DAMTP 94-24
SW 9421

DAMTP/94-24
UGVA-DPT 1994/03-844

BICROSSPRODUCT STRUCTURE OF κ-POINCARE GROUP AND NON-COMMUTATIVE GEOMETRY

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March 1994

ABSTRACT We show that the κ-deformed Poincaré quantum algebra proposed for elementary particle physics has the structure of bicrossproduct $U(so(1,3))\bowtie T$. The algebra is a semidirect product of the classical Lorentz group $so(1,3)$ acting in a deformed way on the momentum sector T. The novel feature is that the coalgebra is also semidirect, with a backreaction of the momentum sector on the Lorentz rotations. Using this, we show that the κ-Poincare acts covariantly on a κ-Minkowski space, which we introduce. It turns out necessarily to be deformed and non-commutative. We also connect this algebra with a previous approach to Planck scale physics.

1 This is a note on the κ-Poincare algebra as introduced in [1][2] and studied extensively with a view to applications in elementary particle physics[3][4][5][6][7]. The idea behind this particular deformation, which is obtained by contraction[8], is that it is one of the weakest possible deformations of the usual Poincare group as a Hopf algebra. Hence it provides an ideal testing-ground for possible applications in particle physics. The momenta remain commutative

$$[P_\mu, P_\nu] = 0 \tag{1}$$

and the rotation part of the Lorentz sector is also not deformed. Because of the mildness of the deformation, many particle constructions and predictions can be obtained easily.

[1] SERC Fellow and Fellow of Pembroke College, Cambridge
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Here we want to argue that in spite of this success, any application of the κ-Poincare group to physics leads necessarily into non-commutative geometry. This is because until now it has not been possible to define an algebra of Minkowski space co-ordinates $\{x_\mu\}$ on which the κ-Poincare acts as a Hopf algebra. Recall that when usual groups act on algebras, one has

$$g{\triangleright}(ab) = (g{\triangleright}a)(g{\triangleright}b), \quad g{\triangleright}1 = 1 \tag{2}$$

and the natural analogue of this for Hopf algebras is

$$h{\triangleright}(ab) = (h_{(1)}{\triangleright}a)(h_{(2)}{\triangleright}b), \quad h{\triangleright}1 = \epsilon(h)1 \tag{3}$$

where $\Delta h = h_{(1)} \otimes h_{(2)} = \sum_i h_{(1)i} \otimes h_{(2)i}$ is the coproduct. Without such a covariant action, one cannot make any products of the space-time generators x_μ in a κ-Poincare invariant way. This affects not only the many-particle theory but any expressions involving, for example, x^2. It means that until now, the actual coproduct structure has only been applied in connection with momentum space and not spacetime itself . Since the coproduct of the κ-Poincare is non-cocommutative, one cannot expect that it acts on the usual commutative algebra of functions on Minkowski space: it needs to be non-commutative or 'quantum'.

Here we provide the correct notion of κ-Minkowski space and the action of κ-Poincare on it. We also understand the structure of the κ-Poincare as a deformation of the usual semidirect product structure. This then makes tractable the problem of representing it covariantly on the κ-Minkowski.

The abstract structure of the κ-Poincare turns out to be an example of a class of non-commutative non-cocommutative Hopf algebras (quantum groups) introduced some years ago by the first author in an algebraic approach to Planck-scale physics[9][10][11]. The context here was quite different, namely the Hopf algebra of observables of a quantum system rather than as a symmetry object. Thus we find in fact that the κ-Poincare algebra P_κ has *two* different physical interpretations, one as a quantum symmetry group and the other as a quantised phase space. Thus, we find

$$P_\kappa = U(so(1,3)){\bowtie\!\!\!\blacktriangleleft} T = U(so(1,3)){\bowtie\!\!\!\blacktriangleleft}\mathbb{C}(X) \tag{4}$$

where in the first picture T is the κ-deformed enveloping algebra of the momentum sector of the Poincare. In the second picture it is the algebra of functions of a classical but curved momentum part X of phase space. This second point of view is recalled briefly in the last section of this note.

2. The κ-Poincare algebra P_κ, (antihermitian generators of translations P_μ, rotations M_i and boosts $\bar{N}_i$; κ real; $i,j,k=1,2,3; \mu,\nu=0,1,2,3$) is [2] :

$$[P_\mu, P_\nu] = 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \tag{5}$$

$$[M_i, P_j] = \epsilon_{ijk} P_k, \quad [M_i, P_0] = 0, \tag{6}$$

$$[M_i, \bar{N}_j] = \epsilon_{ijk} \bar{N}_k, \tag{7}$$

$$[\bar{N}_i, P_0] = P_i, \quad [\bar{N}_i, P_j] = \delta_{ij} \kappa \sinh \frac{P_0}{\kappa}, \tag{8}$$

$$[\bar{N}_i, \bar{N}_j] = -\epsilon_{ijk}(M_k \cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k P \cdot M). \tag{9}$$

The coproducts are given by:

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta M_i = M_i \otimes 1 + 1 \otimes M_i, \tag{10}$$

$$\Delta P_i = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i, \tag{11}$$

$$\Delta \bar{N}_i = \bar{N}_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes \bar{N}_i + \frac{\epsilon_{ijk}}{2\kappa}(P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} M_j \otimes P_k). \tag{12}$$

The starting point of our structure theorem is the observation that P_κ contains $T = \{P_\mu\}$ as a sub-Hopf algebra and projects onto $U(so(1,3))$ also as a Hopf algebra map:

$$T \overset{i}{\hookrightarrow} P_\kappa \overset{\pi}{\rightarrow} U(so(1,3)). \tag{13}$$

The map π consists of setting $P_\mu = 0$ and mapping M_i and $\bar{N}_i$ to their classical counterparts in the Lorentz group. It is easy to see that i, π are classical counterparts as

$$\pi(M_i) = M_i, \quad \pi(\bar{N}_i) = N_i. \tag{14}$$

To this, we add now the maps

$$T \overset{p}{\leftarrow} P_\kappa \overset{j}{\hookleftarrow} U(so(1,3)), \quad \pi \circ j = \mathrm{id}, \quad p \circ i = \mathrm{id} \tag{15}$$

where j is an algebra homomorphism and p is a linear map which is a coalgebra homomorphism

$$(p \otimes p) \circ \Delta = \Delta \circ p, \quad \epsilon \circ p = \epsilon. \tag{16}$$

Moreover,

$$(id \otimes j) \circ \Delta = (\pi \otimes \mathrm{id}) \circ \Delta \circ j \tag{17}$$

which says that j intertwines the coaction of $U(so(1,3))$ on itself by Δ and its coaction on P_κ by $(\pi\otimes \mathrm{id})\circ\Delta$. Likewise

$$p(a)t = p(ai(t)), \quad a\in P_\kappa, \quad t\in T \tag{18}$$

which says that p intertwines the action of T on itself by right-multiplication, with its action on P_κ by i and multiplication in P_κ.

Indeed, we define

$$j(N_i)\equiv \mathcal{N}_i = \bar{N}_i e^{-\frac{P_0}{2\kappa}} - \frac{\epsilon_{ijk}}{2\kappa} M_j P_k e^{-\frac{P_0}{2\kappa}}, \quad j(M_i) = M_i \tag{19}$$

which one can show to be an algebra homomorphism. The new generators $\mathcal{N}_i$ have coproducts

$$\Delta\mathcal{N}_i = \mathcal{N}_i\otimes 1 + e^{-\frac{P_0}{\kappa}}\otimes\mathcal{N}_i + \frac{\epsilon_{ijk}}{\kappa} P_j e^{-\frac{P_0}{2\kappa}}\otimes M_k \tag{20}$$

after which (17) is clear. We also define p as the map that sets $M_i = \bar{N}_i = 0$ and the properties (16), (18) are then clear.

Now, the data (13)-(18) say precisely that P_κ is a Hopf algebra extension of $U(so(1,3))$ by T. The general theory of Hopf algebra extensions has been introduced in [11] [12] [13] (the latter two covered the general case) and one knows that such extensions are semidirect products. There is also the possibility of cocycles but these vanish when j is an algebra homomorphism and p a coalgebra one, as in our case. We deduce from this theory that (a) the classical Lorentz algebra acts on T from the right by

$$t{\triangleleft}h = j(Sh_{(1)})tj(h_{(2)}), \quad \forall t\in T, \quad h\in U(so(1,3)) \tag{21}$$

and (b) T coacts back on $U(so(1,3))$ from the left by

$$\beta(\pi(a)) = p(a_{(1)})Sp(a_{(3)})\otimes\pi(a_{(2)}), \quad \forall\pi(a)\in U(so(1,3)). \tag{22}$$

In both formulae S denotes the appropriate antipode while $\Delta^2 a = a_{(1)}\otimes a_{(2)}\otimes a_{(3)}$ in the second formula. In both cases, the formulae are not obviously well-defined, but $t{\triangleleft}h$ as stated necessarily lies in (the image under i of) T, while β does not depend on $a\in P_\kappa$ but only its image $\pi(a)$.

In our case we have

$$P_0{\triangleleft}M_i = 0, \quad P_i{\triangleleft}M_j = \epsilon_{ijk}P_k, \quad P_0{\triangleleft}N_i = -P_i e^{-\frac{P_0}{2\kappa}} \equiv -\mathcal{P}_i, \tag{23}$$

the generators $\mathcal{P}_i = P_i e^{-\frac{P_0}{2\kappa}}$ are quite natural here, and in terms of these the action becomes

$$\mathcal{P}_i{\triangleleft}M_j=\epsilon_{ijk}\mathcal{P}_k,\quad \mathcal{P}_i{\triangleleft}N_j=-\delta_{ij}(\frac{\kappa}{2}(1-e^{-\frac{2P_0}{\kappa}})+\frac{1}{2\kappa}\vec{\mathcal{P}}^2)+\frac{1}{\kappa}\mathcal{P}_i\mathcal{P}_j \tag{24}$$

as computed for other reasons in [14]. Our present point of view is not that this is the quantum adjoint action in P_κ but simply that the classical $U(so(1,3))$ acts on the Hopf algebra T in this way. Meanwhile, the coaction comes out as

$$\beta(M_i)=1\otimes M_i,\quad \beta(N_i)=e^{-\frac{P_0}{\kappa}}\otimes N_i+\frac{\epsilon_{ijk}}{\kappa}\mathcal{P}_j\otimes M_k \tag{25}$$

on the generators. Here β is not an algebra homomorphism but its values on products of generators can be computed too from (22).

Finally, the general extension theory says that our P_κ is built up in its structure from this data $(T,U(so(1,3)),\triangleleft,\beta)$. Namely, its algebra is a semidirect product defined abstractly by $i(T)$ and $j(U(so(1,3))$ as subalgebras and cross relations

$$i(t)j(h)=j(h_{(1)})i(t{\triangleleft}h_{(2)}),\quad \forall h\in U(so(1,3)),\quad t\in T. \tag{26}$$

Its coalgebra is defined in a dual way as

$$\Delta i(t)=i(t_{(1)})\otimes i(t_{(2)}),\quad \Delta j(h)=j(h_{(1)})(i\otimes j)\circ\beta(h_{(2)}). \tag{27}$$

In our case the cross relations become

$$[P_0,M_i]=P_0{\triangleleft}M_i,\quad [\mathcal{P}_i,M_j]=\mathcal{P}_i{\triangleleft}M_j,\quad [P_0,\mathcal{N}_i]=P_0{\triangleleft}N_i,\quad [\mathcal{P}_i,\mathcal{N}_j]=\mathcal{P}_i{\triangleleft}N_j \tag{28}$$

which, combined with i,j above being algebra homomorphisms, gives our κ-Poincare algebra as

$$[P_0,\mathcal{P}_i]=0,\quad [M_i,M_j]=\epsilon_{ijk}M_k,\quad [\mathcal{N}_i,\mathcal{N}_j]=-\epsilon_{ijk}M_k \tag{29}$$

$$[P_0,M_i]=0,\quad [\mathcal{P}_i,M_j]=\epsilon_{ijk}\mathcal{P}_k \tag{30}$$

$$[P_0,\mathcal{N}_i]=-\mathcal{P}_i,\quad [\mathcal{P}_i,\mathcal{N}_j]=-\delta_{ij}(\frac{\kappa}{2}(1-e^{-\frac{2P_0}{\kappa}})+\frac{1}{2\kappa}\vec{\mathcal{P}}^2)+\frac{1}{\kappa}\mathcal{P}_i\mathcal{P}_j \tag{31}$$

which is analagous to [14]. The coproducts become

$$\Delta\mathcal{N}_i=\mathcal{N}_i\otimes 1+e^{-\frac{P_0}{\kappa}}\otimes\mathcal{N}_i+\frac{\epsilon_{ijk}}{\kappa}\mathcal{P}_j\otimes M_k,\quad \Delta M_i=M_i\otimes 1+1\otimes M_i. \tag{32}$$

In terms of $\mathcal{P}_i$ the coproduct structure of T itself is

$$\Delta P_0=P_0\otimes 1+1\otimes P_0,\quad \Delta\mathcal{P}_i=\mathcal{P}_i\otimes 1+e^{-\frac{P_0}{\kappa}}\otimes\mathcal{P}_i. \tag{33}$$

Thus the new generators $\{P_0, \mathcal{P}_i, \mathcal{N}_i, M_i\}$ provide a natural description of P_κ as a Hopf algebra bicrossproduct $U(so(1,3)){\triangleright\!\!\!\blacktriangleleft} T$ according to the general construction introduced in [11]. The symbol ${\triangleright\!\!\!\blacktriangleleft}$ denotes that one factor acts on the other and the other coacts back on the first. Usually in the theory of groups and Hopf algebras one considers only an action or coaction, but it was argued in [11] that in physics actions tend to have 'reactions' and this turns out to be the case here when $\kappa < \infty$.

Indeed, in [9][15] one finds an example of the form $U(su(2)){\triangleright\!\!\!\blacktriangleleft} T$ where T is the Hopf algebra of functions on $\mathbb{R}^3$ with a deformed coproduct corresponding to curvature from the point of view there, and the action is a deformation of the usual rotations of $\mathbb{R}^3$. This was one of the first non-trivial non-commutative non-cocommutative Hopf algebras, though not as widely known as the celebrated Hopf algebras of Drinfeld and Jimbo. The P_κ is quite similar to this but deformed in the action of the boosts rather than of rotations.

3. We are now in a position to introduce a natural notion of κ-Minkowski space on which our P_κ acts covariantly. Indeed, since T is the enveloping algebra of translations, it is natural to take for κ-Minkowski its dual T^* which will also be an algebra and on which T necessarily acts covariantly as quantum vector fields. We then show that the whole of P_κ acts on it.

The structure of T^* is completely determined by the axioms of a Hopf algebra duality

$$< t, xy > = < t_{(1)}, x >< t_{(2)}, y >, \quad < ts, x > = < t, x_{(1)} >< s, x_{(2)} >, \quad \forall t, s \in T, \quad x, y \in T^* \tag{34}$$

Indeed, since T is the (commutative limit) of the borel subalgebra $U_q(b_-)$ of $U_q(su_2)$ and, as is well-known in that context, its dual is of the same form[16]. Thus, we take for T^* the generators x_μ and relations and coproduct

$$[x_i, x_j] = 0, \quad [x_i, x_0] = \frac{x_i}{\kappa}, \quad \Delta x_\mu = x_\mu \otimes 1 + 1 \otimes x_\mu. \tag{35}$$

For T we again prefer the generators $\mathcal{P}_i, P_0$ and then the duality pairing can be written compactly as

$$< f(\mathcal{P}_i, P_0), :\psi(x_i, x_0): > = f(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_0})\psi(0,0) \tag{36}$$

where $:\psi(x_i, x_0):$ denotes a function ψ of the generators *with all powers of* x_0 *to the right.* One can see [17] for the usefulness of this way of working with this kind of Hopf algebra. Apart from this ordering, we see that the pairing is completely along the classical lines of the pairing of the enveloping algebra of $\mathbb{R}^4$ with the Hopf algebra of functions on $\mathbb{R}^4$, which is by letting the translation generators act and evaluating at zero.

Now the canonical action of T on T^* is

$$t{\triangleright}x =< x_{(1)}, t > x_{(2)}, \quad \forall x \in T^*, \quad t \in T \tag{37}$$

which in our case works out as

$$\mathcal{P}_i{\triangleright} : \psi(x_i, x_0) :=: \frac{\partial}{\partial x_i}\psi(x_i, x_0) :, \quad P_0{\triangleright} : \psi(x_i, x_0) :=: \frac{\partial}{\partial x_0}\psi(x_i, x_0) : \tag{38}$$

i.e. by the classical way but remembering the Wick-ordering.

Next, $U(so(1,3)$ also acts on T^*. This is because it acts from the right on T and this action therefore dualises to an action from the left on T^*:

$$< t, h{\triangleright}x >=< t{\triangleleft}h, x >, \quad \forall t \in T, \quad h \in U(so(1,3)), \quad x \in T^* \tag{39}$$

which computes in our case as

$$M_i{\triangleright}x_j = \epsilon_{ijk}x_k, \quad M_i{\triangleright}x_0 = 0, \quad N_i{\triangleright}x_j = -\delta_{ij}x_0, \quad N_i{\triangleright}x_0 = -x_i. \tag{40}$$

It is not obvious, but the general theory of bicrossproduct Hopf algebras ensures that the canonical action of T on itself by multiplication and $U(so(1,3))$ by $\triangleleft$ generates an action of the semidirect product algebra P_κ on T. This therefore dualises to an action on T^* generated by the actions of these subalgebras. So $P_0, \mathcal{P}_i$ as above and $M_i, \mathcal{N}_i$ acting like M_i, N_i in (40). are a canonical representation of the P_κ on κ-Minkowski. Their extension to products of the spacetime co-ordinates is via the covariance condition (3) using the coproducts ΔM_i, $\Delta\mathcal{N}_i$ etc. from (32)–(33). Thus,

$$M_i{\triangleright}x_j = \epsilon_{ijk}x_k, \quad M_i{\triangleright}x_0 = 0, \quad \mathcal{N}_i{\triangleright}x_0 = -x_i, \quad \mathcal{N}_i{\triangleright}x_j = -\delta_{ij}x_0, \tag{41}$$

$$\mathcal{N}_i{\triangleright}(x_jx_0) = -\delta_{ij}x_0^2 - x_jx_i, \tag{42}$$

$$\mathcal{N}_i{\triangleright}(x_0x_j) = -\delta_{ij}x_0^2 - x_ix_j + \frac{1}{\kappa}\delta_{ij}x_0, \tag{43}$$

$$\mathcal{N}_i{\triangleright}(x_jx_k) = -\delta_{ij}x_0x_k - \delta_{ik}x_jx_0 + \frac{1}{\kappa}(\delta_{ik}x_j - \delta_{jk}x_i), \tag{44}$$

$$\mathcal{N}_i{\triangleright}(x_0^2) = -x_ix_0 - x_0x_i + \frac{1}{\kappa}x_i. \tag{45}$$

The Lorentz-invariant metric turns out as

$$x_0^2 - \vec{x}^2 + \frac{3}{\kappa}x_0 \tag{46}$$

This covariant action of P_κ on κ-Minkowski space T^* is our main result of this section. It appears to be rather non-trivial to verify it directly. Note that covariance is always true for T on T^* and since T is a subhopf algebra of P_κ, it remains true as its translation sector. The classical boosts do not act covariantly on T^* but their coproduct is different in P_κ due to the coaction β. This modification of the coproduct is just what is needed for the construction to work. The proof is straightforward using the abstract Hopf algebra theory of Section 2.

We therefore have the correct basis for wave-functions ψ on κ-Minkowski space and can proceed with various constructions, retaining at all time covariance under P_κ. This will be explored elsewhere.

4. Our structure theorem for the P_κ has many other consequences for the theory. The first of them is that the theory of bicrossproducts is completely symmetric under the process of taking duals (reversing the roles of products and coproducts). This remarkable 'input-output' symmetry was the main physical motivation for the introduction of the bicrossproduct construction in [11][9][11] and several other papers by the first author.

Thus we can compute the function algebra dual to P_κ at once. It is the bicrossproduct

$$\mathbb{C}(SO(1,3)) \hookrightarrow T^* \blacktriangleright\!\!\triangleleft \mathbb{C}(SO(1,3)) \rightarrow T^* \tag{47}$$

where $\mathbb{C}(SO(1,3))$ is the usual commutative algebra of functions on the Lorentz group, and T^* is our algebra of functions on κ-Minkowski. Thus, this Hopf algebra is a deformation of the algebra of functions on the Poincare group. The maps and action/coaction for this dual construction are given in [11] by dualising the above $\beta, \triangleleft$ respectively according to

$$< h, x \triangleright \Lambda > = < \beta(h), x \otimes \Lambda >, \quad \forall\, h \in U(so(1,3)),\ x \in T^*,\ \Lambda \in \mathbb{C}(SO(1,3)) \tag{48}$$

$$< t \otimes h, \beta(x) > = < t \triangleleft h, x >, \quad \forall t \in T,\ x \in T^*,\ h \in U(so(1,3)). \tag{49}$$

The resulting κ-Poincare function Hopf algebra will be developed in detail elsewhere. It can perhaps be compared with a κ-Poincare Hopf algebra proposed in another context in [18][19]. In our approach it necessarily comes with a duality pairing with P_κ given by (36), the usual pairing between $\mathbb{C}(SO(1,3))$ and $U(so(1,3))$, and the trivial pairing (provided by the counits) between translation and Lorentz sectors.

We conclude with some remarks about the physical interpretation of bicrossproducts in [9] as quantum systems. Returning to our enveloping algebra P_κ we can develop quite a different physical picture. Namely, we think of T not as the enveloping algebra of deformed translations but as the perfectly classical Hopf algebra of *functions* on a classical nonAbelian group X,

$$T = \mathbb{C}(X) \tag{50}$$

where X is the group given by exponentiating the Lie algebra Ξ defined by

$$[x_i, x_0] = \frac{x_i}{\kappa}, \quad [x_i, x_j] = 0. \tag{51}$$

These are just the relations of T^* in Section 3 but we think of them no longer as generating the co-ordinates of some non-commutative space but as generating a Lie algebra. It is easy to exponentiate the Lie algebra to a group X described as a subset of $\mathbb{R}^4$ with a κ-deformed (non-Abelian) addition law. In other words, κ controls now the curvature of our space X. We take this X as the position space (configuration space) of a quantum system.

Next, the Lie algebra Ξ and the Lie algebra $so(1,3)$ fit together to form a 'matched pair' of Lie algebras. The concept (due to the first author in [11][9]) is that each Lie algebra acts on the other in a matching way. In our case $so(1,3)$ acts by $\triangleright$, say, on Ξ via usual infinitesimal Lorentz transformation and Ξ acts back from the right by dualising β from (25) according to the formula

$$\xi \triangleleft x_\mu = < \beta(\xi), x_\mu \otimes \mathrm{id} >, \quad \forall \xi \in so(1,3),\ x_\mu \in \Xi \tag{52}$$

remember that the output of β has its first tensor factor in T, which we evaluate against the generators x_i, x_0 using the pairing (36). The two actions fit together as required for a right-left matched pair:

$$\xi \triangleright [x_\mu, x_\nu] = [\xi \triangleright x_\mu, x_\nu] + [x_\mu, \xi \triangleright x_\nu] + (\xi \triangleleft x_\mu) \triangleright x_\nu - (\xi \triangleleft x_\nu) \triangleright x_\mu \tag{53}$$

$$[\xi, \eta] \triangleleft x_\mu = [\xi \triangleleft x_\mu, \eta] + [\xi, \eta \triangleleft x_\mu] + \xi \triangleleft (\eta \triangleright x_\mu) - \eta \triangleleft (\xi \triangleright x_\mu) \tag{54}$$

for all $\xi, \eta \in so(1,3)$. In our case, we can compute $\triangleleft$ explicitly as

$$M_i \triangleleft x_0 = 0, \quad M_i \triangleleft x_j = 0, \quad N_i \triangleleft x_0 = -\frac{1}{\kappa} N_i, \quad N_i \triangleleft x_j = \frac{1}{\kappa} \epsilon_{ijk} M_k \tag{55}$$

and verify (53)-(54) directly for these Lie algebra representations $\triangleright, \triangleleft$. The N_i, M_i here are the classical $so(1,3)$ generators .

The theory of such Lie algebras acting one eachother in such a way is a rich one[11] and tells us among other things that there is a Lie algebra double semidirect sum $\Xi{\triangleright\!\!\triangleleft} so(1,3)$ containing $\Xi, so(1,3)$ and cross relations

$$[\xi, x_\mu] = \xi{\triangleright} x_\mu + \xi{\triangleleft} x_\mu. \tag{56}$$

Moreover, there are theorems that, at least locally, the Lie algebra matched pair exponentiates into a Lie group matched pair $X, SO(1,3)$ acting on each other in a suitable way. The procedure and general formulae (which are non-linear) have been introduced in [12]. There is also a double cross product group $X{\bowtie}SO(1,3)$, at least locally.

Now, the action of $SO(1,3)$ on X has orbits. Consider particles constrained to move on such orbits. The position obervables are $\mathbb{C}(X)$, the momentum observables are the Lie algebra $so(1,3)$ since its elements generate the flows. The natural quantisation of particles on such homogeneous spaces according to the standard Mackey scheme[20][21] is the cross product algebra $U(so(1,3)){\triangleright\!\!\!<}\mathbb{C}(X)$. This can be made precise using the theory of C^*-algebras. The point is that this cross product contains the algebra of $so(1,3)$ and $\mathbb{C}(X)$, with cross-relations which are the natural covariant form of Heisenberg's commutation relations. Our P_κ is this quantum algebra of observables.

Moreover, the dual of the bicrossproduct is also a bicrossproduct: it is the quantisation of particles moving on the homogeneous spaces which are the orbits in $SO(1,3)$ under the action of X, i.e. precisely with the roles of position and momentum reversed. Thus models of this class, demonstrated here by P_κ exhibit a quantum version of Born reciprocity and are interesting for this reason[9][11]. Moreover, this structure generally forces the action to be deformed, often with event-horizon-like singularities. For example, it was shown in [9] that the extensions of $\mathbb{C}(\mathbb{R} \times \mathbb{R})$ (the classical phase-space in one-dimensions) of this bicrossproduct type had just two free parameters, which we identified heuristically as $\hbar$ and G, the gravitational coupling constant. This work was perhaps one of the first serious attempts to apply Hopf algebras and non-commutative geometry to Planck scale physics, and it is interesting that P_κ has an interpretation in these terms as well as a symmetry in particle physics. This picture of the κ-Poincare algebra will be developed in detail elsewhere.

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