $n = 1, \ldots, 6$

$$v_y \frac{\partial B_{n-1}}{\partial y} + F\frac{\partial B_{n-2}}{\partial v_x} = \mathcal{L}B_n + \sum_{\substack{k,m\geq 1\\ k+m=n}} Q(B_k, B_m), \tag{2.15}$$

where $\mathcal{L}f$ is the linearized Boltzmann operator defined as

$$\mathcal{L}f \equiv 2Q(M, f), \tag{2.16}$$

and we put $B_0 = M$ and $B_{-1} = 0$.

The boundary layer terms are obtained by scaling back to microscopic coordinates around $y = \pm 1$. Setting $y' = \varepsilon^{-1}(y+1)$ and $y'' = \varepsilon^{-1}(1-y)$, with both y' and y'' varying in $[0, 2\varepsilon^{-1}]$, the boundary layer corrections near the wall $y = -1$, b_n^-, have to satisfy, for $n = 1, \ldots, 6$

$$\begin{aligned} v_y \frac{\partial}{\partial y'} b_n^- + F\frac{\partial}{\partial v_x} b_{n-2}^- = \mathcal{L}_- b_n^- + 2Q(\Delta M_-, b_{n-1}^-) + \\ \sum_{\substack{i,j\geq 1\\ i+j=n}} \Big[2Q(B_i, b_j^-) + Q(b_i^-, b_j^-) + Q(b_i^+, b_j^-)\Big], \end{aligned} \tag{2.17}$$

where we put $b_0^{+} = b_{-1}^{\pm} = 0$. Moreover, $M_\pm = M_{\rho(\pm 1),u(\pm 1),T(\pm 1)}$, $\mathcal{L}_\pm f = 2Q(M_\pm, f)$ and $\Delta M_\pm = \varepsilon^{-1}(M - M_\pm)$. The functions b_n^{+} satisfy an analogous set of equations near the boundary $y = 1$.

Equations (2.15) and (2.17) are linear, but coupled together in a complicated way by the boundary conditions which they have to satisfy. We will specify the boundary conditions later on but note here that the boundary layer corrections decay exponentially in the variables y' and y'', in consequence of Proposition 2.1 below. So their effect in the bulk is negligible, and this justifies the interpretation of the $b_n^{\pm}$'s as boundary layer terms.

A slightly different version of above expansion was introduced in [9] for the thermal layer problem. The one used here was introduced in [10]. Their solvability is related to the existence of regular solutions of the hydrodynamical equations and to the "dissipative" properties of the linearized Boltzmann operator. In particular, the boundary layer expansion (2.17) can be solved in terms of the solution of the linear Milne problem, discussed for example in [15].

The remainder

To complete the description of f we have to discuss the remainder f_R, which contains the non-linearities of the problem, although in a weaker form, because it is multiplied by a positive power of ε. f_R satisfies the equation

$$v_y \frac{\partial f_R}{\partial y} + \varepsilon F \frac{\partial f_R}{\partial v_x} = \frac{1}{\varepsilon}\mathcal{L} f_R + \mathcal{L}^1 f_R + \varepsilon^2 Q(f_R, f_R) + \varepsilon^3 A \tag{2.18}$$

with

$$\mathcal{L}^1 f_R = 2Q(\sum_{n=1}^{6} \varepsilon^{n-1} f_n, f_R) \tag{2.19}$$

and A given by

$$A = -[v_y \frac{\partial B_6}{\partial y} + \varepsilon F \frac{\partial f_6}{\partial v_x} + F \frac{\partial f_5}{\partial v_x}] + [2Q(\Delta M_+, b_6^{+}) + 2Q(\Delta M_-, b_6^{-})] + \sum_{\substack{6 \geq k,m \geq 1 \\ k+m \geq 7}} \varepsilon^{k+m-7} Q(f_k, f_m). \tag{2.20}$$

Boundary conditions

It is quite easy to satisfy (2.3) and (2.4) to zero order in ε, because M is already a Maxwellian and the temperature and velocity field were chosen to fit with the Maxwellians $\overline{M}_\pm$. Only the density has to be adjusted. Higher order terms are more involved. In fact the B_n satisfy (2.15) which do not involve boundary conditions. So they do not reduce to $\alpha_n^{\pm}\overline{M}_\pm$ on the boundary and one is forced to introduce

boundary layer corrections. The idea is that one introduces at one of the boundaries, say $y = 1$, the correction b_1^{+} to so that $B_1 + b_1^{+}$ is proportional to $\overline{M}_+$ for $v_y < 0$. The same has to be done at $y = -1$. This changes again f_1 at $y = 1$ by non Maxwellian terms. However, since b_1^{-} decays exponentially fast, the modification is exponentially small in ε^{-1}. Therefore we impose on the f_n the following boundary conditions:

$$f_n(-1, v) = \alpha_n^{-}\overline{M}_-(v) + \gamma_{n,\varepsilon}^{-}(v), \quad v_y > 0$$
$$f_n(1, v) = \alpha_n^{+}\overline{M}_+(v) + \gamma_{n,\varepsilon}^{+}(v), \quad v_y < 0 \tag{2.21}$$

with $\gamma_{n,\varepsilon}^{\pm}(v) = b_n^{\mp}(2\varepsilon^{-1}, v)$ exponentially small in ε^{-1} (see Proposition 2.1 below) and we fix

$$\alpha_n^{\pm} = \pm \int_{v_y ? 0} v_y f_n(\pm 1, v) dv \tag{2.22}$$

Finally, to fulfill (2.3) and (2.4) we impose the following conditions on f_R:

$$f_R(-1, v) = \alpha_R^{-}\overline{M}_-(v) - \sum_{n=1}^{6} \varepsilon^{n-3}\gamma_{n,\varepsilon}^{-} \quad v_y > 0 \tag{2.23}$$

$$f_R(1, v) = \alpha_R^{+}\overline{M}_+(v) - \sum_{n=1}^{6} \varepsilon^{n-3}\gamma_{n,\varepsilon}^{+} \quad v_y < 0 \tag{2.24}$$

The normalization condition (2.9) requires

$$\int_{-1}^{1} dy \langle f_n \rangle = 0, \qquad n = 1, \ldots, 6 \tag{2.25}$$

$$\int_{-1}^{1} dy \langle f_R \rangle = 0 \tag{2.26}$$

Results

The construction of the solution to the linear problems (2.15) and (2.17), with the boundary conditions (2.21) and the normalization conditions (2.25) is not straightforward, but the differences with the case discussed in [10] are minor and we refer to that paper for the proofs, see also [9] where a similar problem was considered for the case of the thermal layer. Here we summarize the properties of the f_n which are important in Proposition 2.1. To state them we define for any non negative integer r the norm

$$|f|_r = \sup_{y \in [-1,1]} \sup_{v \in \mathbb{R}^3} (1 + |v|)^r |f(y, v)| \tag{2.27}$$

and we put

$$q = \max\{|F|, |U_+ - U_-|, |T_+ - T_-|\}. \tag{2.28}$$

Proposition 2.1. *Let q be sufficiently small. Then there are unique smooth functions ρ, T, u and w satisfying (2.11)–(2.14), with derivatives of any order bounded by $O(q)$. Moreover it is possible to determine uniquely the functions B_n and $b_n^{\pm}$, $n = 1, \dots, 6$ satisfying (2.15) and (2.17) so that $f_n = B_n + b_n^+ + b_n^-$ verifies (2.25) and the condition:*

$$\langle v_y f_n \rangle = 0 \quad \text{for } y \in [-1, 1] \tag{2.29}$$

and satisfies (2.21). Furthermore, for any positive r there is a constant c such that:

$$|M^{-\frac{1}{2}} B_n|_r < cq \tag{2.30}$$

$$|M_{\pm}^{-\frac{1}{2}} b_n^{\pm}(\varepsilon^{-1}(1 \mp y)) \exp[-\sigma\varepsilon^{-1}(1 \mp y)]|_r < cq \tag{2.31}$$

for some constant $\sigma > 0$. Finally, the A in (2.18) satisfies

$$\langle A \rangle = 0 \quad \text{for } y \in [-1, 1] \tag{2.32}$$

and

$$|A \exp[-\frac{1}{2} p v^2]|_r < cq, \tag{2.33}$$

with $p = \sup_{y \in [-1,1]} T(y)$.

To complete our picture of the distribution function, it is necessary to get solutions of the error equation (2.18) with the boundary conditions (2.23), (2.24) and the normalization condition (2.26). By (2.29) and (2.6) we have the extra condition

$$\langle v_y f_R \rangle = 0 \quad \text{for } y \in [-1, 1], \tag{2.34}$$

Our main result is the following theorem, where we use the norm:

$$|f|_{r,\theta} = \sup_{y \in [-1,1]} \sup_{v \in R^3} (1 + |v|)^r \exp[\theta v^2] |f(y, v)|. \tag{2.35}$$

Theorem 2.2. *There are positive constants ε_0, θ_0 and q_0 such that, if $\varepsilon < \varepsilon_0$ and $q < q_0$, there is a solution to the boundary value problem (2.18), (2.23), (2.24), (2.26) and (2.34) having the property that for any positive integer r there is a constant $c > 0$ such that*

$$|f_R|_{r,\theta} \le c\varepsilon^{\frac{3}{2}} |A|_{r,\theta} \tag{2.36}$$

for any $\theta < \theta_0$. Moreover the solution is unique in the class C_ε of functions f on $[-1, 1] \times R^3$ such that $\varepsilon^{\zeta} |f|_{r,\theta}$ is bounded uniformly for small ε, positive $\theta < \theta_0$, $\zeta < \frac{1}{2}$ and $r \in Z$.

The proof of this theorem will be sketched in next section. Here we make a few remarks about uniqueness: Theorem (2.2) implies that there is a unique solution to

the Boltzmann equation in terms of a truncated expansion in ε, i.e. a solution with hydrodynamic behavior. But this is not enough to prove uniqueness of the solution to the boundary value problem (2.3)–(2.8). In fact, our result simply means that we have uniqueness in the class C'_ε of the functions f on $[-1, 1] \times R^3$ such that $\varepsilon^{(-3+\zeta)} |f - \varepsilon f_1 - \varepsilon^2 f_2|_{r,\theta}$ is bounded uniformly for small ε, positive $\theta < \theta_0$, $\zeta < \frac{1}{2}$ and $r \in Z$. We do not expect to be able to get uniqueness in a wider class with the present methods.

We also note that the uniqueness in C_ε does not exclude the possibility of solutions of the Boltzmann equation (2.1) which depend also on the space coordinates x and z. The estimates we have at the moment are not sufficient for that. We expect however to be able to prove uniqueness in a class similar to C_ε, with full three dimensional space dependence allowed. This is work in progress.

3. Outline of the proof.

We first discuss how to satisfy the conditions on f_R. We can use the constants α_R^+ and α_R^- to satisfy conditions (2.26) and (2.34) on the reimander f_R. Namely, integrating (2.18) with respect to v, then, by (2.32) and the fact that $\langle Q(f, g) \rangle = 0$ for any f and g, it follows that (2.34) is satisfied for any $y \in [-1, 1]$, once it is satisfied at one point, say $y = 1$. We can then use α_R^+ to fulfill (2.34) at $y = 1$ and α_R^- to satisfy (2.26). To be more explicit, we write f_R as

$$f_R = I(R)M + R \tag{3.1}$$

with

$$I(R) = -m^{-1} \int_{-1}^{1} dy \int_{R^3} dv R(y, v), \tag{3.2}$$

so that (2.26) is satisfied. Recalling that

$$\rho_{\pm}(T_{\pm}/2\pi)^{1/2} \overline{M}^{+}(v) = M(\pm 1, v), \tag{3.3}$$

we choose $\alpha_R^- = (T_-/2\pi)^{-1/2} \rho_-^{-1} I(R)$, so that the function R has to solve the following boundary value problem:

$$v_y \frac{\partial R}{\partial y} + \varepsilon F \frac{\partial R}{\partial v_x} = \frac{1}{\varepsilon} \mathcal{L} R + \mathcal{N} R + \varepsilon^2 \tilde{Q}(R, R) + \varepsilon^3 A, \tag{3.4}$$

$$R(-1, v) = \zeta^- \quad v_y > 0, \tag{3.5}$$

$$R(1, v) = \beta_R \overline{M}_+(v) + \zeta^+ \quad v_y < 0 \tag{3.6}$$

$$\langle v_y R \rangle = 0 \quad \text{for } y \in [-1, 1], \tag{3.7}$$

where the linear operator $\mathcal{N}R$ is given by

$$\mathcal{N}R = \mathcal{L}^1 R + I(R)\Big[\sum_{n=2}^{6} \varepsilon^{n-1}\mathcal{L}f_n + \mathcal{L}b_1 - \varepsilon F\frac{\partial M}{\partial v_x}\Big]. \tag{3.8}$$

The non linear term is given by

$$\tilde{Q}(R,R) = Q(R,R) + 2I(R)\mathcal{L}R. \tag{3.9}$$

and we have put $\zeta^{+} = -\sum_{n=1}^{6}\varepsilon^{n-3}\gamma^{\pm}_{n,\varepsilon}$ and $\beta_R = \alpha^{+}_{R} - \alpha^{-}_{R}$.

To get equation (3.4) we have used the fact that $Q(M,M) = 0$ and the relation (2.15) with $n = 1$. In this way there is no normalization condition on the function R. The quantity α^{-}_{R} represents both the outgoing flux of f_R in $y = -1$ and the integral of R over y and v. This is possible because the impermeability condition for f_R at $y = -1$ is automatically satisfied once it is satisfied at $y = 1$. Since M has vanishing mass flux in the direction y the constant β_R is determined so that R satisfies condition (3.7) at the point $y = 1$, i.e.

$$\beta_R = \int_{v_y>0} v_y R(1,v) + \int_{v_y<0} v_y \zeta^{+}. \tag{3.10}$$

In consequence of this R satisfies (3.7) for all $y \in [-1,1]$.

To construct the solution of (3.4)–(3.7), we first consider the following linear boundary value problem: given D on $[-1,1]\times \mathrm{R}^3$ and $\zeta^{\pm}$ on $\{v \in \mathrm{R}^3 s.t.\ v_y \ 7\ 0\}$, find R such that

$$v_y\frac{\partial R}{\partial y} + \varepsilon F\frac{\partial R}{\partial v_x} = \frac{1}{\varepsilon}\mathcal{L}R + \mathcal{N}R + \varepsilon^2 D, \tag{3.11}$$

with the conditions (3.5), (3.6) and (3.7) or, equivalently, (3.10). Once we get estimates on the solution of this linear problem, it will be easy to solve the nonlinear problem by simple Banach fixed point arguments.

The linear problem presents some extra difficulties with respect to the one considered in [10]. One of them is the presence of the β_R term in the boundary conditions. The other is related to the fact that with different temperatures it is no more true that the infimum of the temperatures is reached on the boundary. This is important because the terms $b^{\pm}_n$ decay in velocities according to $M^{1/2}_{\pm}$ (see equation (2.31)). Now there are $y \in [-1,1]$ such that $M_{+}(v)/M(y,v)$ is unbounded. To control this unboundedness in velocities we need as in [10] to divide the solution into high and low velocity parts and the decomposition has to be done more carefully to avoid introducing new, undesired divergent terms. We make this decomposition using mostly the same notation as in [10] to which we refer for more details.

The linear problem

Let $T_{*} > p = \sup_{y\in[-1,1]} T(y)$ and

$$M_{*} = (2\pi T_{*})^{-3/2}\exp[-v^2/2T_{*}]. \tag{3.12}$$

Then we have $M_{*} \geq cM$ for all (y,v) and some positive c. We look for a solution of Eq.(3.11) in the form

$$R = \sqrt{M}g + \sqrt{M_{*}}h \tag{3.13}$$

where the *low velocity* part g and the *high velocity* part h are defined as the solutions of the following system of coupled equations, whose structure justifies the names:

$$v_y\frac{\partial g}{\partial y} + \varepsilon F\frac{\partial g}{\partial v_x} + (\mu + \varepsilon F\mu')\hat{g} = \varepsilon^{-1}Lg + \varepsilon^{-1}\chi_\gamma\sigma^{-1}K_{*}h + N^1\hat{g} + \Lambda\hat{g}, \tag{3.14}$$

$$g(1,v) = \beta_g\overline{M}_{+}(v)M^{-1/2}(1,v), \qquad v_y < 0, \tag{3.15}$$

$$g(-1,v) = 0, \qquad v_y > 0. \tag{3.16}$$

$$v_y\frac{\partial h}{\partial y} + \varepsilon F\frac{\partial h}{\partial v_x} + \varepsilon F\mu'_{*}h + (\mu + \varepsilon F\mu')\sigma(\bar{g} + g_2) =$$

$$\varepsilon^{-1}(-\nu + \bar{\chi}_\gamma K_{*})h + N_{*}[\sigma(\bar{g} + g_2) + h] + \varepsilon[N^{(2)}_{*}\hat{g} + \Delta\Lambda\hat{g}] + \varepsilon^2 d, \tag{3.17}$$

$$h(1,v) = M_{*}^{-1/2}[\zeta^{+}(v) + \beta_h\overline{M}_{+}(v)], \qquad v_y < 0, \tag{3.18}$$

$$h(-1,v) = M_{*}^{-1/2}\zeta^{-}(v) \qquad v_y > 0. \tag{3.19}$$

We summarize the notation used in the above equations:
Let $\psi_\alpha = M^{\frac{1}{2}}\ \tilde{\psi}_i$, $i = 0,\dots,4$ with $\tilde{\psi}_i$ the collision invariants $1, v_x, v_y, v_z, v^2/2$, suitably normalized to make ψ_i, $i = 0,\dots,4$ an orthonormal set in $L_2(dv)$. We decompose any function g into a hydrodynamic part $\hat{g}+g_2$ and a non hydrodynamic part $\bar{g}$ such that

$$g = \hat{g} + g_2 + \bar{g}, \quad \text{with } g_2 = p_2(y)\psi_2,\ \hat{g} = \sum_{j\neq 2} p_j(y)\psi_j \tag{3.20}$$

The function $\chi_\gamma(v)$ is the characteristic function of the set $\{v \in \mathrm{R}^3 s.t. |v| \leq \gamma\}$ and $\bar{\chi}_\gamma = 1 - \chi_\gamma$ the complementary one.

The operators L and L_{*} are defined by

$$Lf = M^{-1/2}2Q(M, M^{1/2}\ f) = (-\nu + K)f \tag{3.21}$$

$$L_{*}f = M_{*}^{-1/2}2Q(M, M_{*}^{1/2}\ f) = (-\nu + K_{*})f \tag{3.22}$$

and the decompositions in terms of ν, K and K_{*} are the usual Grad decomposition into an unbounded multiplication part $-\nu$ and compact parts. We refer to [16,7,10] for their properties. We choose $\beta_g = \int_{v_y>0} dv v_y M^{1/2}g(1,v)$ and

$\beta_h = \int_{v_y>0} dv v_y M_*^{1/2} h(1,v) + \int_{v_y<0} dv v_y \zeta^+$ to make the y component of the mass flow of g and h through the boundary $y = 1$ vanish. This is no more true at $y \neq 1$ for g and h separately, but only for their combination (3.13). This is very convenient for dealing with such terms.

The rest of the notation is:

$$\mu = v_y \frac{1}{2} \partial_y \log M, \quad \mu' = \frac{1}{2} \partial_{v_r} \log M, \quad \mu'_* = \frac{1}{2} \partial_{v_r} \log M_*, \quad \sigma = \sqrt{\frac{M}{M_*}} \tag{3.23}$$

$$d = M_*^{-1/2} D, \quad \tilde{b}_n^{\pm} = b_n^{\pm} M_{\pm}^{-1/2} \tag{3.24}$$

$$N_* f = M_*^{-1/2} \mathcal{N}(M_*^{1/2} f) \tag{3.25}$$

$$N_*^{(2)} \hat{g} = 2M_*^{-1/2} \Big\{ Q\Big[\sum_{n=2}^{6} \varepsilon^{n-2} f_n, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})M) \Big] - 2F\mu' I(M^{1/2}\hat{g})M \Big\} \tag{3.26}$$

$$N^1 \hat{g} = 2M^{-1/2} \Big\{ Q[B_1, M^{1/2}\hat{g}] + Q\big[b_1^-, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})M)\big] \Big\} \tag{3.27}$$

$$\Lambda \hat{g} = M^{-1/2} 2Q\big[\tilde{b}_1^+ M^{1/2}, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})M)\big] \tag{3.28}$$

$$\Delta\Lambda \hat{g} = -M_*^{-1/2} 2Q\big[\tilde{b}_1^+ \Delta' M_+, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})M)\big] \tag{3.29}$$

and $\Delta' M_+ = \varepsilon^{-1}(M^{1/2} - M_+^{1/2})$.

The main difference with respect to the similar decomposition used in [10] is related to the term $\mathcal{N}(M^{1/2}\hat{g})$, and is due to the problem mentioned above of the speed of decay of $b_n^{\pm}$ for large velocities. We recall that in [10] we obtained an estimate for $\hat{g}$ which was ε^{-1} bigger than the estimate for the other terms and this forced us to put this term in the equation for g instead of moving it to the equation for h. Actually, the bad term is the one related to f_1 which has no extra factors ε. Hence, here we put the terms depending on $f_n, n \geq 2$, in the equation for h. To deal with the equation for g one has to consider the Maxwellian M with the true temperature, and we have a term $b_1^+ M^{-1/2}$ which may diverge for large velocities. Therefore, we retain in the equation for g only the term $\Lambda \hat{g}$, which involves the bounded term $\tilde{b}_1^+$: the rest, $\varepsilon\Delta\Lambda\hat{g}$ is put in the equation for h. This works because $\Delta\Lambda\hat{g}$ is uniformly bounded in ε by the exponential decay of b_1^+ and the regularity of the solution of the hydrodynamical equations.

We start with equations (3.14)–(3.16), considering h as a given function and try to get estimates on g in terms of h. The norm we use is:

$$||f|| = \Big(\int_{[-1,1]\times R^3} dy\, dv (1 + |v|) f^2(y,v) \Big)^{\frac{1}{2}} \tag{3.30}$$

This problem is not the usual boundary value problem, with prescribed incoming flux on the boundary, because β_g depends on the solution itself. It can be reduced to the usual one, but this requires some care. Since the problem is linear, we can write the solution g as $g = g^{(B)} + a g^{(b.c.)}$ where the "bulk part" $g^{(B)}$ solves (3.14)–(3.16) with $\beta_g = 0$ and $g^{(b.c.)}$ solves the same problem, but with $K_* h = 0$ and the corresponding $\beta_g = \rho^{1/2}(T/2\pi)^{1/4}$, which means $g^{(b.c.)}(1,v) = \overline{M}_+(v)^{1/2}$ for $v_y < 0$. The solutions to a suitable integral version of these two problems exist by standard compactness arguments (see for example [15], and we can use the constant a to satisfy $\langle v_y g M^{1/2} \rangle = 0$ at $y = 1$, or, equivalently, $\beta_g = \int_{v_y>0} dv v_y M^{1/2} g(1,v)$. To do this one has to check that

$$\langle v_y g^{(b.c.)}(1,v) M^{1/2}(1,v) \rangle \neq 0. \tag{3.31}$$

The proof of (3.31), which requires most of the considerations necessary to estimate g, and of Proposition 3.1 below are given in the Appendix.

We summarize the estimates on g in the following proposition:

Proposition 3.1. *There exist positive constants ε_0, q_0 and $C_\gamma > 0$ such that, for $\varepsilon < \varepsilon_0$ and $q < q_0$ the solutions to Eq. (3.14)–(3.16) satisfy the bounds*

$$||\bar{g}|| \leq \varepsilon^{-1} C_\gamma ||h|| \tag{3.32}$$

$$||\hat{g}|| \leq \varepsilon^{-2} C_\gamma ||h|| \tag{3.33}$$

$$||g_2|| \leq C_\gamma ||h|| \tag{3.34}$$

In order to find a solution to the "high velocity" problem (3.17)–(3.19), for γ, the velocity cutoff, large enough one can use a simple contraction fixed point argument. We will only prove the estimate we need for h to get the bound for the solution of (3.11).

Equation (3.17) differs from equation (5.4) of [10] because of the presence of the term $N_*^{(2)}\hat{g}$ and $\Delta\Lambda\hat{g}$. More relevant is the difference between the boundary condition (3.18) and equation (5.11) of [10], which requires a more careful analysis. In fact β_h depend on the value of h at the point $y = 1$ and cannot be controlled immediately in terms of $||h||$ which depends on the integral on the variable y. To manage this part we have to use the integral representation already used in [10] to get pointwise estimates. Our result on h are summarized in the following proposition, whose proof is in the Appendix

Proposition 3.2. *Under the conditions of Proposition 3.1, if γ is large enough, there is $c > 0$ s.t.*

$$||h|| \le c\varepsilon^3 ||d(1 + |v|)^{-1}|| + c\varepsilon^{1/2}\{|h_-| + |h_+|\} \tag{3.35}$$

This bound, together with Proposition 3.1, implies:

$$||\bar{g}|| \le \varepsilon^2 c||(1 + |v|)^{-1} d|| + c\varepsilon^{-1/2}\{|h_-| + |h_+|\}, \tag{3.36}$$

$$||\hat{g}|| \le \varepsilon c||(1 + |v|)^{-1} d|| + c\varepsilon^{-3/2}\{|h_-| + |h_+|\}, \tag{3.37}$$

$$||g_2|| + ||h|| \le \varepsilon^3 c||(1 + |v|)^{-1} d|| + c\varepsilon^{1/2}\{|h_-| + |h_+|\}. \tag{3.38}$$

Once one has the L_2 estimates for h and g, pointwise estimates can be obtained as in Section 6 of [10], using again the estimate (A.25) for β_h and a similar one for β_g. This provides finally the estimates for the solution R of the linear problem (3.11) with conditions (3.5), (3.6) and (3.7):

$$|R|_{r,\theta} \le c\varepsilon^{\frac{1}{2}} |D|_{r-1,\theta} + c\varepsilon^{-2}[|\zeta^-|_{r,\theta} + |\zeta^+|_{r,\theta}]. \tag{3.39}$$

The nonlinear problem

The estimate (3.39) is all we need to deal with the non linear problem (3.4)-(3.7). We replace the boundary value problem (3.4)-(3.7) with

$$v_y \frac{\partial R_k}{\partial y} + \varepsilon F \frac{\partial R_k}{\partial v_x} = \frac{1}{\varepsilon}\mathcal{L}R_k + \mathcal{N}R_k + \varepsilon^2 \tilde{Q}(R_{k-1}, R_{k-1}) + \varepsilon^3 A, \tag{3.40}$$

$$R_k(-1, v) = \zeta^- \qquad v_y > 0, \tag{3.41}$$

$$R_k(1, v) = \beta_{R_k} \overline{M}_+(v) + \zeta^+ \qquad v_y < 0 \tag{3.42}$$

$$\langle v_y R_k \rangle = 0 \quad \text{for } y \in [-1, 1], \tag{3.43}$$

for $k \ge 1$ and $R_0 = 0$. Choose $D = \tilde{Q}(R_{k-1}, R_{k-1}) + \varepsilon A$. The inequality

$$|M^{-\frac{1}{2}} Q(f, g)|_{r-1} \le c|M^{-\frac{1}{2}} f|_r |M^{-\frac{1}{2}} g|_r \tag{3.44}$$

for any f and g (see [17]) and the estimate (3.39) imply

$$|R_k|_{r,\theta} \le c\varepsilon^{\frac{3}{2}} |A|_{r,\theta} + O(e^{-\frac{1}{\varepsilon}}) \tag{3.45}$$

uniformly in k for ε small enough. The convergence of the sequence is obtained by considering, for $k \ge 1$, $W_k = R_k - R_{k-1}$. The corresponding boundary value

problem, for $k \ge 2$ is:

$$v_y \frac{\partial W_k}{\partial y} + \varepsilon F \frac{\partial W_k}{\partial v_x} = \frac{1}{\varepsilon}\mathcal{L}W_k + \mathcal{N}W_k + \varepsilon^2 \tilde{Q}(R_{k-1} + R_{k-2}, W_{k-1}) + \varepsilon^3 A, \tag{3.46}$$

$$W_k(-1, v) = \zeta^- \qquad v_y > 0, \tag{3.47}$$

$$W_(1, v) = \beta_{W_k} \overline{M}_+(v) + \zeta^+ \qquad v_y < 0 \tag{3.48}$$

$$\langle v_y W_k \rangle = 0 \quad \text{for } y \in [-1, 1], \tag{3.49}$$

Putting $D = \tilde{Q}(R_{k-1} + R_{k-2}, W_{k-1})$ and using again (3.44) and (3.39), it follows that

$$|W_k|_{r,\theta} \le c\varepsilon^2 |W_{k-1}|_{r,\theta}, \tag{3.50}$$

and this implies the convergence if ε is small enough. To prove the uniqueness, let R_1 and R_2 be two solutions of (3.4)-(3.7) and $W = R_1 - R_2$. As above, we get:

$$|W|_{r,\theta} \le c\varepsilon^{\frac{1}{2}-\zeta} |W|_{r,\theta}. \tag{3.51}$$

Therefore, if $\zeta < 1/2$ we have uniqueness for ε small enough. This concludes the proof of Theorem 2.2.

4. Comments.

A few comments are in order, to conclude our discussion.

Boundary conditions.

The assumption of Maxwell boundary conditions has been used in this paper, as well as in [10] to simplify the proof, but we expect that with extra technical effort one can generalize our result to a wider class of boundary conditions, including those described in [5]. The fundamental assumption on the b.c. we need is that there is a unique distribution invariant w.r.t. them and it is a Maxwellian. In this way the non-slip b.c. for the hydrodynamical fields are guaranteed in the limit ε going to 0. For ε fixed there are slip corrections of order ε. The crucial point of our work is that the corresponding boundary layer corrections are of order ε too. Boundary layer corrections of order 1 would arise with more general slip boundary conditions. They would be out of control because the linear theory is not sufficient to deal with them and the nonlinear theory is not available to our knowledge.

Time dependent solutions

The stationary solutions to the rescaled Boltzmann equation are supposedly the limit, as t goes to infinity, of the time dependent solutions. Unfortunately, beyond the case of global equilibrium [18, 19], nothing is known about convergence to stationary solutions. Actually, even the existence of solutions globally in time is far from obvious. Since we want to deal with the hydrodynamical limit we have to consider also the limit as ε goes to zero and the order they have to be taken is a delicate question. In fact, if we scale space and time according to the Euler scaling and take the $\lim_{\varepsilon\to 0}$ before the $\lim_{t\to\infty}$, the latter one will not exist in general, because the hydrodynamical limit, on this scale, destroys the dissipative effects which drive the fluid to a stationary state. On the other hand, if we scale space and time according to the Navier-Stokes limit, $(x \to \varepsilon^{-1}x, \quad t \to \varepsilon^{-2}t)$ the limits are likely to be interchangeble. Therefore the right scaling to discuss the asymptotic behavior of the Boltzmann equation in the hydrodynamical limit is the Navier-Stokes one. On this scale the time dependent analog of eq. (2.2) is

$$\frac{\partial f}{\partial t} + \frac{1}{\varepsilon}\nabla_x f + \frac{1}{\varepsilon^2} G \cdot \nabla_v f = \frac{1}{\varepsilon^2} Q(f,f) \tag{4.1}$$

The first problem one should be able to solve is to get solutions of (4.1) with initial data near local equilibrium, bounded uniformly in ε at least for fixed times. This can be achieved at present only in special situations in which some kind of scaling invariance is recovered.

The most interesting case in which the above problem can be solved is the incompressible limit discussed in [20]. In that paper one scales G as ε^3 and the velocity field at time zero as ε, to guarantee that the velocity field at time t is still of order ε, restoring a scale invariant situation. In [20] only periodic boundary conditions are considered, but a combination of the method presented there and the ideas of this paper should allow us to extend the result to a slab with thermal walls, in the presence of an external force parallel to the walls. The analysis of the stationary behavior follows from the method employed to prove the results of this paper: the velocity field is of order ε with a quadratic profile. We note that in this case one expects the solution of the Boltzmann equation for fixed ε to converge to the global equilibrium as $t \to \infty$, but the interesting part of the solution, the correction of order ε is not under control.

The compressible case corresponds to assuming that $G = O(\varepsilon^2)$ and non conservative (as we do in this paper). Much less is known about the time dependent solutions in this case or even in simpler situations, c.f. [21]. The fact that the stationary solution we obtain is ruled by the stationary Navier-Stokes equation is an indication that the Navier-Stokes scaling is the right one to discuss the long time behavior in the hydrodynamical limit.

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APPENDIX

Proof of Proposition 3.1

The proof follows along the same lines as in [10], so we point out only the differences. We are going to use the following property of L:

$$\int_{-1}^{1} dy \int_{R^3} dv f(y,v) L f(y,v) \le -c_1 ||\bar{f}||^2 \tag{A.1}$$

for a suitable positive c_1. Note that in the r.h.s of (A.1) there is the non hydrodynamical part of f denoted by $\bar{f}$. To take advantage of (A.1), we multiply (3.14) by g and integrate on y and v. As in [10], we get

$$\varepsilon \mathcal{I} + c_1 ||\bar{g}||^2 \le C_\gamma ||h|| \, ||g|| + \varepsilon c q ||\hat{g}|| \, ||\bar{g}|| + \varepsilon^2 c q ||\hat{g}||^2 \tag{A.2}$$

where

$$\mathcal{I} = \frac{1}{2}[\langle v_y g^2(1,v)\rangle - \langle v_y g^2(-1,v)\rangle]. \tag{A.3}$$

and we have estimated the term containing $\Lambda\hat{g}$ as follows:

$$\begin{aligned} \Big| \int dy \int_{R^3} dv g \Lambda \hat{g} \Big| = \Big| \int dy \int_{R^3} dv \bar{g} \Lambda \hat{g} \Big| &\le ||\bar{g}|| \, ||(1+|v|)^{-1} \Lambda \hat{g}|| \\ &\le cq ||\bar{g}|| \, ||\hat{g}||. \end{aligned} \tag{A.4}$$

The first equality is due to the fact that $\langle \psi_\alpha Q(f,g)\rangle = 0$ for $\alpha = 0, \ldots, 4$ and any f and g. For the last step we use (see [22])

$$\int_{R^3} dv \frac{|Q(\sqrt{M}f, \sqrt{M}g)|^2}{(1+|v|)M} \le \int_{R^3} dv (1+|v|)|f|^2 \int_{R^3} dv (1+|v|)|g|^2, \tag{A.5}$$

and the bound (2.31), which assures that $||\tilde{b}_1^1|| \le cq$. We also note that the cancellation $\langle \mu \hat{g}^2 \rangle = 0$, crucial in [10], is still true. Namely, it relies on the fact that $\hat{g}$ is even in v_y (because the part g_2 is taken away), while μ is odd. This only depends on the fact that there is no hydrodynamical flow in the direction y i.e. $U(y)$ has no y-component. Thanks to this we have quadratic terms in $\hat{g}$ in (A.2) only of order ε^2; such a term with a lower power of ε would be uncontrollable with our method.

Another important step in [10] was the fact that $\mathcal{I}$ was a sum of two positive terms due to the ± 1 boundaries. This was a consequence of the fact that the outgoing flow was zero by symmetry on both walls. In the present case the outgoing flow is still zero on the lower plate $y = -1$, while, due to the presence of $\beta_g \neq 0$, a proof is required of the positivity of the contribution coming from $y = 1$. We have:

$$\langle v_y g^2(1,v)\rangle = \int_{v_y>0} dv v_y g^2(1,v) - \beta_g^2 \int_{v_y<0} dv |v_y| \overline{M}_+^2(v) M^{-1}(1,v). \tag{A.6}$$

By the Schwartz inequality,

$$\beta_g^2 = \Big[\int_{v_y>0} dv (v_y M(1,v))^{\frac{1}{2}} (v_y^{\frac{1}{2}} g(1,v))\Big]^2 \leq \int_{v_y>0} dv v_y g^2(1,v) \int_{v_y>0} dv v_y M(1,v) \tag{A.7}$$

and using the relation between $M(1,v)$ and $\overline{M}_+$ and the normalization of $\overline{M}_+$, we get $\langle v_y g^2(1,v)\rangle \geq 0$.

In particular, this means that $-\langle v_y g^2(-1,v)\rangle$, which is positive, is estimated by the r.h.s. of (A.2). Now we are in the same position as in [10] and, from this point on, the estimate of g follows the same lines, so we do not repeat it.

Proof of (3.31)

It can be shown directly, but a simpler proof is obtained by reduction to absurd. In fact, suppose the contrary. Then, since the term $K_\star h$ is put equal to zero, it follows that $\langle v_y g^{(b.c.)}(y,v) M^{1/2}(1,v)\rangle = 0$ for all $y \in [-1,1]$. Therefore, $\int_{v_y>0} dv v_y g^{(b.c.)}(1,v) = 1 = \int_{v_y<0} dv |v_y| g^{(b.c.)}(-1,v)$, because $\int_{v_y>0} dv |v_y| g^{(b.c.)}(-1,v) = 0$ and, by definition, $\int_{v_y<0} dv |v_y| g^{(b.c.)}(1,v) = 1$. Multiplying (3.14), written for $g^{(b.c.)}$, by $g^{(b.c.)}$ and integrating over y and v, for q and ε small, we get, with the same argument used to estimate g, that there is a positive c such that:

$$\int_{v_y>0} dv v_y (g^{(b.c.)})^2(1,v) - 1 + \int_{v_y<0} dv |v_y| (g^{(b.c.)})^2(-1,v) + c\|\tilde{g}^{(b.c.)}\|^2 \leq 0 \tag{A.8}$$

since $\int_{v_y<0} dv |v_y| (g^{(b.c.)})^2(1,v) = 1$. We have also $\int_{v_y>0} dv v_y (g^{(b.c.)})^2(1,v) = 1$. In fact, it is not bigger than 1 by (A.8). On the other hand, by the Schwartz inequality, as before,

$$1 = \int_{v_y<0} |v_y| g^{(b.c.)} \overline{M}_+ \leq \Big(\int_{v_y<0} |v_y| (g^{(b.c.)})^2\Big)^{1/2}. \tag{A.9}$$

Hence, by (A.8), $\int_{v_y<0} dv |v_y| (g^{(b.c.)})^2(-1,v) = 0$, so $g^{(b.c.)}(-1,v) = 0$ for all v. From (A.8) it also follows that $\|\tilde{g}^{(b.c.)}\| = 0$. Similarly one easily gets the vanishing of the hydrodynamical part and finally $g^{(b.c.)} = 0$. This contradicts the condition $g^{(b.c.)}(1,v) = \overline{M}_+(v)$ for $v_y < 0$ and concludes the argument.

Proof of Proposition 3.2

The arguments of [10] provide the bound

$$\|(1+|v|)^{-1} N_\star^{(2)} \hat{g}\| \leq cq\|\hat{g}\|. \tag{A.10}$$

The estimate of $\Delta\Lambda\hat{g}$ is slightly more involved. From (A.5), with $M = M_\star$, we get

$$\|(1+|v|)^{-1} \Delta\Lambda\hat{g}\| \leq \|\hat{g}\| \, \|M_\star^{-1/2} \tilde{b}_1^+ \Delta' M_+\|. \tag{A.11}$$

The bounds on the hydrodynamical fields allow us to estimate $M_\star^{-1/2} \Delta' M_+$. In fact we have

$$\begin{aligned} & M_\star^{-1/2} |\Delta M_+(y',v)| \\ & = \Big|\Big[\frac{\partial \log\rho}{\partial y} - \tilde{v}_x T^{-1} \frac{\partial u}{\partial y} - \frac{T^{-2}}{2}(\tilde{v}^2 - 3T)\frac{\partial T}{\partial y}\Big]\Big|_{y=y^\star} y''\Big| \exp\{-\lambda v^2\} \\ & \leq cqy'' \exp\{-\lambda v^2\}, \end{aligned} \tag{A.12}$$

with a suitable $y^\star \in [-1,y]$ and $\tilde{v}$ the vector $(v_x - u, v_y, v_z)$ and $\lambda = (4p^{-1} - (4T_\star)^{-1}$. Remember that $y'' = \varepsilon^{-1}(1-y)$, p is the sup of the temperatures and $T_\star > p$. The estimates (2.31) and above imply that

$$\|(1+|v|)^{-1} \Delta\Lambda\hat{g}\| \leq cq\|\hat{g}\|.$$

Now we come to the bound for β_h. We recall the notation used in [10]: consider the equation

$$v_y \frac{\partial f}{\partial y} + \varepsilon F \frac{\partial f}{\partial v_x} + \varepsilon^{-1} \nu \, f = \varepsilon^{-1} Z \tag{A.13}$$

with the boundary conditions

$$f(-1,v) = f_-, \ v_y > 0; \quad f(1,v) = f_+, \ v_y < 0. \tag{A.14}$$

Define

$$\Phi_{y,y'} = \int_{y'}^{y} dz \, \nu(z, v_x + \frac{\varepsilon F}{v_y}(z-y), v_y, v_z), \tag{A.15}$$

$$U_\varepsilon Z(y,v) = \frac{1}{\varepsilon v_y} \int_{-1}^{y} dy' Z(y', v_x + \frac{\varepsilon F}{v_y}(y'-y), v_y, v_z) \exp\big[-\frac{\Phi_{y,y'}}{\varepsilon v_y}\big], \tag{A.16}$$

for $v_y > 0$ and

$$U_\varepsilon Z(y,v) = -\frac{1}{\varepsilon v_y} \int_{y}^{1} dy' G(y', v_x + \frac{\varepsilon F}{v_y}(y'-y), v_y, v_z) \exp\big[\frac{\Phi_{y',y}}{\varepsilon v_y}\big]. \tag{A.17}$$

for $v_y < 0$. Moreover, put

$$V_\varepsilon^- f^- = \chi(v_y > 0) f^-(v_x + (y+1)\varepsilon F/v_y, v_y, v_z) \exp\left[-\frac{\Phi_{y,-1}}{\varepsilon v_y} \right] \tag{A.18}$$

$$V_\varepsilon^+ f^+ = \chi(v_y < 0) f^+(v_x + (y-1)\varepsilon F/v_y, v_y, v_z) \exp\left[\frac{\Phi_{1,y}}{\varepsilon v_y} \right] \tag{A.19}$$

The solution of Eq. (A.13), (A.14) can be written as

$$f = V_\varepsilon^+ f^+ + V_\varepsilon^- f^- + U_\varepsilon Z \tag{A.20}$$

We now write equation (3.17) for h as (A.13) with

$$\begin{aligned} Z = & -\varepsilon F \mu_*' h + (\mu + \varepsilon F \mu')\sigma(\bar{g} + g_2) + \varepsilon^{-1} \chi_\gamma K_* h \\ & + N_*[\sigma(\bar{g} + g_2) + h] + \varepsilon[N_*^{(2)} \hat{g} + \Delta\Lambda\hat{g}] + \varepsilon^2 d \end{aligned} \tag{A.21}$$

and the boundary conditions (3.18), (3.19).

Equation (A.20) allows to express β_h in terms of $U_\varepsilon Z$ and the restriction of $h(-1, v)$ to $v_y > 0$. We have the estimate

$$\begin{aligned} \left| \int_{v_y>0} v_y M_*^{1/2} h(1, v) \right| = & \left| \int_{v_y>0} v_y M_*^{1/2} \{ h_-(v_x + (y+1)\varepsilon F/v_y, v_y, v_z) \right. \\ & \left. \exp\left[-\frac{\Phi_{1,-1}}{\varepsilon v_y} \right] + U_\varepsilon Z(1, v) \} \right| \le |h_-| + \left| \int_{v_y>0} v_y M_*^{1/2} U_\varepsilon Z \right|, \end{aligned} \tag{A.22}$$

with $h_\pm = \zeta^\pm M_*^{-1/2}$ and $|h_\pm| = \sup_{v_y \gtrless 0} |h_\pm(v)|$.

By (A.20) and (A.17), using the Schwartz inequality and

$$\int_{-1}^{1} dy (\varepsilon v_y)^{-1} \nu \exp\{-(\varepsilon v_y)^{-1} \Phi_{1,y}\} < 1, \quad v_y > 0. \tag{A.23}$$

we get

$$\left| \int_{v_y>0} v_y M_*^{1/2} U_\varepsilon Z \right| \le \varepsilon^{-1/2} c \int_{v_y>0} |v_y|^{1/2} M_*^{1/2} \left[\int_{-1}^{1} dy \nu^{-1} Z^2 \right]^{1/2} \tag{A.24}$$

Finally, using again the Schwartz inequality and recalling the expression of β_h we get

$$\beta_h \le c[\varepsilon^{-1/2} \|\nu^{-1} Z\| + |h_-| + |h_+|]. \tag{A.25}$$

Define $\mathcal{J} = \langle v_y h^2(1, v) \rangle - \langle v_y h^2(-1, v) \rangle$. Using the boundary conditions for h we have

$$\int_{v_y<0} |v_y| h^2(-1, v) + \int_{v_y>0} |v_y| h^2(1, v) | \le \mathcal{J} + c[\beta_h^2 + |h_+|^2 + |h_-|^2]. \tag{A.26}$$

By the bound (A.25) on β_h we conclude that

$$\mathcal{J} \ge -c\left[\varepsilon^{-1} \|\nu^{-1} Z\| + |h_-|^2 + |h_+|^2\right]. \tag{A.27}$$

Using the last bound and following the same procedure as in [10] we can get Proposition 3.2

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